CLASS GROUPS OF CYCLIC GROUPS
OF SQUARE FREE ORDER

BY

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Abstract. Let G be a finite cyclic group of square free order. Let Cl(ZG) denote the projective class group of the integral group ring ZG. Our main theorem describes explicitly the quotients of a certain filtration of Cl(ZG). The description is in terms of class groups and unit groups of the rings of cyclotomic integers involved in ZG. The proof is based on a Mayer-Vietoris sequence.

1. Introduction. In this note G denotes a cyclic group of square free order b. For every positive integer d dividing b, let ζd be a primitive dth root of unity in a fixed algebraic closure of the field of rational numbers. Wherever the variable d appears, its domain will be some subset of the set of positive divisors of b. For any ring R, R* will denote the group of units and Cl(R) the projective class group of R. For any d ≠ 1, let πd denote the product of all primes in Z\{d\} dividing d. Let ZG denote the integral group ring of G. Our result is the following. It is proved in §2.

(1.1) Theorem. The group Cl(ZG) can be filtered so that the quotients are \( \prod_{d|b} Cl(Z[ζ_d]) \) and the cokernels of the natural maps \( Z[ζ_d]* \rightarrow (Z[ζ_d]/π_d)^* \) for \( d|b, d ≠ 1 \).

The author wishes to thank the referee for showing how to simplify the original proof of Theorem (1.1). The original proof, which was based on a canonical form theorem for matrices over Z with characteristic polynomial \( x^b - 1 \), proved only a slightly weaker version of the theorem. Reiner-Ullom [3] using a Mayer-Vietoris sequence different from the one in this paper obtained a lower bound on \( |Cl(ZG)| \) when \( b \) is the product of two odd primes.

2. Proof of Theorem (1.1). The cartesian diagram

\[
\begin{array}{ccc}
ZG & \rightarrow & \prod_d Z[ζ_d] \\
\downarrow & & \downarrow \\
ZG/ \prod_d bZ[ζ_d] & \rightarrow & \prod_d Z[ζ_d]/(b)
\end{array}
\]
leads to an exact Mayer-Vietoris sequence the latter part of which (see [2]) reduces to
\[
(ZG/bZG)* \oplus \prod_d (Z[\zeta_d])*^{a+\beta} \rightarrow \prod_d (Z[\zeta_d]/(b))^* \rightarrow \text{Cl}(ZG) \rightarrow \prod_d \text{Cl}(Z[\zeta_d]) \rightarrow 0
\]
where \((ZG/\prod_d bZ[\zeta_d])*\) has been replaced by \((ZG/bZG)*\) which maps onto it [1, Lemma 2]. The Mayer-Vietoris sequence immediately implies a filtration of \text{Cl}(ZG) in which the quotients are \(\prod_d \text{Cl}(Z[\zeta_d])\) and \(\text{cok}(\alpha + \beta)\). It remains to analyze \(\text{cok}(\alpha + \beta)\).

