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BP TORSION IN FINITE H-SPACES

BY

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ABSTRACT. Let p be odd and (X, μ) a 1-connected mod p finite H-space. It is shown that for $n \geqslant 1$ the Morava K-theories, $k(n)_*(X)$ and $k(n)^*(X)$, have no higher v_n torsion. Also examples are constructed to show that v_1 torsion in $BP\langle 1\rangle^*(X)$ can be of arbitrarily high order.

1. Introduction. Let p be a prime. Let $\mathbf{Z}_{(p)}$ be the integers localized at the prime p and let $\mathbf{Z}/p$ be the integers reduced mod p. By an H-space (X, μ) we will mean a pointed topological space X which has the homotopy type of a connected CW complex of finite type together with a basepoint preserving map $\mu : X \times X \to X$ with two-sided homotopy unit. By a (mod p) finite H-space (X, μ), we will mean an H-space such that $H^*(X; \mathbf{Z}/p)$ is a finite-dimensional $\mathbf{Z}/p$ module. If (X, μ) is a 1-connected (mod p) finite H-space, then $H_*(X) = H_*(X; \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$ has no higher p torsion for p odd (see [**19**]) and a similar result is conjectured to be true for $p = 2$ (see [**13**]). By no higher p torsion we mean that $px = 0$ for all $x \in$ Torsion $H_*(X)$.

Let $BP_*(X)$ be the Brown Peterson homology of X (see [**2**] and [**25**]). Then $BP_*(X)$ is a module over $BP_*(\mathrm{pt}) = \mathbf{Z}_{(p)}[v_1, v_2, \ldots]$ (dim $v_s = 2p^s - 2$). Hence, besides p torsion, we can speak of v_s torsion in $BP_*(X)$ for $s \geqslant 1$. Based on the above result for ordinary homology, we ask the following obvious question. For each $s \geqslant 0$ (let $v_0 = p$) can there be higher v_s torsion in $BP_*(X)$ when (X, μ) is a 1-connected (mod p) finite H-space? That is, is it necessary true that, for any $x \in BP_*(X)$ and any $s \geqslant 0$, $v_s^n x = 0$ for $n > 1$ always implies $v_s x = 0$. In general the answer is negative (see §8). However, if we pass to certain theories associated with BP, then restrictions on torsion can be obtained.

For each $n \geqslant 1$ let $k(n)_*(X)$ be the connected Morava K-theory of X (see [**12**]). It is a module over $k(n)_*(\mathrm{pt}) = \mathbf{Z}/p[v_n]$ (dim $v_s = 2p^s - 2$). There is a canonical ring homomorphism $BP_*(X) \to k(n)_*(X)$ which maps v_n to v_n and v_i to 0 for $i \neq n$. The only type of torsion which can occur in $k(n)_*(X)$ is v_n torsion. We will show

THEOREM 1:1. *Let p be odd and let (X, μ) be a 1-connected* (mod p) *finite H-space. Then $k(n)_*(X)$ has no higher v_n torsion. That is, $v_n x = 0$ for all $x \in$* Torsion $k(n)_*(X)$.

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If we let $v_0 = p$ and interpret $k(0)_*(X)$ as $H_*(X)$, then 1:1 is an obvious analogue to the torsion result for $H_*(X)$. Our proof of 1:1 is also analogous to that used to prove lack of higher torsion in $H_*(X)$. We employ a Bockstein spectral sequence argument. For each $n \geqslant 1$ we have an exact couple.

$$\begin{array}{ccc} k(n)_*(X) & \xrightarrow{v_n} & k(n)_*(X) \\ & \Delta_n \nwarrow \quad \swarrow \rho_n & \\ & H_*(X; \mathbf{Z}/p) & \end{array} \qquad (T_n)$$

where v_n denotes multiplication by v_n and ρ_n is the canonical map from $k(n)_*(X)$ to $H_*(X; \mathbf{Z}/p)$ ("reduction mod v_n"). This exact couple induces a Bockstein spectral sequence $\{B^r\}$ and we prove 1:1 by showing that $B^2 = B^\infty$ (see §3). The differential d^1 acting on $B^1 = H_*(X; \mathbf{Z}/p)$ can be identified with the Milnor cohomology operation Q_n. Thus our proof of 1:1 also yields the following lifting result (indeed it is equivalent to 1:1).

THEOREM 1:2. *Let p be odd and let (X, μ) be a* 1-*connected* mod p *finite H-space. Then $x \in H_*(X; \mathbf{Z}/p)$ lies in the image of $\rho_n : k(n)_*(X) \to H_*(X; \mathbf{Z}/p)$ if and only if $Q_n(x) = 0$.*

These results can be converted from homology to cohomology. This follows from the fact that, for a finite H-space, X^+ ($= X$ plus a disjoint basepoint) is its own S-dual (see [**6**]). Thus there are natural isomorphisms $\tilde{h}^*(X) = \tilde{h}_*(X)$ for $h = H\mathbf{Z}/p$ and $k(n)$ (see [**1**]). In particular, the following statements are dual to 1:1 and 1:2.

THEOREM 1:3. *Let p be odd and let (X, μ) be a* 1-*connected* (mod p) *finite H-space. Then $k(n)^*(X)$ has no higher v_n torsion.*

THEOREM 1:4. *Let p be odd and let (X, μ) be a* 1-*connected* (mod p) *finite H-space. Then $x \in H^*(X; \mathbf{Z}/p)$ lies in the image of $\rho_n : k(n)^*(X) \to H^*(X; \mathbf{Z}/p)$ if and only if $Q_n(x) = 0$.*

The proofs of Theorems 1:1 and 1:2 occupy §§2–6. Let $K(n)_*(X)$ be the Morava K-theory of X. It is a module over $K(n)_*(\mathrm{pt}) = \mathbf{Z}/p[v_n, v_n^{-1}]$ and can be obtained from $k(n)_*(X)$ by localizing with respect to v_n. We prove 1:1 and 1:2 by studying $K(n)_*(X)$. We begin by studying $H^*(X; \mathbf{Z}/p)$ as a Q_n differential Hopf algebra (see §2). We then use this information to calculate $K(n)_*(X)$ as a $\mathbf{Z}/p[v_n, v_n^{-1}]$ module in two distinct ways. First of all, we dualize our information for $H^*(X; \mathbf{Z}/p)$ and calculate $B^2 =$ the homology of $H_*(X; \mathbf{Z}/p)$ ($= B^1$) with respect to the differential Q_n. Letting $N = \operatorname{rank} B^2$ (as a $\mathbf{Z}/p$ module) we have the inequality $N \geqslant \operatorname{rank} K(n)_*(X)$ as a $\mathbf{Z}/p[v_n, v_n^{-1}]$ module with equality if and only if $B^2 = B^\infty$ (see §3). Secondly, we use our information for $H^*(X; \mathbf{Z}/p)$ to calculate the algebra $H_*(\Omega X; \mathbf{Z}/p)$ via an Eilenberg-Moore spectral sequence (see §4). We then calculate the algebra $K(n)_*(\Omega X)$ (see §5) and use this information to deduce, via another Eilenberg-Moore type spectral sequence, that $\operatorname{rank} K(n)_*(X) \geqslant N$ (see §6). Our two approaches now yield that $B^2 = B^\infty$.

