UNIQUENESS OF PRODUCT AND COPRODUCT DECOMPOSITIONS IN RATIONAL HOMOTOPY THEORY

BY

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ABSTRACT. Let $X$ be a nilpotent rational homotopy type such that (1) $S(X)$, the image of the Hurewicz map has finite total rank, and (2) the basepoint map of $M$, a minimal algebra for $X$, is an element of the Zariski closure of $\text{Aut}(M)$ in $\text{End}(M)$ (i.e. $X$ has "positive weight"). Then (A) any retract of $X$ satisfies the two properties above, (B) any two irreducible product decompositions of $X$ are equivalent, and (C) any two irreducible coproduct decompositions of $X$ are equivalent.

Introduction. This paper is primarily concerned with the uniqueness of irreducible factors in decompositions of rational homotopy types as products and as coproducts (one-point unions). Uniqueness is demonstrated with respect to both products and coproducts, for simply-connected, positive weight, rational homotopy types for which the spherical cohomology is finite dimensional.

There are no known examples of rational homotopy types which fail to satisfy either unique factorization property. Indeed, we conjecture that no such examples exist. This contrasts sharply with the situation in certain more structured geometric contexts; specifically, the noncancellation examples due to Hilton and Roitberg (see [10]) illustrate the failure of unique factorization with respect to products among differentiable manifolds, topological spaces and integral homotopy types. A similar situation exists for coproducts. Hilton and others have constructed examples of integral homotopy types which exhibit noncancellation with respect to the formation of pointed coproducts.

For product decompositions of simply-connected rational homotopy types having positive weights, previous results [3] confirmed unique factorization in the "finitary" case ("finitary" means that either the homotopy or the cohomology are required to be finite dimensional). The "finitary" restriction is removed in this paper, and replaced by the much weaker hypothesis that the Hurewicz homomorphisms are eventually trivial (i.e., have a finite image for all sufficiently high degrees). Moreover, the results reported here represent a significant generalization of the topological results obtained in [1], where (product) unique factorization is proved for formal, simply-connected, rational homotopy types having rational cohomology finitely generated as an algebra.
Recall that a simply-connected, rational homotopy type may be viewed as a “minimal model” $M$; here, $M$ is a differential graded algebra, whose underlying algebra is a simply-connected, free, graded-commutative $Q$-algebra, and whose differential is a degree 1, graded derivation with decomposable image. For details regarding minimal models [8], [9] or [17] should be consulted.

$\text{End}(M)$, the set of differential graded algebra endomorphisms of $M$, is an algebraic variety (defined over $Q$) and it is equipped with a Zariski topology. $\text{Aut}(M)$, the set of invertible endomorphisms, is an open subset of $\text{End}(M)$. The minimal model $M$ is said to have positive weight, if the trivial (basepoint) endomorphism is in the Zariski closure of $\text{Aut}(M)$ in $\text{End}(M)$.

Many familiar spaces satisfy the positive weight condition. For example, all $H$-spaces and co-$H$-spaces, many homogeneous spaces, all formal spaces, and all nonsingular complex varieties [13] have positive weight.

The unique factorization problem may be posed more generally by considering nilpotent rational homotopy types. In this case, for products, the same methods yield an affirmative solution. However, any nilpotent rational homotopy type which is not simply-connected must be irreducible with respect to coproduct. Thus, the unique factorization of nilpotent rational homotopy types with respect to coproducts trivially reduces to the simply connected case.

1. Products, coproducts and splitting idempotents. In this preliminary section an easily verifiable categorical criterion is discussed. Definition (1.1)–(1.4) below is motivated by the question: Under what conditions are product and coproduct splittings determined by idempotents?

Let $\Delta$ be a category with $C$ and $P$ objects of $\Delta$.

Recall that a product structure on $P$ is a collection of morphisms $\{\pi_\alpha: P \rightarrow P_\alpha|\alpha \in J\}$ such that

$$\text{Hom}_\Delta(X, P) \rightarrow \prod_{\alpha \in J} \text{Hom}_\Delta(X, P_\alpha)$$

is a bijection for all objects $X$ of $\Delta$.

Dually, a coproduct structure on $C$ is a collection of morphisms $\{i_\alpha: C_\alpha \rightarrow C|\alpha \in I\}$ such that

$$\text{Hom}_\Delta(C, X) \rightarrow \prod_{\alpha \in I} \text{Hom}_\Delta(C_\alpha, X)$$

is a bijection for all objects $X$ of $\Delta$.

**Definition.** A category $\Delta$ is called $I$-split, if it satisfies

1. $\Delta$ has a zero object.
2. $\Delta$ has finite products and finite coproducts.
3. If $f \in \text{Hom}_\Delta(A, A)$ satisfies $f \circ f = f$ then there is a factorization

$$A \xrightarrow{f} A$$

$\xleftarrow{p}$

such that $p \circ i = 1_B$.
For each object \( A \) of \( \Delta \) there is a finite cardinal \( n(A) \) such that if \( A = \prod_{i \in I} A_i \) (or \( A = \coprod_{i \in I} A_i \)), then \( \text{card}(I) < n(A) \).

For the remainder of this section \( \Delta \) will always denote an \( I \)-split category.

Note. The factorization of (1.3) above is unique up to canonical equivalence in \( \Delta \) because

\[
B \xrightarrow{f} A \xrightarrow{\mu} A
\]

is an equalizer and

\[
A \xrightarrow{\mu} A \rightarrow B
\]

is a coequalizer.

For any object \( A \) of \( \Delta \) let \( E_A = \text{Hom}_\Delta(A, A) \) and \( G_A = \text{Aut}_\Delta(A) \).

Because of (1.3) above a product structure (resp., coproduct structure) on \( P \), \( P \rightarrow \prod P_a \) (resp., \( \prod P_a \rightarrow P \)) is equivalent to a collection \( \{ e_a \} \subseteq E_P \) such that

- (1.5) \( e_a \circ e_a = e_a \) for each \( a \).
- (1.6) \( e_a \circ e_\beta = 0 \) if \( a \neq \beta \).
- (1.7) \( P \rightarrow \prod P_a \) (resp., \( \prod P_a \rightarrow P \)) is an isomorphism where \( e_a = i_a \circ p_a \) is the factorization of (1.3).

