THE C. NEUMANN PROBLEM AS A COMPLETELY INTEGRABLE SYSTEM ON AN ADJOINT ORBIT

BY

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Abstract. It is shown by purely Lie algebraic methods that the C. Neumann problem—the motion of a material point on a sphere under the influence of a quadratic potential—is a completely integrable system of Euler-Poisson equations on a minimal-dimensional orbit of a semidirect product of Lie algebras.

1. The C. Neumann problem. The motion of a point on the sphere \( S^{n-1} \) under the influence of a quadratic potential \( U(x) = \frac{1}{2} A x \cdot x, \quad x \in \mathbb{R}^n, \quad A = \text{diag}(a_1, \ldots, a_n) \) is a completely integrable Hamiltonian system. For \( n = 3 \) this has been shown by C. Neumann in 1859 [12] and for arbitrary \( n \) by K. Uhlenbeck [16], R. Devaney [3], J. Moser [10], [11], M. Adler, and P. van Moerbeke [2]. In this paper we show how this problem fits naturally in the framework of Euler-Poisson equations [4], [5], [14], [17] proving that the C. Neumann problem is a Hamiltonian system on a minimal-dimensional adjoint orbit in a semidirect product of Lie algebras. Thus its complete integrability will follow entirely from Lie algebraic considerations.

The equations of motion are

\[
\dot{x}_i = -a_i x_i + \lambda x_i, \quad i = 1, \ldots, n, \tag{1.1}
\]

where the Lagrange multiplier \( \lambda = A x \cdot x - ||x||^2 \) is chosen such that \( x \in S^{n-1} \) during the motion. Set \( \dot{x} = y \) and get the equivalent system to (1.1)

\[
\dot{x}_i = y_i, \quad \dot{y}_i = -a_i x_i + (A x \cdot x - ||y||^2)y_i, \quad ||x|| = 1, \quad x \cdot y = 0. \tag{1.2}
\]

The following crucial remark that motivated the present investigation is due to K. Uhlenbeck and can be verified without any difficulties.

Lemma 1.1. Put \( X = (x_i x_j), P = (y_i x_j - x_i y_j) \). System (1.2) is equivalent to

\[
\dot{X} = [P, X], \quad \dot{P} = [X, A], \quad ||x|| = 1, \quad x \cdot y = 0. \tag{1.3}
\]

Remark that if one replaces \( X \) and \( A \) by \( X - \text{Id}/n \) and \( A - (\text{Tr}(A))\text{Id}/n \) respectively, where \( \text{Id} \) is the \( n \times n \) identity matrix, equations (1.3) remain unchanged. From now on we shall assume that in (1.3) this change has been made so that \( X, P, A \in \text{sl}(n) \). The next section gives a Lie algebraic interpretation to these equations.

Received by the editors March 12, 1980.

1980 Mathematics Subject Classification. Primary 58F07, 70H05; Secondary 53C15, 17B99.

Key words and phrases. Complete integrability, adjoint orbit, Hamiltonian system, Euler-Poisson equations, Kirillov-Kostant-Souriau symplectic structure.

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2. The equations of motion as a Hamiltonian system on an adjoint orbit. We start by recalling a few facts about the ad-semidirect product $\mathfrak{g}_{ad} \times \mathfrak{g}$ of a semisimple Lie algebra $\mathfrak{g}$ with the abelian Lie algebra $\mathfrak{g}$ having underlying vector space $\mathfrak{g}$. If $(\xi, \eta_1), (\xi_2, \eta_2) \in \mathfrak{g}_{ad} \times \mathfrak{g}$, their bracket is defined by

$$[(\xi, \eta_1), (\xi_2, \eta_2)] = ([\xi, \xi_2], [\xi_1, \eta_2] - [\xi_2, \eta_1]).$$  \hspace{1cm} (2.1)

If $\kappa$ denotes a bilinear, symmetric, nondegenerate, bi-invariant, two-form on $\mathfrak{g}$, the form $\kappa$, called the semidirect product of $\kappa$ with itself and defined by

$$\kappa((\xi, \eta_1), (\xi_2, \eta_2)) = \kappa(\xi, \eta_1) + \kappa(\xi_2, \eta_1)$$  \hspace{1cm} (2.2)

is a bilinear, symmetric, nondegenerate, bi-invariant, two-form on $\mathfrak{g}_{ad} \times \mathfrak{g}$.