There are many potential extensions of the above results. Most are not possible. First of all Theorems 1:1–1:4 do not extend to the case $p = 2$. The work of Hodgkins in III of [**10**] shows that the exceptional Lie groups $X = E_7$ and E_8 have higher v_1 torsion in $k(1)^*(X)$ when $p = 2$. However Theorems 1:1–1:4 may be true if we restrict our attention to 1-connected (mod 2) finite H-spaces such that $H^*(X; \mathbf{Z}/2)$ is primitively generated. Results similar to those obtained in §2 are likely. But it is not known if the multiplicative properties of $k(n)$ and $K(n)$ used in §§5 and 6 are true for $p = 2$. Thus we should also point out that different proofs of 1:1 and 1:2 may be possible. If the type of implication theorems proven by Browder for the classical Bockstein spectral sequence (see [**3**]) have analogues for the spectral sequences used in this paper then $B^2 = B^\infty$ is a simple consequence of the results of §2.

Next, Theorems 1:1 and 1:3 do not extend to *BP* theories in which more than one type of torsion is allowable. In §§7–9 we will study the case of $BP\langle 1\rangle$ theory. It seems typical of the general situation. Recall that $BP\langle 1\rangle^*(\mathrm{pt}) = \mathbf{Z}_{(p)}[v_1]$ (see [**11**]). For $p = 2$ let X be the exceptional Lie group G_2. For p odd let X be the finite H-space constructed by Harper in [**8**]. We will prove

THEOREM 1:5. *For any integer $m \geqslant 1$, $BP\langle 1\rangle^*(\prod_{i=1}^m X)$ contains an element x satisfying $v_1^{m-1}x \neq 0$ while $v_1^m x = 0$.*

On the other hand generalizations of 1:2 and 1:4 do seem possible. One possible generalization would be that for mod odd 1-connected finite H-spaces $x \in H^*(X; \mathbf{Z}/p)$ lies in the image of the Thom map $BP^*(X) \to H^*(X; \mathbf{Z}/p)$ if and only if $Q_0(x) = Q_1(x) = \cdots = 0$. This result has been verified for the simple, simply-connected, compact Lie groups by Yagita (see [**27**] and [**28**]).

Throughout this paper we will always assume, unless otherwise indicated, that *p is an odd prime and that (X, μ) is a 1-connected* (mod p) *finite H-space.*

We close this section with some remarks on the Steenrod algebra and on Hopf algebras. Regarding the Steenrod algebra $A^*(p)$, we will rely on Milnor's treatment from [**22**]. Besides the usual left action of $A^*(p)$ on $H^*(X; \mathbf{Z}/p)$, we will also use the left action of $A^*(p)$ on $H_*(X; \mathbf{Z}/p)$ obtained by duality. That is $\langle \chi(\theta)a, x\rangle = (-1)^{|\theta|\,|a|}\langle a, \theta x\rangle$ for any $\theta \in A^*(p)$, $a \in H_*(X; \mathbf{Z}/p)$ and $x \in H^*(X; \mathbf{Z}/p)$. ($\chi : A^*(p) \to A^*(p)$ is the canonical antiautomorphism.) It is with respect to this action that we have the already mentioned identity $d^1 = Q_n$ and $B^1 = H_*(X; \mathbf{Z}/p)$ for the Bockstein spectral sequence arising from T_n. As for the action of $A^*(p)$ on $H^*(X; \mathbf{Z}/p)$, observe that it is an unstable action. That is, $\mathcal{P}^n$ acts trivially in dimensions $< 2n$ while $\mathcal{P}^n(x) = x^p$ for all x in dimension $2n$.

Regarding Hopf algebras, the general reference is [**23**]. We will deal with Hopf algebras over both $\mathbf{Z}/p$ and $\mathbf{Z}/p[v_n, v_n^{-1}]$. Given an H-space (X, μ) then $H^*(X; \mathbf{Z}/p)$ and $H_*(X; \mathbf{Z}/p)$ have natural structures as $\mathbf{Z}/p$ Hopf algebras over the Steenrod algebra. The structures are induced by μ and the diagonal map $\Delta : X \to X \times X$ and are dual to one another. Similarly $K(n)^*(X)$ and $K(n)_*(X)$ are dual $\mathbf{Z}/p[v_n, v_n^{-1}]$ Hopf algebras (see [**26**] for the properties of $K(n)$). Given a Hopf algebra A, we will use $P(A)$ and $Q(A)$ to denote primitives and indecomposables,

respectively. If A and A^* are dual Hopf algebras, then $Q(A)$ and $P(A^*)$ are dual in the sense of a quotient module of A being dual to a submodule of A^*. If A is a Hopf algebra over $A^*(p)$, then both $P(A)$ and $Q(A)$ inherit $A^*(p)$ structures from $H^*(X; \mathbf{Z}/p)$. Furthermore, the duality between $Q(A)$ and $P(A^*)$ is in the sense of $A^*(p)$ modules. If A is a commutative, associative Hopf algebra over $\mathbf{Z}/p$ then A is isomorphic, as an algebra, to a tensor product $\otimes A_i$ of Hopf algebras over $\mathbf{Z}/p$ where each A_i is generated as an algebra by a single element a_i. Such a decomposition is called a Borel decomposition of A and the elements $\{a_i\}$ are called generators of the decomposition. An element $x \in A$ is said to be of height n if $x^{n-1} \neq 0$ while $x^n = 0$. The height of a Borel generator a_i is 2 if a_i has odd degree and a power of p or ∞ if a_i has even degree.

Unless otherwise indicated we will work in the category of graded $\mathbf{Z}/p$ modules. In particular tensor products will be over $\mathbf{Z}/p$ unless otherwise indicated. Given a graded set S we will use $E(S)$, $\mathbf{Z}/p[S]$ and $\Gamma(S)$ to denote the graded exterior, polynomial and divided polynomial Hopf algebras over $\mathbf{Z}/p$ generated by S. We will use $T(S)$ to denote the polynomial Hopf algebra with all elements truncated at height p.

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2. The structure of $H^*(X; \mathbf{Z}/p)$. In this section we will discuss the Hopf algebra structure of $H^*(X; \mathbf{Z}/p)$. Our results are extensions of those obtained in **[19]**. We begin with the Steenrod module structure of $Q = Q(H^*(X; \mathbf{Z}/p))$.

$$Q^{\text{even}} = \sum_{n>1} \beta_p \mathcal{P}^n Q^{2n+1}. \tag{2:1}$$

(2:2) If $Q^{2n} \neq 0$ then $n = p^k + \cdots + p + 1$ or $n = p^{k+1} + \cdots + p^{l+1} + p^{l-1} + \cdots + p + 1$ $(k \geqslant l \geqslant 1)$.

In particular $Q^{2n} \neq 0$ implies n has a p-adic expansion of the form $n = p^{s_1} + \cdots + p^{s_k}$ $(s_1 > s_2 > \cdots > s_k)$. Such an integer is said to be binary (with respect to p). We will now show that the Steenrod module structure of Q^{even} can be read off the p-adic expansion of the various n such that $Q^{2n} \neq 0$. Given $n = \sum n_s p^s$ let $\omega(n) = \sum n_s$. In particular, if n is binary then $\omega(n)$ is the number of nonzero terms in its p-adic expansion.

