THE C. NEUMANN PROBLEM AS A COMPLETELY INTEGRABLE SYSTEM ON AN ADJOINT ORBIT
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Abstract. It is shown by purely Lie algebraic methods that the C. Neumann problem—the motion of a material point on a sphere under the influence of a quadratic potential—is a completely integrable system of Euler-Poisson equations on a minimal-dimensional orbit of a semidirect product of Lie algebras.

1. The C. Neumann problem. The motion of a point on the sphere $S^{n-1}$ under the influence of a quadratic potential $U(x) = \frac{1}{2} A x \cdot x$, $x \in \mathbb{R}^n$, $A = \text{diag}(a_1, \ldots, a_n)$ is a completely integrable Hamiltonian system. For $n = 3$ this has been shown by C. Neumann in 1859 [12] and for arbitrary $n$ by K. Uhlenbeck [16], R. Devaney [3], J. Moser [10], [11], M. Adler, and P. van Moerbeke [2]. In this paper we show how this problem fits naturally in the framework of Euler-Poisson equations [4], [5], [14], [17] proving that the C. Neumann problem is a Hamiltonian system on a minimal-dimensional adjoint orbit in a semidirect product of Lie algebras. Thus its complete integrability will follow entirely from Lie algebraic considerations.

The equations of motion are
\[ \dot{x}_i = -a_i x_i + \lambda x_i, \quad i = 1, \ldots, n, \quad (1.1) \]
where the Lagrange multiplier $\lambda = A x \cdot x - ||x||^2$ is chosen such that $x \in S^{n-1}$ during the motion. Set $\dot{x} = y$ and get the equivalent system to (1.1)
\[ \dot{x}_i = y_i, \quad \dot{y}_i = -a_i x_i + (A x \cdot x - ||y||^2)x_i, \quad ||x|| = 1, \quad x \cdot y = 0. \quad (1.2) \]

The following crucial remark that motivated the present investigation is due to K. Uhlenbeck and can be verified without any difficulties.

Lemma 1.1. Put $X = (x_i x_j)$, $P = (y_i x_j - x_i y_j)$. System (1.2) is equivalent to
\[ \dot{X} = [P, X], \quad \dot{P} = [X, A], \quad ||x|| = 1, \quad x \cdot y = 0. \quad (1.3) \]

Remark that if one replaces $X$ and $A$ by $X - \text{Id}/n$ and $A - (\text{Tr}(A))\text{Id}/n$ respectively, where $\text{Id}$ is the $n \times n$ identity matrix, equations (1.3) remain unchanged. From now on we shall assume that in (1.3) this change has been made so that $X, P, A \in \mathfrak{sl}(n)$. The next section gives a Lie algebraic interpretation to these equations.

Received by the editors March 12, 1980.

1980 Mathematics Subject Classification. Primary 58F07, 70H05; Secondary 53C15, 17B99.

Key words and phrases. Complete integrability, adjoint orbit, Hamiltonian system, Euler-Poisson equations, Kirillov-Kostant-Souriau symplectic structure.
2. The equations of motion as a Hamiltonian system on an adjoint orbit. We start by recalling a few facts about the ad-semidirect product $\mathfrak{g}_{ad} \times \mathfrak{g}$ of a semisimple Lie algebra $\mathfrak{g}$ with the abelian Lie algebra $\mathfrak{g}$ having underlying vector space $\mathfrak{g}$. If $(\xi_1, \eta_1), (\xi_2, \eta_2) \in \mathfrak{g}_{ad} \times \mathfrak{g}$, their bracket is defined by

$$\left[ (\xi_1, \eta_1), (\xi_2, \eta_2) \right] = \left( [\xi_1, [\xi_2, \eta_1] - [\xi_2, \eta_1]], [\xi_1, \eta_2] - [\xi_2, \eta_1] \right). \quad (2.1)$$

If $\kappa$ denotes a bilinear, symmetric, nondegenerate, bi-invariant, two-form on $\mathfrak{g}$, the form $\kappa$, called the semidirect product of $\kappa$ with itself and defined by

$$\kappa_s((\xi_1, \eta_1), (\xi_2, \eta_2)) = \kappa(\xi_1, \eta_2) + \kappa(\xi_2, \eta_1) \quad (2.2)$$

is a bilinear, symmetric, nondegenerate, bi-invariant, two-form on $\mathfrak{g}_{ad} \times \mathfrak{g}$.