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. The Ad-semidirect product $G_{Ad} \times \mathfrak{g}$ is a Lie group with underlying manifold $G \times \mathfrak{g}$ and composition law

$$(g_1, \xi_1)(g_2, \xi_2) = (g_1 g_2, \xi_1 + \text{Ad}_{g_2} \xi_2).$$  \hspace{1cm} (2.3)

Note that the identity element is $(e, 0)$ and the inverse $(g, \xi)^{-1} = (g^{-1}, -\text{Ad}_{g^{-1}} \xi)$. The Lie algebra of $G_{Ad} \times \mathfrak{g}$ is $\mathfrak{g}_{ad} \times \mathfrak{g}$. The adjoint action of the Lie group $G_{Ad} \times \mathfrak{g}$ on $\mathfrak{g}_{ad} \times \mathfrak{g}$ is given by

$$\text{Ad}_{(g, \theta)}(\xi, \eta) = (\text{Ad}_g \xi, \text{Ad}_g \eta + [\theta, \text{Ad}_g \xi]).$$  \hspace{1cm} (2.4)

In the considerations that follow, the orbit symplectic structure plays a central role (see Abraham and Marsden [1] and Ratiu [14] for proofs). If a Lie algebra $\mathfrak{g}$ has a bilinear, symmetric, nondegenerate, bi-invariant two-form $\kappa$,

$$\omega(\text{Ad}_g \xi)([\eta, \text{Ad}_g \xi], [\xi', \text{Ad}_g \xi]) = -\kappa([\eta, \xi], \text{Ad}_g \xi)$$  \hspace{1cm} (2.5)

for $\xi, \eta, \xi' \in \mathfrak{g}$, $g \in G$, defines the canonical symplectic structure on the adjoint orbit $G \cdot \xi$ through $\xi$. If $E, E' : \mathfrak{g} \to \mathbb{R}$, the Hamiltonian vector field of $E|G \cdot \xi$ is given by

$$X_{E|G \xi}(\text{Ad}_g \xi) = -[(\text{grad} E)(\text{Ad}_g \xi), \text{Ad}_g \xi]$$  \hspace{1cm} (2.6)

and the Poisson bracket of $E|G \cdot \xi$, $E'|G \cdot \xi$ is

$$\{E|G \cdot \xi, E'|G \cdot \xi\}(\text{Ad}_g \xi) = -\kappa(\text{grad} E)(\text{Ad}_g \xi), (\text{grad} E')(\text{Ad}_g \xi)]$$  \hspace{1cm} (2.7)

where grad denotes the gradient with respect to $\kappa$.

For the semidirect product these formulas become

$$\omega(\xi, \eta)([(\xi, \eta), (\xi_1, \xi'_1)], [(\xi, \eta), (\xi_2, \xi'_2)]) = -\kappa((\xi, \eta), [(\xi_1, \xi'_1), (\xi_2, \xi'_2)),$$  \hspace{1cm} (2.8)

$$X_E(\xi, \eta) = -((\text{grad}_2 E)(\xi, \eta), \xi), [(\text{grad}_2 E)(\xi, \eta), \eta] + [(\text{grad}_1 E)(\xi, \eta), \xi],$$  \hspace{1cm} (2.9)

$$\{E, E'(\xi, \eta) = -\kappa(\xi, [(\text{grad}_2 E)(\xi, \eta), (\text{grad}_1 E')(\xi, \eta)]$$

$$- \kappa(\eta, [(\text{grad}_1 E)(\xi, \eta), (\text{grad}_2 E')(\xi, \eta)]$$

$$- \kappa(\eta, [(\text{grad}_2 E)(\xi, \eta), (\text{grad}_2 E')(\xi, \eta)],$$  \hspace{1cm} (2.10)

where $(\text{grad}_1, \text{grad}_2)$ denotes the usual gradient with respect to $\kappa \times \kappa$; note that the gradient with respect to $\kappa$ is $(\text{grad}_2, \text{grad}_1)$.
Assume that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$, where $\mathfrak{h}$ is a vector subspace and $\mathfrak{k}$ a Lie subalgebra of $\mathfrak{g}$, $\mathfrak{h}$ having $N$ as underlying closed Lie subgroup of $G$. Denote by $\Pi_\mathfrak{h}$, $\Pi_\mathfrak{k}$ the projections of $\mathfrak{g}$ on $\mathfrak{h}$ and $\mathfrak{k}$ respectively. Then $\mathfrak{g}^* = \mathfrak{h}^* \oplus \mathfrak{k}^*$. The nondegeneracy of $\kappa$ on $\mathfrak{g}$ defines the isomorphisms $\mathfrak{k}^* \cong \mathfrak{h}^*$, $\mathfrak{k} \cong \mathfrak{h} \mathfrak{k}^*$ and thus the coadjoint action of $N$ on $\mathfrak{k}^*$ induces an action of $N$ on $\mathfrak{k}^*$ given by

