THE HEWITT REALCOMPACTIFICATION OF PRODUCTS

BY

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ABSTRACT. For a completely regular Hausdorff space $X$, $vX$ denotes the Hewitt realcompactification of $X$. Given a topological property $\mathcal{P}$ of spaces, our interest is in characterizing the class $\mathcal{R}(\mathcal{P})$ of all spaces $X$ such that $v(X \times Y) = vX \times vY$ holds for each $\mathcal{P}$-space $Y$. In the present paper, we obtain such characterizations in the case that $\mathcal{P}$ is locally compact and in the case that $\mathcal{P}$ is metrizable.

Introduction. All spaces considered in this paper are assumed to be completely regular Hausdorff and all maps are continuous. The Hewitt realcompactification $vX$ of a space $X$ is the unique realcompactification of $X$ to which each real-valued continuous function on $X$ admits a continuous extension. For details of Hewitt realcompactifications, the reader is referred to [8]. An important problem in the theory concerns when the relation $v(X \times Y) = vX \times vY$ is valid. Following [23], [30], we denote by $\mathcal{R}$ (resp. $\mathcal{R}(\mathcal{P})$) the class of all spaces $X$ such that $v(X \times Y) = vX \times vY$ holds for each space $Y$ (resp. each $\mathcal{P}$-space $Y$), where $\mathcal{P}$ is a given property of spaces. It is known that: (Comfort [4], [5]) a locally compact, realcompact space of nonmeasurable cardinal belongs to $\mathcal{R}$; (Hušek [12], [14] and McArthur [23]) every member of $\mathcal{R}$ is realcompact; (Hušek [13], [14]) every member of $\mathcal{R}$ is of nonmeasurable cardinal; [28] every member of $\mathcal{R}$ is locally compact. These facts characterize $\mathcal{R}$ as precisely the class of locally compact, realcompact spaces of nonmeasurable cardinals. Further, in [30], the author has tried to characterize $\mathcal{R}(\mathcal{P})$ for various properties $\mathcal{P}$ of spaces, and has proved that $\mathcal{R} = \mathcal{R}$ (metacompact) $= \mathcal{R}$ (subparacompact). It is the purpose of this paper to continue our study along this line, in particular, the following results are established:

(A) Both $\mathcal{R}$ (locally compact) and $\mathcal{R}$ (Moore) coincide with the class of all spaces of nonmeasurable cardinals whose Hewitt realcompactifications are locally compact.

(B) The class $\mathcal{R}$ (metrizable) consists precisely of all weak cb*-spaces, in the sense of Isiwata [20], of nonmeasurable cardinals.

In §1, we present a technical theorem which is useful in finding a pair $X$, $Y$ of spaces for which $v(X \times Y) = vX \times vY$ fails. Our result (A) is then proved. We also give a positive answer to the following question of Hušek [13, p. 326]: Do there...
exist minimal cardinals $m$, $n$ for which $|X| = m$, $|Y| = n$ and $\nu(X \times Y) \neq \nu X \times \nu Y$. In §2, we prove the analogue for $\nu X$ of the corollary to Glicksberg's theorem [9, Theorem 1]: For onto maps $f_i: X_i \to Y_i$ ($i = 1, 2$), $\beta(Y_1 \times Y_2) = \beta Y_1 \times \beta Y_2$ holds whenever $\beta(X_1 \times X_2) = \beta X_1 \times \beta X_2$, where $\beta X$ is the Stone-Čech compactification of $X$. It is shown that some additional conditions must be imposed in order that the analogous "$\nu$" theorem holds. In §3, we apply our theory to prove (B), and also show that $\mathcal{R}$ (locally compact, metrizable) is precisely the class of all spaces of nonmeasurable cardinals. When studying the relation $\nu(X \times T) = \nu X \times \nu T$ with a metrizable factor $T$, the central issue is the weak $cb^*$ property in another factor $X$. It is proved that, in case $X$ satisfies the countable chain condition and $T$ is metrizable, the relation holds if and only if (i) either $X$ or $T$ is of nonmeasurable cardinal and (ii) either $X$ is a weak $cb^*$-space or $T$ is locally compact. Finally a number of problems are posed in §4.

Throughout the paper, $m$ and $n$ denote cardinal numbers, and $m^+$ denotes the smallest cardinal greater than $m$. We let $w$, $d$, $c$ and $\chi$ denote the following cardinal functions: weight, density, cellularity and character (cf. [7]). $|A|$ denotes the cardinality of a set $A$, and $m_1$ stands for the first measurable cardinal. Since $m_1$ (if it exists) is greater than any nonmeasurable cardinal, that $|A|$ is nonmeasurable is denoted by $|A| < m_1$. We also denote by $C(X)$ the set of real-valued continuous functions on a space $X$. For general terminology, see [7] and [8].

1. Characterizations of $\mathcal{R}$ (locally compact) and $\mathcal{R}$ (Moore). Two subsets $A$ and $B$ of a space $X$ are said to be completely separated in $X$ if there is $f \in C(X)$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. A family $\{F_a\}$ of subsets of a space $X$ is called expandable if there is a locally finite family $\{G_a\}$ of open sets in $X$ with $F_a \subseteq G_a$ for each $a$. We introduce a new class of expandable families.

1.1. Definition. A family $\{F_a | a \in A\}$ of subsets of a space $X$ is $D(m)$-expandable if there exists a locally finite family $\{G_a | a \in A\}$ of open sets in $X$ with $F_a \subseteq G_a$ for each $a \in A$ and each $F_a$ is the union of at most $m$ subsets each of which is completely separated from $X - G_a$.

If $n > m$, then a $D(m)$-expandable family is $D(n)$-expandable. As a space is completely regular Hausdorff, every expandable family in $X$ is $D(|A'|)$-expandable, and a uniformly locally finite family defined in [17] is $D(1)$-expandable (cf. [27]). Recall from [17] that a space $X$ is pseudo-$m$-compact if each locally finite family of nonempty open sets in $X$ has cardinality less than $m$. Pseudocompact spaces are known to be precisely pseudo-$\aleph_0$-compact spaces. The following theorem plays an essential role in our discussions.

