THE $\aleph_2$-SOUSLIN HYPOTHESIS

BY

RICHARD LAVER$^1$ AND SAHARON SHELAH$^2$

ABSTRACT. We prove the consistency with $CH$ that there are no $\aleph_2$-Souslin trees.

The $\aleph_2$-Souslin hypothesis, $\text{SH}_{\aleph_2}$, is the statement that there are no $\aleph_2$-Souslin trees. In Mitchell's model [5] from a weakly compact the stronger statement holds (Mitchell and Silver) that there are no $\aleph_2$-Aronszajn trees, a property which implies that $2^{\omega_0} > \omega_1$.

THEOREM. Con($ZFC + \text{there is a weakly compact cardinal}$) implies

Con($ZFC + 2^{\omega_1} = \aleph_1 + \text{SH}_{\aleph_1}$).

In the forcing extension, $2^{\omega_1}$ is greater than $\aleph_2$, and can be arbitrarily large. Analogues of this theorem hold with $\aleph_2$ replaced by the successor of an arbitrary regular cardinal. Strengthenings and problems are given at the end of the paper.

Let $\mathfrak{M}_0$ be a ground model in which $\kappa$ is a weakly compact cardinal. The extension which models $\text{SH}_{\kappa}$ and $CH$ is obtained by iteratively forcing $> \kappa^+$ times with certain $\kappa$cc, countably closed partial orders, taking countable supports in the iteration. For $\alpha > 1$, $(\mathcal{P}_\alpha, <)$ is the ordering giving the first $\alpha$ steps in the iteration. $\mathcal{P}_\alpha$ is a set of functions with domain $\alpha$.

Let $L_{\kappa, \kappa}$ be the Levy collapse by countable conditions of each $\beta \in [\kappa_1, \kappa)$ to $\kappa_1$ (so $\kappa$ is the new $\kappa_2$). Then $\mathcal{P}_1$ (isomorphic to $L_{\kappa, \kappa}$) is $\{f: \text{dom } f = 1, f(0) \in L_{\kappa, \kappa}\}$, ordered by $f < g$ iff $f(0) < g(0)$. To define $\mathcal{P}_{\beta+1}$, choose a term $A_\beta$ in the forcing language of $\mathcal{P}_\beta$ for a countably closed partial ordering (to be described later) and let $\mathcal{P}_{\beta+1} = \{f: \text{dom } f = \beta + 1, f \uparrow \beta \in \mathcal{P}_\beta, \exists \bar{f} \in A_\beta \}$, ordered by $f < g$ iff $f \uparrow \beta < g \uparrow \beta$ and $g \uparrow \beta \# f \uparrow \beta (f(\beta) < g(\beta))$. For $\alpha$ a limit ordinal, $\mathcal{P}_\alpha = \{f: \text{dom } f = \alpha, f \uparrow \beta \in \mathcal{P}_\beta$ for all $\beta < \alpha$, and $f(\beta)$ is (the term for) $\emptyset$, the least element of $A_\beta$, for all but $< \kappa_0\beta$'s}, ordered by $f < g$ iff for all $\beta < \alpha, f \uparrow \beta < g \uparrow \beta$.

Each $\mathcal{P}_\alpha$ is countably closed. We are done as in Solovay-Tennenbaum [7] if the $A_\beta$'s can be chosen so that each $\mathcal{P}_\alpha$ has the $\kappa$cc, and therefore that every $\kappa_2$ ($= \kappa$)-Souslin tree which crops up gets killed by some $A_\beta$.

