THE COHOMOLOGY ALGEBRAS OF FINITE DIMENSIONAL HOPF ALGEBRAS

BY

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Abstract. The cohomology algebra of a finite dimensional graded connected cocommutative biassociative Hopf algebra over a field \( K \) is shown to be a finitely generated \( K \)-algebra. Counterexamples to the analogue of a result of Quillen (that nonnilpotent cohomology classes should have nonzero restriction to some abelian sub-Hopf algebra) are constructed, but an elementary proof of the validity of this "detection principle" for the special case of finite sub-Hopf algebras of the mod 2 Steenrod algebra is given. As an application, an explicit formula for the Krull dimension of the cohomology algebras of the finite skeletons of the mod 2 Steenrod algebra is given.

If \( A \) is an augmented algebra over the field \( K \), the cohomology algebra \( H^*(A) \) is defined as \( \text{Ext}_A(K, K) \). If \( A \) is finite dimensional as a \( K \)-vector space, \( H^*(A) \) may still fail to be a finitely generated \( K \)-algebra, e.g., Löfwall [12]. However, if \( A = K[G] \), the group algebra of a finite group, then \( H^*(A) \) is finitely generated, Evens [6] and Venkov [21]. Cocommutative Hopf algebras are one generalization of group algebras, and connected graded cocommutative Hopf algebras are closely analogous to finite \( p \)-groups. It is the intent of this work to push this analogy as far as possible. The first positive result is that finite generation holds in this context also (all Hopf algebras mentioned in this work are biassociative and either commutative or cocommutative).

Theorem A. If \( A_* \) is a finite dimensional graded connected cocommutative \( K \)-Hopf algebra, then \( H^{**}(A_*) = \text{Ext}^*_A(K, K) \) is a finitely generated \( K \)-algebra.

The strategy of the proof is essentially that developed by Adams [1] and Liulevicius [10] for the computation of the cohomology of small sub-Hopf algebras of the Steenrod algebras. One resolves \( A_* \) as a sequence of iterated central extensions of Hopf algebras. Each such extension has an associated spectral sequence, and some hold on the differentials is provided by the transgression theorem relating Steenrod operations on the "fiber" and "base". The only philosophical difference between the present plan and that of [1], [10] is that precise computational results are sacrificed for the sake of generality.

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Another analogue of a result of Evens [6] is a relative version of Theorem A that includes as Corollary C the analogue of a result of Swan [20]:

**Theorem B.** If $A_\ast$ is a finite dimensional graded connected cocommutative $K$-Hopf algebra and $B_\ast$ is a sub-Hopf algebra of $A_\ast$, then $H^{**}(B_\ast)$ is a finitely generated $H^{**}(A_\ast)$ module under the restriction map.

**Corollary C.** (i) If $i: B_\ast \to A_\ast$ is the inclusion of a sub-Hopf algebra with $A_\ast$ as in Theorem B, then the restriction map $H^{t,i}(i): H^{t,i}(A_\ast) \to H^{t,i}(B_\ast)/\text{nilpotents}$ is nonzero in infinitely many positive bidegrees.

(ii) If $\Phi: C_\ast \to A_\ast$ is onto, then $\Phi$ is an isomorphism if and only if $H^{t,i}(\Phi): H^{t,i}(A_\ast) \to H^{t,i}(C_\ast)/\text{nilpotents}$ is onto for all $(s, t)$ with $s + t$ sufficiently large.

**Corollary D.** The Krull dimension of $H^{**}(A_\ast)$ is finite and greater than or equal to the Krull dimension of $H^{**}(B_\ast)$, for $B_\ast$ any sub-Hopf algebra of $A_\ast$.

Finally, pursuing the analogy with $p$-groups even further, we force it beyond its capacity. Quillen [16] proved that for finite groups, an element of $H^*(G, F_p)$ is nilpotent if and only if its restriction to each $p$-elementary subgroup of $G$ is zero. H. Miller speculated to the author that some approximation of this result might be valid in the context of Hopf algebras. A connected cocommutative commutative Hopf algebra $E_\ast$ over a perfect field of characteristic $p$ is said to be elementary if $(E_\ast \setminus K)^p \equiv 0$, and the Hopf algebra $A_\ast$ is said to have the detection property if each nonnilpotent cohomology class has a nonzero restriction to at least one elementary sub-Hopf algebra of $A_\ast$. All the steps of the Quillen-Venkov proof [17] of the detection property for finite groups are valid for the Hopf algebra setting, except that the analogue of a key result of Serre [18] characterizing $p$-elementary groups cohomologically fails. Hence one obtains only a sufficient condition for the detection property to hold.

**Counterexample E.** For each prime $p$, there exists a finite dimensional graded connected cocommutative Hopf algebra $B_\ast$ over $F_p$ and a nonnilpotent cohomology class $u_B$ which restricts to zero on every abelian sub-Hopf algebra of $B_\ast$. If $p$ is odd, $B_\ast$ may be taken to be a sub-Hopf algebra of the cyclic reduced powers in the mod $p$ Steenrod algebra.

In spite of these counterexamples to the universal validity of the detection property, the sufficient condition derived from the Quillen-Venkov proof can be directly verified in favorable cases:

**Theorem F.** Any finite sub-Hopf algebra of the mod 2 Steenrod algebra $\mathcal{A}(2)$ has the detection property.

Theorem F was originally proved by W. H. Lin [9] from a somewhat different point of view.

**Corollary G.** (i) If $A_\ast$ has the detection property, then the Krull dimension of $H^{**}(A_\ast)$ is the maximal rank of the elementary sub-Hopf algebras of $A_\ast$, where the rank of $E_\ast$ is $\dim_K(H^{1,e}(E_\ast))$. 

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(ii) The Krull dimension of the cohomology algebra of any finite sub-Hopf algebra of $\mathcal{H}(2)$ can be calculated explicitly. For example, if $\mathcal{H}_n$ denotes the sub-Hopf algebra of $\mathcal{H}(2)$ generated by $Sq^1, \ldots, Sq^{2^n}$, the Krull dimension of its cohomology algebra is $e + r(2n + 5 - 3r)/2$, where $r$ is the greatest integer in $(2n + 5)/6$ and $e = 0$ unless $3$ divides $n$, in which case it is $1$.

Extensions of Quillen’s results to $H^*(G, M)$ for $M$ a $G$-module have been recently obtained by J. Alperin and L. Evens. These new results appear to have analogues for modules over Hopf algebras with the detection property, but a discussion of this material is postponed to a sequel to the present work.