$$N \times \mathfrak{k} \ni (n, \xi) \mapsto \Pi_\mathfrak{k} \text{Ad}_n \xi \in \mathfrak{k}$$

where $\Pi_\mathfrak{k}: \mathfrak{g} \to \mathfrak{k}$ denotes the canonical projection defined by the direct sum decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$. Thus the orbit $N \cdot \xi$ equals

$$N \cdot \xi = \{ \Pi_\mathfrak{k}(\text{Ad}_n \xi) | n \in N \} \subseteq \mathfrak{k}, \quad \xi \in \mathfrak{k}^*,$$

whose tangent space at $\xi \in N \cdot \xi$ is

$$T_\xi (N \cdot \xi) = \{ \Pi_\mathfrak{k}[\xi^\top, \eta] | \eta \in \mathfrak{h} \} \subseteq \mathfrak{h}.$$ (2.12)

The symplectic structure on $N \cdot \xi$ equals, by (2.5),

$$\omega_\xi (\bar{\xi})(\Pi_\mathfrak{k}[\eta^\top, \bar{\xi}], \Pi_\mathfrak{k}[\bar{\xi}^\top, \bar{\xi}]) = -\kappa([\eta, \bar{\xi}], \bar{\xi}), \quad \bar{\xi} \in N \cdot \xi \subseteq \mathfrak{k}^*,$$ (2.13)

and the Hamiltonian vector field defined by $E|N \cdot \xi$, $E: \mathfrak{g} \to \mathbb{R}$, is, by (2.6),

$$X_{E|N \cdot \xi}(\bar{\xi}) = -\Pi_\mathfrak{k}[\Pi_\mathfrak{k}(\text{grad} E)(\bar{\xi})], \quad \bar{\xi} \in N \cdot \xi \subseteq \mathfrak{k}^*.$$ (2.14)

Finally the Poisson bracket of $E|G \cdot \xi$, $E'|G \cdot \xi$ is given by (2.7),

$$\{ E|G \cdot \xi, E'|G \cdot \xi \} (\bar{\xi}) = -\kappa(\Pi_\mathfrak{k}(\text{grad} E)(\bar{\xi}), \Pi_\mathfrak{k}(\text{grad} E')(\bar{\xi})), \bar{\xi} \in N \cdot \xi \subseteq \mathfrak{k}^*.$$ (2.15)

All previous considerations naturally live on the duals but this is the form we shall use for the C. Neumann problem; see Ratiu [14] for a parallel description on duals.

We shall apply all previous results to a specific Lie algebra. Let $\mathfrak{g} = \mathfrak{sl}(n) \oplus \mathfrak{sl}(n)$, $G = \mathfrak{sl}(n) \operatorname{Ad} \times \mathfrak{sl}(n)$, $\mathfrak{h} = \mathfrak{so}(n) \times \mathfrak{sym}$, $\mathfrak{k} = \mathfrak{so}(n) \times \mathfrak{sym}$, where $\mathfrak{sym} \subseteq \mathfrak{sl}(n)$ denotes the vector space of all symmetric matrices. Clearly $\mathfrak{h}$ is a Lie subalgebra and $\mathfrak{k}$ a vector subspace of $\mathfrak{g}$, $N$ a Lie subgroup of $G$ with Lie algebra $\mathfrak{k}$. Thus by our general considerations $N$ acts on $\mathfrak{k}$. It is easy to check that with respect to $\kappa$, where $\kappa(\mathfrak{A}, \mathfrak{B}) = -\frac{1}{2} \text{Tr}(\mathfrak{A}\mathfrak{B})$, $\mathfrak{k}^* = \mathfrak{h}$, $\mathfrak{k} \cong \mathfrak{h}$. In what follows we shall determine explicitly a particular $N$-orbit; note first that in the case above $\Pi_\mathfrak{k}$ in formula (2.11) is not necessary, i.e. the action of $N$ on $\mathfrak{k}^*$ is given by (2.4).