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. The Ad-semidirect product $G_{ Ad } \times \mathfrak{g}$ is a Lie group with underlying manifold $G \times \mathfrak{g}$ and composition law

$$(g_1, \xi_1)(g_2, \xi_2) = (g_1g_2, \xi_1 + \text{Ad}_g\xi_2). \quad (2.3)$$

Note that the identity element is $(e, 0)$ and the inverse $(g, \xi)^{-1} = (g^{-1}, -\text{Ad}_g^{-1}\xi)$. The Lie algebra of $G_{ Ad } \times \mathfrak{g}$ is $\mathfrak{g}_{ad} \times \mathfrak{g}$. The adjoint action of the Lie group $G_{ Ad } \times \mathfrak{g}$ on $\mathfrak{g}_{ad} \times \mathfrak{g}$ is given by

$$\text{Ad}_{(g, \theta)}(\xi, \eta) = (\text{Ad}_g\xi, \text{Ad}_g\eta + [\theta, \text{Ad}_g\xi]). \quad (2.4)$$

In the considerations that follow, the orbit symplectic structure plays a central role (see Abraham and Marsden [1] and Ratiu [14] for proofs). If a Lie algebra $\mathfrak{g}$ has a bilinear, symmetric, nondegenerate, bi-invariant two-form $\kappa$,

$$\omega(\text{Ad}_g\xi)([\eta, \text{Ad}_g\xi], [\xi, \text{Ad}_g\xi]) = -\kappa([\eta, \xi], \text{Ad}_g\xi) \quad (2.5)$$

for $\eta, \xi \in \mathfrak{g}, g \in G$, defines the canonical symplectic structure on the adjoint orbit $G \cdot \xi$ through $\xi$. If $E, E' : \mathfrak{g} \to \mathbb{R}$, the Hamiltonian vector field of $E|G \cdot \xi$ is given by

$$X_{E|G \cdot \xi}(\text{Ad}_g\xi) = -[(\text{grad} E)(\text{Ad}_g\xi), \text{Ad}_g\xi] \quad (2.6)$$

and the Poisson bracket of $E|G \cdot \xi, E'|G \cdot \xi$ is

$$\{ E|G \cdot \xi, E'|G \cdot \xi \}(\text{Ad}_g\xi) = -\kappa([\text{grad} E)(\text{Ad}_g\xi), (\text{grad} E')(\text{Ad}_g\xi)], \text{Ad}_g\xi) \quad (2.7)$$

where grad denotes the gradient with respect to $\kappa$.

For the semidirect product these formulas become

$$\omega(\xi, \eta)([(\xi, \eta), (\xi_1, \xi_1)], [(\xi, \eta), (\xi_2, \xi_2)]) = -\kappa_s(\xi, \eta), [(\xi_1, \xi_1), (\xi_2, \xi_2)), \quad (2.8)$$

$$X_E(\xi, \eta) = -[(\text{grad}_2 E)(\xi, \eta), \xi], [(\text{grad}_2 E)(\xi, \eta), \eta] + [(\text{grad}_1 E)(\xi, \eta), \xi], \quad (2.9)$$

$$\{ E, E' \}(\xi, \eta) = -\kappa(\xi, [\text{grad}_2 E)(\xi, \eta), (\text{grad}_1 E')(\xi, \eta)])$$

$$-\kappa(\eta, [\text{grad}_1 E)(\xi, \eta), (\text{grad}_2 E')(\xi, \eta)])$$

$$-\kappa(\eta, [\text{grad}_2 E)(\xi, \eta), (\text{grad}_1 E')(\xi, \eta)]) \quad (2.10)$$

where $(\text{grad}_1, \text{grad}_2)$ denotes the usual gradient with respect to $\kappa \times \kappa$; note that the gradient with respect to $\kappa_s$ is $(\text{grad}_2, \text{grad}_1)$. 