If $T$ is a tree then $(T)_\lambda$ is the $\lambda$th level of $T$, $(T)_\lambda = \bigcup_{\mu < \lambda} T_\mu$. Regarding the previous problem, it is a theorem of Mitchell that if $CH$ and $\diamondsuit(\alpha < \omega_2: cf(\alpha) = \aleph_1)$ hold, then there are countably closed $\aleph_2$-Souslin trees $T_n$, $n < \omega$, such that for
each $m < \omega$, $\otimes_{n<m} T_n$ has the $\mathfrak{N}_{2\text{cc}}$, but $\otimes_{n<\omega} T_n$ does not have the $\mathfrak{N}_{2\text{cc}}$. We give for interest his proof modulo the usual Jensen methods. At stage $\mu < \omega_2$ construct each $(T_n)_\mu$ normally above $(T_n)_{<\mu}$. If $\mu = \nu + 1$ let each $x \in (T_n)_\mu$ have at least two successors in $(T_n)_{<\mu}$. If $cf(\mu) = \omega$ let all branches in $(T_n)_{<\mu}$ go through. If $cf(\mu) = \omega_1$ make sure that the antichain given by the $\diamond$-sequence for $\otimes_{n<m} T_n$ is taken care of, and choose $\langle c_{\mu n} : n < \omega \rangle \in \otimes_{n<\omega} (T_n)_\mu$ so that if $\mu' < \mu$, $cf(\mu') = \omega_1$, then $\langle c_{\mu'n} : n < \omega \rangle \not\prec \langle c_{\mu n} : n < \omega \rangle$. We also carry along the following induction hypothesis: if $v < \mu$, $\langle x_n : n < \omega \rangle \in \otimes_{n<\omega} (T_n)_v$, $m < \omega$, $\langle y_n : n < m \rangle \in \otimes_{n<m} (T_n)_\mu$, $x_n < y_n (n < m)$ and $\langle x_n : n < \omega \rangle \not\prec \langle c_{\lambda n} : n < \omega \rangle$, for all $\lambda < \omega$ with $cf(\lambda) = \omega_1$, then there are $y_n \in (T_n)_\mu (m < n < \omega)$ with $x_n < y_n$, such that $\langle y_n : n < \omega \rangle \not\prec \langle c_{\lambda n} : n < \omega \rangle$, for all $\lambda < \mu$ with $cf(\lambda) = \omega_1$.

If $\delta$ is inaccessible, then forcing with $L_{\kappa,\delta}$ (whence $2^{\kappa_0} = \mathfrak{N}_1$, $2^{\kappa_1} = \mathfrak{N}_2 = \delta$, and $\diamond \{ \alpha < \kappa_2 : cf(\alpha) = \omega_1 \}$ hold) followed by forcing with the $\otimes_{n<\omega} T_n$ constructed previously, gives a countably closed length $\omega$ iteration of countably closed, $\delta\text{cc}$ partial orderings which does not have $\delta\text{cc}$.

The previous theorem does not rule out that an iteration of $\mathfrak{N}_{2}$-Souslin trees can give $CH$ and $SH_{\kappa_1}$: in this paper, though, the $\mathfrak{N}_{2}$-Souslin trees are killed by a different method. Let $T$ be an $\mathfrak{N}_{2}$-Souslin tree (we may assume without loss of generality that $T$ is normal and $\text{Card}(T)_1 = \mathfrak{N}_1$). The antichain partial order $A_T$ is defined to be $\{(x \subseteq T : x a countable antichain, \text{root } T \notin x), \subseteq\}$. Now $A_T$ need not have the $\mathfrak{N}_{2\text{cc}}$, as shown by the following result of the first author: $\text{Con}(ZFC)$ implies $\text{Con}(ZFC + \text{"there is an } \mathfrak{N}_{2}\text{-Souslin tree } T \text{ and a sequence } \langle d_{\alpha n} : n < \omega \rangle \text{ from } (T)_\alpha, \text{ for each } \alpha < \omega_2, \text{ such that if } \alpha < \beta, \text{ there is an } m < \omega \text{ with } d_{\alpha n} < d_{\beta n}, \text{ for all } n > m\}$.) Namely, start with a model of $CH$. Determine in advance that, say, $(T)_\alpha = [\omega_1,\omega_1(\alpha + 1))$ and that $d_{\alpha n} = \omega_1\alpha + n$. Conditions are countable sub-trees $S$ of $T$ such that if $S \cap (T)_\alpha \neq \emptyset$ then $\{d_{\alpha n} : n < \omega\} \subseteq S$, which meet the requirements on the $d_{\beta n}$'s.

Devlin [2] has shown that such a tree exists in $L$.

We show now that if each $A_\beta$ is an $A_T$, $T$ an $\mathfrak{N}_{2}$-Souslin tree, then each $\mathcal{P}_\alpha$ has the $\kappa\text{cc}$, which will prove the theorem (we actually just use that Card $T < \kappa$ the cardinal designated as the new $2^{\kappa_1}$ and $T$ has no $\omega_2$-paths; see remarks at the end). This theorem was originally proved by the first author when $\kappa$ is measurable; that the assumption can be weakened to weak compactness of $\kappa$ is due to the second author.

We consider now only the case $\alpha < \kappa^+$ (which will suffice, assuming $2^\kappa = \kappa^+$ in $\mathcal{M}$, for $CH + SH_{\kappa_1} + 2^{\kappa_1} = \mathfrak{N}_3$); $\alpha$ arbitrary will be dealt with at the end.