If $y, z \in \mathbb{R}^n$, denote by $y \otimes z$ the matrix having entries $y_jz_j$ and remark that if $g \in \text{SO}(n)$, $g(y \otimes z)g^{-1} = (gy) \otimes (gz)$. Let $z = (1, \ldots, 1) / \sqrt{n}$ and take $(z \otimes z - \text{Id} / n, 0) \in \mathfrak{k}^*$. Let $g \in \text{SO}(n)$ be arbitrary and denote $x = gz$. Then $||x|| = ||z|| = 1$ and $g(z \otimes z - \text{Id} / n)g^{-1} = x \otimes x - \text{Id} / n$ which is a matrix $X$ having all off-diagonal entries equal to $x_jx_j$ and diagonal entries $x_j^2 - 1 / n$. Thus the first component of the $N$-orbit through $(z \otimes z - \text{Id} / n, 0)$ is the matrix $X$ occurring in Lemma 1.1. We compute the second component. If $\theta \in \text{sym}$, $X_{\theta} = x_1x_j$, $X_{\theta} = x_j^2 - 1 / n$, then

$$[\theta, X]_\theta = \left( \sum_{k=1}^n \theta_{jk}x_k - x_iC(x, \theta) \right)x_j - x_i \left( \sum_{k=1}^n \theta_{jk}x_k - x_jC(x, \theta) \right).$$
where $C(x, \theta) = \sum_{k=1}^{n} x_i x_k \theta_{ik}$. Put $y_j = \sum_{k=1}^{n} (\theta_{ik} x_k - x_i C(x, \theta))$ and remark that if $y = (y_1, \ldots, y_n)$, $x \cdot y = 0$ since $||x|| = 1$. Thus the second component of the $N$-orbit consists of matrices $P \in so(n)$, $P_{ij} = y_i x_j - x_j y_i$, $x \cdot y = 0$. We showed hence that this $N$-orbit consists of pairs $(X, P) \in sym \times so(n)$ with $X, P$ defined as in Lemma 1.1.

Remark that the correspondence $(X, P) = \lambda(x, y)$ defines a diffeomorphism of this orbit onto the tangent bundle $TS^{n-1}$ of the unit sphere $S^{n-1}$ in $\mathbb{R}^n$. A tangent vector at $(X, P)$ to this orbit is $[(X, P), (\xi, \eta)]$ for $(\xi, \eta) \in so(n) \times sym$ and is of the form $(V, W) \in sym \times so(n)$, where $V_i = y_i x_i + x_i y_i$, $v_i = \sum_{k=1}^{n} x_k \xi_{ik}$, $W_j = w_j x_j - x_j w_j + y_j v_j - y_j v_j$, $w_i = \sum_{k=1}^{n} (y_k \xi_{ik} - x_k \eta_{ik}) + x_i \sum_{k=1}^{n} x_k x_k \eta_{ik}$ as a short calculation shows. Thus the tangent map of $\lambda$ is given by $(V, W) \mapsto (v, w)$, where $v = (v_1, \ldots, v_n)$, $w = (w_1, \ldots, w_n)$.

$TS^{n-1}$ has a natural symplectic structure induced by the canonical symplectic form $-\sum_{i=1}^{n} dx_i \wedge dy_i$ of $\mathbb{R}^n$. By (2.8) the canonical symplectic structure $\omega$ on the orbit is given by

$$\omega(X, P)(([(X, P), (\xi^1, \eta^1)]), [(X, P), (\xi^2, \eta^2)]) = \kappa \left( [[(\xi^2, \eta^2), (\xi^1, \eta^1)], (X, P)] \right).$$

Let $V^i, W^i, \xi^i, \eta^i$ be defined by $\xi^i, \eta^i, i = 1, 2$. We have by bi-invariance of $\kappa$, antisymmetry of $\xi^2$, symmetry of $\eta^2$, and by (2.8), (2.14)

$$\lambda^* \omega((x, y)((v^1, w^1), (v^2, w^2)) = \omega((V^1, W^1), (V^2, W^2))$$

$$\begin{align*}
&= -\kappa((\xi^2, \eta^2), (V^1, W^1)) \\
&= \frac{1}{2} \text{Tr}(\xi^2 W^1) + \frac{1}{2} \text{Tr}(\eta^2 V^1) \\
&= \sum_{k=1}^{n} \left( w_k^1 \sum_{i=1}^{n} x_i \xi_{ik}^2 \right) - \sum_{k=1}^{n} \left( v_k^1 \sum_{i=1}^{n} y_i \xi_{ik}^2 \right) + \sum_{k=1}^{n} \left( v_k \sum_{i=1}^{n} x_i \eta_{ik}^2 \right) \\
&= -\sum_{k=1}^{n} \left( v_k^1 w_k^2 - w_k^1 v_k^2 \right) \\
&= \left( -\sum_{k=1}^{n} dx_k \wedge dy_k \right)((v^1, w^1), (v^2, w^2)).
\end{align*}$$