Even in cases where $A_*$ is not related to the mod $p$ Steenrod algebras, there is a topological motivation for the study of $H**(A_*)$. If $X$ is a simply connected CW complex such that the loop space $\Omega X$ has finite $\mathbb{Z}/p\mathbb{Z}$ cohomology, then the Eilenberg-Moore or Rothenberg-Steenrod spectral sequence converging to $H^*(X, \mathbb{Z}/p\mathbb{Z})$ has its $E_2$-term isomorphic to $\text{Ext}^*_{A_*(\Omega X, \mathbb{Z}/p\mathbb{Z})}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$. By Theorem A, this $E_2$-term is a finitely generated algebra. However, it is not yet known that the spectral sequence degenerates at any finite stage, so the question of finite generation of $H^*(X, \mathbb{Z}/p\mathbb{Z})$ is still an important open question.

Finally, $H**(A_*)$ is a function of only the algebra structure of $A_*$. J. Moore has asked if it is possible to abstract the properties of $A_*$ forced by the Hopf algebra structure, e.g., the existence of a central series, in a nontautological way, so that Theorem A would be valid for this wider class of algebras. The present answer is no; the entire line of proof rests on the existence of Steenrod operations and on the transgression theorem. Corollary C(i) is essentially equivalent to Theorem A, and no large class of examples for which Corollary C(i) is valid springs to mind.

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1. The case $\text{char } K = 0$; The central series. The reader might well believe that the restrictions placed on the algebra structure of $A_*$ by the concommitant coalgebra structure are so severe that Theorem A is trivially forced. This skepticism is perhaps justified if $\text{char } K = 0$, but in general the hypothesis manifests itself in a more subtle fashion.

**Proposition 1.1.** If $A_*$ is as in Theorem A and $\text{char } K = 0$, then $A_*$ is isomorphic to a tensor product of exterior algebras. Thus Theorem A is valid in this case.

**Proof.** Since $\text{char } K = 0$, the cocommutativity of $A_*$ implies that $A_*$ is primitively generated, Milnor-Moore [15]. Since $A_*$ is finite dimensional, any nonzero primitive is odd dimensional, and hence the commutator of any two primitives is zero. Therefore $A_*$ is commutative, and the Borel Structure Theorem implies the structure of $A_*$; $H**(A_*)$ is the polynomial algebra on $n$ generators, for $n$ the $K$-dimension of the indecomposables of $A_*$. This is essentially the original argument of H. Hopf.
In general, the only immediate consequence of the Hopf algebra hypothesis is

**Proposition 1.2.** If $A_\ast \neq K$ is as in Theorem A, there exists a nontrivial central monogenic sub-Hopf algebra $C_\ast$.

**Proof.** The general case has the same proof as the special case given in Liulevicius [10 p. 28]. Denote by $A^\ast$ the graded $K$-linear dual of $A_\ast$. Let $n$ be the largest dimension in which the indecomposable quotient $QA^\ast$ is nonzero. Let $I^\ast$ be the Hopf ideal of $A^\ast$ generated by the elements of degree strictly less than $n$. Define $B_\ast$ as the linear dual of $A^\ast/I^\ast$. A diagram chase shows that $B_\ast$ is central in $A_\ast$. Take $C_\ast$ to be the monogenic Hopf algebra generated by a nonzero element of $B_\ast$ of lowest positive degree.

We will later need the result that in the category of connected cocommutative Hopf algebras morphisms have unique kernels:

**Proposition 1.3.** If $\Phi: A_\ast \to B_\ast$ is a surjective morphism of graded connected cocommutative $K$-Hopf algebras, there exists a unique sub-Hopf algebra $N_\ast$ of $A_\ast$ such that $\ker \Phi = A_\ast N_\ast$, the left ideal generated by the elements of positive dimension in $N_\ast$. If $C_\ast$ is any sub-Hopf algebra of $A_\ast$ such that $C_\ast \cap N_\ast = K$, then $\Phi$ restricted to $C_\ast$ is one-to-one.

**Proof.** This is a restatement of Theorem 4.9 of Milnor-Moore [15]. $N_\ast$ is the linear dual to $A^\ast/A^* B^*$, and $A_\ast$ is isomorphic to $N_\ast \otimes_K B_\ast$ as a left $N_\ast$-module. If $C_\ast \cap \ker \Phi \neq 0$, any lowest dimensional nonzero element in the intersection is primitive, and together with $N_\ast$ would generate a strictly larger sub-Hopf algebra contained in $\ker \Phi$. This contradicts $A_\ast \approx N_\ast \otimes_K B_\ast$.

2. Steenrod operations and spectral sequences. Since $A_\ast$ is graded, the cohomology algebra is bigraded by $H^{s,t}(A_\ast) = \text{Ext}^{s,t}_A(K,K)$, where $s$ is the homological degree and $t$ is the internal grading. $H^{*,*}(A_\ast)$ can in theory be computed from the cobar construction on $A^\ast$, Adams [1]: $B^*(A_\ast)$ is the free tensor algebra on the $K$-vector space of elements of positive degree of $A^\ast$. The generators of $B^*(A_\ast)$ are denoted by $[z]$, for $z$ in $A^\ast$, and the differential on generators is specified as $d[z] = \Sigma[z_i'] [z_i'']$ where the reduced coproduct of $z$ is $\Sigma z_i' \otimes z_i''$. $d$ is extended to a graded differential on $B^*(A_\ast)$, where the grading is by total degree.

Steenrod operations appear in the guise of cup-$i$ products on the cohomology of cocommutative Hopf algebras in Adams [1] and are defined explicitly in Liulevicius [10] by applying Steenrod's construction to the cobar construction. The spectral sequence associated to a central extension of cocommutative Hopf algebras appears in Adams [1] and Liulevicius [10]. We need basically the existence and formal properties of the Steenrod operations, the spectral sequence of a central extension, and the transgression formula. All this material is in May [14] and we follow that choice of notation and indexing. The only liberty taken is to extend the operations to the case $K$ of characteristic $p$, but not equal to the prime field; then the operations are $\mathbb{Z}/p\mathbb{Z}$-linear, but not $K$-linear.
Proposition 2.1. Let $A_\bullet$ be a cocommutative Hopf algebra over $K$, char $K > 2$. There exist operations $\{\beta, \beta^i \}$ for $i > 0$ with the following properties:

(i) $\beta$ acting on $H^i(A_\bullet)$ has bidegree $((2i - t)(p - 1), t(p - 1))$ and $\beta^i$ has bidegree $((2i - t)(p - 1) + 1, t(p - 1))$.

(ii) $\beta^i = 0$ if $2i < t$ or $2i > s + t$; $\beta^2 = 0$ if $2i < t$ or $2i > s + t$; $\beta^i x_{st} = x^p$ if $s + t = 2i$.

(iii) The usual Cartan formulae and Adem relations hold if $\beta$ is considered as an independent homomorphism and not necessarily the identity.

(iv) The operations are natural for maps of Hopf algebras.