Product structures (resp., coproduct structures) will be called \( \prod \)-splittings (resp., \( \coprod \)-splittings). A \( \prod \)-splitting (or \( \coprod \)-splitting) will be called irreducible if each factor \( P_a \) is nontrivial and has no nontrivial \( \prod \)-splitting (resp., \( \coprod \)-splitting).

Two splittings are equivalent if they have isomorphic factors.

Observe that \( G_A \times E_A \rightarrow E_A \), \( (g, f) \mapsto gf^{-1} \) is a group action for each object \( A \) of \( \Delta \). If the \( \prod \)-splitting \( P \rightarrow \prod P_a \) (resp., \( \coprod \)-splitting \( \coprod P_a \rightarrow P \)) is given by \( \{ e_a \} \) as in (1.5)–(1.7), then the \( \prod \)-splitting (resp., \( \coprod \)-splitting) obtained from \( \{ ge_ag^{-1} \} \) is equivalent to the one obtained from \( \{ e_a \} \) for any \( g \in G_A \).

1.8. Lemma. Let \( \{ e_a \} \subseteq E_A \) be a \( \prod \)-splitting (resp., \( \coprod \)-splitting) of \( A \) and let \( f \in E_A \) satisfy \( f = f \circ f \) and \( f \circ e_a = e_a \circ f \) for all \( a \). Then \( \{ g_a \} \subseteq E_B \) is a \( \prod \)-splitting (resp., \( \coprod \)-splitting) of \( B \), where

\[
A \xrightarrow{f} A
\]

from (1.3) and \( g_a = q \circ e_a \circ j \).

Proof. \( \{ g_a \} \subseteq E_B \) is a collection of commuting idempotents. Let

\[
\begin{array}{ccc}
B & \overset{\mu_a}{\longrightarrow} & B \\
& & \overset{\nu_a}{\uparrow} \\
& & B_a
\end{array}
\]

and

\[
\mu_a \circ \nu_a = 1_{B_a}.
\]
Also let $s_a = p_a \circ j \circ r_a : B_a \rightarrow A_a$, $r_a = \mu_a \circ q \circ i_a : A_a \rightarrow B_a$. Observe that $r_a \circ s_a = 1_{B_a}$. Thus we have

\[
\begin{array}{cccc}
\Pi B_a & \Pi \{s_a\} & \Pi A_a & \Pi \{r_a\} & \Pi B_a \\
(r_a) \downarrow & \downarrow (i_a) & \downarrow (r_a) & \\
B & A & B \\
(\kappa_a) \downarrow & \downarrow (p_a) & \downarrow (\kappa_a) & \\
\Pi B_a & \Pi \{s_a\} & \Pi A_a & \Pi \{r_a\} & \Pi B_a \\
\end{array}
\]

where $(\Pi (r_a)) \circ (\Pi (s_a)) = 1_{\Pi B_a}$, $q \circ j = 1_B$, and $(\Pi (r_a)) \circ (\Pi (s_a)) = 1_{\Pi B_a}$.

Thus the result follows in either case because a retract of an isomorphism is an isomorphism. Q.E.D.

1.9. Proposition. Let $A$ be an object of $\Delta$. Suppose $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_m)$ are irreducible $\Pi$-splittings (resp., $\Pi$-splittings) of $A$. If $e_a \circ f_\beta = f_\beta \circ e_\alpha$ for all $\alpha$ and $\beta$, then $m = n$ and the splittings of $A$ are equivalent.

The proof of Proposition 1.9 is a straightforward application of Lemma 1.8 and is left to the reader. (A model for the proof may be found in the proof of Theorem 2 in [2].)

We say that two splittings (either $\Pi$ or $\Pi$) $(e_a)$ and $(f_\beta)$ of $A$ are compatible if there exists $g \in G_A$ such that $e_a \circ g \circ f_\beta \circ g^{-1} = g \circ f_\beta \circ g^{-1} \circ e_\alpha$ for each $\alpha$ and $\beta$.

1.10. Corollary. Let $A$ be an object of $\Delta$. If all irreducible $\Pi$-splittings (resp., $\Pi$-splittings) of $A$ are compatible, then they are all equivalent. In fact, any two such splittings are conjugate.

Corollary 1.10 follows directly from Proposition 1.9.

Remark. Assume that $\Delta$ is skeletally small. Let $\Delta'$ be the set of isomorphism classes of objects of $\Delta$. Then the conclusion of Corollary 1.10 is equivalent to the assertion that $\Delta'$ is a free commutative semigroup under $\Pi$ (resp., $\Pi$). (Here, "skeletally small" means that the class of isomorphism classes of objects in the category form a set.)

2. Splittings of minimal algebras. In this section and the next we prove our main results. Most of the discussion is focussed on proving the following two assertions.

(A) The category $\mathcal{P}$ of minimal algebras with positive weights is $I$-split (Definition (1.1)–(1.4)).

(B) Every object $M$ of $\mathcal{P}$ satisfies the assumptions of Corollary 1.10.

Before proceeding to the details we outline the structure of the proof of (B).

Observe that $\text{Aut}(M) \subseteq \text{End}(M)$ is an open imbedding of an affine $Q$-group in an affine variety (cf. Remark 3.1, below).

If $M$ is an object of $\mathcal{P}$ and $M \rightarrow \prod M_a$ is an isomorphism, then (up to homotopy) $\{e_a\} \subseteq \overline{T} \subseteq \text{Aut}(M)$ (Zariski closure) where

$$M \xrightarrow{e_a} M \xleftarrow{p_a} M_a \xrightarrow{i_a} M$$
as in (1.7) and $T$ is a $Q$-split torus. (This is part of the reason for our categorical strategy of §1.) Thus any splitting adheres to a maximal $Q$-split torus.

Maximal $Q$-split tori are conjugate. Thus any two irreducible splittings are compatible; in fact, they are conjugate.

2.1. Notation and definitions. A differential graded algebra (d.g.a.) is a pair $(A, d_A)$ where $A = \bigoplus_{n \geq 0} A^n$ is a nonnegatively graded, associative algebra over $Q$ with identity, such that $a \cdot b = (-1)^{pq} b \cdot a$ for $a \in A^p$ and $b \in A^q$, and $d_A$ is a degree 1 derivation such that $d_A^2 = 0$.

A morphism of d.g.a.'s is a graded algebra map which commutes with the differentials.

A d.g.a. $A$ is called connected if $A^0 = Q$ and simply-connected if in addition $A^1 = 0$.