Fix $\alpha$ for the rest of the proof. We assume by induction that

(1) For each $\beta < \alpha$, $\mathcal{P}_\beta$ has the $\kappa\text{cc}$.

(One more induction hypothesis is listed later.)

For $\beta < \alpha$, let $T_\beta$ be the $\beta$th $\mathfrak{N}_{2}$-Souslin tree, so $\mathcal{P}_{\beta+1} = \mathcal{P}_\beta \overset{\otimes}{\times} A_\beta$, where $A_\beta = A_{T_\beta}$. Assume without loss of generality that for each $\lambda < \kappa$,

$$(T_\beta)_\lambda \subseteq [\omega_1\lambda, \omega_1(\lambda + 1)).$$

An $f \in \mathcal{P}_\beta$, $\beta < \alpha$, is said to be determined if there is in $\mathcal{M}$ a sequence $\langle z_\gamma : \gamma \in \text{dom } f - \{0\} \rangle$ of countable sets of ordinals such that for all $\gamma \in \text{dom } f - \{0\}$,
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If \( f_\gamma : n < \omega \to \mathbb{R} \) is a sequence of determined members of $\mathcal{P}_\beta$, with \( f_n < f_{n+1} \), then the coordinatewise union $f_\omega$ of the $f_n$'s is seen to be a determined member of $\mathcal{P}_\beta$ extending each $f_n$. From this it may be seen, by induction on $\beta < \alpha$, that the set of determined members of $\mathcal{P}_\beta$ is cofinal in $\mathcal{P}_\beta$. Redefine each $\mathcal{P}_\beta$ then to consist just of the determined conditions. Clearly Card $\mathcal{P}_\beta < \kappa$, for all $\beta < \alpha$.

For $f, g \in \mathcal{P}_\beta$, $f \sim g$ means that $f$ and $g$ are compatible.

Fix for the rest of the proof a one-one enumeration $\alpha = \{ \alpha_\mu : \mu \in S \}$, for some $S \subseteq \kappa$ (this induces a similar enumeration of each $\beta < \alpha$, the induction hypothesis (2) for $\beta$ below, is with respect to this induced enumeration). For notational simplicity we now assume that $S$ is some $\kappa' < \kappa$.

\[
\gamma = 0 \Rightarrow h(\gamma) = f(\gamma) \uparrow (\omega_1 \times \lambda), \quad \gamma > 0 \Rightarrow h(\gamma) = f(\gamma) \cap \lambda.
\]

The function $f|\lambda$ need not be a condition, but for $g \in \mathcal{P}_\beta$, we will still write $f|\lambda < g$ to mean that $f|\lambda$ is coordinatewise a subset of $g$. Let $\mathcal{P}_\beta|\lambda = \{ f \in \mathcal{P}_\beta : f|\lambda = f \}$.

Suppose $0 < \beta < \alpha$, $\lambda < \kappa$. Define

\[
\sharp^\beta_\lambda(f, g, h) \iff f, g \in \mathcal{P}_\beta, f|\lambda = g|\lambda = h,
\]

\[
\dagger^\beta_\lambda(f, h) \iff f \in \mathcal{P}_\beta, h \in \mathcal{P}_\beta|\lambda \text{ and for every } h' \geq h \text{ with } h' \in \mathcal{P}_\beta|\lambda, h' \sim f,
\]

\[
\delta^\beta_\lambda(f, g, h) \iff \dagger^\beta_\lambda(f, h) \text{ and } \delta^\beta_\lambda(g, h).
\]

For $P \subseteq Q$, $P \subseteq_{\text{reg}} Q$ means that $P$ is a regular subordering of $Q$, that is, any two members of $P$ compatible in $Q$ are compatible in $P$, and every maximal antichain of $P$ is a maximal antichain of $Q$. If $\mathcal{P}_\beta|\lambda \subseteq_{\text{reg}} \mathcal{P}_\beta$, then $\delta^\beta_\lambda(f, g, h)$ states that $h \uparrow \gamma_{\mathcal{P}_\beta} \uparrow [f] \neq 0$.

Recall that the sets of the form $\{ \lambda < \kappa : (R, A, \in, A \cap R) \vdash \Phi \}$, where $A \subseteq \kappa$, $\Phi$ is $\tau_1$, and $(R, A, \in, A) \vdash \Phi$, belong to a normal uniform filter $\mathcal{T}_w$, the weakly compact filter on $\kappa$ (see [9], [8]). The second thing we assume by induction is

(2) for all $\beta < \alpha$, for $\mathcal{T}_w$-almost all $\lambda < \kappa$, for all $f, g, h$, $\sharp^\beta_\lambda(f, g, h)$ implies that for some $h' \geq h$, $\delta^\beta_\lambda(f, g, h')$.