For $p = 2$, the usual reindexing is required for the proper statement of Proposition 2.1. There is also a reindexing that is convenient for the operations on $H^{*+2r}$, $\beta^i = \beta^{i+t}$ on $H^{*-2t}$, and similarly for $\beta^i$. The Cartan formulae and Adem relations are satisfied for the new operations also. One operation of particular interest is the new $\beta^0$. On the cobar construction it is represented by the map $[z] \rightarrow [z^p]$ on the generators.

Proposition 2.2. If $M_\bullet$ is the monogenic Hopf algebra

(i) $\Lambda[x_{2n-1}]$ ($\Lambda[x_{n-1}]$ if $p = 2$) or

(ii) $F_p[x_{2n}]/(x^p)$ for $p > 2$, then $H^{**}(M_\bullet)$ is

(a) $\Lambda[z] \otimes F_p[u]$, where bidegree $z$ is $(1, 2n - 1)$, $(1, n)$ for $p = 2$,

(b) $\Lambda[z] \otimes F_p[u]$, where bidegree $z = (1, 2n)$ and bidegree $u = (2, 2np)$

and the nonzero Steenrod operations which are not compositions are

(a) $\beta^n z = z^n$, $(Sq^rz = z^2)$,

(b) $\beta^{np+1} u = u^p$, $\beta^{np} z = ru$, for $r \neq 0$.

Proof. The structure of the cohomology algebras is standard, e.g., Liulevicius [10], while the operations are forced from the axioms, except for $\beta^{np}$. By analogy to the topological case in $H^*(BZ/pZ, Z/pZ)$, one might think that there should be a Bockstein connecting $z$ to $u$, and $\beta^{np}$ is the only possibility on dimensional grounds. Less wistfully, the iterated $p$-fold Massey product $\langle z, z, \ldots, z \rangle$ is shown in May ([14], following Kraines [8]) to contain $-\beta^{np} z$ for any $z$ in $H^{12n}$. A short calculation with the cobar construction shows that in this particular case, the obvious representative of $\langle z, z, \ldots, z \rangle$ is the definition of $u$ given in Liulevicius [10], up to sign. The indeterminancy is zero, so the proposition is established.

Proposition 2.3 [1], [10], [13]. Let $K \rightarrow C_\bullet \rightarrow A_\bullet \rightarrow B_\bullet \rightarrow K$ be a central extension of graded connected cocommutative Hopf algebras. There is a first quadrant cohomology spectral sequence of algebras with $E_2$-term $H^{**}(B_\bullet) \otimes H^{**}(C_\bullet)$ and abutting to $H^{**}(A_\bullet)$. The Steenrod operations defined via the Cartan formula on $E_2$ commute with the differentials in the sense that $d_r \beta x = \beta d_r x$, for $\theta$ a Steenrod operation of homological degree $r - j$, and $x$ in $E_j$. That is, $\theta x$ and $\theta d_r x$ are $d_{r,i}$-cycles for $i > 0$, and the equation holds in $E_r$. In particular, if $x$ transgresses to $y$, then $\beta x$ transgresses to $\beta y$.

Actually, the spectral sequence is trigraded, but since the differentials preserve the internal degree, we suppress this degree whenever possible. We have also a
spectral sequence in the case that \( C_\ast \) is not central in \( A_\ast \). \( \text{Ext}^{\ast \ast}_{B_\ast}(K, \text{Ext}^{\ast \ast}_{C_\ast}(K, K)) \)
abutting to \( H^{\ast \ast}(A_\ast) \), as in Cartan-Eilenberg [5]. This is used in later sections, but is not required for the proof of finite generation.

**Example 2.4.** The simplest nontrivial example of the use of Propositions 2.1, 2.2 and 2.3 is provided by the sub-Hopf algebra of the mod 2 Steenrod algebra generated by \( \text{Sq}^1 \) and \( \text{Sq}^2 \). This algebra, denoted as \( A_1 \), has relations \((\text{Sq}^1)^2 = 0, (\text{Sq}^2)^2 = \text{Sq}^3\text{Sq}^1, \) and the commutator of \( \text{Sq}^1 \) and \( \text{Sq}^2 \), \( \text{Sq}^3 \), is nonzero. \( \text{Sq}^3 \) is central in \( A_1 \). The \( E_2 \)-term of the spectral sequence is \( F_2[h_0, h_1, h_0h_1] \), and \( d_2h_0 = h_0h_1 \) where the tridegrees are \((1, 0, 1), (1, 0, 2), \) and \((0, 1, 3)\) respectively. \( \text{Sq}^0h_0 = h_1, \text{Sq}^0h_1 = 0 \). The behavior of the rest of the spectral sequence is now determined by the transgression theorem: \( E_3 = F_2[h_0, h_1, (h_0h_1)]/((h_0h_1)), d_3 \) is determined on \( h_0^2 \) as \( \text{Sq}^0h_0h_1 = h_1^2 \), and \( E_4 \) is the module over \( F_2[h_0, h_1, (h_0h_1)]/((h_0h_1), h_1^2) \) generated by 1 and \( h_0h_1^2 \). By the transgression theorem, \( h_0h_1^2 \) survives to \( E_5 \), where \( d_5h_0^2 = \text{Sq}^3\text{Sq}^1h_0h_1 = 0 \) in \( E_5 \). The extra indecomposable \( h_0h_2 \) is a \( d_r \)-cycle for all \( r \), so the spectral sequence collapses after stage four. In general, the spectral sequences considered need not collapse at the same point that all differentials on the “fiber” vanish. The possibility of the creation at each stage of the spectral sequence of new indecomposables on which higher differentials are nontrivial lends content to Theorem A.

This example is well known and can be computed directly via resolutions, e.g., Liulevicius [11]. The above use of the transgression theorem for the computation was shown to me by H. Miller.

3. **The proof of Theorem A.** For connected graded algebras over a field \( K \), finite generation as a \( K \)-algebra is equivalent to the Noetherian condition. The following lemma, used basically in all Noetherian arguments involving spectral sequences, was pointed out to me by L. Evens. It is the key to dealing with the creation of new indecomposables in the spectral sequence in an implicit way.

**Lemma 3.1.** If the first quadrant spectral sequence \( \{E_r, d_r\} \) is a \( R_\ast \) module for some Noetherian ring \( R_\ast \), and \( E_2 \) is a finitely generated \( R_\ast \) module, then \( E_\infty \) is a finitely generated \( R_\ast \) module.

**Proof.** We have the sequence of \( R_\ast \) submodules of \( E_2 \), \( 0 \subset B_2 \subset \cdots \subset B_\ast \subset \cdots \subset B_\infty \subset Z_\infty \subset \cdots \subset Z_\ast \), where \( Z_r \) is the set of elements that survive to stage \( r \), and \( B_r \) consists of those elements that bound by stage \( r \). Since \( E_2 \) is a Noetherian \( R_\ast \) module, \( Z_\infty \) is also a Noetherian \( R_\ast \) module, and therefore \( E_\infty = Z_\infty / B_\infty \) is Noetherian and hence finitely generated.