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Assume that \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r} \), where \( \mathfrak{g} \) is a vector subspace and \( \mathfrak{r} \) a Lie subalgebra of \( \mathfrak{g} \), \( \mathfrak{h} \) having \( N \) as underlying closed Lie subgroup of \( G \). Denote by \( \Pi_{\mathfrak{h}}, \Pi_{\mathfrak{r}} \) the projections of \( \mathfrak{g} \) on \( \mathfrak{h} \) and \( \mathfrak{r} \) respectively. Then \( \mathfrak{g}^* = \mathfrak{h}^* \oplus \mathfrak{r}^* \). The nondegeneracy of \( \kappa \) on \( \mathfrak{g} \) defines the isomorphisms \( \mathfrak{r}^\perp = \mathfrak{h}^* \), \( \mathfrak{h}^\perp = \mathfrak{r}^* \) and thus the coadjoint action of \( N \) on \( \mathfrak{r}^* \) induces an action of \( N \) on \( \mathfrak{r}^\perp \) given by
\[
N \times \mathfrak{r}^\perp \ni (n, \xi) \mapsto \Pi_{\mathfrak{r}^\perp} \text{Ad}_n \xi \in \mathfrak{r}^\perp \tag{2.11}
\]
where \( \Pi_{\mathfrak{r}^\perp} : \mathfrak{g} \to \mathfrak{r}^\perp \) denotes the canonical projection defined by the direct sum decomposition \( \mathfrak{g} = \mathfrak{h}^\perp \oplus \mathfrak{r}^\perp \). Thus the orbit \( N \cdot \xi \) equals
\[
N \cdot \xi = \{ \Pi_{\mathfrak{r}^\perp} (\text{Ad}_n \xi) \mid n \in N \} \subseteq \mathfrak{r}^\perp, \quad \xi \in \mathfrak{r}^\perp, \tag{2.12}
\]
whose tangent space at \( \tilde{\xi} \in N \cdot \xi \) is
\[
T_{\tilde{\xi}} (N \cdot \xi) = \{ \Pi_{\mathfrak{r}^\perp} [\tilde{\xi}, \eta] \mid \eta \in \mathfrak{r} \} \subseteq \mathfrak{r}^\perp. \tag{2.13}
\]
The symplectic structure on \( N \cdot \xi \) equals, by (2.5),
\[
\omega_N (\tilde{\xi}) (\Pi_{\mathfrak{r}^\perp} [\eta, \xi], \Pi_{\mathfrak{r}^\perp} [\xi, \bar{\xi}]) = -\kappa ([\eta, \xi], \bar{\xi}), \quad \tilde{\xi} \in N \cdot \xi \subseteq \mathfrak{r}^\perp, \tag{2.14}
\]
and the Hamiltonian vector field defined by \( E \mid G \cdot \xi, E' \mid G \cdot \xi \) is, by (2.6),
\[
X_{E \mid N \cdot \xi} (\tilde{\xi}) = -\Pi_{\mathfrak{r}^\perp} \left[ \Pi_{\mathfrak{r}^\perp} (\text{grad } E)(\tilde{\xi}), \tilde{\xi} \right], \quad \tilde{\xi} \in N \cdot \xi \subseteq \mathfrak{r}^\perp. \tag{2.15}
\]
Finally the Poisson bracket of \( E \mid G \cdot \xi, E' \mid G \cdot \xi \) is given by (2.7),
\[
\{ E \mid G \cdot \xi, E' \mid G \cdot \xi \} (\tilde{\xi}) = -\kappa ([\Pi_{\mathfrak{r}^\perp} (\text{grad } E)(\tilde{\xi}), \Pi_{\mathfrak{r}^\perp} (\text{grad } E')(\tilde{\xi})], \tilde{\xi}), \tag{2.16}
\]
for \( \tilde{\xi} \in N \cdot \xi \subseteq \mathfrak{r}^\perp \). All previous considerations naturally live on the duals but this is the form we shall use for the C. Neumann problem; see Ratiu [14] for a parallel description on duals.

We shall apply all previous results to a specific Lie algebra. Let \( \mathfrak{g} = \mathfrak{s}l(n)_{\text{ad}} \times \mathfrak{s}l(n), G = \mathfrak{s}l(n)_{\text{ad}} \times \mathfrak{s}l(n), \mathfrak{r} = \mathfrak{s}o(n) \times \mathfrak{sym}, N = \mathfrak{s}o(n) \times \mathfrak{sym}, \mathfrak{h} = \mathfrak{sym} \times \mathfrak{s}o(n), \) where \( \mathfrak{sym} \subset \mathfrak{s}l(n) \) denotes the vector space of all symmetric matrices. Clearly \( \mathfrak{r} \) is a Lie subalgebra and \( \mathfrak{h} \) a vector subspace of \( \mathfrak{g} \), \( N \) a Lie subgroup of \( G \) with Lie algebra \( \mathfrak{r} \). Thus by our general considerations \( N \) acts on \( \mathfrak{r}^\perp \). It is easy to check that with respect to \( \kappa \), where \( \kappa (A, B) = -\frac{1}{2} \text{Tr}(AB), \mathfrak{r}^\perp = \mathfrak{r}, \mathfrak{r}^\perp = \mathfrak{r} \). In what follows we shall determine explicitly a particular \( N \)-orbit; note first that in the case above \( \Pi_{\mathfrak{r}^\perp} \) in formula (2.11) is not necessary, i.e. the action of \( N \) on \( \mathfrak{r}^\perp \) is given by (2.4).