This shows that $\lambda$ is a symplectic diffeomorphism:

$$\lambda^* \omega = \left( -\sum_{i=1}^{n} dx_i \wedge dy_i \right)|TS^{n-1}.$$

Let $L: sl(n) \to sl(n)$ be given by $L(\xi) = -\xi$. $L$ is clearly a $\kappa$-symmetric isomorphism. The following Euler-Poisson Hamiltonian (see [4], [5], [15], [17] for motivations) $E(\xi, \eta) = \frac{1}{2} \kappa(\eta, L(\eta)) + \kappa(A, \xi) + \kappa(A, \xi)$ for $A \in sl(n)$ a fixed diagonal matrix, induces a Hamiltonian vector field on the $N$-orbit through $(z \otimes z - \text{Id}/n, 0)$ given by (2.15), $(X, P) \mapsto ([P, X], [X, A])$, i.e. we get equations (1.3). Hence we proved the following.
THEOREM 2.1. The $N = so(n) \times \text{sym-orbit through } (z \otimes z - \text{Id}/n, 0)$ in $\mathfrak{so}^\perp = \mathfrak{sl} = \text{sym} \times so(n)$, $z = (1, \ldots, 1)/\sqrt{n}$, consists of all pairs $(X, P)$, $X_i = x_i x_j$ for $i \neq j$, $X_i = x_i^2 - 1/n$, $P_{ij} = y_i x_j - x_j y_i$, $||x|| = 1$, $x \cdot y = 0$. With the Kirillov-Kostant-Souriau symplectic structure this $(2n-2)$-dimensional orbit is symplectically diffeomorphic via $(X, P) \mapsto (x, y)$ to $TS^{n-1}$ with the symplectic structure induced from $\mathbb{R}^{2n}$ by $-\sum_{i=1}^{n} dx_i \wedge dy_i$. The Hamiltonian $E(X, P) = -\frac{1}{2} \kappa(P, P) + \kappa(A, X)$ defines on this orbit the equations of motion of the C. Neumann problem

$$\dot{X} = [P, X], \quad \dot{P} = [X, A], \quad ||x|| = 1, \quad x \cdot y = 0.$$  
(2.17)

Remark. M. Adler and P. van Moerbeke [2] have independently observed that (2.17) is a Hamiltonian system in a semidirect product.

3. The complete set of integrals and their involution.

Lemma 3.1. The equations $\dot{X} = [P, X], \quad \dot{P} = [X, A], \quad ||x|| = 1, \quad x \cdot y = 0$ are equivalent to

$$(-X + P\lambda + A\lambda^2)^2 = [-X + P\lambda + A\lambda^2, -P - A\lambda]$$

for any parameter $\lambda$.

The proof is a straightforward verification. It follows that the functions $(1/2(k + 1))\text{Tr}(-X + P\lambda + A\lambda^2)^{k+1}$ are conserved on the flow of (3.1). If $t \mapsto (X(t), P(t))$ denotes the flow of (2.17), then $t \mapsto X(t) + P(t)\lambda + A\lambda^2$ is the flow of (3.1) and we conclude that the coefficients of $\lambda$ in the expansion of $(1/2(k + 1))\text{Tr}(-X + P\lambda + A\lambda^2)^{k+1}$ are conserved along the flow of (2.17). Let $f_k(X, P)$ be the coefficient of $\lambda^{2k}$ in this expansion for $k = 1, \ldots, n - 1$. We shall prove in this section that all $f_k$ are in involution. The method of the proof follows Ratiu [13], [15] closely.

Theorem 3.2. Let $\mathfrak{g}$ be a Lie algebra with a bilinear, symmetric, nondegenerate, bi-invariant, two-form $\kappa$. Let $f, g: \mathfrak{g} \to \mathbb{R}$ satisfy $\{\text{grad } f, \text{grad } g\} = 0$ for all $\xi \in \mathfrak{g}$. Denote $f_\alpha(\xi, \eta) = f(\xi + a\eta + a^2\varepsilon)$, $g_\beta(\xi, \eta) = g(\xi + b\eta + b^2\varepsilon)$ for $\varepsilon \in \mathfrak{g}$ fixed and $a, b$ arbitrary parameters. Then $f_\alpha, g_\beta$ Poisson commute in the bracket of $\mathfrak{g}$ defined by its symplectic decomposition in adjoint orbits.