**Proposition 3.2.** If \( M_\ast \) is a monogenic Hopf algebra of height at most \( p \), where \( p \) is char \( K \), then for any central extension \( K \rightarrow M_\ast \rightarrow A_\ast \rightarrow B_\ast \rightarrow K \) of cocommutative connected Hopf algebras with \( H^{\ast \ast}(B_\ast) \) finitely generated as a \( K \)-algebra, \( H^{\ast \ast}(A_\ast) \) is finitely generated as a \( K \)-algebra.

**Proof.** \( M_\ast \) is either \( \Lambda[x_{2n-1}] \) or \( K[x_{2n}] / (x^p) \) for \( p \) odd, or \( \Lambda[x_{n-1}] \) for \( p = 2 \). Thus the structure of \( H^{\ast \ast}(M_\ast) \) is described by Proposition 2.2. The Steenrod operations given there imply that the \( p^\ast \)th powers of the polynomial generator of
$H^{**}(M_\bullet)$ transgress to $E^*_2\cdot^0$, the base. But $H^{**}(B_\bullet) = E^*_2\cdot^0$ is Noetherian by assumption. If one defines the increasing sequence of ideals $\{I_n\}$ so that $I_n$ is the kernel of $E^*_2\cdot^0 \to E^*_n\cdot^0$, then $I_N = I_{N+1} = \ldots$ for all $N \gg 0$. The $p$'th powers of the polynomial generator of $H^{**}(M_\bullet)$ therefore transgress to zero, for $s > S \gg 0$. Define $R_\bullet$ to be the subring of $E_2$ generated by $E^*_2\cdot^0$ and the $p^S$'th power of the polynomial generator of $H^{**}(M_\bullet)$. Then $R_\bullet$ consists of infinite cycles, and $E_2$ is a finitely generated $R_\bullet$ module. By Lemma 3.1, $E_\infty$ is finitely generated as a $R_\bullet$ module and hence as a $K$-algebra. But $H^{**}(A_\bullet)$ is complete with respect to the exhaustive filtration giving rise to the spectral sequence, so the fact that its associated graded algebra is Noetherian implies that $H^{**}(A_\bullet)$ is Noetherian, and hence a finitely generated $K$-algebra, Bourbaki [4, 3.2.9, Corollary 2 to Proposition 12].

**Proof of Theorem A.** We induce on dim$_K(A_\bullet)$. Theorem A is trivially true for dim$_K(A_\bullet)$ less than or equal to $p$. If dim$_K(A_\bullet)$ is greater than $p$, $A_\bullet$ must contain a nontrivial central monogenic sub-Hopf algebra of height at most $p$, so that dim$_K(A_\bullet/M_\bullet)$ is less than dim$_K(A_\bullet)$. Applying the inductive hypothesis, Proposition 3.2 applies, so $H^{**}(A_\bullet)$ is finitely generated. The proof of Theorem A given above is analogous to the $p$-group case given by Golod [7], but was derived from the proof in Evens [6]. The Steenrod operations and the transgression theorem are used as a poor man's substitute for a transfer argument of Evens.

**Proof of Theorem B.** We induce over dim$_K(A_\bullet)$. If dim$_K(A_\bullet) < \text{char } K$, the result is clear. Now assume that the result is valid for all pairs $(A'\hat{\bullet}, B'\hat{\bullet})$ for which dim$_K(A'\hat{\bullet}) < \text{dim}_K(A_\bullet)$. By Proposition 1.1, there is a nontrivial central monogenic sub-Hopf algebra $C_\bullet$ of $A_\bullet$ with height at most $p$. Hence we have a diagram of central extensions.

$$
K \to B_\bullet \cap C_\bullet \to B_\bullet \to B_\bullet /\left( B_\bullet \cap C_\bullet \right) \to K
$$
(1)

$$
K \to \quad C_\bullet \to A_\bullet \to A_\bullet /\left( C_\bullet \right) \to K
$$
(2)

If $B_\bullet \cap C_\bullet = K$, then by Proposition 1.3, $\Phi$ is one-to-one, and the result follows from the inductive hypothesis, since image $\Phi^* \subset \text{image } i^*$. If $B_\bullet \cap C_\bullet = C_\bullet$, we apply the main argument of the proof of Theorem A again. The $E_2$-term of the spectral sequence for extension (2) is a finitely generated $R_\bullet$ module, for $R_\bullet$ the subalgebra of infinite cycles generated by $H^{**}(A_\bullet/C_\bullet)$ and a sufficiently large $p^S$'th power of the polynomial generator of $H^{**}(C_\bullet)$. Applying the inductive hypothesis, the $E_2$-term of the spectral sequence of extension (1) is also a finitely generated $R_\bullet$-module. By Lemma 3.1, the $E_2$-term of spectral sequence (1) is a finitely generated $R_\bullet$ module, and hence finitely generated over the associated graded algebra to $H^{**}(A_\bullet)$. By Bourbaki [4, 3.2.9, Proposition 12], $H^{**}(B_\bullet)$ is a finitely generated $H^{**}(A_\bullet)$ module.

Before the proofs of Corollaries C and D we need to recall the definition of Krull dimension and some basic facts about Krull dimension, e.g., Chapter 5 of Matsumura [13]. If $R_\bullet$ is a graded commutative ring, consider chains of homogeneous prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n$, where the inclusions are proper. If $R_\bullet$ is
finitely generated over a field \( k \), there exists a finite maximal length \( n \), called the Krull dimension or simply the dimension of \( R_* \). If \( R_* = k[X_1, \ldots, X_n] \), a graded polynomial algebra, then dimension \( R_* = n \). If \( \sqrt{0} \) denotes the ideal of nilpotent elements, then \( R_*/\sqrt{0} \) has the same dimension as \( R_* \) for \( R_* \) f.g. over \( k \). If \( R_* \to R'_* \) is monic so that \( R'_* \) is a f.g. \( R_* \) module, \( \dim R'_* = \dim R_* \) [13, Theorem 20, p. 81]. If \( \phi: R_* \to R'_* \) is surjective, then \( \dim R'_* < \dim R_* \).

Thus the Krull dimension is a rough measure of the size of \( R_* \). In fact, if \( \dim R_* = n \), the E. Noether Normalization Theorem implies that there exists a polynomial subalgebra on \( n \) generators such that \( R_* \) is finitely generated as a module over the subpolynomial algebra [13, p. 91, Corollary 13]. If \( R_* = k[X_1, \ldots, X_n]/\mathfrak{a} \), then the radical of \( \mathfrak{a} \), \( \sqrt{\mathfrak{a}} = \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_r \), where \( \mathfrak{a}_j \) are prime homogeneous ideals. Then \( k[X_1, \ldots, X_n]/\mathfrak{a_j} \) is an integral domain, and \( \dim R_* = \max_j(\text{Tr} \cdot \deg_k k[X_1, \ldots, X_n]/\mathfrak{a}_j) \). That is, geometrically the variety corresponding to \( R_* \) is a union of irreducible varieties of dimension < \( \dim R_* \).