If $\beta < \alpha$, $\lambda < \kappa$, say that $(T_\beta)_{< \lambda}$ is determined by $\mathcal{P}_\beta|\lambda$ if for each $\theta$, $\tau$ in $(T_\beta)_{< \lambda}$ there is a $\mathcal{P}_\beta$-maximal antichain $R$ of conditions deciding the ordering between $\theta$ and $\tau$ in $T_\beta$, such that $R \subseteq \mathcal{P}_\beta|\lambda$.

**Lemma 1.** There is a closed unbounded set of $\lambda < \kappa$ such that for all $\mu < \lambda$, $(T_\alpha)_{< \lambda}$ is determined by $\mathcal{P}_\alpha|\lambda$.

**Proof.** This is a consequence of the strong inaccessibility of $\kappa$ and the assumption that each $\mathcal{P}_\beta$, $\beta < \alpha$, has $\text{kcc}$.
Lemma 2. For $\mathcal{P}_\text{we}$-almost all $\lambda < \kappa$,
(a) $\lambda$ is strongly inaccessible.
(b) For all $\mu < \lambda$, $\mathcal{P}_\alpha^\lambda | \lambda$ has the $\lambda$-cc.
(c) For all $\mu < \lambda$, $\mathcal{P}_\alpha^\lambda | \lambda \subseteq \mathcal{P}_\alpha$.
(d) For all $\mu < \lambda$, $\mathcal{P}_\alpha^\lambda | \lambda = \aleph_2$.
(e) For all $\mu < \lambda$, $\mathcal{P}_\alpha^\lambda (T_{\alpha_\gamma} \prec \lambda)$ is an $\aleph_2$-Souslin tree.

Proof. By $\pi_1^\lambda$ reflection and the normality of $\mathcal{P}_\text{we}$.

Lemma 3. Let $\beta < \alpha$, $\lambda < \kappa$, $\mathcal{P}_\beta | \lambda \subseteq \mathcal{P}_\beta$.
(a) If $f \in \mathcal{P}_\beta$, $j \in \mathcal{P}_\beta | \lambda$, and $f \sim j$, then there is an $h > j$ with $\star^\lambda(f, h)$.
(b) If $\star^\lambda(f, g, h)$ and $D, E$ are cofinal subsets of $\mathcal{P}_\beta$, then there exists $\langle f', g', h' \rangle > \langle f, g, h \rangle$ with $\star^\lambda(f', g', h')$, $f' \subseteq D$, $g' \subseteq E$, $h < f'$, $g'$.

Proof. These are standard facts about forcing.

The following is T. Carlson's version of the lemma we originally used here.

Lemma 4. Suppose $\lambda$ satisfies Lemma 1 and Lemma 2(c), $\mu < \lambda$, and $\star^\lambda(f, h)$. Then $f(\mu) < h$.

Proof. Otherwise there is a $v < \lambda$ with $\alpha_v < \alpha$, and a $\theta \in f(\alpha_v) \cap \lambda$ such that $\theta \not\in h(\alpha_v)$. We have that $h \upharpoonright \alpha_v \vdash \theta$ is $T_{\alpha_v}$-incomparable with each member of $h(\alpha_v)$; otherwise $\star^\lambda(f, h)$ would be contradicted. Pick an $h' \in \mathcal{P}_\alpha | \lambda$, $h' > h \upharpoonright \alpha_v$, and a $\theta' < \lambda$ such that $h' \vdash \theta'$. Let $\tilde{h}$ be $h' \cup \langle h(\alpha_v) \cup \{\theta'\} \rangle$. Then $h \in \mathcal{P}_\alpha | \lambda$, $h < \tilde{h}$, and $h \sim f$, a contradiction.

Lemma 5. Suppose $\lambda$ satisfies Lemma 1 and Lemma 2(c), $\mu < \lambda$, and $\star^\lambda(f, g, h)$. Then there is an $\langle f', g', h' \rangle > (f, g, h)$ with $\#(f', g', h')$.

Proof. Choose $(f, g, h) = (f_0, g_0, h_0) < \cdots < (f_n, g_n, h_n) < \cdots$ so that $\star^\lambda(f_n, g_n, h_n)$, $h_n < f_{n+1}$, $h_n < g_{n+1}$. This is done by repeated applications of Lemma 3(a), (b). Then Lemma 4 implies that the coordinatewise union $(f', g', h')$ of the $(f_n, g_n, h_n)$'s is as desired.