1.2. Theorem. Let $X \times Y$ be $C$-embedded in $X \times \nu Y$. If there exists a $D(m)$-expandable family $\mathcal{F}$ in $Y$, with $|\mathcal{F}| = n$, such that $\bigcap \{\text{cl}_\nu F | F \in \mathcal{F}\} \neq \emptyset$, then each point $x \in X$, with $\chi(x, X) < n$, has a pseudo-$m$-compact neighborhood.

Proof. Suppose on the contrary that there exists a point $x_0 \in X$, with $\chi(x_0, X) < n$, which has no pseudo-$m$-compact neighborhood. Let $\{G_\lambda | \lambda \in \Lambda\}$ be a neighborhood base at $x_0$ in $X$ with $|\Lambda| = n$. Then, for each $\lambda \in \Lambda$, $\text{cl}_X G_\lambda$ is not pseudo-$m$-compact, and thus there is a locally finite family $\{G_{\mu, \lambda} | \mu \in M_\lambda\}$ of
nonempty open sets in $\text{cl}_X G_\lambda$ with $|M_\lambda| = m$. Setting $G_\mu = G_\mu \cap G_\lambda$ for each
$\mu \in M_\lambda$, we have a locally finite family $\{G_\mu|\mu \in M_\lambda\}$ of nonempty open sets
in $X$. It can be assumed without loss of generality that $x_0 \notin \bigcup \{G_\mu|\mu \in M_\lambda\}$. For each
$\mu \in M_\lambda$, pick $x_\mu \in G_\mu$, and choose $f_\mu \in C(X)$ such that $f_\mu(x_\mu) = 0$ and
$f_\mu(X - G_\mu) = \{1\}$. On the other hand, since $|\mathbb{F}| = n$, we may write $\mathbb{F} = 
\{F_\lambda|\lambda \in \Lambda\}$. Then there is a locally finite family $\{H_\lambda|\lambda \in \Lambda\}$ of open sets
in $Y$ with $F_\lambda \subset H_\lambda$ for each $\lambda \in \Lambda$. Each $F_\lambda$ is a union of $m$ subsets each of
which is completely separated from $Y - H_\lambda$, and so we express it by $F_\lambda = 
\bigcup \{F_\mu|\mu \in M_\lambda\}$, i.e., there is $g_\mu \in C(Y)$ such that $g_\mu(F_\mu) = \{0\}$ and
$g_\mu(Y - H_\lambda) = \{1\}$. For each $\lambda \in \Lambda$ and each $\mu \in M_\lambda$, let us set

$$J_\lambda = \{x_\mu \times F_\mu, \quad K_\mu = G_\mu \times H_\lambda,$$

$$h_\mu((x,y)) = \min\{1, f_\mu(x) + g_\mu(y)\}, \quad (x,y) \in X \times Y.$$

Then $h_\mu \in C(X \times Y)$, $h_\mu(J_\mu) = \{0\}$ and $h_\mu((X \times Y) - K_\mu) = \{1\}$. It is easily
checked that $\mathbb{K} = \{K_\mu|\mu \in M_\lambda, \lambda \in \Lambda\}$ is locally finite in $X \times Y$. Therefore if
we define a function $h$ on $X \times Y$ by

$$h(p) = \inf\{h_\mu(p)|\mu \in M_\lambda, \lambda \in \Lambda\}, p \in X \times Y,$$

then $h$ is continuous. Let us choose $y_0 \in \bigcap \{\text{cl}_Y F_\lambda|\lambda \in \Lambda\}$; then $y_0 \notin \nu Y - Y$,
because $\mathbb{F}$ is locally finite in $Y$. Now we show that $h$ admits no continuous
extension to the point $p_0 = (x_0, y_0) \in X \times \nu Y$. Let $U \times V$ be a given basic
neighborhood of $p_0$ in $X \times \nu Y$. There is $\lambda \in \Lambda$ with $G_\lambda \subset U$, and $V \cap F_\mu = \emptyset$
for some $\mu \in M_\lambda$. Choose $y \in V \cap F_\mu$. Then both $p_1 = (x_\mu, y)$ and $p_2 = (x_\nu, y)$
belong to $U \times V$ and $h(p_1) = 0$, while $h(p_2) = 1$. This shows that $h$ does not
extend continuously to $p_0$, which contradicts the assumption that $X \times Y$ is C-em-
bedded in $X \times \nu Y$. Hence the proof is complete.

1.3. Corollary. Let $X \times Y$ be C-embedded in $X \times \nu Y$. If there exists a locally
finite family $\mathbb{K}$ of nonempty open sets in $Y$, with $|\mathbb{K}| = n$, such that

$$\bigcap \{\text{cl}_Y H|H \in \mathbb{K}\} \neq \emptyset,$$

then each point $x \in X$, with $\chi(x, X) < n$, has a pseudo-c(Y)-compact neighborhood.

Proof. Let $\mathbb{K} = \{H_\lambda|\lambda \in \Lambda\}$, and choose $y_0 \in \bigcap \{\text{cl}_Y H_\lambda|\lambda \in \Lambda\}$. For each
$\lambda \in \Lambda$, by Zorn's lemma, there is a maximal disjoint family $\mathbb{F}_\lambda$ of nonempty open
sets in $H_\lambda$ such that each $F \in \mathbb{F}_\lambda$ is completely separated from $Y - H_\lambda$. Let us set
$F_\lambda = \bigcup \{F|F \in \mathbb{F}_\lambda\}$. For each $\lambda \in \Lambda$, the maximality of $\mathbb{F}_\lambda$ implies that $y_0 \in$
$\text{cl}_Y F_\lambda$. Since $|\mathbb{F}_\lambda| < c(Y)$, $\{F_\lambda|\lambda \in \Lambda\}$ is a $D(c(Y))$-expandable family in $Y$, with
$|\Lambda| = n$, such that $\bigcap \{\text{cl}_Y F_\lambda|\lambda \in \Lambda\} \neq \emptyset$. Thus the corollary follows from Theo-
rem 1.2.