Proof. Clearly $(\text{grad } f_\alpha)(\xi, \eta) = (\text{grad } f)(\xi + a\eta + a^2\varepsilon)$, $(\text{grad } g_\beta)(\xi, \eta) = a(\text{grad } f)(\xi + a\eta + a^2\varepsilon)$ and similarly for $g_\beta$. By (2.10)

$$\{f_\alpha, g_\beta\}(\xi, \eta) = -\kappa([\text{grad } f_\alpha, \text{grad } g_\beta], \xi) - \kappa([\text{grad } g_\beta, \text{grad } f_\alpha], \eta) - \kappa([\text{grad } f_\alpha, \text{grad } f_\beta], \eta, \text{grad } g_\beta)(\xi, \eta) = 0$$

$$(\text{grad } g)(\xi + \eta + b\varepsilon) = (b^2/(a - b))\kappa([\xi + \eta + a^2\varepsilon, \text{grad } f](\xi + \eta + a^2\varepsilon), (\text{grad } g)(\xi + \eta + b^2\varepsilon)) + (a^2/(a - b))\kappa([\xi + b\eta + b^2\varepsilon, \text{grad } f](\xi + \eta + a^2\varepsilon), (\text{grad } g)(\xi + \eta + b^2\varepsilon)) = 0$$
by hypothesis. By continuity \( \{ f_a, g_b \} = 0 \) holds also for \( a = b \).

**Remark.** The condition \( \{ (\text{grad } f)(\xi), \xi \} = 0 \) for all \( \xi \in \mathfrak{g} \) is the infinitesimal version of Ad-invariance of \( f \) as an easy computation shows.

**Theorem 3.3.** Let \( G \) be a Lie group, \( N \) a closed subgroup, with Lie algebras \( \mathfrak{g} \) and \( \mathfrak{r} \) respectively. Assume \( \mathfrak{g} = \mathfrak{r} \oplus \mathfrak{r} \), \( \mathfrak{r} \) a vector subspace, \( [\mathfrak{r}, \mathfrak{r}] \subseteq \mathfrak{r} \), and that \( \mathfrak{g} \) has a bilinear, symmetric, nondegenerate, bi-invariant, two-form \( \kappa \). Assume that \( f, g: \mathfrak{g} \to \mathbb{R} \) Poisson commute on \( \mathfrak{g} \), i.e.

\[
\kappa(\{ (\text{grad } f)(\xi), (\text{grad } g)(\xi) \}, \xi) = 0,
\]

for all \( \xi \in \mathfrak{g} \). If either

1. \( \mathfrak{r} \) is a Lie subalgebra, or
2. \( \Pi_\mathfrak{g} \Pi_\mathfrak{r} (\text{grad } f)(\eta), \Pi_\mathfrak{r} (\text{grad } g)(\eta) \) = 0 for all \( \eta \in \mathfrak{r} \),

then on any \( N \)-orbit in \( \mathfrak{r} \), the functions \( f, g \) Poisson commute.

**Proof.** Let \( \eta \in \mathfrak{r} \). By hypothesis and (2.16) we get

\[
0 = -\kappa(\{ (\text{grad } f)(\eta), (\text{grad } g)(\eta) \}, \eta) = \{ f|_{N^{-} \eta}, g|_{N^{-} \eta} \}(\eta) - \kappa(\Pi_\mathfrak{g} \Pi_\mathfrak{r} (\text{grad } f)(\eta), \Pi_\mathfrak{r} (\text{grad } g)(\eta), \eta).
\]

The second term vanishes in either hypothesis 1 or 2.

**Remark.** Both theorems have identical versions on duals and \( \kappa \) is not needed there.

These two general theorems prove the involution of the functions \( f_k \) in the following way. In Theorem 3.2 take \( \mathfrak{g} = \mathfrak{sl}(n) \) and let

\[
\phi_k(\xi, \eta) = (1/2(k + 1)) \text{Tr}( - \xi + \eta \lambda_{k+1} + A \lambda^2_{k+1} )^{k+1}.
\]