Definition. Suppose $\lambda < \kappa$, $\mu < \lambda$, $f, g \in \mathcal{P}_\alpha$, and suppose $\theta, \tau$ are nodes of $(T_{\alpha_\gamma} \prec \lambda) (\theta = \tau$ allowed). Then $\langle f, g \rangle$ is said to $\lambda$-separate $\langle \theta, \tau \rangle$ if there is a $\gamma < \lambda$ and $\theta', \tau' \in (T_{\alpha_\gamma})$, with $\theta' \neq \tau'$, such that $f(\mu) < \theta', g(\mu) > T_{\alpha_\gamma} \tau'$.

Lemma 6. Suppose $\lambda$ satisfies Lemmas 1 and 2, $\mu < \lambda$, $\star^\lambda(f, g, h)$, $(\theta, \tau) \subseteq (T_{\alpha_\gamma} \prec \lambda)$, with $\theta = \tau$ allowed. Then there is an $\langle f', g', h' \rangle > \langle f, g, h \rangle$ such that $\star^\lambda(f', g', h')$ and $\langle f', g' \rangle$ $\lambda$-separates $\langle \theta, \tau \rangle$.

Proof. Claim. There are $f_0, f_1 > f$, $\tilde{h} > h$, with $\star^\lambda(f_0, \tilde{h})$, $\star^\lambda(f_1, \tilde{h})$, such that $\langle f_0, f_1 \rangle$ $\lambda$-separates $\langle \theta, \theta \rangle$ via a $\langle \theta_1, \theta_2 \rangle \in (T_{\alpha_\gamma})$, for some $\gamma < \lambda$.

Proof. Consider the result of taking a generic set $G_{\alpha_\gamma} | \lambda$ over $\mathcal{P}_\alpha | \lambda$ which contains $h$. In $\mathcal{M}[G_{\alpha_\gamma}], (T_{\alpha_\gamma} \prec \lambda)$ is a $\lambda$ (= $\aleph_2$)-Souslin tree. In the further extension $\mathcal{M}[G_{\alpha_\gamma}], \theta$ determines a $\lambda$-path through $(T_{\alpha_\gamma} \prec \lambda)$. Since this path is not in $\mathcal{M}[G_{\alpha_\gamma}]$, there must be $\tilde{h} \in G_{\alpha_\gamma} | \lambda$, $\tilde{h} > h$, $f_0, f_1 > f$, $\gamma < \lambda$, $\theta_0, \theta_1 \in (T_{\alpha_\gamma})$, $\theta_0 \neq \theta_1$, with
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$h \vdash f_0 \models \theta \, <_{\kappa_2} \theta, h' \vdash f_1 \models \theta \, <_{\kappa_2} \theta$, such that $\ast_{\lambda}(f_0, h)$ and $\ast_{\lambda}(f_1, h')$. This gives the claim.

Now, by Lemma 3, choose $(g', h') > (g, h)$ and a $\tau' \in (T_{\kappa_2})_{\gamma}$ so that $\ast_{\lambda}(g', h')$ and $g' \models \tau' <_{\kappa_2} \tau$. Pick $i \in \{0, 1\}$ with $\tau' \neq \theta_i$. Let $f' = f_i, \theta' = \theta_i$. Then $(f', g', h')$ are as desired. This proves the lemma.

We claim that the induction hypotheses (1) and (2) automatically pass up to $\alpha$ if $\operatorname{cf}(\alpha) > \omega$. Namely, (1) holds at $\alpha$ by a $\Delta$-system argument. For (2), suppose that for an $\mathcal{T}_{\kappa_2}$-positive set $W$ of $\lambda$'s there is a counterexample $<f_\lambda, g_\lambda, h_\lambda>$. Let $N_\lambda = (\operatorname{support} f_\lambda \cup \operatorname{support} g_\lambda)$. If $\operatorname{cf}(\alpha) \neq \kappa$ then for some $\beta < \alpha$ and $\mathcal{T}_{\kappa_2}$-positive $V \subseteq W$, $\lambda \in V$ implies $N_\lambda \subseteq \beta$, and we are done. If $\operatorname{cf}(\alpha) = \kappa$, pick a closed unbounded set $C \subseteq \kappa$ such that $\langle \sup\{\alpha_\gamma : \gamma < \lambda\} : \lambda \in C \rangle$ is increasing, continuous and cofinal in $\alpha$ and an $\mathcal{T}_{\kappa_2}$-positive $V \subseteq W \cap C$ such that for some $\beta < \alpha$ and all $\lambda \in V, N_\lambda \cap \sup\{\alpha_\gamma : \gamma < \lambda\} \subseteq \beta$, then apply (2) at $\beta$.