1.4. Remark. Let us say that a family $\mathcal{F}$ of subsets of a space $X$ converges to
$x \in X$ if each neighborhood of $x$ contains some member of $\mathcal{F}$, and that a subspace
$S$ of $X$ is relatively pseudo-$m$-compact in $X$ if each locally finite family $\mathcal{Q}_U$ of
nonempty open sets in $X$ such that $S \cap U \neq \emptyset$ for each $U \in \mathcal{Q}_U$ has cardinality
less than $m$. The conclusion of Theorem 1.2 (resp. Corollary 1.3) can be generalized...
as follows: Each convergent family $\mathcal{F}$ of subsets of $X$, with $|\mathcal{F}| < n$, has a member which is relatively pseudo-$m$-compact (resp. relatively pseudo-$\mathfrak{c}(Y)$-compact) in $X$.

Our next work is to construct spaces $Y$ which have a $D(\mathfrak{N}_0)$-expandable family $\mathcal{F}$ such that $\cap \{cl_y F | F \in \mathcal{F}\} \neq \emptyset$. A space is called 0-dimensional if it has a base consisting of open-and-closed sets. For an ordinal $\alpha$, we denote by $W(\alpha)$ the set of all ordinals less than $\alpha$ topologized with the order topology, and by $\omega_0$ (resp. $\omega_1$) the first infinite (resp. first uncountable) ordinal.

1.5. **Fact.** For every infinite cardinal $n$, there exists a 0-dimensional locally compact space $Y = Y_1(n)$, with $|Y| = w(Y) = n \cdot \omega_1$, that has a $D(\mathfrak{N}_0)$-expandable family $\mathcal{F}$ such that $|\mathcal{F}| = n$ and $\cap \{cl_y F | F \in \mathcal{F}\} \neq \emptyset$.

**Proof.** Let $T_1 = W(\omega_1 + 1) \times W(\omega_0 + 1)$, and let $T_2 = \Lambda \cup \{\infty\}$ be the one point compactification of a discrete space $\Lambda$ of cardinality $n$. We denote a base for the topology on $T_i$ by $\mathcal{B}_i$, for $i = 1, 2$. Let $E = \{(\omega_1, \beta) | \beta < \omega_0\}$, and let $Z'$ be the quotient space obtained from $R = T_1 \times T_2$ by collapsing the set $E \times T_2$ to a point $z(\beta) \in Z'$ for each $(\omega_1, \beta) \in E$. Let $\phi': R \to Z'$ be the quotient map. Let $Z_0$ be the set $Z'$, retopologized by letting $\cup \{\mathcal{B}(B) | B \in \mathcal{B}_1\}$ be a base, where

$\mathcal{B}(B) = \begin{cases} \{\phi'(B \times T_2)\} & \text{if } B \cap E \neq \emptyset, \\ \{\phi'(B \times B') | B' \in \mathcal{B}_2\} & \text{if } B \cap E = \emptyset. \end{cases}$

Then the natural map $\phi: R \to Z_0$ is continuous, and hence $Z_0$ is compact. Let us set

$Z = Z_0 - \phi((\{y, \omega_0\}, \infty) | y < \omega_1)).$

The space $Z$ is a 0-dimensional locally compact space with $|Z| = w(Z) = n \cdot \omega_1$. Since $z(\beta)$ is a $P$-point for each $\beta < \omega_0$, it is easily checked that $Z$ is $C$-embedded in $Z \cup \{z_0\}$, where $z_0 = z(\omega_0)$, and so $z_0 \in \nu Z - Z$ by [8, 8.6]. Setting $D_\lambda = \phi((\{y, \omega_0\}, \lambda) | y < \omega_1))$ for each $\lambda \in \Lambda$, we have a discrete family $\{D_\lambda | \lambda \in \Lambda\}$ of closed subsets in $Z$ such that $z_0 \in \cup \{cl_z D_\lambda | \lambda \in \Lambda\} \neq \emptyset$. Define a subspace $Y$ of the product space $Z \times W(\omega_0 + 1)$ by

$Y = (Z \times \{\omega_0\}) \cup (\cup \{D_\lambda \times W(\omega_0 + 1) | \lambda \in \Lambda\}).$

Then $Y$ is a 0-dimensional locally compact space with $|Y| = w(Y) = n \cdot \omega_1$, because $Y$ is a closed subspace of $Z \times W(\omega_0 + 1)$. It remains to show the existence of a $D(\mathfrak{N}_0)$-expandable family in $Y$ satisfying the stated conditions. Since $Z \times \{\omega_0\}$ is $C$-embedded in $Y$, it follows from [8, 8.10(a)] that $\nu Z = \nu(Z \times \{\omega_0\}) \subset \nu Y$, and hence we may consider $z_0$ as an element of $\nu Y - Y$. Setting $F_\lambda = D_\lambda \times W(\omega_0)$ for each $\lambda \in \Lambda$, we have a discrete family $\mathcal{F} = \{F_\lambda | \lambda \in \Lambda\}$ of open sets in $Y$ such that $z_0 \in \cap \{cl_y F_\lambda | \lambda \in \Lambda\}$. Then, since each $F_\lambda$ is a union of countably many open-and-closed subsets in $Y$, $\mathcal{F}$ is a $D(\mathfrak{N}_0)$-expandable family in $Y$. Hence $Y$ is proved to be the desired space.

1.6. **Fact.** For every infinite cardinal $n$, there exists a 0-dimensional Moore space $Y = Y_2(n)$, with $|Y| = w(Y) = n \cdot \exp \mathfrak{N}_0$ that has a $D(\mathfrak{N}_0)$-expandable family $\mathcal{F}$ such that $|\mathcal{F}| = n$ and $\cap \{cl_y F | F \in \mathcal{F}\} \neq \emptyset$. 

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Proof. In [30], for every infinite cardinal $n$, we constructed a 0-dimensional Moore space $Z = Z(n)$, with $|Z| = w(Z) = n \cdot \exp N_0$, that has a discrete family $\mathcal{D}$ of closed subsets such that $|\mathcal{D}| = n$ and $\cap \{\text{cl}_vD | D \in \mathcal{D}\} \neq \emptyset$. The desired space $Y_2(n)$ can be made from $Z(n)$ by the same procedure as in the proof of 1.4.