If \( y, z \in \mathbb{R}^n \), denote by \( y \otimes z \) the matrix having entries \( y_i z_j \) and remark that if \( g \in \mathfrak{s}o(n), g(y \otimes z)g^{-1} = (gy) \otimes (gz) \). Let \( z = (1, \ldots, 1)/\sqrt{n} \) and take \( z \otimes z - \text{Id}/n, 0 \) \( \in \mathfrak{r}^\perp \). Let \( g \in \mathfrak{s}o(n) \) be arbitrary and denote \( x = gz \). Then \( \|x\| = \|z\| = 1 \) and \( g(z \otimes z - \text{Id}/n)g^{-1} = x \otimes x - \text{Id}/n \) which is a matrix \( X \) having all off-diagonal entries equal to \( x_i x_j \) and diagonal entries \( x_i^2 - 1/n. \) Thus the first component of the \( N \)-orbit through \( (z \otimes z - \text{Id}/n, 0) \) is the matrix \( X \) occurring in Lemma 1.1. We compute the second component. If \( \theta \in \mathfrak{sym}, X_\theta = x_i x_j, X_\theta = x_i^2 - 1/n, \) then
\[
[\theta, X]_\theta = \left( \sum_{k=1}^{n} \theta_{jk} x_k - x_i C(x, \theta) \right) x_j - x_i \left( \sum_{k=1}^{n} \theta_{jk} x_k - x_j C(x, \theta) \right),
\]

where \( C(x, \theta) = \sum_{k,i} x_i x_k \theta_{ik} \). Put \( y_i = \sum_{k=1}^n (\theta_{ik} x_k - x_i C(x, \theta)) \) and remark that if \( y = (y_1, \ldots, y_n) \), \( x \cdot y = 0 \) since \( \|x\| = 1 \). Thus the second component of the \( N \)-orbit consists of matrices \( P \in \text{so}(n), P_{ij} = y_i x_j - x_j y_i, \) \( x \cdot y = 0 \). We showed hence that this \( N \)-orbit consists of pairs \((X, P) \in \text{sym} \times \text{so}(n)\) with \( X, P \) defined as in Lemma 1.1.

Remark that the correspondence \((X, P) = \lambda(x, y)\) defines a diffeomorphism of this orbit onto the tangent bundle \( TS^{n-1} \) of the unit sphere \( S^{n-1} \) in \( \mathbb{R}^n \). A tangent vector at \((X, P)\) to this orbit is \([X, P] \), \((\xi, \eta) \in \text{so}(n) \times \text{sym}\) and is of the form \((V, W) \in \text{sym} \times \text{so}(n)\), where \( V_i = v_i x_i + x_i v_i, v_i = \sum_{k=1}^n x_k \xi_{ki}, W_i = w_i x_i - x_i w_i + y_i v_i - y_i v_i, w_i = \sum_{k=1}^n (y_k \xi_{ki} - x_k \eta_{ki}) + x_i \sum_{k=1}^n x_k x_k \eta_{ik} \) as a short calculation shows. Thus the tangent map of \( \lambda \) is given by \((V, W) \mapsto (v, w)\), where \( v = (v_1, \ldots, v_n), w = (w_1, \ldots, w_n)\).

\( TS^{n-1} \) has a natural symplectic structure induced by the canonical symplectic form \(-\sum_{i=1}^n dx_i \wedge dy_i\) of \( \mathbb{R}^n \). By (2.8) the canonical symplectic structure \( \omega \) on the orbit is given by

\[
\omega(X, P)([X, P], [X, P]) = \kappa([\xi^1, \eta^1], [\xi^2, \eta^2]) = \kappa((\xi^1, \eta^1), (\xi^2, \eta^2)) = \kappa((\xi^1, \eta^1), (\xi^2, \eta^2)).
\]

Let \( V_i, W_i, V_i', W_i' \) be defined by \( \xi^i, \eta^i, i = 1, 2 \). We have by bi-invariance of \( \kappa \), antisymmetry of \( \xi^i \), symmetry of \( \eta^i \), and by (2.8), (2.14)

\[
(\lambda_* \omega)(x, y)((v^1, w^1), (v^2, w^2)) = \omega(X, P)((V^1, W^1), (V^2, W^2)) = -\kappa((\xi^2, \eta^2), (V^1, W^1))
\]

\[
= \frac{1}{2} \text{Tr}(\xi^2 W^1) + \frac{1}{2} \text{Tr}(\eta^2 V^1)
\]

\[
= \sum_{k=1}^n \left( v_k^1 \sum_{i=1}^n x_i \xi_{ik}^2 - \frac{n}{2} \sum_{k=1}^n (v_k^1 \sum_{i=1}^n y_i \xi_{ik}^2) + \sum_{k=1}^n \frac{n}{2} \sum_{i=1}^n x_i \eta_{ik}^2 \right)
\]

\[
= -\sum_{k=1}^n (v_k^1 w_k^2 - w_k^1 v_k^2)
\]

\[
= \left( -\sum_{k=1}^n dx_k \wedge dy_k \right)((v^1, w^1), (v^2, w^2)).
\]