(2:3) The function ω defines a splitting of Q^{even} as a Steenrod module. That is, $Q^{\text{even}} = \bigoplus_{s \geqslant 2} M_s$ where $M_s = \sum_{\omega(n)=s} Q^{2n}$.

(2:4) Given $n = p^{k+1} + \cdots + p^{l+1} + p^{l-1} + \cdots + p + 1$ let

$$m = \begin{cases} p^{k+1} + \cdots + p^{l+2} + p^l + \cdots + p + 1 & \text{if } k > l, \\ p^k + \cdots + p + 1 & \text{if } k = l. \end{cases}$$

Then $Q^{2n} = \mathcal{P}^{p^l} Q^{2m}$.

To visualize these structure theorems write down the sequence

$$(p+1, p^2+1)(p^2+p+1, p^3+p+1, p^3+p^2+1)(p^3+p^2+p+1, \ldots)$$

representing the possible n for which $Q^{2n} \neq 0$. Each set of brackets contains the

dimensions occurring in a particular M_s. Thus no Steenrod power acts across any bracket. On the other hand, any two adjacent dimensions within a given set of brackets are connected by the appropriate Steenrod power. For example

$$p^2 + p + 1 \xrightarrow{\mathcal{P}^{p^2}} p^3 + p + 1 \xrightarrow{\mathcal{P}^{p}} p^3 + p^2 + 1.$$

These results on Q can be used to obtain theorems about the algebra structure of $H^*(X; \mathbf{Z}/p)$. Since $H^*(X; \mathbf{Z}/p)$ is commutative and associative it has a Borel decomposition $\otimes A_i$ with generators $\{a_i\}$. Since the Borel generators project to a $\mathbf{Z}/p$ basis of Q, it follows that 2:2 restricts the dimensions in which generators can lie. Also, since the pth power of any $x \in H^*(X; \mathbf{Z}/p)$ can be identified with a Steenrod power operating on x, 2:3 and 2:4 can be used to restrict the height of each generator.

$$\begin{aligned} &\text{If } |a_i| = 2(p^k + \cdots + p + 1) \text{ then } a_i^{p^2} = 0. \\ &\text{If } |a_i| = 2(p^{k+1} + \cdots + p^{l+1} + p^{l-1} + \cdots + p + 1) \text{ then} \\ &a_i^p = 0. \end{aligned} \tag{2:5}$$

For all of the above results, see [**19**]. We will now use these results to study how the Milnor elements $\{Q_s\}_{s \geqslant 0}$ in $A^*(p)$ act on $H^*(X; \mathbf{Z}/p)$. Our basic result is

THEOREM 2:6. *$Q^{2n} = Q_s Q^{2n-2p^s+1}$ if $n \geqslant p^s$.*

Most of the remaining part of this section will be spent in proving 2:6. We first observe that the operations $\{Q_s\}_{s \geqslant 0}$ satisfy the relations:

$$Q_s Q_t = -Q_t Q_s \quad \text{for } s, t \geqslant 0, \tag{2:7}$$

$$Q_s \mathcal{P}^t = \mathcal{P}^t Q_s - Q_{s+1} \mathcal{P}^{t-p^s} \quad \text{for } s, t \geqslant 0 \tag{2:8}$$

(here, as elsewhere, we will assume $\mathcal{P}^i = 0$ for $i < 0$). By repeated applications of 2:8 we obtain the relations

$$Q_s \mathcal{P}^t = \sum_{i \geqslant 0} (-1)^i \mathcal{P}^{t-(p^s + \cdots + p^{s+i-1})} Q_{s+i} \quad \text{for } s, t \geqslant 0. \tag{2:9}$$

We will prove 2:6 by induction on n, so we can make the following assumption:

INDUCTION HYPOTHESIS. Theorem 2:6 is true in dimensions $< 2n$.

We break our proof of 2:6 for $2n$ into a number of cases.

Case I. $n = p^k + \cdots + p + 1$. By an inductive argument on s we can prove that for $0 \leqslant s \leqslant k$ we have the relation

$$Q^{2(p^k + \cdots + p + 1)} = Q_s \mathcal{P}^{p^{k-1} + \cdots + p^s} Q^{2(p^{k-1} + \cdots + p + 1) + 1}. \tag{$*$}$$

The initial case is 2:1 since $Q_0 = \beta_p$. Assuming s we deduce $s + 1$ by using 2:8. (The term $\mathcal{P}^{p^{k-1} + \cdots + p^s} Q_s$ disappears by 2:2.)

Case II. $n = p^{k+1} + \cdots + p^{l+1} + p^{l-1} + \cdots + p + 1$ *and* $s \leqslant l - 1$. Let m be as in 2:4. Thus $Q^{2n} = \mathcal{P}^{p^l} Q^{2m}$. Also, by the induction hypothesis, $Q^{2m} = Q_s Q^{2m-2p^s+1}$. Therefore

$$Q^{2n} = \mathcal{P}^{p^l} Q_s Q^{2m-2p^s+1} = Q_s \mathcal{P}^{p^l} Q^{2m-2p^s+1}. \tag{$**$}$$

The second equality follows from 2:9 (the terms $\mathcal{P}^{p^l-(p^s + \cdots + p^{s+i-1})} Q_{s+i}$ disappear by 2:2).

Case III. $n = p^{k+1} + \cdots + p^{l+1} + p^{l-1} + \cdots + p + 1$ *and* $s \geqslant l + 1$. Since we must have $n \geqslant p^s$ we can actually assume that $l + 1 \leqslant s \leqslant k + 1$. Again let m be as in 2:4. Thus $Q^{2n} = \mathcal{P}^{p^l} Q^{2m}$. First of all, suppose $k > l$. Then $m \geqslant p^{k+1} \geqslant p^s$. Hence, by the induction hypothesis, $Q^{2m} = Q_s Q^{2m-2p^s+1}$. Then

$$Q^{2n} = \mathcal{P}^{p^l} Q_s Q^{2m-2p^s+1} = Q_s \mathcal{P}^{p^l} Q^{2m-2p^s+1}. \qquad (***)$$

The second equality follows from 2:8 (the term $Q_{s+1}\mathcal{P}^{p^l-p^s}$ disappears since $p^l - p^s < 0$). Secondly, suppose $k = l$. Then $s = l + 1$ and $m = p^l + \cdots + p + 1$. By the induction hypothesis $Q^{2m} = Q_l Q^{2m-2p^l+1}$. Thus

$$Q^{2n} = \mathcal{P}^{p^l} Q_l Q^{2m-2p^l+1} = Q_{l+1} Q^{2m-2p^l+1}. \qquad (****)$$

The second equality follows from 2:8 (the term $Q_l \mathcal{P}^{p^l}$ disappears since $\mathcal{P}^{p^l}$ acts trivially in dimensions $< 2p^l$).