The map \(\alpha\) is the restriction of the natural map \(\gamma: ZG/bZG \rightarrow \prod_d Z[\zeta_d]/(b)\). Let \(F_p\) be the field of \(p\) elements. The \(p\)-component of \(\gamma\) is
\[
F_p G \rightarrow \prod_{(d, p) = 1} F_p[\zeta_d] \times \prod_{(d, p) = 1} F_p[\zeta_{pd}].
\]
(2.1)
Let \(f_d(x)\) be the \(d\)th cyclotomic polynomial. Using the fact that \(f_{pd} \equiv f_d^{-1} \mod p\), (2.1) can be rewritten
\[
F_p[x]/\left(\prod_{(p, d) = 1} f_d(x)^p\right) \rightarrow \prod_{(p, d) = 1} F_p[x]/(f_d(x)) \times \prod_{(p, d) = 1} F_p[x]/(f_d(x)^{p^{-1}}).
\]
Call this \(A \rightarrow B \times C\). Identify the isomorphic rings \(B\) and \(C/\text{rad}\, C\), and define a map \(B^* \times C^* \rightarrow B^*\) by \((b, c) \mapsto b^{-1} c\). Then
\[
A^* \rightarrow (B \times C)^* \rightarrow B^* \rightarrow 1
\]
is exact. Putting \(p\)-components back together gives an exact sequence
\[
(ZG/bZG)* \rightarrow \prod_{d} (Z[\zeta_d]/(b))^* \rightarrow \prod_{(p, d) = 1} (F_p[\zeta_d])^* \rightarrow 1.
\]
Using the fact that \(F_p[\zeta_d] \cong F_p[\zeta_{pd}]/\text{rad}\) when \((p, d) = 1\), and combining \(p\)-components by the Chinese Remainder Theorem (CRT hereafter) yields an isomorphism
\[
\phi: \prod_{(p, d) = 1} F_p[\zeta_d]^* \rightarrow \prod_{d \neq 1} (Z[\zeta_d]/\pi_d)^*.
\]
Thus there is an exact sequence
\[
(ZG/bZG)* \rightarrow \prod_{d \neq 1} (Z[\zeta_d]/(b))^* \rightarrow \prod_{d \neq 1} (Z[\zeta_d]/\pi_d)^* \rightarrow 1
\]
where \(h = \phi \circ \delta\). This determines \(\text{cok}\, \alpha\).

Next the map \(h \circ \beta: \prod_d Z[\zeta_d]^* \rightarrow \text{cok}\, \alpha\) must be determined. Fix \(d\) and let \(p\) be a prime dividing \(d\). Set \(D = d/p\). Define a ring isomorphism
\[
\tau_{d^2}: Z[\zeta_d]/(\pi_p) = p\text{-component of } Z[\zeta_d]/\pi_d
\]
by \(\zeta_d \mod(p) \mapsto \zeta_d \mod(\pi_p)\) where \((\pi_p)\) denotes the ideal \(\pi_p\) (in \(Z[\zeta_d]\)) induced up to \(Z[\zeta_d]\). For \(x \in Z[\zeta_d]/\pi_d\), let \(x_p\) denote the image of \(x\) in the \(p\)-component of \(Z[\zeta_d]/\pi_d\). By the CRT, \(x\) is completely determined by the \(x_p, p\) ranging over all primes dividing \(d\). Let \(u = (u_d) \in \prod_d Z[\zeta_d]^*\), and \(w = h \circ \beta(u)\). It is straightforward to check that
where \( u_d' \) is the image of \( u_d \) in \( \mathbb{Z}[\xi_d]/(\pi_p) \) under the canonical map, and \( \mu_{Dd} \) is the image in \( \mathbb{Z}[\xi_d]/(\pi_p) \) of \( u_d^{-1} \) under

\[
Z[\xi_d] \rightarrow Z[\xi_D]/(p) \Rightarrow Z[\xi_d]/(\pi_p).
\]

Fix an ordering \( d_1, d_2, \ldots, d_t \) of the distinct positive divisors of \( b \) such that the number of primes dividing \( d_i \) is less than or equal to the number of primes dividing \( d_j \) when \( i < j \). Make the obvious notation change in subscripts. For example \( \xi_d \) is now denoted \( \xi_i \). Let \( G_i = \mathbb{Z}[\xi_i]^*, \ 1 < i < t, \) and \( H_i = (\mathbb{Z}[\xi_i]/\pi_i)^*, \ 2 < i < t. \) Let \( G = \prod G_i, \ H = \prod H_i. \)

(2.3) **Lemma.** Let \( a \in G_k. \) If \( k > 1, \) let \( a \equiv 1 \mod \pi_k. \) Then there is some \( g = (g_i) \) in the kernel of \( h \circ \beta: G \rightarrow H \) with \( g_i = 1 \) for \( i < k, \) and \( g_k = a. \)

**Proof.** Define a ring homomorphism \( T_{ji}: Z[\xi_j] \rightarrow Z[\xi_i] \) when \( d_j \mid d_i \) by \( \xi_j \rightarrow \xi_i^n \) where \( n \equiv 1 \mod d_j \) and \( n \equiv 0 \mod(d_i/d_j). \) Let \( g_i = T_{ki}(a) \) if \( d_k \mid d_i, \ g_i = 1 \) otherwise. Clearly \( g_i = 1 \) for \( i < k. \) Let \( w = h \circ \beta(g). \) It remains to show that \( w = 1. \)