Finally one can consider the Poincaré series of \( R_* \), \( P_T(R_*) = \sum_{n=0}^{\infty} \dim R_n T^n \). If \( R_* \) is a f.g. algebra, \( \dim R_* = \text{order of pole of } P_T(R_*) \) at \( T = 1 \), Quillen [16]. This is not a particularly useful definition for the purposes of this paper but it does have the advantage of brevity.

**Proof of Corollary C.** (i) Suppose that \( H^{i+t}(i) \) is zero for all \( (s, t) \) with \( s + t \) sufficiently large. Then the Krull dimension of image \( i^* \) is zero, since image \( i^*/\text{nilpotents} \) is \( K \). But if \( B_* \) is not \( K \), then \( B_* \) contains a central monogenic sub-Hopf algebra \( C_* \). Clearly \( H^{**}(C_*) \) is not finitely generated as a module over image \( i^* = K \), contradicting Theorem B, so it must not be the case that \( H^{i+t}(i) \) is zero mod nilpotents for all \( (s, t) \) with \( s + t \) sufficiently large. Part (ii) is proved in the next section as Proposition 4.2.

**Proof of Corollary D.** Since \( H^{**}(B_*) \) is a finitely generated \( H^{**}(A_*) \) module, \( H^{**}(B_*) \) is integral over \( H^{**}(A_*) \), and \( \dim H^{**}(B_*) = \dim \text{image } i^* < \dim H^{**}(A_*) \).

4. Homological characterization of isomorphisms and elementary Hopf algebras.

\( A_*, B_* \ldots \) continue to denote finite dimensional graded connected cocommutative Hopf algebras over \( K \). We first have an easy analogue of a theorem of Stallings [19].

**Proposition 4.1.** If \( \Phi: A_* \to B_* \) such that \( H^{1,*}(\Phi) \) is an isomorphism and \( H^{2,*} \) is a monomorphism, then \( \Phi \) is an isomorphism.

**Proof.** The proposition is valid for graded connected algebras, but the proof here is valid only for the Hopf algebra case. Denote by \( N_* \) the Hopf algebra kernel of \( \Phi \). Since \( H^{1,*}(\Phi) \) is monic, \( \Phi \) is onto, and we have an extension: \( K \to N_* \to A_* \to B_* \to K \). Grading the spectral sequence [5] \( \{ H^{**}(B_*; H^{**}(N_*)) \to H^{**}(A_*) \} \) by homological degree, we have \( \delta_2: H^{0,*}(B_*; H^{1,*}(N_*)) \to H^{2,*}(B_*) \) must be identically zero, since \( H^{2,*}(\Phi) \) is monic. Thus \( H^{0,*}(B_*; H^{1,*}(N_*)) \) survives to \( E_\infty \). But \( H^{1,*}(A_*) = H^{1,*}(B) \oplus H^{1,*}(B_*) \oplus H^{1,*}(N_*) \), so \( H^{1,*}(\Phi) \) onto implies that \( H^{0,*}(B_*; H^{1,*}(N_*)) = 0 \). Since \( B_* \) is nilpotent, \( H^{1,*}(N_*) = 0 \), and \( N_* = K \).
Proposition 4.2. If $\Phi: A_\bullet \rightarrow B_\bullet$ is onto, then $\Phi$ is an isomorphism if and only if $H^{s+t}(\Phi)$ is onto $H^{s+t}(A_\bullet)/\text{nilpotents}$ for all $(s, t)$ with $s + t$ sufficiently large.

Proof. $\Rightarrow$ Trivial.
$\Leftarrow$ Let $N_\bullet$ be the Hopf algebra kernel of $\Phi$. Then $i^*: H^{s+t}(A_\bullet) \rightarrow H^{s+t}(N_\bullet)/\text{nilpotents}$ is zero for all $(s, t)$ with $s + t > 0$. By Corollary C(i), we must have $N_\bullet = K$, since otherwise $i^*$ would be nontrivial. Hence $\Phi$ is an isomorphism.

Definition 4.3. The Hopf algebra $E_\bullet$ is elementary $\iff E_\bullet$ is commutative and $(E_\bullet)_p = 0$. The coalgebra structure is unspecified but cocommutative.

At this point, we restrict to considering $K$ a perfect field of characteristic $p$, since we will use the Borel Structure Theorem.

Proposition 4.4. The coalgebra structure of an elementary Hopf algebra $E_\bullet$ over a perfect field of characteristic $p$ is uniquely determined by $H^{1,\bullet}(E_\bullet)$ as a $\hat{\mathcal{P}}^0$ module. Any $\hat{\mathcal{P}}^0$ module structure in which $\hat{\mathcal{P}}^0$ acts nilpotently, and with the proper degree, is realizable.

Proof. Since $E_\bullet$ is finitely generated, $\hat{\mathcal{P}}^0$ is a nilpotent transformation on $H^{1,\text{even}}(E_\bullet)$. $H^{1,\text{even}}(E_\bullet)$ is thus the direct sum of cyclic $\hat{\mathcal{P}}^0$ modules of the form $\langle x_{2n}, \hat{\mathcal{P}}^0 x_{2n}, \ldots, (\hat{\mathcal{P}}^0)^r x_{2n} \rangle$, where $\langle (\hat{\mathcal{P}}^0)^r x_{2n} = 0 \rangle$, for some $\{x_{2n}\} \in H^{1,2n}(E_\bullet)$. Each such cyclic module corresponds to a tensor product factor in $(E_\bullet)^\bullet$ of the form $K[x_{2n}]/(x_{2n}^p)$, so the coalgebra structure can be read off from the $\hat{\mathcal{P}}^0$ module structure, and conversely.

Proposition 4.5. For any $A_\bullet$, there exists a $\psi: A_\bullet \rightarrow E_\bullet$ elementary, such that $H^{s+t}(\psi)$ is an isomorphism.

Proof. If there is such a map, it factors through the abelianization of $A_\bullet$, and it must be $A_\bullet(ab)/p$th powers. An explicit construction is also possible: Choose a graded vector space basis $\{f_1, \ldots, f_n\}$ of $\text{Alg}(A_\bullet, M_\bullet)$, where $M_\bullet = \Lambda_K[x_{2n-1}]$ or $K[x_{2n}]/(x_{2n}^p)$.