Note that if $A$ is connected then $A$ has a unique augmentation $\epsilon_A : A \to Q$.

For the purposes of this paper we make the following definition.

A minimal algebra is a d.g.a. $M$ such that $M$ is simply-connected, $M^n$ is finite-dimensional for all $n$, $M$ is free as a graded commutative algebra, and $d_M(M^n) \subseteq M^+ \cdot M^+$ (the image of $d_M$ is decomposable).

A minimal algebra $M$ has positive weights if there is a direct sum decomposition $M = \bigoplus_{a,n \geq 0} aM^n$ such that $M^n = \bigoplus_{a \geq 0} aM^n$, $d_M(aM^n) \subseteq aM^{n+1}$, $aM \cdot \beta M \subseteq a + \beta M$ and $\alpha M = M^0$. Such a direct sum decomposition is called a positive weight splitting.

In general a given minimal algebra may possess many distinct positive weight splittings, or none at all.

2.2. Remark. A very satisfying topological interpretation of this condition is given in [4]. One striking feature is the following: Let $X$ be a finitary simply-connected C.W. complex and let $M_X$ be its minimal algebra. Then $M_X$ has positive weights if and only if there is a prime $p$ and a map $\lambda_p : X \to X$ such that $\lambda_p : M_X \to M_X$ is an isomorphism and

$$\lambda_p \otimes 1 : \pi_\ast(X) \otimes \mathbb{Z}/p\mathbb{Z} \to \pi_\ast(X) \otimes \mathbb{Z}/p\mathbb{Z}$$

is the zero morphism. Furthermore, this is independent of the prime $p$.

2.3. The category $\mathcal{P}$. Let $M$ be a minimal algebra and let $Z(M) = \{x \in M | dx = 0\}$, $M^+ = \bigoplus_{k \geq 0} M^k$ and $Q(M) = M^+/(M^+ \cdot M^+)$. There is a linear map

$$Z^+(M) \to M^+ \to Q(M)$$

which is inclusion followed by projection (the "Hurewicz map"). Let $S(M) = \text{Image}(\pi \circ i)$.

The category $\mathcal{P}$ has as its objects the minimal algebras $M$ such that $M$ has positive weights (2.1), and $\dim_Q S(M) < \infty$. The morphisms of $\mathcal{P}$ are the d.g.a. homotopy classes of d.g.a. maps.

Note that d.g.a. homotopies can be defined and computed with either a path or a cylinder object. This is, if $f, g : M \to N$ are d.g.a. maps then $f$ is homotopic to $g$.
(f ≃ g) if either

\[
\begin{align*}
F &\ni (e_0 e_1) \\
M &\to N \times N \\
\langle f, g \rangle
\end{align*}
\]

commutes for some d.g.a. map \( F \), or

\[
\begin{align*}
\lambda_0 \lambda_1 &\uparrow \\
M \otimes M &\to N \\
\langle f, g \rangle
\end{align*}
\]

commutes for some d.g.a. map \( G \).

An explicit development of both these notions and their interrelationship can be found in Chapters 5 and 11 of [9].

In order to prove that the category \( \mathcal{P} \) is \( J \)-split (so that we are able to apply Corollary 1.10) we shall have to exploit the good behaviour of homotopies under the passage to inverse limits (a property peculiar to the rational homotopy categories).

2.4. Lemma. Let \( f, g : M \to N \) be d.g.a. maps, where \( M \) is minimal and both \( M \) and \( N \) are finite type. Then \( f \simeq g \) if and only if \( f_n \simeq g_n \) for all \( n \), where \( f_n = f|_{M_n} \) and \( M_n \) is the subalgebra generated by elements of degree \(< n \).

Proof. In [6] the functors \( A : S \to \mathrm{DGA} \) and \( F : \mathrm{DGA} \times \mathrm{DGA} \to S \) are defined where \( S \) is the category of simplicial sets.

\( A \) is the simplicial de Rham functor and \( F \) is the simplicial function space construction for \( \mathrm{DGA} \). Further [6, p. 25] the adjunction

\[
\Phi : \text{Hom}_{\mathrm{DGA}}(M, A(W) \otimes N) \to \text{Hom}_S(W, F(M, N))
\]

is proved under mild assumptions (\( W \) finite or \( M \) finite type). The following facts are then deduced:

(i) If \( i : M \to M' \) is a cofibration, then \( F(M', N) \to F(M, N) \) is a Kan fibration [6, p. 26].

(ii) If \( M \) is minimal then \( [M, N] = \pi_0 F(M, N) \) where \([ , ]\) denotes d.g.a. homotopy classes [6, p. 32].

An induction argument (similar to Lemma 2.7 of [16]) shows that for \( M' \) finitely generated minimal, \( \pi_r(F(M', N), f) \) is a \( Q \)-nilpotent group for \( r > 1 \) (\( N \) finite type).

We can write for \( M \) finite type minimal \( Q \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq \text{ind lim}_n M_n = M \).

By (i) above we obtain a tower of Kan fibrations \( F(M, N) = \text{proj lim}_n F(M_n, N) \to \cdots \to F(M_n, N) \to \cdots \to * \).

If further we let \( f_n \in F(M_n, N) \) be the basepoint we have the following exact sequence of sets (see [7, p. 254])

\[
* \to \text{proj lim}_n \pi_1(F(M_n, N)) \to \pi_0(F(M, N)) \to \text{proj lim}_n \pi_0(F(M_n, N)) \to *.
\]

But \( \text{proj lim}_n \pi_1(F(M_n, N)) \simeq * \) because the inverse tower of nilpotent groups \( \{ \pi_1(F(M_n, N), f_n) \} \) satisfies a Mittag-Leffler condition. Thus,

\[
[M, N] = \pi_0(F(M, N)) \simeq \text{proj lim}_n \pi_0(F(M_n, N)) \simeq \text{proj lim}_n [M_n, N].
\]
Remark. The proof above relies on simplicial techniques. Alternatively, a topological argument, based on Lemma 2.7 of [16], can be constructed using elementary obstruction theory. It is curious that no elementary algebraic proof is available.

2.5. Proposition. The category \( \mathcal{P} \) is \( I \)-split.

Proof. \( Q \), the d.g.a. concentrated in degree zero, is the zero object of \( \mathcal{P} \).