Case IV. $n = p^{k+1} + \cdots + p^{l+1} + p^{l-1} + \cdots + p + 1$ *and* $s = l$. We want to show $Q^{2n} = Q_l Q^{2n-2p^l+1}$. We will deduce our result by using secondary operations. (Unstable) secondary operations are associated to (unstable) Adem relations. In dimensions $\leqslant 2r + 2$ we have the unstable relation

$$Q_0 \mathcal{P}^{r+p^{l-1}} Q_{l-1} = -Q_l Q_0 \mathcal{P}^r. \qquad (2{:}10)$$

Relation 2:10 is a simple consequence of 2:7 and 2:8 (see §3 of **[13]**). This unstable relation gives rise to an unstable "secondary operation" ϕ defined on Q (see **[18]**). Let $B(q)$ be the sub-Hopf algebra of $H^*(X; \mathbf{Z}/p)$ generated over $A^*(p)$ by $\Sigma_{i \leqslant q} H^i(X; \mathbf{Z}/p)$. This filtration $\{B(q)\}$ of $H^*(X; \mathbf{Z}/p)$ defines a filtration $\{F_q Q\}$ of Q. Our secondary operation ϕ is used to prove the following:

LEMMA 2:11. *Given* $\bar{x} \in Q^{2r+1}$ *and* $\bar{y} = Q_{l-1}(\bar{x}) \in Q^{2r+2p^{l-1}}$, *pick* q *such that* $\bar{y} \in F_{q+1}Q$, $\bar{y} \notin F_q Q$. *Then pick* $a \in P_{2r+2p^{l-1}}(H_*(X; \mathbf{Z}/p))$ *such that* $\langle a, \bar{y} \rangle \neq 0$ *and* $\langle a, B(q) \rangle = 0$ (*such choices are always possible*). *If we can find a representative* $x \in H^{2r+1}(X; \mathbf{Z}/p)$ *for* $\bar{x}$ *such that* $Q_0 \mathcal{P}^r(x) \in B(q) \cdot B(q)$ *then* $Q_l(a) \neq 0$.

This lemma is an application of Theorem 3:1:1 of **[19]** to the Adem relation 2:10. Actually 3:1:1 of **[19]** allows a second possible conclusion, namely $a^p \neq 0$. However Theorem 5:4:1 of **[19]** eliminates this possibility.

Recall that $n = p^{k+1} + \cdots + p^{l+1} + p^{l-1} + \cdots + p + 1$. We will apply Lemma 2:11 with $r = n - p^{l-1}$ and $q = 2(p^{k-1} + \cdots + p + 1)$. To apply the lemma pick an arbitrary $0 \neq \bar{y} \in Q^{2n}$.

(i) $\bar{y} \in F_{q+1}Q$. It follows from the relations in $(*)$ and $(**)$ for $s = l - 1$ that we can trace $\bar{y}$ back to an element $\bar{z} \in Q^{2(p^{k-1} + \cdots + p + 1)+1}$. More exactly we have the identities

$$\begin{aligned} Q^{2n} &= \mathcal{P}^{p^l} \mathcal{P}^{p^{l+1}} \cdots \mathcal{P}^{p^k} Q_{l-1} \mathcal{P}^{p^{k-1} + \cdots + p^{l-1}} Q^{2(p^{k-1} + \cdots + p + 1)+1} \\ &= Q_{l-1} \mathcal{P}^{p^l} \mathcal{P}^{p^{l+1}} \cdots \mathcal{P}^{p^k} \mathcal{P}^{p^{k-1} + \cdots + p^{l-1}} Q^{2(p^{k-1} + \cdots + p + 1)+1}. \end{aligned}$$

This concludes the proof of (i).

(ii) $\bar{y} \notin F_q Q$. Every element of $A^*(p)$ is a polynomial in the elements $\{\mathcal{P}^s\}_{s \geqslant 1}$ and $\{Q_s\}_{s \geqslant 0}$. Indeed, by 2:9, every element of $A^*(p)$ can be written in terms of the monomials $\{\mathcal{P}^{n_1} \cdots \mathcal{P}^{n_i} Q_{m_1} \cdots Q_{m_j}\}$. Thus, if $\bar{y}$ can be traced back to an element

in dimension $\leqslant 2(p^{k-1} + \cdots + p + 1)$ then it can be done using the operations $\mathcal{P}^{n_1} \cdots \mathcal{P}^{n_i} Q_{m_1} \cdots Q_{m_j}$. Now $\bar{y} \in M_k$. By 2:3 any nonzero element of M_k can only be traced back to other elements in M_k via the operations $\mathcal{P}^{n_1} \cdots \mathcal{P}^{n_i}$. On the other hand nonzero elements of M_k cannot be traced back to dimensions $\leqslant 2(p^{k-1} + \cdots + p + 1)$ using the operations $Q_{m_1} \cdots Q_{m_j}$. For, by [3], $Q_s Q_t = 0$ on Q for any $s, t \geqslant 0$. Thus we need only consider the operations $\{Q_s\}_{s \geqslant 0}$. And, considering the dimensions in which nontrivial elements of M_k lie (see 2:2), and subtracting the dimension of Q_s $(= 2p^s - 1)$, we always end up with integers < 0 or $\geqslant 2(p^{k-1} + \cdots + p + 1) + 1$. This concludes the proof of (ii).

Using the relations in (i) we can pick x and $\bar{x}$. Pick $\bar{z} \in Q^{2(p^{k-1} + \cdots + p + 1) + 1}$ such that

$$\bar{y} = Q_{l-1} \mathcal{P}^{p^l} \cdots \mathcal{P}^{p^k} \mathcal{P}^{p^{k-1} + \cdots + p^{l-1}}(z).$$

Let $z \in H^{2(p^{k-1} + \cdots + p + 1) + 1}(X; \mathbf{Z}/p)$ be any representative for $\bar{z}$. Let

$$x = \mathcal{P}^{p^l} \cdots \mathcal{P}^{p^k} \mathcal{P}^{p^{k-1} + \cdots + p^{l-1}}(z) \quad \text{and} \quad \bar{x} = \mathcal{P}^{p^k} \cdots \mathcal{P}^{p^l} \mathcal{P}^{p^{k-1} + \cdots + p^{l-1}}(\bar{z}).$$

By definition

(iii) $\bar{y} = Q_{l-1}(\bar{x})$.

Furthermore

(iv) $Q_0 \mathcal{P}^r(x) \in B(q) \cdot B(q)$.

By 2:1 and 2:2 $Q_0 \mathcal{P}^r(\bar{x}) = 0$ in Q, that is $Q_0 \mathcal{P}^r(x)$ is decomposable. Also, since $u^*(z) = z \otimes 1 + 1 \otimes z + z'$ where $z' \in B(q) \otimes B(q)$, it follows that $u^*(x) = x \otimes 1 + 1 \otimes x + x'$ where $x' \in B(q) \otimes B(q)$. The proof of (iv) follows from these two facts via standard arguments (see Lemma 3:1:2 of [19]).