This shows that \( \lambda \) is a symplectic diffeomorphism:

\[
\lambda_* \omega = \left( -\sum_{i=1}^n dx_i \wedge dy_i \right)|TS^{n-1}.
\]

Let \( L: \text{sl}(n) \to \text{sl}(n) \) be given by \( L(\xi) = -\xi \). \( L \) is clearly a \( \kappa \)-symmetric isomorphism. The following Euler-Poisson Hamiltonian (see [4], [5], [15], [17] for motivations) \( E(\xi, \eta) = \frac{1}{2} \kappa(\eta, L(\eta)) + \kappa(A, \xi) \) for \( A \in \text{sl}(n) \) a fixed diagonal matrix, induces a Hamiltonian vector field on the \( N \)-orbit through \((z \otimes z - \text{Id}/n, 0)\) given by (2.15), \((X, P) \mapsto ([P, X], [X, A]), i.e. we get equations (1.3). Hence we proved the following.
THE C. NEUMANN PROBLEM

Theorem 2.1. The $N = \mathfrak{so}(n) \times \text{sym-orbit through } (z \otimes z - \text{Id}/n, 0)$ in $\mathfrak{so}^\perp = \mathfrak{so}$ consists of all pairs $(X, P)$, $X_{ij} = x_ix_j$ for $i \neq j$, $X_{ii} = x_i^2 - 1/n$, $P_{ij} = y_ix_j - x_iy_j$, $||x|| = 1$, $x \cdot y = 0$. With the Kirillov-Kostant-Souriau symplectic structure this $(2n-2)$-dimensional orbit is symplectically diffeomorphic via $(X, P) \mapsto (x, y)$ to $TS^{n-1}$ with the symplectic structure induced from $\mathbb{R}^{2n}$ by $-\sum_{i=1}^{n} dx_i \wedge dy_i$. The Hamiltonian $E(X, P) = \frac{1}{2} \kappa(P, P) + \kappa(A, X)$ defines on this orbit the equations of motion of the C. Neumann problem

\[ \dot{X} = [P, X], \quad \dot{P} = [X, A], \quad ||x|| = 1, \quad x \cdot y = 0. \quad (2.17) \]

Remark. M. Adler and P. van Moerbeke [2] have independently observed that (2.17) is a Hamiltonian system in a semidirect product.

3. The complete set of integrals and their involution.

Lemma 3.1. The equations $\dot{X} = [P, X], \quad \dot{P} = [X, A], \quad ||x|| = 1, \quad x \cdot y = 0$ are equivalent to

\[ (-X + P\lambda + A\lambda^2) = [-X + P\lambda + A\lambda^2, -P - A\lambda] \quad (3.1) \]

for any parameter $\lambda$.

The proof is a straightforward verification. It follows that the functions $(1/2(k+1))\text{Tr}(-X + P\lambda + A\lambda^2)^{k+1}$ are conserved on the flow of (3.1). If $t \mapsto (X(t), P(t))$ denotes the flow of (2.17), then $t \mapsto -X(t) + P(t)\lambda + A\lambda^2$ is the flow of (3.1) and we conclude that the coefficients of $\lambda$ in the expansion of $(1/2(k+1))\text{Tr}(-X + P\lambda + A\lambda^2)^{k+1}$ are conserved along the flow of (2.17). Let $f_k(X, P)$ be the coefficient of $\lambda^{2k}$ in this expansion for $k = 1, \ldots, n-1$. We shall prove in this section that all $f_k$ are in involution. The method of the proof follows Ratiu [13], [15] closely.

Theorem 3.2. Let $\mathfrak{g}$ be a Lie algebra with a bilinear, symmetric, nondegenerate, bi-invariant, two-form $\kappa$. Let $f, g: \mathfrak{g} \rightarrow \mathbb{R}$ satisfy $[(\text{grad } f)(\xi), \xi] = 0, [(\text{grad } g)(\xi), \xi] = 0$ for all $\xi \in \mathfrak{g}$. Denote $f_a(\xi, \eta) = f(\xi + a\eta + a^2\epsilon), \quad g_b(\xi, \eta) = g(\xi + b\eta + b^2\epsilon)$ for $\epsilon \in \mathfrak{g}$ fixed and $a, b$ arbitrary parameters. Then $f_a, g_b$ Poisson commute in the bracket of $\mathfrak{g}_{ad} \times \mathfrak{g}$ defined by its symplectic decomposition in adjoint orbits.