\( \mathcal{P} \) has finite coproducts. For if \( M \) and \( N \) are objects of \( \mathcal{P} \) then so too is \( M \otimes N \). If \( M = \bigoplus_{a>0} aM \) and \( N = \bigoplus_{\beta>0} \beta N \) are positive weight splittings of \( M \) and \( N \) respectively, then \( M \otimes N = \bigoplus_{\gamma>0} \left( \bigoplus_{a+\beta=\gamma} (aM \otimes \beta N) \right) \) is a positive weight splitting of \( M \otimes N \) (where \( \gamma(M \otimes N) = \bigoplus_{a+\beta=\gamma} aM \otimes \beta N \)).

\( \mathcal{P} \) has finite products. For if \( M \) and \( N \) are objects of \( \mathcal{P} \) let \( MXN \) be the product of simply-connected d.g.a.'s. That is,

\[
\begin{array}{c}
M \times_Q N \\
N
\end{array}
\]

\[
\begin{array}{c}
P_N \downarrow \\
\epsilon_N
\end{array}
\]

\[
\begin{array}{c}
M \\
Q
\end{array}
\]

is a pull-back diagram. \( M \times_Q N \) is not minimal, but by Theorem 6.1 of [9] there is a minimal algebra \( M[N \) and a map \( \rho: M[N \rightarrow M \times_Q N \) such that \( \rho*: H^*(M[N) \rightarrow H^*(M \times_Q N) \) is an isomorphism.

Thus we have

\[
\begin{array}{c}
M \times_Q N \\
M \times Q N
\end{array}
\]

\[
\begin{array}{c}
P_M \\
P_M
\end{array}
\]

\[
\begin{array}{c}
M \\
Q
\end{array}
\]

where \( P_M \) and \( P_N \) are defined by the commutativity of the diagram.

If \( L \) is a minimal algebra and \( f: L \rightarrow M \) and \( g: L \rightarrow N \) are d.g.a. maps, then there is a unique d.g.a. map \( \phi: L \rightarrow M \times Q N \) such that

\[
P_M \cdot \phi = f \quad \text{and} \quad P_N \cdot \phi = g.
\]

By Proposition 6.4 of [6] the lifting problem

\[
\begin{array}{c}
M \times_Q N \\
\rightarrow
\end{array}
\]

has a solution \((f, g): L \rightarrow M[N \] unique up to d.g.a. homotopy.

If \( f \simeq f' \) and \( g \simeq g' \) then \((f, g) \simeq (f', g') \). For let

\[
F: L \rightarrow M, \quad F: f \simeq f',
\]

\[
G: L \rightarrow N, \quad G: g \simeq g'.
\]

Then \( L \rightarrow L \times L \rightarrow M \times Q N \) is a homotopy: \((f, g) \simeq (f', g') \). Thus

\[
\begin{array}{c}
M \times_Q N \\
M
\end{array}
\]

\[
\begin{array}{c}
P_M \\
P_M
\end{array}
\]

\[
N
\]

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is a product diagram in the homotopy category of minimal algebras. $M[N]$ is an object of $\mathcal{P}$ since $S(M[N]) = S(M) \oplus S(N)$ and by Lemma 3.13 below we can “lift positive weight splittings over weak equivalences” in this situation. (If $M = \bigoplus_{a \geq 0} aM$ and $N = \bigoplus_{\beta \geq 0} \beta N$ are positive weight splittings, then $M \times_Q N = Q \oplus (\bigoplus_{\gamma \geq 0} (\gamma M \oplus \gamma N))$ is a positive weight splitting of $M \times_Q N$. Now apply Lemma 3.13 to the map $\rho: M[N] \to M \times_Q N$.)

If $M$ is an object of $\mathcal{P}$ and $f \in \text{Hom}_\mathcal{P}(M, N)$ satisfies $f \circ f = f$, then there is a factorization (in $\mathcal{P}$)

\[
\begin{array}{ccc}
M & \rightarrow & M \\
p \searrow & & \nearrow i \\
\downarrow & & \downarrow \\
N & \rightarrow & N
\end{array}
\]

such that $p \circ i = 1_M$. This follows from 3.12 below.

If $M$ is an object of $\mathcal{P}$ then by definition (see 2.3) $\dim_Q S(M) < \infty$. So if $M \cong M_1 \amalg M_2$ (or $M_1[N] \amalg M_2$) then $S(M) \cong S(M_1) \oplus S(M_2)$. Also if $S(M) \cong (0)$ then $M \cong Q$. Thus put $n(M) = \dim_Q S(M)$ and $M$ will satisfy the conditions of (1.4).

Summing up, we have shown that $\mathcal{P}$ satisfies conditions (1.1)–(1.4). Q.E.D.

Because of Proposition 2.5 our questions about unique factorization in $\mathcal{P}$ have been reduced to the study of conjugacy properties of idempotents (by Corollary 1.10). This necessitates the study of the topological properties of $\text{Aut}(M) \subseteq \text{End}(M)$ for each object $M$ of $\mathcal{P}$.

3. Algebraic groups of automorphisms. If $A$ is a finitely generated d.g.a. then $\text{End}(A)$ is an affine $Q$-variety and $\text{Aut}(A)$ is an affine $Q$-group. Further, $\text{Aut}(A) \subseteq \text{End}(A)$ is an open imbedding.

This follows from the following elementary observations:

$\text{End}(A)$ is faithfully represented on a finite dimensional subspace of $A$.

Preservation of the multiplicative structure is a quadratic relation (hence algebraic) and commutation with $d_A$ is a linear relation.

$\text{Aut}(A) = \text{det}^{-1}(Q \setminus \{0\})$ is open because $\text{det}: \text{End}(A) \to Q$ is continuous.

3.1. Remark. Properly speaking $\text{Aut}(A)$ and $\text{End}(A)$ are not algebraic varieties, but are the $Q$-rational points of the $Q$-varieties $\text{Aut}(A \otimes_Q K)$ and $\text{End}(A \otimes_Q K)$, respectively, where $K$ is an algebraic closure of $Q$. This is the point of view adopted in [5] which is our basic reference for the theory of algebraic groups.

Our slight heresy shall not cause difficulties. (To translate from our terminology to that of [5], one need only insert the words, “the $Q$-rational points of”, where appropriate.)