Define

$$\Phi: A_\bullet \rightarrow A_\bullet \otimes \cdots \otimes A_\bullet \rightarrow \otimes M_\bullet(f_i).$$

Then $H^{1,\bullet}(\Phi)$ is an isomorphism by construction. Put the coalgebra structure on $\otimes M_\bullet(f_i)$ induced by the $\hat{\mathcal{P}}^0$-module structure on $H^{1,\text{even}}(A_\bullet)$.

Proposition 4.6. The following conditions are equivalent:

(a) $A_\bullet$ is elementary.
(b) $H^{s+t}(A_\bullet)/\text{nilpotents}$ is generated by $H^{1,\text{odd}}$ and $\beta \hat{\mathcal{P}}^0/H^{1,\text{even}}$ for all sufficiently large dimensions.
(c) The degree 2 monomials in $H^{1,\text{odd}}, H^{1,\text{even}}$ and $\beta \hat{\mathcal{P}}^0/H^{1,\text{even}}$ are linearly independent.

Proof. $a \Rightarrow b$ and $c \Rightarrow a$ by Proposition 4.1 applied to $\Phi: A_\bullet \rightarrow E_\bullet$. $b \Rightarrow a$ by Proposition 4.2 applied to $\Phi: A_\bullet \rightarrow E_\bullet$.

Proposition 4.5 is precisely the point at which the analogy with finite $p$-groups begins to falter. Serre [18] gives a third equivalent condition, deduced from (c) via
Steenrod operations. "d": The $p$-group $G$ is elementary if and only if $u_G = \prod_{v \in G} \beta v$ in $H^1(G)$ is not nilpotent. Condition "d" is not valid for the cohomology of Hopf algebras.

5. Detection of cohomology classes. The aim of this section is to trace through the proof of Quillen's Theorem given in Quillen-Venkov [17], translating into the Hopf algebra setting. The entire translation is successful, except for the reference to Serre [18] mentioned at the end of §4. The outcome then is a sufficient condition for a Hopf algebra to have the detection property with respect to its elementary sub-Hopf algebras. In the important special case of sub-Hopf algebras of $G(2)$, this sufficient condition is easily verified, (see §6), and in general the condition points out what properties a counterexample must have. $K$ is to be a finite field of char $p$.

**Proposition 5.1.** (1) If $v \in \ker G^{-1} \cap H^{1,2n}(A_*)$ there exists a Hopf algebra morphism $\Phi_v: A_* \to K[x_{2n}]/(x_{2n}^p)$ such that $\Phi_v(z) = v$, for $z$ the generator of $H^{1,2n}(K[x]/(x^p))$.

(2) If $v \in H^{1,2n-1}(A_*)$, there exists $\Phi_v: A_* \to \Lambda_K[x_{2n-1}]$ such that $\Phi_v(z) = v$, for $z$ the generator of $H^{1,2n-1}(\Lambda_K[x_{2n-1}])$.

In either case, define $B_v(v)$ as the Hopf algebra kernel of $\Phi_v$, and $i_v$ as the inclusion $B_v(v) \to A_*$. 

**Proof.** (1) $H^{1,*}(A_*)$ is naturally isomorphic to the primitives of $A^*$, so regard $v$ as a primitive of $A^*$. $v$ generates a sub-Hopf algebra $V_*$ of $A^*$, and since $\bar{G}_0 v = v^p = 0$, $V_*$ is a truncated at height $p$ polynomial algebra. Hence its dual is also a truncated at height $p$ polynomial algebra. $\Phi_v$ is then dual to $V_* \to A^*$. Case (2) is similar.

**Proposition 5.2.** If $v \in \ker G^{-1} \cap H^{1,2n}(A_*)$ or $(H^{1,2n-1}(A_*))$ respectively, and $u \in H^{**}(A_*)$ such that $i_v^* u = 0$, then $u^N \in H^{**}(A_*)\beta F(v) or H^{**}(A_*)(u)$ respectively, for some $N > 0$.

**Proof.** This is the exact analogue of the first lemma of Quillen-Venkov [17]. Consider the spectral sequence associated to $K \to B_v(v) \to A_* \to K[x_{2n}]/(x^p) \to K$.

Each $E_*$ is a module over $H^{**}(K[x_{2n}]/(x^p)) = \Lambda_K[z] \otimes K[w]$, where $w = \beta \bar{G}_0 z$. Multiplication by $w$ on $E_1$ is a surjection for $s_1 > 0$ and an injection for $s_1 > 1$, by the periodicity of the cohomology of $M_* = K[x]/(x^p)$. By induction one proves that

$$w: E^{s_1,s_2}_r \to E^{s_1+2,s_2+1}_r \to 2np$$

is surjective for $s_1 > 0$ and injective for $s_1 > r - 1$. Hence $E^{s_1,s_2}_\infty \to E^{s_1+2,s_2+1}_\infty$ is surjective for $s_1 > 0$. Now

$$E^{s_1,s_2}_\infty = F_{s_1}H^{s_1+2s_2}(A_*)/F_{s_1+1}H^{s_1+2s_2}(A_*)$$
where $F^*_s H^\bullet(A_\bullet)$ is the filtration induced on $H^\bullet(A_\bullet)$ from the filtration on $B^\bullet(A_\bullet)$. By decreasing induction over $s_1$,

$$w_1 F_{s_1} H^{s_1 + s_2}(A_\bullet) = F_{s_1 + 2} H^{s_1 + s_2 + 2 + 2\pi}(A_\bullet).$$

If $i^*_u u = 0$, $u \in F_1 H^\bullet(G)$, so $u^N \in w H^\bullet(A_\bullet)$ where $N > 2$ and $tN > 2n$. The case for $p$ equals 2 or the quotient an exterior algebra is virtually the same. Alternately, the long exact sequence of Theorem 3.2 of [3] applies.

**Definition 5.3.** (i) $A_\bullet$ has the detection property if for each $u \in H^\bullet(A_\bullet)$ such that $u$ restricts to zero on every elementary sub-Hopf algebra, $u$ is nilpotent.

(ii) The fundamental class of $A_\bullet$ is $\prod v \prod \beta \tilde{\delta}^0 u$, where the product is taken over all nonzero $v$ in $H^{1,\text{odd}}(A_\bullet)$ and all nonzero $u$ in $\ker \tilde{\delta}^0 \cap H^{1,\text{even}}(A_\bullet)$. Denote this class by $u_A$.

**Lemma 5.4.** If $i: B_\bullet \rightarrow A_\bullet$ is the inclusion of a proper sub-Hopf algebra, then $i^* u_A = 0$.

**Proof.** Since $B_\bullet \neq A_\bullet$, ker $H^{1,*}(i) \neq 0$. Let $x \in \ker i^*$. If $x$ has odd internal degree, it appears in the product $u_A$, and $i^* u_A$ is clearly zero. If $x$ has even internal degree, there exists $r > 0$ such that $(\tilde{\delta}^0)^r x \neq 0$ is in $\ker i^* \cap \ker \tilde{\delta}^0$. Then $\beta \tilde{\delta}^0 (\tilde{\delta}^0)^r x$ is in $\ker i^*$, and is a term in the product, so $i^* u_A = 0$.