Then \( \{ \phi_k, \phi_l \} = 0 \) on \( \mathfrak{sl}(n) \) for any parameters \( \lambda_{k+1}, \lambda_{l+1} \), i.e. \( \phi_{k+1} \) is constant on the flow defined by \( \phi_{k+1} \) no matter what \( \lambda_{k+1}, \lambda_{l+1} \) are, i.e. the coefficients of \( \lambda_{k+1} \) in \( \phi_{k+1} \) are constant on the flow defined by \( \phi_{k+1} \) for all \( \lambda_{l+1} \). Hence \( \{ f_k, f_l \} = 0 \) for all \( \lambda \) and thus \( \{ f_k, f_l \} = 0 \) for any \( k, l \). In Theorem 3.3 take \( \mathfrak{g} = \mathfrak{sl}(n) \), \( \mathfrak{r} = \mathfrak{sym} \times \mathfrak{so}(n) \), \( \mathfrak{r} = \mathfrak{so}(n) \times \mathfrak{sym} \) and remark that \( [\mathfrak{g}, \mathfrak{r}] \subset \mathfrak{r} \). \( f_k, f_l \) Poisson commute on \( \mathfrak{g} \) by what we just proved, so in order to conclude that they Poisson commute on the \( N \)-orbit through \( (z \otimes z - \text{Id}/n, 0) \) we have to check condition (2) of Theorem 3.3 for \( \eta = (X, P) \in \mathfrak{r} = \mathfrak{g} \). An easy computation shows that

\[
f_k(X, P) = \frac{1}{2(k + 1)} \text{Tr} \left[ - \sum_{i=0}^{k} A^i X A^{k-i} + \sum_{i+j+l = k-1, i,j,l > 0} \sum_{i+j+l = k-1} A^i P A^j P A^l \right]
\]

so that the gradient of \( f_k \) with respect to \( \kappa \) is

\[
(\text{grad } f_k)(X, P) = \left( - \sum_{i=0}^{k-1} A^i P A^{k-1-i}, A^k \right) \in \mathfrak{so}(n) \times \mathfrak{sym} = \mathfrak{r}
\]

and hence \( \Pi_\mathfrak{r} (\text{grad } f_k)(X, P) = 0 \).

**Theorem 3.4.** The functions \( f_k, k = 1, \ldots, n - 1 \), are constants of the motion in involution for the C. Neumann problem. \( f_1 = -E \), \( E \) = energy function.
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Remark. Equation (3.1) is Hamiltonian in the Kac-Moody extension of $sl(n)$; see Adler and van Moerbeke [2].

4. Independence. Throughout this section we assume that $A = \text{diag}(a_1, \ldots, a_n)$ has all entries distinct.

Let $\mathcal{V} = \text{span}\{X_k(X,P) | k = 1, \ldots, n - 1\}$. We have to show that generically $\dim(\mathcal{V}) = n - 1$.

Denote by $U_{ki}$ the coefficient of $\lambda^i$ in the expansion of $(-X + P\lambda + A\lambda^2)^k$. From (3.3) it follows that $(\text{grad} f_k(X,P) = (-U_{k,2k-1}, U_{k,2k})$, so that $\mathcal{V} = \text{span}_{(A\lambda)} A_0$, where $A_0 = \text{span}\{(-U_{k,2k-1}, U_{k,2k}) | k = 1, \ldots, n - 1\}$. Since $U_{k,2k} = A^k$ and $A$ has all entries distinct we conclude that $\{A^k | k = 1, \ldots, n - 1\}$ are linearly independent in $sl(n)$ and thus $\dim(\mathcal{V}) = n - 1$; in particular $\dim(\mathcal{V}) < \dim(\mathcal{V}_0) = n - 1$ which was already obvious from the definition of $\mathcal{V}$.

Let $\mathcal{V}_j = \text{span}\{(-U_{k,2k-1-2j}, U_{k,2k-2j}) | k = j, \ldots, n - 1\}$ where we make the convention that any $U_{ki}$ with $i < 0$ is identical zero; thus $\dim(\mathcal{V}_j) < n - j, j = 1, \ldots, n - 1$. Denote $\mathcal{V}_j = \text{ad}_{(X,P)} A_j, j = 0, 1, \ldots, n - 1$, so that $\mathcal{V} = \mathcal{V}_0$.

Lemma 4.1. The linear map $f_{A,P}: sl(n) \times sl(n) \rightarrow \text{sl}(n)$ defined by

$$f_{A,P}(\xi, \eta) = ([\eta, P] - [\xi, A], [A, \eta])$$

is injective on all $\mathcal{V}_j, j = 1, \ldots, n - 1$, for generic $(X, P)$.

This is a direct, but somewhat lengthy verification (see [15] for a more complicated similar proof).