3.2. A positive weight splitting on a finitely generated minimal algebra $M = \bigoplus_{a \geq 0} aM$ induces a morphism of $Q$-algebraic groups ($Q$-group) $\lambda: Q^* \to \text{Aut}(M)$ where $\lambda(t)(x) = t^a \cdot x$ for $x \in aM$, and $t \in Q^*$. Further, $\lambda$ extends to a morphism of varieties $\tilde{\lambda}: Q \to \text{End}(M)$ such that $\tilde{\lambda}(0) = O_M$.

Thus, if $M$ is an object of $\mathcal{P}$, then $O_M \in \text{Aut}(M)$ (Zariski closure) if $M$ is finitely generated.

Conversely, if $O_M \in \text{Aut}(M)$ then $M$ is in $\mathcal{P}$ (for $M$ finitely generated).
The proof of this assertion will only be outlined. A general discussion of toroidal symmetry of minimal algebras is advanced in [14].

Step 1. If \( O_M \in \overline{\text{Aut}(M)} \) then \( O_M \in \overline{T} \) where \( T \subseteq \text{Aut}(M) \) is any maximal \( Q \)-split torus. This follows from the following more general result about certain representation of algebraic groups.

Let \( G \subseteq \text{Gl}(V) \subseteq \text{Hom}_Q(V, V) \) be an algebraic subgroup such that \( O_V \in \overline{G} \).

Then \( O_V \in \overline{T} \) where \( T \subseteq G \) is any maximal \( Q \)-split torus. A detailed proof is contained in [14].

Step 2. If \( O_V \in \overline{T} \) (as above) then there is a one-parameter subgroup (1-p.s.g.) \( \lambda: Q^* \to T \) such that \( \lambda \) extends to \( \overline{\lambda}: Q \to \overline{T} \) with \( \overline{\lambda}(0) = O_V \).

This follows from the following result proved in [11].

If \( T \to \overline{T} \) is a toroidal imbedding (\( T \to \overline{T} \) is an open dominant imbedding and, \( T \times T \to T \) extends to an action \( T \times \overline{T} \to \overline{T} \), then for any orbit \( X \) in \( \overline{T} \setminus T \) there is a 1-p.s.g. \( \lambda: Q^* \to T \) extending to \( \overline{\lambda}: Q \to \overline{T} \) with \( \overline{\lambda}(0) \in X \).

Thus, the assertion above follows because \( O_V = T \cdot O_V \) is the unique point in its orbit.

Step 3. If \( \lambda: Q^* \to T \subseteq \text{Aut}(M) \) is a 1-p.s.g. that extends to \( \overline{\lambda}: Q \to \text{End}(M) \) with \( \overline{\lambda}(0) = O_M \), then this induces a positive weight splitting on \( M \).

This requires an elementary proof from linear algebra. For each \( t \in Q^* \), \( \lambda(t) \) is \( Q \)-diagonalizable with eigenvalues \( t^a \) for various integers \( a \). Thus \( M = \bigoplus_a \mathcal{M} \), where \( \mathcal{M} \) is the eigenspace of \( \lambda(t) \) with eigenvalue \( t^a \). All weights are nonnegative because \( \lambda \) extends to \( \overline{\lambda} \), \( \mathcal{M} = M^0 \) because \( \overline{\lambda}(0) = O_M \).

Thus, we have established the following characterization of finitely generated minimal algebras \( M \) with positive weights:

\( M \) has positive weights if and only if \( O_M \in \overline{\text{Aut}(M)} \) (Zariski closure) within \( \text{End}(M) \).

3.3. Proposition. If \( M \) is a finitely generated object of \( \mathcal{G} \) and \( N \) is a retract of \( M \), then \( N \) is in \( \mathcal{G} \).

Proof. From the remarks above, we need only prove that \( O_N \in \overline{\text{Aut}(N)} \) (because retracts of minimal algebras are minimal).

Let \( N \xrightarrow{i} M \xrightarrow{r} N \) satisfy \( r \circ i = 1_N \). This induces

\[ \text{End}(M) \to \text{End}(N), \quad \phi(f) = r \circ f \circ i, \]

a morphism of varieties. Now \( O_M \in \overline{\text{Aut}(M)} \). Thus \( O_M \in (\overline{\text{Aut}(M)})^0 \) (the irreducible component of \( 1 \)). \( \phi((\overline{\text{Aut}(M)})^0) \subseteq \text{End}(N) \) is irreducible, as is its closure, \( \phi((\overline{\text{Aut}(M)})^0) \). Moreover,

\[ 1_N \in (\overline{\text{Aut}(N)}) \cap \phi((\overline{\text{Aut}(M)})^0) = S. \]

Further, \( S \subseteq \overline{\phi((\overline{\text{Aut}(M)})^0)} \) is open. Thus,

\[ O_N \in \phi((\overline{\text{Aut}(M)})^0) \subseteq \overline{\phi((\overline{\text{Aut}(M)})^0)} = S. \]

Therefore, \( O_N \in \overline{\text{Aut}(N)} \). Q.E.D.
Thus, to complete the proof of Proposition 2.5 we still have to prove the following two assertions.

(3.4) (Homotopy) idempotents split in \( \mathcal{P} \).

(3.5) Proposition 3.3 is true without assuming \( M \) is finitely generated. (Thus \( \mathcal{P} \) is closed under retracts.)

Assertion (3.4) follows easily with the aid of Lemma 2.4. Let \( e: M \to M \) represent an idempotent homotopy class (thus, \( e \sim e \circ e \)). Now let

\[
N = \bigcap_{n \geq 1} \text{Image } e^n \quad \text{and} \quad J = \bigcup_{n \geq 1} \text{kernel } e^n.
\]

Then \( N \subseteq M \) is a d.g. subalgebra and \( J \subseteq M \) is a d.g. ideal. Further \( M = N \oplus J \). Thus

\[
N \xrightarrow{i} M \xrightarrow{p} N \text{ is a retract.}
\]

A routine verification shows that

\[
i \circ p \big|_{M_n} \simeq e \big|_{M_n} \quad \text{for all } n > 0.
\]

By Lemma 2.4 \( i \circ p \simeq e \).

The proof of assertion (3.5) requires the following discussion of pro-algebraic groups.