**Proposition 5.5.** If every sub-Hopf algebra of $A_\bullet$ of the form $B_\bullet(v)$, for $v \neq 0$ in $\ker \tilde{\delta}^0 \cap H^{1,*}(A_\bullet)$, has the detection property, then $A_\bullet$ has the detection property if and only if either $A_\bullet$ is elementary or $u_A$ is nilpotent.

**Proof.** By Lemma 5.4, $u_A$ restricts to zero on any proper sub-Hopf algebra, so if $A_\bullet$ has the detection property and is not elementary, $u_A$ is nilpotent. Conversely, if $u$ restricts to zero on each elementary sub-Hopf algebra of $A_\bullet$, this is also true for its restrictions $i^*_v u$, $v \in H^{1,*}(A_\bullet) \cap \ker \tilde{\delta}^0$. Therefore, since there are a finite number of such $v$, and $i^*_v$ is nilpotent for each $v$, we can assume $i^*_v u^N = 0$ for all such $v$, for $N \gg 0$. By Proposition 5.2, $u^M \in H^\bullet(A_\bullet) u_A$ for $M \gg 0$. But since $u_A$ is nilpotent, $u$ is nilpotent.

**Corollary 5.6.** If every sub-Hopf algebra $B_\bullet$ of $A_\bullet$ is either elementary or has $u_B$ nilpotent, then $A_\bullet$ has the detection property.

**Corollary 5.7.** If $A_\bullet$ has the detection property, then Krull dimension $H^\bullet(A_\bullet)$ is the maximum of the ranks of the elementary sub-Hopf algebras of $A_\bullet$.

**Proof.** Let $(E_\bullet(j))$ be the maximal elementary sub-Hopf algebras of $A_\bullet$. If $A_\bullet$ has the detection property, then the diagonal map $H^\bullet(A_\bullet) \rightarrow \bigoplus_j H^\bullet(E_\bullet(j))$ is a monomorphism modulo nilpotents. Hence the Krull dimension of $H^\bullet(A_\bullet)$ is less than that of the direct sum, which is the maximum of that of $H^\bullet(E_\bullet(j))$. The Krull dimension of $H^\bullet(E_\bullet(j))$ is the rank of $E_\bullet(j)$, since modulo nilpotents, the cohomology of $E_\bullet$ is a polynomial algebra on rank $E_\bullet$ generators. Corollary D gives $\max_j \text{rank } (E_\bullet(j))$ as a lower bound for Krull dimension $H^\bullet(A_\bullet)$. Hence this is an equality.
6. Counterexamples and finite sub-Hopf algebras of $\mathcal{A}(2)$. We need the explicit determination of the sub-Hopf algebras of the mod $p$ Steenrod algebra $\mathcal{A}(p)$ given by Adams-Margolis [2] and Anderson-Davis for $p = 2$ [3]. The characterization is that the only finite sub-Hopf algebras are the obvious ones. More explicitly

**Proposition 6.1.** Each finite sub-Hopf algebra $B_*$ of $\mathcal{A}(p)$ determines functions $e: \{1, 2, \ldots \} \rightarrow \{0, 1, 2, \ldots \}$ and $k: \{0, 1, 2, \ldots \} \rightarrow \{1, 2\}$ such that

1. $e(r) > \min(e(r - i) - i, e(i))$ for $0 < i < r$,
2. if $k(i + j) = 1$, then either $e(i) < j$ or $k(j) = 1$ for all $i > 1, j > 0$,
3. $e(r) = 0$ and $k(r) = 1$ for almost all $r$.

$B_*$ is isomorphic to the dual of the quotient of $\mathcal{A}(p)^* = \{F_p[\xi_1, \ldots, \xi_n, \ldots] \otimes \Lambda[\tau_0, \tau_1, \ldots]\}$

by the ideal generated by $\{\xi_1^{p^m}, \ldots, \xi_i^{p^m}, \ldots, \tau_0^{k_0}, \ldots, \tau_i^{k_i}, \ldots\}$ (for $p = 2$, set $k(i) \equiv 1$). Conversely, any functions $e$ and $k$ satisfying (1), (2) and (3) determine a finite sub-Hopf algebra of $\mathcal{A}(p)$, denoted by $B_*(e, k)$ in the following.

**Lemma 6.2.** Let $B_*(e, k)$ be as described in Proposition 6.1. If $\xi_1, \ldots, \xi_{s-1}$ are zero but $\xi_s \neq 0$, in $(B_*(e, k))^*$, and $\xi_s, \ldots, \xi_{n-1}$ are primitive, then the reduced coproduct $\tilde{\psi}_{\xi_n} = \xi_n^{p^m} \otimes \xi_n$. If all $\xi_i$ are primitive, $e(i) < r$ for all $i$.

**Proof.** $\xi_r, \ldots, \xi_{2r-1}$ are primitive since $\xi_1, \ldots, \xi_{r-1}$ are zero. By induction, $e(r + i) < r$ if $i < n - 2r$. Hence in the reduced coproduct

$$\tilde{\psi}_{\xi_n} = \xi_n^{p^m} \otimes \xi_n + \xi_n^{p^m} \otimes \xi_{n-1} + \cdots + \xi_n^{p^m} \otimes \xi_{n-1}$$

and all the terms except the first are zero.

**Counterexample 6.3.** $B_*(e, k)$ for $p$ odd with the exponent sequences $e(1) = 1$, $e(2) = 2$, $e(3) = 1$, and $e(3 + i) = 0$; $k(i) \equiv 1$ does not have the detection property. Hence the sub-Hopf algebra of $\mathcal{A}(p)$ generated by $\mathcal{P}^1$, $\mathcal{P}^p$, and $\mathcal{P}^{p^2}$ does not have the detection property.

**Proof.** View $B_*(e, k)$ as the central extension dual to $F_p \rightarrow F_p[\xi_1, \xi_2] / (\xi_1^p, \xi_2^p) \rightarrow F_p[\xi_1, \xi_2, \xi_3] / (\xi_1^p, \xi_2^p, \xi_3^p) \rightarrow F_p[\xi_3] / (\xi_3^p) \rightarrow F_p$.