Proof. Clearly $(\text{grad}_1 f_a)(\xi, \eta) = (\text{grad } f)(\xi + a\eta + a^2\epsilon), \quad (\text{grad}_2 f_a)(\xi, \eta) = a(\text{grad } f)(\xi + a\eta + a^2\epsilon)$ and similarly for $g_b$. By (2.10)

\[
[ f_a, g_b ](\xi, \eta) = -\kappa([ (\text{grad}_1 f_a)(\xi, \eta), (\text{grad}_2 g_b)(\xi, \eta) ], \xi) - \kappa([ (\text{grad}_2 f_a)(\xi, \eta), (\text{grad}_1 g_b)(\xi, \eta) ], \xi) - \kappa([ (\text{grad}_2 f_a)(\xi, \eta), (\text{grad}_2 g_b)(\xi, \eta) ], \eta)
\]

\[
= -\kappa((a + b)\xi + a\eta, [ (\text{grad } f)(\xi + a\eta + a^2\epsilon), (\text{grad } g)(\xi + b\eta + b^2\epsilon) ]) = \frac{b^2}{(a - b)}\kappa([ \xi + a\eta + a^2\epsilon, (\text{grad } f)(\xi + a\eta + a^2\epsilon), (\text{grad } g)(\xi + b\eta + b^2\epsilon) ]
\]

\[
+ (a^2/(a - b))\kappa([ \xi + b\eta + b^2\epsilon, (\text{grad } g)(\xi + b\eta + b^2\epsilon) ], (\text{grad } f)(\xi + a\eta + a^2\epsilon) ) = 0
\]
by hypothesis. By continuity \((f_a, g_b) = 0\) holds also for \(a = b\). □

**Remark.** The condition \([(\text{grad } f)(\xi), \xi] = 0\) for all \(\xi \in \mathcal{O}\) is the infinitesimal version of Ad-invariance of \(f\) as an easy computation shows.

**Theorem 3.3.** Let \(G\) be a Lie group, \(N\) a closed subgroup, with Lie algebras \(\mathfrak{G}\) and \(\mathfrak{R}\) respectively. Assume \(\mathfrak{G} = \mathfrak{R} \oplus \mathfrak{R}\), \(\mathfrak{R}\) a vector subspace, \([\mathfrak{R}, \mathfrak{R}] \subseteq \mathfrak{R}\), and that \(\mathfrak{G}\) has a bilinear, symmetric, nondegenerate, bi-invariant, two-form \(\kappa\). Assume that \(f, g: \mathfrak{G} \to \mathbb{R}\) Poisson commute on \(\mathfrak{G}\), i.e.

\[
\kappa\left([ (\text{grad } f)(\xi), (\text{grad } g)(\xi) ], \xi \right) = 0,
\]

for all \(\xi \in \mathfrak{G}\). If either

1. \(\mathfrak{R}\) is a Lie subalgebra, or
2. \(\Pi_{\mathfrak{R}}[\Pi_{\mathfrak{R}}(\text{grad } f)(\eta), \Pi_{\mathfrak{R}}(\text{grad } g)(\eta)] = 0\) for all \(\eta \in \mathfrak{R}^\perp\),

then on any \(N\)-orbit in \(\mathfrak{R}^\perp\), the functions \(f, g\) Poisson commute.

**Proof.** Let \(\eta \in \mathfrak{R}^\perp\). By hypothesis and (2.16) we get

\[
0 = -\kappa\left([ (\text{grad } f)(\eta), (\text{grad } g)(\eta) ], \eta \right) = \{ f|N \cdot \eta, g|N \cdot \eta \}(\eta) - \kappa(\Pi_{\mathfrak{R}}[\Pi_{\mathfrak{R}}(\text{grad } f)(\eta), \Pi_{\mathfrak{R}}(\text{grad } g)(\eta)], \eta).
\]

The second term vanishes in either hypothesis 1 or 2. □

**Remark.** Both theorems have identical versions on duals and \(\kappa\) is not needed there.

These two general theorems prove the involution of the functions \(f_k\) in the following way. In Theorem 3.2 take \(\mathfrak{G} = \mathfrak{sl}(n)\) and let

\[
\phi_k(\xi, \eta) = (1/2(k + 1))\text{Tr}(-\xi + \eta\lambda_{k+1} + A\lambda_{k+1}^2)^{k+1}.
\]