Thus, we may assume for the rest of the proof that $\alpha$ is a successor ordinal or $\operatorname{cf}(\alpha) = \omega$. Fix $\langle \mu_n : n < \omega \rangle$ such that if $\alpha = \beta + 1$ then each $\mu_n$ is the $\mu$ with $\alpha_\mu = \beta$, and if $\operatorname{cf}(\alpha) = \omega$ then $\langle \alpha_\mu : n < \omega \rangle$ is an increasing sequence converging to $\alpha$.

**Lemma 7.** For $\mathcal{T}_{\kappa_2}$-almost all $\lambda$, the following holds: if $f, g \in \mathcal{P}_\alpha, h \in \mathcal{P}_\alpha|\lambda$ and $\#_{\lambda}(f, g, h)$ then there exists $<f', g', h'> > <f, g, h>$ such that $\#_{\lambda}(f', g', h')$ and such that for each $\mu < \lambda$ with $\alpha_\mu \neq 0$, and each $\theta \in f'(\alpha_\mu) - \lambda, \langle f' \upharpoonright \alpha_\mu, g' \upharpoontright \alpha_\mu \rangle \lambda$-separates $<\theta, \tau>$.

**Proof.** We prove the lemma for $\lambda$, assuming that $\lambda$ satisfies Lemmas 1 and 2, $\lambda > \mu_n (n < \omega)$ and for each $n < \omega, \lambda$ is in the $\mathcal{T}_{\kappa_2}$ set given by induction hypothesis (2) for $\alpha_n$. Construct $<f_n, g_n, h_n>, n < \omega$, so that

(a) $f_n, g_n \in \mathcal{P}_{\alpha_n}, \#_{\lambda}(f_n, g_n, h_n),$

(b) $<f \upharpoonright \alpha_n, g \upharpoontright \alpha_n, h \upharpoontright \alpha_n> < <f_n, g_n, h_n>,$

(c) $<f_n, g_n, h_n> < <f_{n+1}, g_{n+1}, h_{n+1}>,$

(d) if, at stage $n > 1$, $<\theta_n, \tau_n>$ is the $n$th pair (in the appropriate bookkeeping list for exhausting them) with $\theta_0 \in f_n(\alpha_n) - \lambda, \tau_0 \in g_n(\alpha_n) - \lambda, \nu_n < \lambda, \alpha_n < \alpha_\mu$, then

$$<f_n \upharpoonright \alpha_n, g_n \upharpoontright \alpha_n \lambda$$-separates $<\theta, \tau>.$

Let $f_0 = f \upharpoonright \alpha_\mu, g_0 = g \upharpoontright \alpha_\mu, h_0 = h \upharpoontright \alpha_\mu$. Suppose $n > 1$ and $f_{n-1}, g_{n-1}, h_{n-1}$ have been constructed. Let

$$f' = f_{n-1} \upharpoonright f \upharpoontright \langle \alpha_{\mu_{n-1}}, \alpha_\mu \rangle, \quad g' = g_{n-1} \upharpoonright g \upharpoontright \langle \alpha_{\mu_{n-1}}, \alpha_\mu \rangle, \quad h' = h_{n-1} \upharpoonright h \upharpoontright \langle \alpha_{\mu_{n-1}}, \alpha_\mu \rangle.$$

Then $\#_{\lambda}(f', g', h')$. By induction hypothesis (2), there is an $h_n > h'$ such that $\ast_{\lambda}(f', g', h_n)$. By Lemma 6, there is $<f''_n, g''_n, h''_n> > <f'_n, g'_n, h'_n>$ such that $\ast_{\lambda}(f''_n, g''_n, h''_n)$ and

$$<f''_n \upharpoonright \alpha_n, g''_n \upharpoontright \alpha_n \lambda$$-separates $<\theta_n, \tau_n>.$

Finally, by Lemma 5 we may choose $<f_n, g_n, h_n> > <f''_n, g''_n, h''_n>$ so that $\#_{\lambda}(f_n, g_n, h_n)$. License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Taking \( f', g', h' \) to be the coordinatewise unions of the \( f_n \)'s, \( g_n \)'s, \( h_n \)'s gives the lemma.

We now verify the two induction hypotheses.

(1) \( \mathcal{P}_\alpha \) has the kcc.