Then the $E_2$-term of the spectral sequence of the extension is

$$\Lambda[\xi_1, \xi_2, \xi_3] \otimes F_p[u_1, u_2, u_3, u_4]$$

where $\beta \mathcal{P}^0[\xi_1] = u_1$, $\beta \mathcal{P}^0[\xi_2] = u_2$, $\beta \mathcal{P}^0[\xi_3] = u_3$, and $\beta \mathcal{P}^0[\xi_i] = u_4$. $d_2$ is determined by $d_2[\xi_1] = [\xi_1^p][\xi_1]$. The $d_2$-cycles are the module over $\Lambda[\xi_1, \xi_2, \xi_3] \otimes F_p[u_1, u_2, u_3, u_4]$ generated by $1$, $[\xi_1^p][\xi_3]$, and $[\xi_1][\xi_3]$, so $E_3$ is this module with the relations generated by $[\xi_1^p][\xi_3]$. The module generators other than 1 have bidegree $(1, 1)$ in the spectral sequence and hence are $d_3$-cycles for all $r$. $d_2[u_4] = d_2[\beta \mathcal{P}^0[\xi_3] = \beta \mathcal{P}^0[\xi_1^p][\xi_3] = u_3 \cdot 0 + 0 \cdot u_1 = 0$ by the Cartan formula for $\beta \mathcal{P}^0$. That is, the spectral sequence collapses at $E_3$, since all higher differentials vanish for dimensional reasons. Thus the element $u_B = (u_1 u_2)^p$ is not nilpotent in $H^{**}(B_*(e, k))$, since it is not nilpotent in the associated graded algebra to $H^{**}(B_*(e, k))$, $E_\infty$. This
$B_*(e, k)$ is a sub-Hopf algebra of the Hopf algebra generated by 1, $\mathcal{O}^1$, $\mathcal{O}^p$, and $\mathcal{O}^p^i$ corresponding to the exponent sequence (3, 2, 1, 0, \ldots), (1, 1, \ldots). Therefore the Krull dimension of $H^{**}(B_*(3, 2, 1, 0, \ldots; 1, 1, \ldots))$ is greater than or equal to the Krull dimension of $H^{**}(B_*(1, 1, 1, 1, \ldots; 1, 1, \ldots))$, which is 4. The two Hopf algebras have the same elementary sub-Hopf algebras,

$B_*(1, 1, 1, 1, \ldots; 1, 1, \ldots)$ and $B_*(0, 2, 1, 0, \ldots; 1, 1, \ldots)$,

each of which has rank 3. Therefore, by the contrapositive to Corollary 5.7, $B_*(3, 2, 1, 0, \ldots; 1, 1, \ldots)$ does not have the detection property.

**Theorem 6.4.** Any finite sub-Hopf algebra of $\mathcal{O}(2)$ has the detection property.

**Proof.** By Lemma 6.2, the first nontrivial coproduct in $(B_*(e))^*$ is $\psi_{\xi_n} = \xi_{m-r} \otimes \xi_{m-r}$, where $\xi_n$ and $\xi_{m-r}$ are primitive. I claim that by the application of Steenrod operations, the relation $[\xi_{m-r}, \xi_n] = 0$ in $H^2(B_*)$ generates a relation of the form $(x^N) = 0$ for $x, y \in H^1(B_*) \cap \ker Sq^0$. That is, $u_B^N = 0$.

Case 1. $e(r) = e(n - r) - r$. Apply $(Sq^0)^{n-r-1}$ to obtain

$$[\xi_{2n-r-1}][\xi_{2n-r-1}] = 0.$$ 

Case 2. $e(r) - k = e(n - r) - s$, for $k > 0$. Apply $Sq^2$ to obtain

$$[\xi_{2n-r-1}] = 0.$$ 

Case 3. $e(r) = e(n - r) - r - k$, for $k > 0$. This is similar to Case 2.

Thus, each finite sub-Hopf algebra $B_*$ of $\mathcal{O}(2)$ is either elementary or $u_B$ is nilpotent. By Corollary 5.6, each finite sub-Hopf algebra of $\mathcal{O}(2)$ has the detection property, by induction over its sub-Hopf algebras.

**Counterexample 6.5.** For $p = 2$, the Hopf algebra with dual $A[x_1, x_2, x_3, x_4, x_5]$, degree $x_i = i$, $\psi x_i = 0$, $i < 5$, $\psi x_5 = x_1 \otimes x_4 + x_2 \otimes x_3$ does not have the detection property.

**Proof.** In the spectral sequence of $(x^*) \rightarrow A_\ast \rightarrow A_\ast///(x^*)$, the $E_2$-term is $E_\infty$, and $E_\infty = F_2[z_1, z_2, z_3, z_4, z_5]/(z_1z_4 + z_2z_3)$ is an integral domain. Hence $u_A = z_1z_2z_3z_4$ is not nilpotent.

S. Priddy has observed that 6.5 is the universal enveloping algebra for its restricted Lie algebra of primitives, and hence it is a counterexample to a detection principle for connected graded Lie algebras, with respect to abelian sub-Lie algebras.

**Proposition 6.6.** The Krull dimension of $H^{**}(\mathcal{O}_n)$, for $\mathcal{O}_n$ the sub-Hopf algebra of $\mathcal{O}(2)$ given by the exponent sequence $(n + 1, n, n - 1, \ldots, 1, 0, \ldots)$, is given by the formula $g_n(r) = e + r(2n + 5 - 3r)$, where $r$ is the greatest integer in $(2n + 5)/6$ and $e$ is 0 if $n$ is not divisible by 3, and 1 if $n$ is divisible by 3.

**Proof.** The work has been done; it remains only to count. The maximal elementary sub-Hopf algebras have exponent sequences $(0, 0, \ldots, 0, r, r, \ldots, r - 1, r - 2, \ldots, 1, 0, \ldots)$ where $r$ first occurs in the $r$th position, and $r - 1$ occurs in the $(n - r + 3)$th position. We have only to add up the entries and maximize over $r$ to compute the Krull dimension. The continuous maximum occurs at $(2n + 5)/6$, and the discrete maximum at either the greatest integer in $(2n + 5)/6$ or
\((2n + 5)/6) + 1\), since the function \(g_n(r)\) is quadratic in \(r\). Analysis of the cases gives the definition of \(e\).

**Proposition 6.7.** If \(B_\ast \subseteq D_\ast\) are finite sub-Hopf algebras of the mod 2 Steenrod algebra with the same family of maximal elementary sub-Hopf algebras, then the restriction map is monic mod nilpotents.

**Proof.** This was proved by W. H. Lin [9] as the main argument in his proof of Theorem 6.4 for the special case that \(B_\ast\) is the intersection of \(D_\ast\) with the (nonfinite) sub-Hopf algebra of \(\mathfrak{e}(2)\) with exponent sequence \((1, 2, 3, 4, 5, \ldots)\). In the logic of the present paper however, it is a corollary of 6.4.

In view of the counterexamples, the seeking of detection theorems for general Hopf algebras seems futile. However, there remains the hope that for finite sub-Hopf algebras of the mod \(p\) Steenrod algebra there exist suitable families of sub-Hopf algebras which detect nonnilpotent cohomology elements.

**References**


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