Now pick $a \in P_{2n}(H_*(X; \mathbf{Z}/p))$ such that $\langle a, \bar{y} \rangle \neq 0$. The argument used to prove (ii), if dualized, shows that $\langle a, B(q) \rangle = 0$ for any choice of a. We now conclude from Lemma 2:11 plus (i)–(iv) that $Q_l(a) \neq 0$. Since $\bar{y}$ and a are arbitrary we conclude, by dualizing, that $Q^{2n} = Q_l Q^{2n - 2p^l + 1}$. This concludes our proof of Theorem 2:6. □

We now prove a corollary of 2:6. Let $\zeta: H^*(X; \mathbf{Z}/p) \to H^*(X; \mathbf{Z}/p)$ be the Frobenius map defined by $\zeta(x) = x^p$ for all $x \in H^*(X; \mathbf{Z}/p)$. Then $\zeta(H^*(X; \mathbf{Z}/p))$ is a sub-Hopf algebra of $H^*(X; \mathbf{Z}/p)$. By 2:5, $\zeta(H^*(X; \mathbf{Z}/p)) = T(V)$ for some set V. Fix $n \geqslant 0$. We will prove

COROLLARY 2:12. *We can select V such that the elements of V in dimension $\geqslant 2p^n$ lie in the image of Q_n.*

(When we say $x^p = Q_n(y)$ we are thinking of $\zeta(H^*(X; \mathbf{Z}/p))$ as a sub-Hopf algebra of $H^*(X; \mathbf{Z}/p)$. Thus y can be any element in $H^*(X; \mathbf{Z}/p)$.) We first describe how to obtain algebra generators for $\zeta(H^*(X; \mathbf{Z}/p))$.

LEMMA 2:13. *If a set $W \subset H^*(X; \mathbf{Z}/p)$ projects to a basis of Q^{even} then $\zeta(W)$ generates $\zeta(H^*(X; \mathbf{Z}/p))$ as an algebra.*

PROOF. Suppose we have a set W which projects to a basis of Q^{even}. Let Z be the image of W in the quotient algebra $A = H^*(X; \mathbf{Z}/p)/\zeta(H^*(X; \mathbf{Z}/p))$. Then, by definition, $A = E(Y) \otimes T(Z)$ as an algebra for some set Y. By 2:5 the Frobenius

map factors through the quotient map $H^*(X; \mathbf{Z}/p) \to A$. That is, we have a commutative diagram

$$\begin{array}{ccc} H^*(X; \mathbf{Z}/p) & \xrightarrow{\zeta} & \zeta(H^*(X; \mathbf{Z}/p)) \\ \downarrow & \nearrow \zeta' & \\ A & & \end{array}$$

where ζ' is uniquely determined. Since ζ is surjective, ζ' is also surjective. Thus, using the map ζ', we can regard $\zeta(H^*(X; \mathbf{Z}/p))$ as a quotient algebra of $A = E(Y) \otimes T(Z)$. Since the elements of Y map trivially under ζ' (they are odd dimensional), it follows that $\zeta(H^*(X; \mathbf{Z}/p))$ is actually a quotient algebra of $T(Z)$. It now follows that $\zeta(W)$ generates $\zeta(H^*(X; \mathbf{Z}/p))$ as an algebra. □

LEMMA 2:14. *There exists a set $W \subset H^*(X; \mathbf{Z}/p)$ projecting to a basis of Q^{even} such that the elements of $\zeta(W)$ in dimension $\geqslant 2p^n$ lie in the image of Q_n.*

PROOF. Consider dimension $2s$. If $s \neq p^k + \cdots + p + 1$ for any k, then by 2:5 we can pick the elements of W from kernel ζ. If $s = p^k + \cdots + p + 1$ for $k \geqslant n$, then by 2:6 we can select the elements of W from Image Q_n. And if $y = Q_n(x)$ it follows that $y^p = Q_n(xy^{p-1})$. If $s = p^{n-1} + \cdots + p + 1$, then by 2:6 we can select the elements of W from Image Q_{n-1}. And if $y = Q_{n-1}(x)$, it follows that

$$y^p = \mathcal{P}^{p^{n-1}+\cdots+p+1}(y) = \mathcal{P}^{p^{n-1}+\cdots+p+1}Q_{n-1}(y) = Q_n\mathcal{P}^{p^{n-2}+\cdots+p+1}(y).$$

The last equality follows from 2:8. (The term $Q_{n-1}\mathcal{P}^{p^{n-1}+\cdots+p+1}$ disappears since $\mathcal{P}^{p^{n-1}+\cdots+p+1}$ acts trivially in dimensions $< 2(p^{n-1} + \cdots + p + 1)$.)

Corollary 2:12 now follows from the previous two lemmas. For, having obtained W as in 2:14, we can eliminate any extra algebra generators from $\zeta(W)$ and reduce to V. □

3. Bockstein spectral sequence. In this section we will discuss the Bockstein spectral sequence $\{B^r\}$ arising from the exact sequence (T_n) in §1. In particular we will give a precise statement of what we need to prove in order to show that $B^2 = B^\infty$. The Bockstein spectral sequence satisfies the following properties:

(3:1) $B^1 = H_*(X; \mathbf{Z}/p)$ with differential $d^1 = Q_n$.

(3:2) $B^\infty = k(n)_*(x)/T_n + v_n k(n)_*(X)$ where $T_n =$ the v_n torsion submodule of $k(n)_*(X)$.

(3:3) Image $d^s = \rho_n T_n(s)$ where $T_n(s) = \{x \in T_n | v_n^s x = 0\}$.Thus the sth differential d^s detects the v_n torsion elements of order s which generate direct summands in $k(n)_*(X)$ (as a $\mathbf{Z}/p[v_n]$ module). In particular $B^2 = B^\infty$ implies $v_n x = 0$ for all $x \in T_n$.

Let $N =$ rank B^2 (as a $\mathbf{Z}/p$ module). Then we have the inequalities

$$\begin{aligned} N &\geqslant \text{rank } B^\infty \quad (\text{as a } \mathbf{Z}/p \text{ module}) \\ &= \text{rank } k(n)_*(X)/T_n \quad (\text{as a } \mathbf{Z}/p[v_n] \text{ module}) \\ &= \text{rank } K(n)_*(X) \quad (\text{as a } \mathbf{Z}/p[v_n, v_n^{-1}] \text{ module}). \end{aligned}$$

Furthermore, we have equalities if and only if rank $K(n)_*(X) \geqslant N$. In other words

(3:4) $B^2 = B^\infty$ if and only if rank $K(n)_*(X) \geqslant N$.

The next three sections will be spent in proving rank $K(n)_*(X) \geqslant N$. Obviously, we must first determine N. Choose a Borel decomposition $\otimes A_i$ of $H^*(X; \mathbf{Z}/p)$. Let

a = the number of odd-dimensional generators,

b = the number of even-dimensional generators having dimension $< 2p^n$ plus the number of nonzero pth powers of generators having dimensions $< 2p^n$.

Regarding b, observe that it is possible to count a generator but not its pth power. We will spend the rest of this section in proving

LEMMA 3:5. $N = 2^a p^b$.

By the duality between $H_*(X; \mathbf{Z}/p)$ and $H^*(X; \mathbf{Z}/p)$ as differential Q_n Hopf algebras, it suffices to calculate the homology of $H^*(X; \mathbf{Z}/p)$ with respect to Q_n. We will use the biprimitive spectral sequence arguments of [**4**]. Given a connected Hopf algebra over $\mathbf{Z}/p$, let $\bar{A}$ be the elements of positive dimension and let $\psi^k: A \to \otimes_{i=1}^k A$ be the map defined by the recursive formula $\psi^2 = \bar{\psi}$, the reduced comultiplication, while $\psi^k = (\psi^{k-1} \otimes 1)\bar{\psi}$. We can define a decreasing filtration $\{F_n(A)\}$ of A by the rule

$$F_0(A) = A, \qquad F_n(A) = \bar{A}^n, \quad \text{the } n\text{-fold decomposables of } A, \text{ for } n > 0.$$

Let $E_0(A)$ be the associated graded Hopf algebra. We can define an increasing filtration $\{F^n(A)\}$ by the rule

$$F^0(A) = \mathbf{Z}/p, \qquad F^n(A) = \text{kernel } \psi^{n+1} \quad \text{for } n > 0.$$

Let $E^0(A)$ be the associated graded Hopf algebra. We define the biprimitive form of A to be E^0E_0A. The Hopf algebra E^0E_0A is primitively generated and is isomorphic, as an algebra, to $E(X) \otimes T(Y)$ for some sets X, Y. For any Borel decomposition $\otimes A_i$ of A with generators $\{a_i\}$, the set $X \cup Y$ is represented by the elements $\{a_i\}$ plus the nontrivial iterated pth powers (see 2:7 of [**4**]).