**Proof.** Given \( f_\lambda \in \mathcal{P}_\alpha, \lambda < \kappa \). For each \( \lambda \) which satisfies Lemmas 1, 2 and 7, with \( \lambda > \mu_n \ (n < \omega) \), apply Lemma 7 to the triple \( \langle f_\lambda, f_\lambda, f_\lambda \lambda \rangle \), obtaining a triple \( \langle f_\lambda^*, f_\lambda^*, f_\lambda \lambda \rangle \) (so \( f_\lambda \leq f_\lambda^*, f_\lambda^*, f_\lambda \lambda = f_\lambda^* | \lambda = f_\lambda^* | \lambda \)).

Let

\[ B_\lambda = (\text{support } f_\lambda^* \cup \text{support } f_\lambda^*) \cap \{ \alpha_\mu : \mu < \lambda \}. \]

If \( 0 \neq \alpha_\mu \in B_\lambda \), write

\[ f_\lambda^*(\alpha_\mu) - \lambda = \{ \theta_\mu, n < r_\mu \}, \quad r_\mu < \omega, \]

\[ f_\lambda^*(\alpha_\mu) - \lambda = \{ \tau_\mu, m < s_\mu \}, \quad s_\mu < \omega. \]

To each pair \( \langle \theta_\mu, \tau_\mu \rangle, n < r_\mu, m < s_\mu \), \( \langle f_\lambda^*, f_\lambda^* \rangle \) assigns a separating pair \( \langle \theta_\mu, \tau_\mu, n < r_\mu, m < s_\mu \rangle \in \lambda \times \lambda \).

Let \( J_\lambda = (\text{dom } f_\lambda^*(0) \cup \text{dom } f_\lambda^*(0)) - (\omega_1 \times \lambda) \).

By the normality of \( \mathcal{P}_\alpha \), there is an \( \mathcal{P}_\alpha \)-positive set \( U \) such that on \( U \), the sets \( B_\lambda, r_\mu, s_\mu, \theta_\mu, \tau_\mu, \lambda \) are independent of \( \lambda \), and such that if \( \lambda, \lambda' \in U, \lambda < \lambda' \), then \( \langle \text{support } f_\lambda^*, \text{support } f_\lambda^* \rangle \cap (\text{support } f_\lambda^*, \text{support } f_\lambda^*) = B_\lambda, \) and \( J_\lambda \cap J_{\lambda'} = \emptyset \).

By induction on \( \gamma < \alpha \) it is seen that if \( \lambda, \mu \in U \) and \( \lambda < \mu \), then \( f_\lambda^* \sim f_\mu^* \).

Namely, there is no trouble with coordinates in the support of at most one of these functions; coordinates in both supports, being in \( B_\lambda \), are taken care of by the construction. Since \( f_\lambda < f_\lambda^* \) and \( f_\mu < f_\mu^* \), we are done.

The following strengthening of kcc for \( \mathcal{P}_\alpha \) has thus been proved: if for an \( \mathcal{P}_\alpha \)-positive set \( W \) of \( \lambda \)'s, \( \#^\alpha(f_\lambda, g_\lambda, h_\lambda) \), then there is an \( \mathcal{P}_\alpha \)-positive \( U \subseteq W \) and \( \langle f_\lambda, g_\lambda, h_\lambda \rangle, \lambda \in U \), such that \( \langle f_\lambda, g_\lambda, h_\lambda \rangle \leq \langle f_\lambda^*, g_\lambda^*, h_\lambda^* \rangle, \#^\alpha(f_\lambda^*, g_\lambda^*, h_\lambda^*) \), and so that if \( \lambda, \mu \in W, \lambda < \mu \), then \( f_\lambda \sim g_\mu \) in the strong sense that the coordinatewise union of \( f_\lambda^* \) and \( g_\mu \) is a condition extending both \( f_\lambda^* \) and \( g_\mu^* \).

Lastly, we prove the second induction hypothesis for \( \alpha \).

(2) For \( \mathcal{P}_\alpha \)-almost all \( \lambda < \kappa \), for all \( f, g, h, \#^\alpha(f, g, h) \) implies that for some \( h' > h \), \( \#^\alpha(f, g, h') \).

**Proof.** Otherwise for an \( \mathcal{P}_\alpha \)-positive set \( W \) of \( \lambda \)'s there exists a counterexample \( \langle f_\lambda, g_\lambda, h_\lambda \rangle \). We may assume that for each \( \lambda \in W \), \( \lambda > \mu_n \ (n < \omega) \) and \( \lambda \) satisfies Lemmas 1, 2 and 7. Furthermore, since we have already proved that \( \mathcal{P}_\alpha \) has the kcc, we may assume that for each \( \lambda \in W \), \( \mathcal{P}_\alpha | \lambda \subseteq \text{reg } \mathcal{P}_\alpha \) and \( \mathcal{P}_\alpha | \lambda \) has the kcc. If \( f_\lambda \) or \( g_\lambda \) equals \( h_\lambda \), we are done, so assume, for each \( \lambda \in W \), that \( f_\lambda, g_\lambda \notin \mathcal{P}_\alpha | \lambda \).