Lemma 4.2. The following relations hold for any $k = 1, \ldots, n - 1$:

$$- [U_{k,2k-j}, X] + [U_{k,2k-j-1}, P] + [U_{k,2k-j-2}, A] = 0.$$}

This is obvious if one notes that the expression above is the coefficient of $\lambda^j$ in the expansion of $((-X + P\lambda + A\lambda^2)^k, -X + P\lambda + A\lambda^2) = 0$.

We have thus by the two prior lemmas

$$\text{ad}_{(X,P)}(-U_{k,2k-1-2j}, U_{k,2k-2j})$$

$$= ([U_{k,2k-1-2j}, X], [X, U_{k,2k-2j}] + [U_{k,2k-1-2j}, P])$$

$$= ([U_{k,2k-2-2j}, P] + [U_{k,2k-3-2j}, A], [A, U_{k,2k-2j-2}]$$

$$= f_{A,P}(-U_{k,2k-2j-3}, U_{k,2k-2j-2}),$$

i.e. $f_{A,P}(\mathcal{V}_j) \subseteq \text{ad}_{(X,P)}(\mathcal{V}_j) = \mathcal{V}_j$. $f_{A,P}$ injective implies $\dim(\mathcal{V}_j) \geq \dim(\mathcal{V}_j+1), j = 0, 1, \ldots, n - 1$. Assume from now on that for any $j = 1, \ldots, n - 1, X^j \neq 0$; this condition is generically satisfied. Since $U_{j,0} = (-1)^j X^j$ we conclude $\text{ad}_{(X,P)}(0, U_{j,0}) = (0, 0)$ and hence $\dim(\mathcal{V}_j) \geq 1 + \dim(\mathcal{V}_j)$ for $j > 1$. Hence we obtain

$$\dim(\mathcal{V}_j) \geq 1 + \dim(\mathcal{V}_{j+1}), \quad j = 1, \ldots, n - 1, \mathcal{V}_n = 0. \quad (4.1)$$

Clearly $\mathcal{V}_{n-1} = \text{span}(0, U_{n-1,0})$ so that $\dim(\mathcal{V}_{n-1}) = 1$. Repeated application of (4.1) yields then $\dim(\mathcal{V}_j) \geq n - 1$, which combined with $n - 1 > \dim(\mathcal{V}_0) > \dim(\mathcal{V})$ gives $\dim(\mathcal{V}) = n - 1$. 

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Theorem 4.3. Let $A = \text{diag}(a_1, \ldots, a_n)$ have all entries distinct. The C. Neumann problem is a completely integrable Hamiltonian system, $n - 1$ generically independent integrals in involution being given by

$$f_{k+1}(X, P) = \frac{1}{2(k + 1)} \text{Tr} \left( -\sum_{i=0}^{k} A^i X A^{k-i} + \sum_{i+j+l = k-1 \atop i, j, l > 0} A^i P A^{j} P A^{l} \right).$$

Remarks. (1) The geodesic spray on an ellipsoid in $\mathbb{R}^n$ with all axes distinct is also completely integrable and given by the Euler-Poisson equations on the same orbit

$$\dot{X} = [Q, X], \quad \dot{P} = [Q, P] + [X, X^{-1}],$$

for $Q_y = -P_y/a_i a_j$, with Hamiltonian $E(X, P) = -\frac{1}{2} \kappa(P, Q) + \kappa(X, A^{-1})$. It has the same integrals $f_k$ since the previous equations can be written as

$$(-X + P\lambda + A\lambda^2) = [-X + P\lambda + A\lambda^2, -Q - A^{-1}\lambda].$$

This follows easily from the work of Moser [10], [11] and has been independently observed by Adler and van Moerbeke [2] who also linearize the flow.

(2) The geodesic spray on $S^n$ corresponds to $A = 0$ in the C. Neumann problem, or to $A = 1d$ in the ellipsoidal problem. The Euler-Poisson equations on the same orbit are $\dot{X} = [P, X], \dot{P} = 0$ and the integrals in involution are

$$\mu_k(X, P) = \frac{1}{2}(k + 1) \text{Tr}(P^{k+1}), \quad k = \text{odd}.$$

The Hamiltonian is $-f_1(X, P) = -\frac{1}{2} \kappa(P, P)$. The prior proof of independence can be easily modified step-by-step to show that $X_k, k = 1, \ldots, n - 1$, are generically independent.

Acknowledgements. I want to thank K. Uhlenbeck for telling me equations (1.3) which started the present investigation. A conversation with C. Moore on semidirect products is gratefully acknowledged.

Bibliography


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