Then \(\{\phi_k, \phi_l\} = 0\) on \(\text{sl}(n)_{\text{ad}} \times \text{sl}(n)\) for any parameters \(\lambda_{k+1}, \lambda_{l+1}\), i.e. \(\phi_{k+1}\) is constant on the flow defined by \(\phi_{k+1}\) no matter what \(\lambda_{k+1}, \lambda_{l+1}\) are, i.e. the coefficients of \(\lambda_{k+1}\) in \(\phi_{k+1}\) are constant on the flow defined by \(\phi_{k+1}\) for all \(\lambda_{l+1}\). Hence \(\{f_k, \phi_l\} = 0\) for all \(\lambda_k\) and thus \(\{f_k, f_l\} = 0\) for any \(k, l\). In Theorem 3.3 take \(\mathfrak{G} = \mathfrak{sl}(n)_{\text{ad}} \times \text{sl}(n), \mathfrak{R} = \text{sym} \times \text{so}(n), \mathfrak{R} = \text{so}(n) \times \text{sym}\) and remark that \([\mathfrak{R}, \mathfrak{R}] \subseteq \mathfrak{R}, f_k, f_l\) Poisson commute on \(\mathfrak{G}\) by what we just proved, so in order to conclude that they Poisson commute on the \(N\)-orbit through \((z \otimes z - \text{Id}/n, 0)\) we have to check condition (2) of Theorem 3.3 for \(\eta = (X, P) \in \mathfrak{R}^\perp = \mathfrak{R}\). An easy computation shows that

\[
f_k(X, P) = \frac{1}{2(k + 1)}\text{Tr}\left(-\sum_{i=0}^{k} A_i^T X A_i^{k-i} + \sum_{i+j+l = k-1} A_i^T PA_i^T \right)
\]

so that the gradient of \(f_k\) with respect to \(\kappa\), is

\[
(\text{grad } f_k)(X, P) = \left(-\sum_{i=0}^{k-1} A_i^T PA_i^{k-1-i}, A^k\right) \in \text{so}(n) \times \text{sym} = \mathfrak{R}
\]

and hence \(\Pi_{\mathfrak{R}}(\text{grad } f_k)(X, P) = 0\).

**Theorem 3.4.** The functions \(f_k, k = 1, \ldots, n - 1\), are constants of the motion in involution for the C. Neumann problem. \(f_1 = -E, E = \text{energy function.}\)
Remark. Equation (3.1) is Hamiltonian in the Kac-Moody extension of $sl(n)$; see Adler and van Moerbeke [2].

4. Independence. Throughout this section we assume that $A = \text{diag}(a_1, \ldots, a_n)$ has all entries distinct.

Let $\mathcal{V} = \text{span}\{X_k(X, P)|k = 1, \ldots, n - 1\}$. We have to show that generically $\dim(\mathcal{V}) = n - 1$.

Denote by $U_{ki}$ the coefficient of $\lambda^i$ in the expansion of $(-X + P\lambda + A\lambda^2)^k$. From (3.3) it follows that $\text{grad} f_k(X, P) = (-U_{2k-1,2k}, U_{k,2k})$, so that $\mathcal{V} = \text{ad}(X, P)\mathcal{V}_0$, where $\mathcal{V}_0 = \text{span}\{(-U_{2k-1,2k}, U_{k,2k})|k = 1, \ldots, n - 1\}$. Since $U_{k,2k} = A^k$ and $A$ has all entries distinct we conclude that $\{A^k|k = 1, \ldots, n - 1\}$ are linearly independent in $sl(n)$ and thus $\dim(\mathcal{V}_0) = n - 1$; in particular $\dim(\mathcal{V}) < \dim(\mathcal{V}_0) = n - 1$ which was already obvious from the definition of $\mathcal{V}$.

Let $\mathcal{V}_j = \text{span}\{(-U_{2k-2j-3,2k-2j}|k = j, \ldots, n - 1\} where we make the convention that any $U_{ki}$ with $i < 0$ is identical zero; thus $\dim(\mathcal{V}_j) < n - j$, $j = 1, \ldots, n - 1$. Denote $\mathcal{V}_j = \text{ad}(X, P)\mathcal{V}_j$, $j = 0, 1, \ldots, n - 1$, so that $\mathcal{V} = \mathcal{V}_0$.

Lemma 4.1. The linear map $f_{A, P}: sl(n) \times sl(n) \rightarrow f(n) \times f(n)$ defined by $f_{A, P}(x, y) = ([x, y], [A, x'] + [x, x'])$ is injective on all $\mathcal{V}_j$, $j = 1, \ldots, n - 1$, for generic $(X, P)$.

This is a direct, but somewhat lengthy verification (see [15] for a more complicated similar proof).