Now \( w = (w_i), \ i = 2, 3, \ldots, t. \) Taking \( p \)-components of \( Z[\xi_i]/\pi_i, \) it suffices to show that \( (w_i)_p = 1 \) for \( p \mid d_i. \) By (2.2), \( (w_i)_p \) is the product of the images in \( Z[\xi_i]/(\pi_i) \) of \( g_i \) and \( g_i^{-1} \) where \( d_i = d_j/p. \) The following facts are now needed.

(1) \( [T_{ji}(x)] \mod(\pi_p) = T_{ji}(x \mod p). \)

(2) \( T_{ji} \circ T_{kj} = T_{ki} \) when \( d_k \mid d_j. \)

(3) The composite

\[
Z[\xi_k] \xrightarrow{T_{ki}} Z[\xi_i] \rightarrow Z[\xi_i]/(\pi_p)
\]

factors through \( Z[\xi_k] \rightarrow Z[\xi_k]/\pi_k \) when \( d_k \mid d_i, \ p \mid d_k. \)

If \( d_k \mid d_i, \) then by (2.2) and facts (1) and (2) the images of \( g_i^{-1} \) and \( g_i \) cancel in \( Z[\xi_i]/(\pi_p). \) If \( d_k \mid d_i, \) and \( d_k \mid d_j, \) then the image of \( g_i \) is 1 by (2.2), fact (3), and the fact that \( a \equiv 1 \mod \pi_k, \) while \( g_j = 1 \) by definition. If \( d_k \mid d_i, \ g_i = 1 \) and \( g_j = 1 \) by definition. Thus, in all cases, \( (w_i)_p = 1, \) and the lemma is proved.

Define filtrations on \( X = G, H \) by

\[
F_k(X) = \{ x \in X | x_i = 1 \text{ for } i < k \},
\]

and let \( F_k(Q) \) be the filtration induced on \( Q = \text{cok}(h \circ \beta). \)

(2.4) **Proposition.** For \( k = 2, 3, \ldots, t, \)

\[
F_k(Q)/F_{k+1}(Q) \cong H_k/G'_k
\]

where \( G'_k \) denotes the image of \( G_k \) in \( H_k \) under the canonical map.

**Proof.** One verifies that

\[
F_k(Q)/F_{k+1}(Q) \cong F_k(H)/[F_{k+1}(H) + (F_k(H) \cap \text{Im } F_1(G))] \quad (2.5)
\]

where \( \text{Im} \) means image under \( h \circ \beta. \) By Lemma (2.3),

\[
F_k(H) \cap \text{Im } F_1(G) = \text{Im } F_k(G).
\]

Therefore the right-hand side of (2.5) becomes

\[
F_k(H)/F_{k+1}(H) + \text{Im } F_k(G).
\]
This is isomorphic to \( H_k / G_k \), and the proof is complete.

Theorem (1.1) follows from Proposition (2.4) because \( Q \cong \text{cok}(\alpha + \beta) \).

Remark. Theorem (1.1) can be strengthened slightly. Let \( Q_d \) be the cokernel of the natural map \( \mathbb{Z}[\xi_d]^* \to (\mathbb{Z}[\xi_d] / \pi_d)^* \). For \( d \mid b \), let \( \Delta(d) \) be the number of distinct primes dividing \( d \). Then the filtration of \( Q \) in the proof of Theorem (1.1) can be made coarser so that the quotients are the groups \( \prod \{ Q_d \mid \Delta(d) = k \} \) for \( 1 \leq k \leq \Delta(b) \). This is because if \( \Delta(d_i) \) is constant for \( k \leq i < r \), then, modulo \( F_i(G) \) and \( F_i(H) \), the restriction of \( h \circ \beta \) to \( F_k(G) \) splits as the direct product of the natural maps \( G_i \to H_i, \ k \leq i < r \). The required analogue of Proposition (2.4) is proved without difficulty by imitating the proof of Proposition (2.4).

References


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