(3.6) If \( M \) is not finitely generated, then \( \text{Aut}(M) \) and \( \text{End}(M) \) are no longer algebraic varieties. However, the situation can be remedied as follows. Every object \( M \) of \( \mathcal{P} \) has a canonical series

\[
Q = M_1 \subseteq M_2 \subseteq \cdots \subseteq \text{ind lim}_n M_n = M
\]

(where \( M_n \) is the subalgebra generated by elements of degree \(< n \)) which induces

\[
\begin{align*}
\text{Aut}(M_2) & \leftarrow \cdots \leftarrow \text{proj lim}_n \text{Aut}(M_n) = \text{Aut}(M) \\
\text{End}(M_2) & \leftarrow \cdots \leftarrow \text{proj lim}_n \text{End}(M_n) = \text{End}(M).
\end{align*}
\]

Thus \( \text{Aut}(M) \) and \( \text{End}(M) \) are pro-algebraic varieties (inverse limits of algebraic varieties). In order to make use of this observation we shall need the following facts from algebraic group theory and general topology.

Let \( G \) be a \( Q \)-group. If \( \mathfrak{W}_G = \{ F \mid F \text{ is a finite union of cosets of } Q \text{-closed subgroups of } G \} \cup \{ \emptyset \} \) then \( \mathfrak{W}_G \) is the set of closed sets for a \( T_1 \) compact topology on \( G \), the \( \mathfrak{W} \)-topology. If \( \phi: G \to H \) is a morphism of \( Q \)-groups then \( \phi \) is continuous for the \( \mathfrak{W} \)-topology. If Kernel \( \phi \) is a unipotent subgroup of \( G \) then \( \phi(G) \) is a \( Q \)-closed subgroup of \( H \) [5, p. 363]. Consequently, for such \( \phi: G \to H \), \( \phi \) is a closed map in the \( \mathfrak{W} \)-topology.

3.7. Projective Limit Theorem. If \( \{ X_i, \pi_{ij} \} \) is an inverse system of topological spaces and continuous maps such that

(a) for each \( i \), \( X_i \neq \emptyset \) is compact and \( T_1 \),

(b) \( \pi_{ij}: X_i \to X_j \) is closed for each \( i > j \),

then \( \text{proj lim}_i X_i \) is nonempty.

Such a system will be called a proper system and \( \text{proj lim}_i X_i \) will be called a proper limit. For a proof, see [14, Theorem 1.2.1] or [15, Theorem 3].
(3.8) If \( M \) is a minimal algebra, then the natural map \( S: \text{Aut}(M) \to \text{Aut}(S(M)) \) has a locally unipotent kernel. That is, for all \( \phi \in \ker S \) and all \( x \in M \) there exists \( n_x \) such that \( (\phi - 1)^{n_x}(x) = 0 \).

This is easily proved by induction along the canonical series of \( M \). See for example [17]. Thus we have

\[
\begin{array}{cccc}
\text{Aut}(M) & \xrightarrow{S} & \text{Aut}(S(M)) \\
\downarrow & & \downarrow \\
\text{Aut}(M_{n-1}) & \xrightarrow{S} & \text{Aut}(S(M_{n-1})) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}
\]

If \( S(M) = S(M_N) \) for some \( N \) (for example, if \( M \) is an object of \( \mathfrak{S} \)) we have

\[
\text{Aut}(M_{N+k}) \xrightarrow{r_{N+k,N}} \text{Aut}(M_N) \xrightarrow{S} \text{Aut}(S(M_N))
\]

for all \( k \geq 0 \). Hence, \( \ker r_{N+k,N} \) is unipotent.

3.9. Proposition. Let \( M \) be a minimal algebra such that \( \dim_Q S(M) < \infty \) and let \( G_n = \text{Aut}(M_n) \). Then for sufficiently large integers \( N \),

\[
* \leftarrow G_N \leftarrow G_{N+1} \leftarrow \cdots
\]

is a proper system of topological spaces for the \( \mathfrak{W} \)-topology.

Proof. (3.6), 3.7 and (3.8) above.

Remark. If \( A_k \subseteq G_k \) are nonempty \( \mathfrak{W} \)-closed sets and \( r_{k,l}(A_k) \subseteq A_l \), then \( \{A_k, r_{k,l}|A_k\} \) is also a proper system.

Much of the structure theory of \( Q \)-groups remains valid for proper limits of \( Q \)-groups [14]. In particular, the following theorem is central to our discussion of splitting idempotents of minimal algebras.

3.10. Theorem. If \( \{G_i, \tau_i\} \) is a proper system of \( Q \)-groups, then there exists \( T \subseteq G = \text{proj lim } G_i \) such that

(a) \( T \) is a closed subgroup of \( G \) (in the \( \mathfrak{W} \)-topology).
(b) \( \tau_i(T) \subseteq G_i \) is a \( Q \)-split torus for all \( i \).
(c) \( T \) is maximal with respect to conditions (a) and (b).

If \( S \subseteq G \) satisfies (a), (b) and (c) then there exists \( g \in G \) such that \( gSg^{-1} = T \).

Proof. In the algebraic case \( (G_i = G_j \) for all \( i \) and \( j \)) this is a standard result (see [5, p. 263]). The general case follows from two applications of the Projective Limit Theorem 3.7.
Assume for the moment that \( \pi_{ij}: G_i \to G_j \) is onto for \( i > j \). Then each \( \pi_{ij} \) induces \( \mathfrak{T}(\pi_{ij}): \mathfrak{T}(G_i) \to \mathfrak{T}(G_j) \) where \( \mathfrak{T}(G_i) = \{ T \subseteq G_i | T \) is a maximal \( Q \)-split torus\}. \( \mathfrak{T}(G_i) \) can be given the structure of a homogeneous space over \( G_i \) in such a way that \( \{ \mathfrak{T}(G_i), \mathfrak{T}(\pi_{ij}) \} \) is a proper system. Thus

\[
\text{proj lim } \mathfrak{T}(G_i) \text{ is nonempty.}
\]

Given \( \{ T_i \} \in \text{proj lim } \mathfrak{T}(G_i) \) it is a routine verification to show that \( T = \text{proj lim } T_i \) satisfies (a), (b) and (c).

If \( S \subseteq \text{proj lim } G_i \) satisfies (a), (b) and (c) then \( S_i = \pi_i(S) \subseteq G_i \) is a maximal \( Q \)-split torus for each \( i \), and \( S = \text{proj lim } \pi_i(S) \). But

\[
\text{Tran}_{G_i}(S_i, T_i) = \left\{ g \in G_i | g S_i g^{-1} = T_i \right\}
\]

is closed and nonempty for all \( i \). Thus \( \text{Tran}_G(S, T) = \text{proj lim } \text{Tran}_G(S_i, T_i) \) is nonempty by the Projective Limit Theorem 3.7.