Lemma 4.2. The following relations hold for any $k = 1, \ldots, n - 1$:

$$- [U_{2k-2j}, X] + [U_{2k-2j-1}, P] + [U_{2k-2j-2}, A] = 0.$$ 

This is obvious if one notes that the expression above is the coefficient of $\lambda^j$ in the expansion of $[-X + P\lambda + A\lambda^2]^k, [-X + P\lambda + A\lambda^2] = 0$.

We have thus by the two prior lemmas

$$\text{ad}(X, P)(-U_{2k-2j-3,2k-2j})$$

$$= ([U_{k-2k-2j}, X], [X, U_{k-2k-2j}] + [U_{2k-2k-2j-2}, P])$$

$$= ([U_{k-2k-2j-2}, P] + [U_{2k-2k-2j-2}, A], [A, U_{k-2k-2j-2}])$$

$$= f_{A, P}(-U_{2k-2k-2j-3}, U_{k-2k-2j-2}),$$

i.e. $f_{A, P}(\mathcal{V}_{j+1}) \subseteq \text{ad}(X, P)(\mathcal{V}_j) = \mathcal{V}_j$. $f_{A, P}$ injective implies $\dim(\mathcal{V}_j) > \dim(\mathcal{V}_{j+1})$, $j = 0, 1, \ldots, n - 1$. Assume from now on that for any $j = 1, \ldots, n - 1, X^j \neq 0$; this condition is generically satisfied. Since $U_{j,0} = (-1)^j X^j$ we conclude $\text{ad}(X, P)(0, U_{j,0}) = (0, 0)$ and hence $\dim(\mathcal{V}_j) > 1 + \dim(\mathcal{V}_j)$ for $j > 1$. Hence we obtain

$$\dim(\mathcal{V}_j) > 1 + \dim(\mathcal{V}_{j+1}), \quad j = 1, \ldots, n - 1, \mathcal{V}_n = 0. \quad (4.1)$$

Clearly $\mathcal{V}_n = \text{span}(0, U_{n-1,0})$ so that $\dim(\mathcal{V}_n) = 1$. Repeated application of (4.1) yields then $\dim(\mathcal{V}_n) > n - 1$, which combined with $n - 1 > \dim(\mathcal{V}_0) > \dim(\mathcal{V}_1)$ gives $\dim(\mathcal{V}) = n - 1$. 

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Theorem 4.3. Let $A = \text{diag}(a_1, \ldots, a_n)$ have all entries distinct. The C. Neumann problem is a completely integrable Hamiltonian system, $n - 1$ generically independent integrals in involution being given by

$$f_{k+1}(X, P) = \frac{1}{2(k+1)} \text{Tr} \left\{ -\sum_{i=0}^{k} A^i X A^{k-i} + \sum_{i+j+l = k-1 \atop i,j,l > 0} A^l P A^j A^i \right\}.$$ 

Remarks. (1) The geodesic spray on an ellipsoid in $\mathbb{R}^n$ with all axes distinct is also completely integrable and given by the Euler-Poisson equations on the same orbit

$$\dot{X} = [Q, X], \quad \dot{P} = [Q, P] + [X, A^{-1}],$$

for $Q_g = -P_g/a_g$, with Hamiltonian $E(X, P) = -\frac{1}{2} \kappa(P, Q) + \kappa(X, A^{-1})$. It has the same integrals $f_k$ since the previous equations can be written as

$$(-X + P\lambda + A\lambda^2)^* = [-X + P\lambda + A\lambda^2, -Q - A^{-1}\lambda].$$

This follows easily from the work of Moser [10], [11] and has been independently observed by Adler and van Moerbeke [2] who also linearize the flow.

(2) The geodesic spray on $S^{n-1}$ corresponds to $A = 0$ in the C. Neumann problem, or to $A = \text{Id}$ in the ellipsoidal problem. The Euler-Poisson equations on the same orbit are $\dot{X} = [P, X], \dot{P} = 0$ and the integrals in involution are

$$f_k(X, P) = \begin{cases} \kappa \left( P^k - \left( \frac{1}{n} \text{Tr} \: P^k \right) \text{Id}, X \right), & k = \text{even}, \\ \frac{1}{2(k+1)} \text{Tr}(P^{k+1}), & k = \text{odd}. \end{cases}$$

The Hamiltonian is $-f_1(X, P) = -\frac{1}{2} \kappa(P, P)$. The prior proof of independence can be easily modified step-by-step to show that $X_k, k = 1, \ldots, n - 1$, are generically independent.

Acknowledgements. I want to thank K. Uhlenbeck for telling me equations (1.3) which started the present investigation. A conversation with C. Moore on semidirect products is gratefully acknowledged.

Bibliography


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