Thus, to prove 3:5, we want to show

(3:6) the biprimitive form of $H(A)$ for the case $A = H^*(X; \mathbf{Z}/p)$, $d = Q_n$ is of the form $E(X) \otimes T(Y)$ where

(i) $X^\# = a$,

(ii) $Y^\# = b$.

Associated with the differential Hopf algebra (A, d) is a spectral sequence $\{E_r\}$ where

(3:7) (i) E_1 = the biprimitive form of A,

(ii) E_∞ = the biprimitive form of $H(A)$.

We use the spectral sequence of 3:7 to prove 3:6. Consider the spectral sequence for the case $(H^*(X; \mathbf{Z}/p), Q_n)$. Each term in the spectral sequence is obviously of the form $E_r = E(X_r) \otimes T(Y_r)$ for sets X_r, Y_r. For $E_1 = E^0E_0(H^*(X; \mathbf{Z}/p))$, the set X_1 can be represented by odd-dimensional indecomposables of $H^*(X; \mathbf{Z}/p)$ while Y_1 can be represented by even-dimensional indecomposables of $H^*(X; \mathbf{Z}/p)$ plus their nonzero pth powers. This follows from 2:5. By the type of argument in 4:4 of [**4**], we can select the elements of Y_1 to be permanent cycles in the spectral sequence. Furthermore, those in dimension $\geqslant 2p^n$ will eventually become boundaries in the spectral sequence. This follows from 2:6 and 2:12. It now follows

that each E_r is isomorphic as a differential Hopf algebra to a tensor product $\otimes E_i$ where each E_i is one of the following types: $K = E(x)$, $L = T(y)$, $M = E(x) \otimes T(y)$, $d(x) = y$ ($|x|$ odd, $|y|$ even). Since $H(M) = E(xy^{p-1})$ it follows that $X_r^\# = X_{r+1}^\#$ for all r. Thus $X_\infty^\# = X_1^\# = a$ and 3:6(i) is established. Regarding 3:6(ii), every element of Y_1 in dimension $\geqslant 2p^n$ eventually appears in a factor of type M while the elements of Y_1 in dimension $< 2p^n$ always appear in factors of type L. It follows that $Y_\infty^\# = b$. $\square$

4. Structure of $H_*(\Omega X; \mathbf{Z}/p)$. Let ΩX be the loop space of X. In this section, we summarize some results from **[13]**. We will use our structure theorem describing $H^*(X; \mathbf{Z}/p)$ as a Hopf algebra over $A^*(p)$ to deduce results about the Hopf algebra structure of $H_*(\Omega X; \mathbf{Z}/p)$. The Hopf algebra structure on $H_*(\Omega X; \mathbf{Z}/p)$ is the bicommutative, biassociative one induced by Ωu and the diagonal map Δ: $\Omega X \to \Omega X \times \Omega X$. The relationship between the Hopf algebra structures on $H^*(X; \mathbf{Z}/p)$ and $H_*(\Omega X; \mathbf{Z}/p)$ is established via an Eilenberg-Moore spectral sequence converging to $H^*(\Omega X; \mathbf{Z}/p)$. It is a second quadrant spectral sequence $\{E_r, d_r\}$ of bicommutative, biassociative bigraded Hopf algebras where:

(4:1) $E_2 = \mathrm{Tor}_{H^*(X;\, \mathbf{Z}/p)}(\mathbf{Z}/p; \mathbf{Z}/p)$ as Hopf algebras.

(4:2) $E_\infty = E^0(H^*(X; \mathbf{Z}/p))$ is isomorphic to $H^*(\Omega X; \mathbf{Z}/p)$ as coalgebras where $E^0(H^*(X; \mathbf{Z}/p))$ is the bigraded Hopf algebra associated to a filtration on $H^*(\Omega X; \mathbf{Z}/p)$.

(4:3) d_r is of bidegree $(r, -r + 1)$.

Regarding 4:1 the E_2 term can be calculated from any Borel decomposition $\otimes A_i$ of $H^*(X; \mathbf{Z}/p)$. Given a Borel decomposition let Y be the set of odd-dimensional generators and Z be the set of even-dimensional generators. Then

(4:4) $E_2 = E(sZ) \otimes \Gamma(sY) \otimes \Gamma(tZ)$ where sx has bidegree $(-1, |x|)$ and tx has bidegree $(-2, p^n|x|)$ if x is of height p^n. In particular the elements $sY \cup sZ$ establish an isomorphism

$$s\colon Q(H^*(X; \mathbf{Z}/p)) \simeq \mathrm{Tor}^{-1,*}_{H^*(X;\, \mathbf{Z}/p)}(\mathbf{Z}/p; \mathbf{Z}/p). \tag{4:5}$$

Since the elements of the -1 stem are permanent cycles in the spectral sequence it follows from 4:5 that the loop map $\Omega^* : Q(H^*(X; \mathbf{Z}/p)) \to P(H^*(\Omega X; \mathbf{Z}/p))$ has an obvious definition in terms of the spectral sequence.

Regarding differentials, the only differential which acts nontrivially is d_{p-1}. Using 4:5 the differential d_{p-1} can be characterized in terms of the Steenrod powers $Q_0\mathcal{P}^n$ acting on $Q(H^*(X; \mathbf{Z}/p))$, namely $d_{p-1}\gamma_p sx = sQ_0\mathcal{P}^n x$ if $|x| = 2n + 1$. Thus the action of d_{p-1} is entirely determined by 2:1. In particular E_{p-1} can be written as a tensor product $\otimes E_i$ of differential Hopf algebras where each E_i is one of the following types: $K = \Gamma(x) \otimes E(y)$, $d\gamma_p(x) = y$, $L = \Gamma(x)$. Calculating the coalgebra structure of $H(E_{p-1}) = E_p = E_\infty = H^*(\Omega X\colon \mathbf{Z}/p)$ and dualizing to the algebra structure of $H_*(\Omega X; \mathbf{Z}/p)$ we conclude that:

PROPOSITION 4:6. *$H_*(\Omega X; \mathbf{Z}/p) = \mathbf{Z}/p[\chi_1]/I \otimes \mathbf{Z}/p[\chi_2]$ where I is the ideal generated by $\{x_i^p | x_i \in \chi_1\}$.*

Moreover we have an explicit relationship between the elements of χ_1 and those in $Q(H^*(X; \mathbf{Z}/p))$.