Hence, the proof is complete once we have justified the assumption that each \( \pi_{ij}: G_i \to G_j \) is onto.

Observe that \( G \supseteq \text{proj lim } \mathfrak{T}(G) \) and that \( \pi_i(G) = \cap_{j \geq i} \pi_{ij}(G) \).

Thus \( G \) is a proper limit of the proper system \( \{ \pi_i(G), \pi_{ij} \} \). Q.E.D.

For the first application of Theorem 3.10 we complete the proof of the Proposition 2.5.

3.11. Proposition. \( M \) is an object of \( \mathfrak{P} \) if and only if \( M_n \) is an object of \( \mathfrak{P} \) for all \( n \) and \( \text{dim}_Q \text{S}(M) < \infty \).

Proof. If \( M \) is in \( \mathfrak{P} \), then any positive weight splitting on \( M \) restricts to a positive weight splitting on \( M_n \).

Conversely, if \( M_n \) is in \( \mathfrak{P} \) for all \( n \) and \( \text{dim}_Q \text{S}(M) < \infty \) we shall construct a 1-p.s.g. \( \lambda: Q^* \to \text{Aut}(M) \) such that \( \lambda \) extends to \( \bar{\lambda}: Q \to \text{Aut} M \) with \( \bar{\lambda}(0) = O_M \).

By Theorem 3.10 and Proposition 3.9 there exist tori \( T_i \subset \text{Aut } M_i \) for each \( i > 0 \) such that \( \text{proj lim } T_i = T \) is a maximal torus in \( \text{Aut}(M) \). Since tori are faithfully represented on \( S(M) \) (which is finite dimensional) there exists \( N \) such that \( \text{dim } T_n = \text{dim } T_N = S(M_n) = S(M_N) \) for all \( n > N \). Since \( \text{ker } r_{k,N} \) is unipotent for \( k > N \) it follows that \( r_{i,j}: T_i \to T_j \) is an isomorphism for \( i > j > N \).

By assumption that exist \( \lambda_i: Q^* \to T_i \ (i > N) \) such that \( \lambda_i \) extends to \( \bar{\lambda}_i: Q \to \bar{T}_i \) (Zariski closure) with \( \bar{\lambda}_i(0) = O_{M_i} \).

For \( i > N \) we have

\[
\begin{array}{c}
\vdots \\
\downarrow = \\
T_{i+2} \subseteq \text{Aut}(M_{i+2}) \\
\downarrow = \\
T_{i+1} \subseteq \text{Aut}(M_{i+1}) \\
\downarrow = \\
Q^* \rightarrow T_i \subseteq \text{Aut}(M_i)
\end{array}
\]
Thus, there exist unique liftings which make the whole diagram commute. Since $S^n(M) = 0$ for all $n > N$, $\lambda = \text{proj lim} \lambda_i$ extends to $\bar{\lambda}: Q \to \bar{T} = \text{proj lim} \bar{T}_i$ with $\bar{\lambda}(0) = O_M$ (since all eigenvalues of $\lambda$ are multiplicatively generated on $S(M)$).

Q.E.D.

3.12. Conclusion of Proposition 2.5. Suppose $N \to M \to N$ satisfies $r \circ i = 1_N$, where $M$ is in $\mathcal{P}$. Then $N_n \to M_n \to N_n$, $r_n \circ i_n = 1_{N_n}$. By Proposition 3.3, $N_n$ is in $\mathcal{P}$ for all $n$. Then by Proposition 3.11, $N$ is in $\mathcal{P}$.

Hence we have completed the proof that $\mathcal{P}$ is an $I$-split category.

In order to prove unique factorization in $\mathcal{P}$ we need to prove that any two irreducible $\|\$-splittings (resp., $\|\$-splittings) of a fixed object $M$ of $\mathcal{P}$ are compatible. This too we deduce with the aid of Proposition 2.5.

Let $A$ be a simply-connected d.g.a. of finite type such that $\rho: M_A \to A$ is onto, where $M_A$ is minimal and $\rho^*: H^*(M_A) \to H^*(A)$ is an isomorphism. Let $T \subseteq \text{Aut}(A)$ be a $Q$-split torus and let

$$B = \{(\alpha, \beta) \in (\text{Aut}(M_A)) \times T | \rho \circ \alpha = \beta \circ \rho\}.$$  

3.13. Lemma. In the pull-back diagram

$$\begin{array}{ccc}
B & \to & T \\
\downarrow q & & \downarrow \rho^* \\
\text{Aut}(M_A) & \to & \text{Hom}(M_A, A)
\end{array}$$

(a) $q$ is one-to-one.
(b) $r$ is onto.
(c) $B \to T$ splits as a morphism of $Q$-groups.

Proof. If $q(\alpha, \beta) = q(\alpha, \beta')$ then $\beta \circ \rho = \beta' \circ \rho$. Thus $\beta = \beta'$ because $\rho$ is onto. This proves (a).

Let $\beta \in T$. Then in

$$\begin{array}{ccc}
M_A & \to & M_A \\
\rho \downarrow & & \downarrow \rho \\
A & \to & \beta \quad A
\end{array}$$

the dotted arrow exists because $\rho$ is a fibration and a weak equivalence. (See [6].) This proves (b).

Kernel($r$) = \{($\alpha, \beta$)$|$ $\beta = 1$\}.

That is,

$$\begin{array}{ccc}
M_A & \to & M_A \\
\rho \downarrow & & \downarrow \rho \\
A & \to & A
\end{array}$$

if $\alpha \in \text{Kernel}(r)$. 

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Thus $\alpha \simeq 1$ because the lifting problem has a solution unique up to homotopy. In particular, $\text{Kernel}(r)$ is unipotent. Thus by the Proposition on p. 219 of [5], $B \rightarrow T$ splits because $B$ is a $Q$-split $Q$-group. Q.E.D.