For later use, we quote a theorem due to Hušek [13]:

1.7. Theorem (Hušek). Let $Q$ be a discrete space. Then $v(P \times Q) = vP \times vQ$ holds if and only if either $|P| < m_1$ or $|Q| < m_1$ (i.e., either $|P|$ or $|Q|$ is nonmeasurable).

We are now in a position to prove main theorems of this section. For the notion of locally pseudocompact spaces see [5]. We remark that the assumption $|X| < m_1$ of Theorem 1.8 is useful only for the implications (a) $\rightarrow$ (b) and (a) $\rightarrow$ (c).

1.8. Theorem. The following conditions on a space $X$ with $|X| < m_1$ are equivalent:

(a) $X$ is locally pseudocompact.

(b) $X \times Y$ is $C$-embedded in $X \times vY$ for each 0-dimensional locally compact space $Y$ with $w(Y) < \chi(X) \cdot \aleph_1$.

(c) $X \times Y$ is $C$-embedded in $X \times vY$ for each 0-dimensional Moore space $Y$ with $w(Y) < \chi(X) \cdot \exp N_0$.

Proof. We proved in [28] that if $X$ is a locally pseudocompact space of nonmeasurable cardinal, then $X \times Y$ is $C$-embedded in $X \times vY$ for each $k$-space $Y$. Since both locally compact spaces and Moore spaces are $k$-spaces, (a) $\rightarrow$ (b) and (a) $\rightarrow$ (c) follow from this result. To prove (b) $\rightarrow$ (a) ((c) $\rightarrow$ (a)) suppose on the contrary that $X$ is not locally pseudocompact at $x_0 \in X$. Let $Y$ be the space $Y_1(n)$ ($Y_2(n)$) constructed in 1.5 (1.6), where $n = \chi(x_0, X)$. Then it follows from Theorem 1.2, that $X \times Y$ is not $C$-embedded in $X \times vY$. This contradiction completes the proof.

1.9. Theorem. The following conditions on a space $X$ are equivalent:

(a) $vX$ is locally compact and $|X| < m_1$.

(b) $v(X \times Y) = vX \times vY$ holds for each 0-dimensional locally compact space $Y$ with $w(Y) < \chi(vX) \cdot \aleph_1$.

(c) $v(X \times Y) = vX \times vY$ holds for each 0-dimensional Moore space $Y$ with $w(Y) < \chi(vX) \cdot \exp N_0$.

Proof. Since (a) $\rightarrow$ (b) and (a) $\rightarrow$ (c) follow from Hušek [14, Corollary (a), p. 177] (cf. also [28]), we prove only (b) $\rightarrow$ (a) and (c) $\rightarrow$ (a). By Theorem 1.8, $vX$ is locally pseudocompact, and hence it is locally compact, because every pseudocompact realcompact space is compact [7, 3.11.1]. Suppose that $|X| > m_1$; then $\chi(vX) > m_1$ by [21, Theorem 2]. If we take for $Y$ a discrete space of cardinality $m_1$, then it follows from Theorem 1.7 that $v(X \times Y) \neq vX \times vY$. This contradicts (b) and (c) simultaneously. Hence the proof is complete.

In [14], Hušek proved that if $X$ satisfies 1.9(a), then $v(X \times Y) = vX \times vY$ holds for each $k$-space $Y$. Therefore Theorem 1.9 tells us that both $\mathcal{R}$ (locally compact)
and \( \beta(X) \) coincide with the class of spaces \( X \) such that \( vX \) is locally compact and \( |X| < \aleph_1 \).

1.10. Remarks. (1) Let \( \Psi \) be the space described in \([8, 51, p. 79]\); \( \Psi \) is constructed as follows: Let \( \mathcal{D} \) be a maximal infinite almost-disjoint family of infinite subsets of the set \( N \) of integers. Then \( |\mathcal{D}| = \exp N \). The space \( \Psi \) is the union of \( N \) with a new set \( D = \{ \omega E | E \in \mathcal{D} \} \) of distinct points endowed with the following topology: Each point of \( N \) is isolated, and a neighborhood of \( \omega E \) is any set containing \( \omega E \) and all but a finite number of points of \( E \). It is well known that \( \Psi \) is a 0-dimensional pseudocompact (and hence \( \beta \Psi = v \Psi \) by \([8, 8A4, p. 125]\)) locally compact Moore space. In \([25]\), Mrówka showed that \( \mathcal{D} \) can be chosen so that \( \beta \Psi \) is the one point compactification. Then, dividing \( D \) into a disjoint family of countable infinite subsets, we have a discrete family \( \mathcal{D} \) of closed subsets in \( \Psi \) such that \( |\mathcal{D}| = \exp N \) and \( \cap \{ \text{cl}_v D | D' \in \mathcal{D} \} \neq \emptyset \). Thus, by the same method as in the proof of 1.5, we can make a 0-dimensional locally compact Moore space \( Y \), with \( w(Y) = \exp N \), that has a \( D(N) \)-expansible family \( \mathcal{T} \) such that \( |\mathcal{T}| = \exp N \) and \( \cap \{ \text{cl}_v F | F \in \mathcal{T} \} \neq \emptyset \). This fact combined with Theorem 1.2 implies that the following condition (d) is also equivalent to 1.9(a) under the assumption that \( \chi(X) < \exp N \).

(d) \( v(X \times Y) = vX \times vY \) holds for each 0-dimensional locally compact Moore space \( Y \) with \( w(Y) < \chi(X) \times \exp N \).

We do not know whether (d) implies 1.9(a) or not in general.

(2) We can apply our theory to answer the following question of Hušek \([13, p. 326]\): Do there exist spaces \( X, Y \) of cardinalities \( \aleph_0 \) and \( \aleph_1 \), respectively, such that \( v(X \times Y) \neq vX \times vY \)? Let \( X \) be the space of rational numbers with the usual topology. We take for \( Y \) the space \( y, (N) \) constructed in 1.4. Then \( |X| = w(X) = \aleph_0 \) and \( |Y| = w(Y) = \aleph_1 \). Since \( X \) is not locally pseudocompact, it follows from Theorem 1.2 that \( v(X \times Y) \neq vX \times vY \).