PROPOSITION 4:7. *Letting* $\chi_1 = \{x_i\}$ *there exist elements* $\{a_i\}$ *in* $Q^{\text{odd}}(H^*(X; \mathbf{Z}/p))$ *such that*

(i) $\{\beta_p \mathcal{P}^{n_i}(a_i)\}$ *is a basis of* $Q^{\text{even}}(H^*(X; \mathbf{Z}/p))$ $(|a_i| = 2n_i + 1)$,

(ii) $\langle x_i, \Omega^*(a_j)\rangle = \delta_{ij}$ (*the Kronecker delta*).

5. The structure of $BP_*(\Omega X)$, $k(n)_*(\Omega X)$, and $K(n)_*(\Omega X)$. In this section we will study the algebra structure of $BP_*(\Omega X)$, $k(n)_*(\Omega X)$, and $K(n)_*(\Omega X)$. We will concentrate on $BP_*(\Omega X)$. The results for $k(n)_*(\Omega X)$ and $K(n)_*(\Omega X)$ will be simple consequences. Each of the theories are multiplicative. The multiplications are related by a commutative diagram

$$\begin{array}{ccccc} BP \wedge BP & \to & k(n) \wedge k(n) & \to & K(n) \wedge K(n) \\ \downarrow & & \downarrow & & \downarrow \\ BP & \to & k(n) & \to & K(n) \end{array}$$

(By [**26**] the outside square and the right-hand square commute. It follows that the left-hand square commutes as well. For if the left hand did not commute then the obstruction $\in k(n)^*(BP \wedge BP)$ would map nontrivially to $K(n)^*(BP \wedge BP)$.) The multiplications for BP and $K(n)$ are commutative, associative, and induce Kunneth formulas (see [**17**] and [**26**]). It is not known whether $k(n)$ satisfies similar properties in general. However, since $BP_*(\Omega X) \to k(n)_*(\Omega X)$ is surjective, $k(n)$ must satisfy these properties for the space ΩX. Thus, the map $\Omega\mu$: $\Omega(X \times X) \to \Omega X$ induces a commutative, associative algebra structure on each of $BP_*(\Omega X)$, $k(n)_*(\Omega X)$ and $K(n)_*(\Omega X)$.

Our arguments for $BP_*(\Omega X)$ are extensions of those in [**15**] and [**16**]. We have surjective maps $BP_*(\Omega X) \xrightarrow{T} H_*(\Omega X) \xrightarrow{\rho} H_*(\Omega X; \mathbf{Z}/p)$. (If we think of $H_*(\Omega X)$ and $H_*(\Omega X; \mathbf{Z}/p)$ as the BP theories $BP(0)_*(\Omega X)$ and $BP(-1)_*(\Omega X)$ then T and ρ can be identified with the maps $\rho(0, \infty)$ and $\rho(-1, 0)$ of §7.) The maps T and ρ are surjective and kernel $T = (v_1, v_2, \ldots)$ while kernel $T = (p, v_1, v_2, \ldots)$. Thus we obtain $H_*(\Omega X)$ and $H_*(\Omega X; \mathbf{Z}/p)$ from $BP_*(\Omega X)$ by factoring out the ideals $(v_1, v_2, \ldots)$ and $(p, v_1, v_2, \ldots)$, respectively. Let $\hat{\chi} = \hat{\chi}_1 \cup \hat{\chi}_2$ be a set of representatives in $BP_*(\Omega X)$ for the elements $\chi = \chi_1 \cup \chi_2$ in $H_*(\Omega X; \mathbf{Z}/p)$. Let D be the set of monomials in the elements of $\hat{\chi}$ of weight $\geqslant 2$ which do not include the pth power of any element from $\hat{\chi}_1$. Then $\hat{\chi} \cup D$ is a $\Lambda = BP_*(\text{pt})$ basis of $BP_*(\Omega X)$. In fact it follows from 4:6 that

PROPOSITION 5:1. *$BP_*(\Omega X)$ is isomorphic, as an algebra, to $\Lambda[\hat{\chi}]/J$ where J is the ideal generated by $\{R_X | x \in \hat{\chi}_1\}$ and each R_X is of the form $R_X = X^p - \sum \lambda_i X_i - \sum \omega_j d_j$ for some $X_i \in \hat{\chi}$, $d_j \in D$, and $\lambda_i, \omega_j \in \Lambda$.*

(Therefore, J defines the relation by which monomials in $\hat{\chi}$ involving pth powers of elements from $\hat{\chi}_1$ can be written in terms of $\hat{\chi} \cup D$.) We have much more precise information on the relation $\{R_X\}$. The rest of our study of $BP_*(\Omega X)$ will consist in obtaining this information. First of all, it follows from 4:6 that, for each $X \in \hat{\chi}_1$,

$$X^p = pU \bmod (v_1, v_2, \ldots) \tag{5:2}$$

for some $U \in BP_*(\Omega X)$. Moreover, the elements $\{U\}$ satisfy an additional property. For each X_i and corresponding U_i, let $x_i = \rho T(X_i)$ and $u_i = \rho T(U_i)$. Let $\{a_i\}$ be the elements in $Q^{\text{odd}}(H^*(X; \mathbf{Z}/p))$ from 4:7. If $|a_i| = 2n_i + 1$, let $b_i = \mathcal{P}^{n_i}(a_i)$.

PROPOSITION 5:3. $\langle u_i, \Omega^*(b_j)\rangle = \delta_{ij}$ (*the Kronecker delta*).

PROOF. Let $c_j = \Omega^*(a_j)$ and let $(\Omega\Delta)^p : H_*(\Omega X; \mathbf{Z}/p) \to \bigotimes_{i=1}^p H_*(\Omega X; \mathbf{Z}/p)$ be the p-fold reduced comultiplication defined as in §4. Then

$$\begin{aligned}\langle u_i, \Omega^*(b_j)\rangle &= \langle u_i, \Omega^*\mathcal{P}^{n_j}(a_j)\rangle = \langle u_i, \mathcal{P}^{n_j}(c_j)\rangle \\ &= \langle u_i, c_j^p\rangle = \langle \Omega\Delta^p(u_i), c_j \otimes \cdots \otimes c_j\rangle.\end{aligned}$$

To show that the last pairing is equal to δ_{ij} let B_s be the sub-Hopf algebra of $H_*(\Omega X; \mathbf{Z}/p)$ generated by $\sum_{i \leqslant s} H_i(\Omega X; \mathbf{Z}/p)$. It follows from the implication arguments which appear in [5] that

(*) if $|x_i| = 2n_i$ then $(\Omega\Delta)^p(u_i) = x_i \otimes \cdots \otimes x_i$ in $\bigotimes_{i=1}^p H_*(\Omega X; \mathbf{Z}/p)//B_{2n_i-1}$.