Let $M$ be an object of $\mathcal{P}$ and suppose $M \rightarrow \prod M_{\alpha}$ is an isomorphism (in $\mathcal{P}$, where morphisms are homotopy classes, as in 2.3). Let $\{g_{\alpha}\} \subseteq \text{End}_{\mathcal{P}}(M)$ be the corresponding splitting idempotents (since $\mathcal{P}$ is $I$-split). Thus

$$
\begin{array}{ccc}
M & \xrightarrow{g_{\alpha}} & M \\
p_{\alpha} & \nearrow & i_{\alpha} \\
M_{\alpha}
\end{array}
$$

commutes in $\mathcal{P}$ and $g_{\alpha} = g_{\alpha} \circ g_{\alpha}$ in $\mathcal{P}$ for all $\alpha$. So if we choose d.g.a. maps $\{e_{\alpha}\} \subseteq \text{End}(M)$, $e_{\alpha}$ representing $g_{\alpha}$ for each $\alpha$, we have $e_{\alpha} \simeq e_{\alpha} \circ e_{\alpha}$.

3.14. LEMMA. Let $M$ and $\{e_{\alpha}\}$ be as above. Then there exist a $Q$-split torus $T \subseteq \text{Aut}(M)$ and a collection $\{e'_{\alpha}\} \subseteq T$ such that

(a) $e'_{\alpha} \simeq e_{\alpha}$ for all $\alpha$, and

(b) $e'_{\alpha} = e'_{\alpha} \circ e'_{\alpha}$ for all $\alpha$.

PROOF. We have a commutative diagram in $\mathcal{P}$

$$
\begin{array}{ccc}
\prod M_{\alpha} & \xrightarrow{(p_{\alpha})} & \prod M_{\alpha} \\
M & \xrightarrow{(p_{\alpha})} & \times Q M_{\alpha}
\end{array}
$$

and a homotopy-commutative diagram of d.g.a. morphisms

$$
\begin{array}{ccc}
M & \xrightarrow{\epsilon_{\alpha}} & M \\
p \downarrow & & \downarrow \rho \\
\times Q M_{\alpha} & \xrightarrow{f_{\alpha}} & \times Q M_{\alpha}
\end{array}
$$

where $f_{\alpha}$ is projection followed by inclusion and $\rho$ represents $(p'_{\alpha})$. Define

$$
\prod Q^{*} \times \cdots \times Q^{*} \xrightarrow{\lambda} \text{Aut}(\times Q M_{\alpha})
$$

$$
\prod \lambda_{\alpha} \xrightarrow{\lambda} \prod \text{Aut}(M_{\alpha})
$$

as follows. Let $M_{\alpha} = \bigoplus_{k \geq 0} k(M_{\alpha})$ be a positive weight splitting of $M_{\alpha}$ and let $\lambda_{\alpha}(t)(x) = t^{k_{x}} \cdot x$ for $x \in k(M_{\alpha})$. $\prod \lambda_{\alpha}$ and $j$ are the obvious maps.

By Lemma 3.13 there is a $Q$-split torus $T \subseteq \text{Aut}(M)$ such that the lifting problem

$$
\begin{array}{ccc}
M & \rightarrow & M \\
p \downarrow & & \downarrow \rho \\
\times Q M_{\alpha} & \xrightarrow{\beta} & \times Q M_{\alpha}
\end{array}
$$

has a unique solution in $T$ for each $\beta \in \lambda(Q^{*} \times \cdots \times Q^{*}) = \Sigma$.

Since each weight splitting is nonnegative, $\{f_{\alpha}\} \subseteq \overline{S} \subseteq \text{Aut}(\times Q M_{\alpha})$ and there are 1-p.s.g.'s $\omega_{\alpha} : Q^{*} \rightarrow S$ which extend to $\omega : Q \rightarrow \overline{S}$ with $\omega_{\alpha}(0) = f_{\alpha}$. Thus each
PRODUCT AND COPRODUCT DECOMPOSITIONS

1-p.s.g. \( \omega_\alpha : Q^* \to S \) has a unique lifting

\[
\begin{array}{ccc}
Q^* & \xrightarrow{\omega_\alpha} & T \\
\downarrow & & \downarrow \cong \\
S & & \\
\end{array}
\]

Each \( \omega_\alpha' \) extends to \( \overline{\omega_\alpha} : Q \to \overline{T} \) again because the weight splittings are nonnegative.

Let \( e_\alpha' = \omega_\alpha(0) \) for each \( \alpha \). Then we have the following strictly commutative diagram of d.g.a. morphisms:

\[
\begin{array}{ccc}
M & \xrightarrow{e_\alpha} & M \\
\rho \downarrow & & \downarrow \rho \\
Q \times M & \xrightarrow{f_\alpha} & Q \times M_
\end{array}
\]

Then \( e_\alpha \simeq e_\alpha' \) because they both solve the same lifting problem and \( e_\alpha' = e_\alpha' \circ e_\alpha' \) because \( e_\alpha' \in T \). Q.E.D.

Thus for any irreducible II-splitting \( \{ g_\alpha \} \) of \( M \) we can find representative \( \{ e_\alpha \} \subseteq \text{End}(M) \) and a maximal \( Q \)-split torus \( T \subseteq \text{Aut}(M) \) such that \( \{ e_\alpha \} \subseteq T \).

Let \( h_\alpha \) for each \( \alpha \). Then we have the following strictly commutative diagram of d.g.a. morphisms:

Another irreducible II-splitting \( \{ f_\alpha \} \) yields a collection \( \{ f_\alpha \} \) of representatives and a maximal \( Q \)-split torus \( S \subseteq \text{Aut}(M) \) such that \( \{ f_\alpha \} \subseteq S \). By Theorem 3.10 there exists \( x \in \text{Aut}(M) \) such that \( x \circ T \circ x^{-1} = S \). Thus in the category \( \mathcal{P} \) the two splittings are compatible. By Corollary 1.10 the two splittings are equivalent. Thus each object \( M \) of \( \mathcal{P} \) satisfies unique factorization with respect to the formation of products. The dual statement (coproducts) is somewhat easier as no lifting lemmas are needed (the coproduct does not have to be constructed). The details will be left to the reader.

3.15. **Theorem.** Let \( M \) be an object of \( \mathcal{P} \). Then any two irreducible II-splittings of \( M \) are equivalent.

3.16. **Theorem.** Let \( M \) be an object of \( \mathcal{P} \). Then any two irreducible II-splittings of \( M \) are equivalent.

**Remark.** The question of unique factorization for \( p \)-local (or \( p \)-adic) positive weight spaces remains open. There is an added challenge to this problem arising from the fact that there exists no \( p \)-local analogue to the algebraic nature of rational homotopy types.

**References**