2. Mapping theorems. In this section, we give mapping theorems which will be used in the next section. As is well known, for a map \( f: X \rightarrow Y \), there exists a continuous extension \( \tilde{v}f: vX \rightarrow vY \) of \( f \) \([8, 8.7]\). If \( f_i: X_i \rightarrow Y_i \) is a map for \( i = 1, 2 \), then the product map \( f = f_1 \times f_2 \) from \( X_1 \times X_2 \) to \( Y_1 \times Y_2 \) is defined by \( f((x_1, x_2)) = (f_1(x_1), f_2(x_2)) \) for \((x_1, x_2) \in X_1 \times X_2 \).

2.1. Theorem. Let \( f_i: X_i \rightarrow Y_i \) (\( i = 1, 2 \)) be onto maps. If \( \tilde{v}f_1 \times \tilde{v}f_2 \) is a quotient map from \( vX_1 \times vX_2 \) onto \( vY_1 \times vY_2 \), then \( v(X_1 \times X_2) = vX_1 \times vX_2 \) implies \( v(Y_1 \times Y_2) = vY_1 \times vY_2 \).

More precisely, we have the following theorem:

2.2. Theorem. Let \( F_i: X_i^* \rightarrow Y_i^* \) (\( i = 1, 2 \)) be onto maps such that \( F = F_1 \times F_2 \) is a quotient map, and let \( X_i \) (resp. \( Y_i = F_i(X_i) \)) be a dense C-embedded subspace of \( X_i^* \) (resp. \( Y_i^* \)). If \( X_1 \times X_2 \) is C-embedded in \( X_1^* \times X_2^* \), then \( Y_1 \times Y_2 \) is C-embedded in \( Y_1^* \times Y_2^* \).

Proof. Let us set \( f_i = F_i|X_i \) (\( i = 1, 2 \)) and \( f = f_1 \times f_2 \). To show that \( Y_1 \times Y_2 \) is C-embedded in \( Y_1^* \times Y_2^* \), let \( g \in C(Y_1 \times Y_2) \). Since \( h = g \circ f \in C(X_1 \times X_2) \), by
our assumption, there exists \( H \in C(X_1^* \times X_2^*) \) such that \( H|(X_1 \times X_2) = h \). We shall show that \((\ast)\) \( H \) takes on the constant value \( t_p \) on \( F^{-1}(p) \) for each \( p \in Y_1^* \times Y_2^* \). Let \( x \in X_1 \); then \( h(x, \cdot) = g(f_1(x), \cdot) \circ f_2 \), where \( h(x, \cdot) = h((x) \times X_2) \). Since \( g(f_1(x), \cdot) \in C(Y_2) \), it has a continuous extension \( G_x \) over \( Y_2^* \). Then, \( X_2 \) being dense in \( X_2^* \), \( H(x, \cdot) = G_x \circ f_2 \). Hence it follows that \( H(x, \cdot) = \) constant on \( \{x\} \times F_2^{-1}(y) \) for each \( y \in Y_2^* \). This implies that \( H \) is constant on \( f_1^{-1}(y_1) \times F_2^{-1}(y_2) \) for each \( (y_1, y_2) \in Y_1 \times Y_2^* \). Similarly, \( H \) is constant on \( f_2^{-1}(y_2) \times F_2^{-1}(y_2) \) for each \( (y_1, y_2) \in Y_1^* \times Y_2^* \). To see \((\ast)\), let \( p = (y_1, y_2) \in Y_1^* \times Y_2^* \). Then it follows from these facts that

\[
H(x, \cdot) = H(x', \cdot) \quad \text{for each } x, x' \in F_1^{-1}(y_1),
\]
\[
H(\cdot, x) = H(\cdot, x') \quad \text{for each } x, x' \in F_2^{-1}(y_2),
\]

and from which \((\ast)\) is proved. Define a function \( G \) on \( Y_1^* \times Y_2^* \) by \( G(p) = t_p \) for each \( p \in Y_1^* \times Y_2^* \). Then \( H = G \circ F \) and \( G|(Y_1 \times Y_2) = g \). Since \( F \) is a quotient map and \( H \) is continuous, it follows from [7, 2.4.2] that \( G \) is continuous. Hence our proof is complete.

Theorem 2.2 remains true if “C-embedded” is replaced by “C*-embedded”. Ishii proved in [18] that if \( f: X \to Y \) is an open perfect onto map, then so is \( vf \). This leads to the following corollary of Theorem 2.1.

2.3. Corollary. If \( f_i: X_i \to Y_i \) is an open perfect map onto \( Y_i \) for \( i = 1, 2 \), then \( \nu(X_1 \times X_2) = \nu X_1 \times \nu X_2 \) implies \( \nu(Y_1 \times Y_2) = \nu Y_1 \times \nu Y_2 \).

The following theorem shows that in Theorem 2.1 the assumption that \( vf_1 \times vf_2 \) is quotient onto cannot be dropped, even when \( f_1 \) is an identity and \( f_2 \) is a perfect map. Recall from [13] that a space \( X \) is pseudo-\( m_1 \)-compact if the cardinality of each locally finite family of nonempty open sets in \( X \) is nonmeasurable.

2.4. Theorem. Among the following conditions on a space \( X \), \( (a) \to (b) \to (c) \) is valid. Conversely, \( (c) \to (a) \) holds if \( |X| < m_1 \).

\((a)\) \( \nu X \) is locally compact.

\((b)\) For each space \( Y \) satisfying \( \nu(X \times Y) = \nu X \times \nu Y \) and each quotient image \( Z \) of \( Y \), \( \nu(X \times Z) = \nu X \times \nu Z \) holds.

\((c)\) As in \((b)\), with “perfect” instead of “quotient”.