Apply Lemma 7 to each \( \langle f_\lambda, g_\lambda, h_\lambda \rangle, \lambda \in W \), getting \( \langle f_\lambda, g_\lambda, h_\lambda \rangle \). Now uniformize as in part (a) to get an \( \mathcal{P}_\alpha \)-positive \( V \subseteq W \) such that if \( \lambda, \mu \in V \) and \( \lambda < \mu \) then \( f_\lambda \sim g_\mu \). Since \( \langle f_\lambda, g_\lambda, h_\lambda \rangle \) is a counterexample to (b), there is a maximal antichain \( H_\lambda \) of \( \{ h \in \mathcal{P}_\alpha | \lambda : h > h_\lambda \} \) such that for each \( h \in H_\lambda, h \sim f_\lambda \) or \( h \sim g_\lambda \). Then \( H_\lambda \) is a maximal antichain of \( \{ h \in \mathcal{P}_\alpha : h \succ h_\lambda \} \), and \( \text{Card } H_\lambda < \lambda \). Pick an \( \mathcal{P}_\alpha \)-positive \( U \subseteq V \) on which \( H_\lambda = H \) is independent of \( \lambda \) and such that for each \( h \in H \), the questions, whether or not \( h \sim f_\lambda, h \sim g_\lambda \), are independent of \( \lambda \). Pick
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$\lambda, \mu \in U, \lambda < \mu$, and let $j > f'_\lambda, g'_\mu$. Now $j > h_\lambda$, and $j \not\in \mathcal{P}_{\alpha} | \lambda$ (whence $j \not\in H$). But for each $h \in H$, either $h \sim g_\lambda$ (whence $h \sim g_\mu$) or $h \sim f'_\lambda$. In either case, $h \sim j$ since $j > f'_\lambda, g'_\mu$, so $H$ is not maximal, a contradiction.

This completes the proof of the theorem.

Denote by an $\omega_2$-tree a tree $T$ of any cardinality with no paths of length $\omega_2$. An $\omega_2$-tree $T$ is special if there is an $f: T \to \omega_1$ such that $x < y$ implies $f(x) \neq f(y)$. By the previous methods, using countable specializing functions instead of countable antichains, the consistency of “$2^{\aleph_1} = \aleph_1$, $2^{\aleph_2} > \aleph_2$, and every $\omega_2$-tree of cardinality $< 2^{\aleph_1}$ is special” is obtained — the analogous theorem for the $\aleph_1$ case being Baumgartner-Malitz-Reinhardt [1]. We can also get this model to satisfy the “generalized Martin’s axioms” (which are consistent relative to just ZFC but which do not imply $SH_{\aleph_2}$) that have been considered by the first author and by Baumgartner (see Tall [8]). Desirable, of course, would be the consistency of a generalized $MA$ which is both simple and powerful.

The partial orderings appropriate for the prior methods can be iterated an arbitrary number of times, giving generalized $MA$ models in which $2^\kappa$ is arbitrarily large. The ordering $\mathcal{R}_{\alpha}$ giving the first $\alpha$ steps of the iteration need not be of cardinality $< \kappa$, but, assuming each $\mathcal{R}_\beta, \beta < \alpha$, has $\text{ccc}$, any sequence $\langle p_\lambda: \lambda < \kappa \rangle$ from $\mathcal{R}_{\alpha}$ is a subset of a sufficiently closed model of power $\kappa$, in which the proof that two $p_\lambda$’s are compatible can be carried out.

Regarding the analog of these results where $\aleph_2$ is replaced by $\gamma^+$—the relevant forcing is $\gamma$-directed closed, so by upward Easton forcing we may guarantee that, for example, $\gamma$ remains supercompact if it was in the ground model.

For results involving consequences of $SH_{\aleph_1}$: with $GCH$, see Gregory [3], [4] ($\text{Con}(SH_{\aleph_1}$ and $GCH$) is open); with just $CH$, see a forthcoming paper by Stanley and the second author.

REFERENCES

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COLORADO, BOULDER, COLORADO 80309
DEPARTMENT OF MATHEMATICS, HEBREW UNIVERSITY, JERUSALEM, ISRAEL