Proof. \((a) \to (b)\). Let \( Y \) be a space satisfying \( \nu(X \times Y) = \nu X \times \nu Y \), and let \( Z \) be the image of \( Y \) under a quotient map \( f \). Since \( \nu X \) is locally compact, by Whitehead’s theorem [7, 3.3.17], \( id_{\nu X} \times f \) is a quotient map, where \( id_{\nu X} \) is the identity map of \( \nu X \). It follows from Theorem 2.2 that \( X \times Z \) is C-embedded in \( \nu X \times Z \). Husek proved in [13] that if \( P \) is a locally compact, realcompact space, then \( \nu(P \times Q) = \nu P \times \nu Q \) if and only if either \( |P| < m_1 \) or \( Q \) is pseudo-\( m_1 \)-compact. If we apply this theorem to our case, then \( |\nu X| < m_1 \) or \( Y \) is pseudo-\( m_1 \)-compact. If \( Y \) is pseudo-\( m_1 \)-compact, so is \( Z \). Hence it follows that \( \nu(\nu X \times Z) = \nu X \times \nu Z \). Thus we have \( \nu(X \times Z) = \nu X \times \nu Z \).

\((b) \to (c)\). Obvious.

\((c) \to (a)\). Suppose that \( |X| < m_1 \) and \( \nu X \) is not locally compact at \( x_0 \in \nu X \).

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Then, by [7, 3.11.1], \( x_0 \) has no pseudocompact neighborhood in \( vX \). Let \( n = \max\{\nu X', \chi(x_0, vX')\} \); then \( n < m_1 \). Let \( \omega_\alpha \) be the initial ordinal of \( n^+ \), and let \( T = W(\omega_\alpha + 1) \times W(\omega_0 + 1) \). Let \( \Lambda \) be a discrete space of cardinality \( n \), and let \( S_0 \) be the quotient space obtained from \( R_0 = T \times \Lambda \) by collapsing the set \( \{ (\omega_\alpha, \beta) \} \times \Lambda \) to a point \( s(\beta) \) for each \( \beta \in E \), where \( E = \{ 2n | n < \omega_0 \} \cup \{ \omega_0 \} \). Let \( g: R_0 \to S_0 \) be the quotient map. Let us set \( S = S_0 - \{ s_0 \} \), where \( s_0 = s(\omega_0) \), and let \( R = R_0 - g^{-1}(s_0) \). Then it is easily checked that \( \nu S = S_0 \) and \( \nu R = R_0 \). If we set

\[
G = g(\{ (\gamma, 2n) | \gamma < \omega_\alpha, n < \omega_0 \} \times \Lambda),
\]
then \( G \) is a cozero-set of \( S_0 \), and hence \( G = G^* \cap S_0 \) for some cozero-set \( G^* \) of \( \beta S_0 \) (= \( \beta S \)). Let us set \( Z = S \cup G^* \). We now need the following lemma:

**2.5. Lemma.** Let \( X \supseteq X_1 \supseteq X_2 \). Suppose that \( X_2 \) is dense in \( X \) and is \( C \)-embedded in \( X_1 \). Then, for each open set \( H \) of \( X \), \( X_2 \cup H \) is \( C \)-embedded in \( X \times H \).

The proof is left to the reader, since it requires only routine verification. We continue the proof of Theorem 2.4. By Lemma 2.5 and [7, 3.11.10], \( \nu Z = S_0 \cup G^* \), and hence \( s_0 \in \nu Z - Z \).

Setting

\[
F_\lambda = g(\{ (\gamma, 2n + 1) | \gamma < \omega_\alpha, n < \omega_0 \} \times \{ \lambda \})
\]
for each \( \lambda \in \Lambda \), we obtain a locally finite family \( \{ F_\lambda | \lambda \in \Lambda \} \) of open sets in \( Z \). Since each \( F_\lambda \) is a countable union of open-and-closed subsets of \( Z \), \( \{ F_\lambda | \lambda \in \Lambda \} \) is a \( D(\mathfrak{K}_0) \)-expandable family in \( Z \) such that \( \cap \{ c_{\nu X} F_\lambda | \lambda \in \Lambda \} \supseteq S_0 \). Since \( \chi(x_0, vX) < |\Lambda| \), it follows from Theorem 1.2 that \( \nu(X \times Z) \neq vX \times \nu Z \). For our end, it suffices to show that \( Z \) is the perfect image of a space \( Y \) satisfying \( \nu(X \times Y) = vX \times \nu Y \). There exists the extension \( \beta g: \beta R_0 \to \beta S_0 \) of \( g \). Let us set \( Y = R \cup H^* \), where \( H^* = (\beta g)^{-1}(G^*) \), and set \( f = (\beta g)|Y \). Then, since \( H^* \) is a cozero-set of \( \beta R_0 \) (= \( \beta R \)), \( \nu Y = R_0 \cup H^* \) by Lemma 2.5 and [7, 3.11.10]. Further it is easily checked that \( f \) is a perfect map from \( Y \) onto \( Z \) and \( Y \) is locally compact. Since \( |Y| < m_1 \), it follows from [5, Theorem 2.1] that \( X \times Y \) is \( C \)-embedded in \( vX \times Y \). It remains to show that \( vX \times Y \) is \( C \)-embedded in \( vX \times vY \). Since \( |\nu X| < n \), a similar argument to that of [8, 8.20] shows that \( vX \times W(\omega_\alpha) \times W(\omega_0 + 1) \) is \( C \)-embedded in \( vX \times W(\omega_\alpha + 1) \times W(\omega_0 + 1) \). Thus \( vX \times R \) is \( C \)-embedded in \( vX \times R_0 \). Since \( vX \times H^* \) is an open set of \( vX \times \beta Y \), it follows from Lemma 2.5 that

\[
(\nu X \times R) \cup (\nu X \times H^*) \quad (= vX \times Y)
\]
is \( C \)-embedded in

\[
(\nu X \times R_0) \cup (\nu X \times H^*) \quad (= vX \times vY).
\]

Hence the proof is complete.

**2.6. Remark.** In case \( |X| \geq m_1 \), (c) \( \rightarrow \) (a) of Theorem 2.4 need not be true. If \( D \) is a discrete space of cardinality \( m_1 \), then, by Theorem 1.6, \( D \) satisfies 2.4(c). But it is known [5, p. 115] that \( \nu D \) is not even a \( k \)-space.

The following corollary is proved by using Theorem 2.4 repeatedly.

**2.7. Corollary.** Let \( f_i: X_i \to Y_i \) (\( i = 1, 2 \)) be quotient onto maps. If both \( vX_1 \) and \( vX_2 \) are locally compact, then \( \nu(X_1 \times X_2) = vX_1 \times vX_2 \) implies \( \nu(Y_1 \times Y_2) = vY_1 \times vY_2 \).
3. Characterizations of $\beta(A)$ (metrizable). A space $X$ is called a weak cb*-space if for each decreasing sequence $\{F_n\mid n < \omega_0\}$ of regular closed sets in $X$ with empty intersection, $\bigcap \{\text{cl}_X F_n\mid n < \omega_0\} = \emptyset$ holds, where a regular closed set is the closure of an open set. This notion was introduced by Isiwata [20] as a common generalization of realcompact spaces and weak cb-spaces in the sense of Mack and Johnson [22]. Since normal countably paracompact spaces, extremally disconnected spaces [8, 1H, p. 22] and pseudocompact spaces (or more generally, $M'$-spaces in the sense of Isiwata [19]) are weak cb-spaces, they are weak cb*-spaces. In this section, we prove the following theorem:

3.1. Theorem. The following conditions on a space $X$ are equivalent:

(a) $X$ is a weak cb*-space and $|X| < m_1$.

(b) $v(X \times T) = vX \times vT$ holds for each metrizable space $T$.

(c) $v(X \times D(d(X)))^o = vX \times vD(d(X))^o$.

Here, $D(d(X))^o$ denotes the product of countably many copies of a discrete space of cardinality $d(X)$. Associated with each space $X$, there exist an extremally disconnected space $E(X)$ and a perfect irreducible map (i.e., a perfect map which takes proper closed subsets onto proper subsets) $e_X$ from $E(X)$ onto $X$. The space $E(X)$ is unique up to homeomorphism and is called the absolute of $X$ (cf. [16], [31]). To prove Theorem 3.1, we make use of the following lemmas. The next lemma follows immediately from [10, Theorem 2.4] and [11, Proposition 1.2]; the first part also appears in [15].

3.2. Lemma. A space $X$ is a weak cb*-space if and only if $vE(X) = E(vX)$ holds. Moreover, in case $vE(X) = E(vX)$, then $e_{vX}$ is the extension of $e_X$ over $vE(X)$.

3.3. Lemma [29]. Let $X$ be a space and $T$ a metrizable space. If either $X$ is extremally disconnected or $T$ is locally compact, then $X \times T$ is z-embedded in $\beta X \times T$ (i.e., each zero-set of $X \times T$ is the restriction to $X \times T$ of a zero-set of $\beta X \times T$).

The next lemma is a corollary of Blair [1, Theorem 7.6]:

3.4. Lemma (Blair). Let $X \times Y$ be z-embedded in $\beta X \times Y$. If either $|X| < m_1$ or $Y$ is pseudo-$m_1$-compact, then $v(X \times Y) = vX \times vY$ holds.

Proof of Theorem 3.1. (a) $\rightarrow$ (b). Let $X$ be a weak cb*-space with $|X| < m_1$ and $T$ a metrizable space. Since $|E(X)| < m_1$, it follows from Lemmas 3.3 and 3.4 that $v(E(X) \times T) = vE(X) \times vT$. By Lemma 3.2 $v e_X$ ($= e_{vX}$) is a perfect map from $vE(X)$ onto $vX$, and so $v e_X \times id_{vT}$ is perfect. Hence it follows from Theorem 2.1 that $v(X \times T) = vX \times vT$.

(b) $\rightarrow$ (c). Obvious.

(c) $\rightarrow$ (a). Suppose on the contrary that $X$ is not a weak cb*-space. Then there is a locally finite sequence $\{G_n\mid n < \omega_0\}$ of open sets in $X$ with $\bigcap \{\text{cl}_X G_n\mid n < \omega_0\} \neq \emptyset$. Since $c(X) < d(X)$, each point of $D(d(X))^o$ has no pseudo-$c(X)$-compact neighborhood, and $\chi(D(d(X))^o) = \aleph_0$. Hence it follows from Corollary 1.3 that $X \times D(d(X))^o$ is not C-embedded in $vX \times D(d(X))^o$. This contradicts (c).
prove that $|X| < m_1$, find a discrete family $\{G_\alpha|\alpha \in A\}$ of nonempty open sets in $D(d(X))^\omega$ with $|A| = d(X)$. Pick $t_\alpha \in G_\alpha$ for each $\alpha \in A$, and set $D = \{t_\alpha|\alpha \in A\}$. Then it is easily checked that $X \times D$ is $C$-embedded in $X \times D(d(X))^\omega$. Since $\nu D \subseteq \nu D(d(X))^\omega$, (c) implies $\nu(X \times D) = \nu X \times \nu D$. Hence it follows from Theorem 1.6 that $|X| < m_1$ or $|D| = d(X) < m_1$. If $|D| < m_1$, then $|X| < m_1$ by [8, 12.5]. Thus the proof is complete.

3.5. REMARKS. (1) If $E(X) \times T$ is $z$-embedded in $\beta E(X) \times T$, then it follows from [6, Proposition 5.1] and [2, Corollary 3.6] that $E(X) \times T$ is $C$-embedded in $\nu E(X) \times T$. Therefore the proof of Theorem 3.1 shows that, more generally, a space $X$ is a weak cb*-space if and only if $X \times T$ is $C$-embedded in $\nu X \times T$ for each metrizable space $T$.

(2) The product $X \times T$ of a weak cb*-space $X$ with a metrizable space $T$ need not be $z$-embedded in $\beta X \times T$. In fact, it was remarked in [29] that $\dim(X \times T) < \dim X + \dim T$ whenever $X \times T$ is $z$-embedded in $\beta X \times T$, while Wage showed in [32] that there exist a Lindelöf space (hence a weak cb*-space) $X$ and a metrizable space $T$ such that $\dim(X \times T) > \dim X + \dim T$.

(3) Lemmas 3.3 and 3.4 can be combined with Theorem 1.6 to yield the following result: $\nu(X \times T) = \nu X \times \nu T$ holds for each locally compact, metrizable space $T$ if and only if $|X| < m_1$.

We conclude this section with a theorem, which gives conditions on $X$ and $Y$ necessary and sufficient that the relation $\nu(X \times Y) = \nu X \times \nu Y$ be valid in a restrictive situation.

3.6. THEOREM. Let $X$ be a space satisfying the countable chain condition (i.e., $c(X) < \aleph_0$) and $T$ a metrizable space. Then $\nu(X \times T) = \nu X \times \nu T$ holds if and only if (i) either $|X| < m_1$ or $|T| < m_1$ and (ii) either $X$ is weak cb* or $T$ is locally compact.

PROOF. Necessity: (i) is proved just like (c) $\rightarrow$ (a) in Theorem 3.1. If $T$ is not locally compact, then $T$ is not locally pseudocompact by [7, 3.10.21 and 4.1.17]. Thus it follows from Corollary 1.3 that $X$ is a weak cb*-space.

Sufficiency: In case $|X| < m_1$, then the proof follows from Theorem 3.1 and Lemmas 3.3 and 3.4. In case $|T| < m_1$, then $T$ is realcompact by [8, 15.20]. It follows from [5, Corollary 2.2] and 3.5(1) that $\nu(X \times T) = \nu X \times \nu T$. Hence the proof is complete.

3.7. REMARK. Theorem 3.6 fails to be valid if we drop the assumption $c(X) < \aleph_0$. To see this, we utilize the space $Q$ of all rational numbers and the space $Y_0$ due to Comfort [4, p. 99]. The space $Y_0$ was constructed as the quotient space obtained from the product space

$$Z = W(\omega_0) \times W(\omega_1 + 1) \times W(\omega_1 + 1)$$

by identifying, for each $n < \omega_0$ and each $\gamma < \omega_1$, the two points $(n, \omega_1, \gamma)$ and $(n + 1, \gamma, \omega_1)$. Let $f: Z \rightarrow Y_0$ be the quotient map, and let us set $X = Y_0 - \{y_0\}$, where $y_0$ is the center point $f((0, \omega_1, \omega_1)) (= f((n, \omega_1, \omega_1)))$. Then he proved that $\nu X = Y_0$, and a similar argument assures us that $\nu(X \times Q) = \nu X \times Q$. Obviously $Q$ is metrizable but not locally compact. It remains to show that $X$ is not a weak
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cb*-space. Setting

\[ F_n = f(\{ i | i > n \} \times W(\omega_1 + 1) \times W(\omega_1 + 1)) \cap X \]

for each \( n < \omega_0 \), we obtain a decreasing sequence \( \{ F_n | n < \omega_0 \} \) of regular closed sets in \( X \) with empty intersection. Then \( y_0 \in \cap \{ \text{cl}_{\mathcal{w}} G_n | n < \omega_0 \} \), and hence \( X \) is not a weak cb*-space.

4. Problems and remarks. Many interesting problems related to our results remain unsolved. Following [20], we say that a space \( X \) is \( v \)-locally compact if \( vX \) is locally compact.

4.1. Characterize \( \mathcal{R}(v\text{-locally compact}) \). It is easy to see that

\[ \mathcal{R}(v\text{-locally compact}) = \mathcal{R}(\text{pseudocompact}). \]

4.2. Characterize \( \mathcal{R}(\text{realcompact}) \). Hušek [12], [14] and McArthur [23] proved that each member \( X \) of this class, with \( |X| < \aleph_1 \), is realcompact; however, the characterization is not yet known in complete form.

4.3. Characterize \( \mathcal{R}(\text{weak cb*}) \). We note that it follows from Lemma 3.2 and Theorem 2.1 that \( \mathcal{R}(\text{weak cb*}) = \mathcal{R}(\text{extremally disconnected}) \). Moreover, since the space \( Y \) constructed in the proof of [23, Theorem 5.2] is a weak cb*-space, every member of \( \mathcal{R}(\text{weak cb*}) \) is realcompact.

4.5. Let \( \phi_i : X_i \to Y_i \) (\( i = 1, 2 \)) be onto maps. When does \( v(\bigcap_{i=1}^{2} Y_i) = vY_1 \times vY_2 \) imply \( v(\bigcap_{i=1}^{2} X_i) = vX_1 \times vX_2 \)?

4.6. Remark. Let \( f : Y \to Z \) be a perfect onto map. Then \( v(\bigtimes \cap T) = vX \times vT \) be valid in the case where \( T \) is a metrizable space.

4.8. Remark. It seems that the classes \( \mathcal{R}(\mathcal{P}) \) considered above have several common properties. Finally, we list some of these below. Each assertion follows from the results in the bracket.
(1) $\mathcal{R}(\mathcal{P})$ includes all locally compact, realcompact spaces of nonmeasurable cardinals [5, Corollary 2.2].

(2) $\mathcal{R}(\mathcal{P})$ is closed under cozero-subspaces [3, 3.2].

(3) $\mathcal{R}(\mathcal{P})$ is closed under open perfect images (Corollary 2.3); more generally, if $\nu f: \nu X \to \nu Y$ is an onto biquotient map, then $Y \in \mathcal{R}(\mathcal{P})$ whenever $X \in \mathcal{R}(\mathcal{P})$ ([24, Theorem 1.2] and Theorem 2.1).

(4) If each $\mathcal{P}$-space is $\nu$-locally compact, then $\mathcal{R}(\mathcal{P})$ is closed under quotient images (Theorem 2.4).

(5) If $X \in \mathcal{R}(\mathcal{P})$ and $Y$ is a locally compact, realcompact space with $|Y| < m_1$, then $X \times Y \in \mathcal{R}(\mathcal{P})$ [5, Corollary 2.2].

REFERENCES


14. ———, Realcompactness of function spaces and $\nu(P \times Q)$, General Topology Appl. 2 (1972), 165–179.


32. M. Wage, An easy counterexample to the inequality \( \dim(X \times Y) < \dim X + \dim Y \) (to appear).

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