Now, to prove 5:3, we may obviously assume that $|u_i| = |\Omega^*(b_j)|$ $(= 2pn_i)$. It follows that $|x_i| = |c_j|$ $(= 2n_i)$. Thus, by 3:10 of [23]

$$\langle B_{2n_i-1}, c_j\rangle = 0, \tag{**}$$

for c_j is primitive while any element of B_{2n_i-1} in dimension $2n_i$ is decomposable. Finally, by 4:7(ii),

$$\langle x_i, c_j\rangle = \delta_{ij}. \tag{***}$$

It follows from (*), (**), and (***) that we have the equality

$$\langle (\Omega\Delta)^p(u_i), c_j \otimes \cdots \otimes c_j\rangle = \delta_{ij}.$$

This concludes the proof of 5:3. □

We now further expand the identity 5:2. The argument uses cohomology operations. Let $\Delta_s = (0, \ldots, 0, 1, 0, \ldots)$ be the sequence with 1 in the sth position. Let $\{r_{\Delta_s}\}_{s \geqslant 1}$ be the operations in $BP^*(BP)$ defined by Quillen (see [2] or [25]). For any space X these operations act on the left of $BP^*(X)$ as differentials. They act on $\Lambda = BP_*(\text{pt})$ by the rule that

$$r_{\Delta_i} v_j = p\delta_{ij} \bmod(v_1, v_2, \ldots). \tag{5:4}$$

This follows from a knowledge of how r_{Δ_i} acts on $\Lambda \otimes_{\mathbf{Z}} Q = Q[m_1, m_2, \ldots]$ (see [29]) plus the Hazewinkel definition of v_j in terms of the elements $\{m_s\}$ (see [9]). For each sequence Δ_s we have the Milnor element $\mathcal{P}^{\Delta_s}$ in $A^*(p)$. The operations r_{Δ_s} and $\mathcal{P}^{\Delta_s}$ are related via the following diagram:

$$\begin{array}{ccc} BP_*(\Omega X) & \xrightarrow{r_{\Delta_s}} & BP_*(\Omega X) \\ \downarrow \rho T & & \downarrow \rho T \\ H_*(\Omega X; \mathbf{Z}/p) & \xrightarrow{-\mathcal{P}^{\Delta_s}} & H_*(\Omega X; \mathbf{Z}/p) \end{array} \tag{5:5}$$

(see [15] for a proof).

If we work modulo $(v_1, v_2, \ldots)^2$, then we can extend 5:2. For each $X \in \hat{\chi}_1$,

$$X^p \equiv pU - \sum_{s \geqslant 1} v_s r_{\Delta_s}(U) + \sum v_s d_s \bmod(v_1, v_2, \ldots)^2 \tag{5:6}$$

where d_s is decomposable.

(We note that the right-hand of 5:6 is a finite sum since $BP_i(X) = 0$ for $i < 0$.) The proof of 5:6 is by induction on s. The case $s = 1$ is Proposition 5:2 of **[15]**. The general case is proved by an argument analogous to that used in proving 5:2 of **[15]**. We use 5:4 and replace the use of the operation r_{Δ_1} and the coefficient v_1 by the operation r_{Δ_s} and the coefficient v_s.

The rest of our study of $BP_*(\Omega X)$ will consist in showing that if we take the expansion 5:6, pass to $Q(BP_*(\Omega X))$ and project down to $Q(H_*(\Omega X; \mathbf{Z}/p))$, then:

PROPOSITION 5:7. *For each $n \geqslant 1$ the set $\{r_{\Delta_n}(U) \mid |U| \geqslant 2p^n\}$ projects to a linearly independent set in $Q(H_*(\Omega X; \mathbf{Z}/p))$.*

Our proof of 5:7 is analogous to the arguments used in **[16]**. Let $Q = Q(H^*(X; \mathbf{Z}/p))$ and $K = Q \cap \text{kernel } \beta_p$. For each U_i, let $u_i = \rho T(U_i)$. We will show

$$\langle u_i, \Omega^* K \rangle = 0 \quad \text{for each } u_i. \tag{5:8}$$

Thus, there is a well-defined pairing between the elements $\{u_i\}$ and the elements in $Q' = Q/K$. By 2:1 Q' has nontrivial elements only in odd dimension. A $\mathbf{Z}/p$ basis of Q' is represented by any set $\{b_j\} \subset Q^{\text{odd}}$ such that $\{\beta_p(b_j)\}$ is a $\mathbf{Z}/p$ basis of Q^{even}. Thus 5:3 produces a $\mathbf{Z}/p$ basis $\{b_j\}$ of Q' such that $\langle u_i, b_j \rangle = \delta_{ij}$. We will show

(5:9) for each $n \geqslant 1$ the map $Q \xrightarrow{\mathcal{P}^{\Delta_n}} Q \to Q'$ is surjective in dimensions $\geqslant 2p^n + 1$.

Thus, in dimension $\geqslant 2p^n + 1$, the basis $\{b_j\}$ can be chosen to be of the form $\{\mathcal{P}^{\Delta_n}(h_j)\}$. Therefore, when we pass to ΩX, it follows that in dimensions $\geqslant 2p^n$ we have the identities

$$\langle \mathcal{P}^n(u_i), \Omega^* h_j \rangle = \langle u_i, \mathcal{P}^{\Delta_n}\Omega^*(h_j) \rangle = \langle u_i, \Omega^* \mathcal{P}^{\Delta_n}(h_j) \rangle = \langle u_i, \Omega^* b_j \rangle = \delta_{ij}$$

(the last equality follows from 5:3). Thus the set $\{\mathcal{P}^{\Delta_n}(u_i)\}$ is linearly independent in dimension $\geqslant 2p^n$. By 5:5 this suffices to prove 5:7. So we are left with proving 5:8 and 5:9.

PROOF OF 5:8. First of all we show

(*) the map ρ: $Q(H^*(X)) \to Q$ maps onto K.

That is, each element of K has a representative in $H^*(X; \mathbf{Z}/p)$ which survives the p torsion Bockstein cohomology spectral sequence $\{B_r\}$. Since $B_2 = B_\infty$ (see 4:6:1 of **[19]**) and $d_1 = \beta_p$ it suffices to show that each element of K has a representative in $H^*(X; \mathbf{Z}/p)$ on which β_p acts trivially. Consider the biprimitive spectral sequence $\{E_r\}$ of 3:7 for $(A, d) = (H^*(X; \mathbf{Z}/p), \beta_p)$. By the discussion in §3 the elements of Q define unique elements in $E_1 = E^0E_0H^*(X; \mathbf{Z}/p)$. It suffices to show that the elements of K survive the spectral sequence. It follows from 2:6 that E_1 is isomorphic, as a differential Hopf algebra, to a tensor product $\otimes E_i$ of differential Hopf algebras where each E_i is one of the following type: $K = E(x) \otimes T(y)$, $dx = y$, $L = E(x)$. Thus E_2 is an exterior algebra and the elements of K define nonzero elements in E_2. By the usual Hopf algebra arguments (see **[4]**) E_2 exterior implies $E_2 = E_\infty$. In particular, K survives to E_∞.

We use (*) to prove 5:8. Pick $k \in K$. Then $k = \rho(l)$ for some $l \in Q(H^*(X))$ and

$$\langle u_i, \Omega^*(k) \rangle = \langle u_i, \rho\Omega^*(l) \rangle = \langle \rho T(u_i), \rho\Omega^*(l) \rangle = \rho\langle T(u_i), \Omega^*(l) \rangle.$$

Thus it suffices to show that $\langle T(u_i), \Omega^*(l)\rangle = 0$. Now $\langle T(u_i), \Omega^*(l)\rangle \in \mathbf{Z}_{(p)}$. Hence it is equivalent to show that $p\langle T(u_i), \Omega^*(l)\rangle = 0$. And

$$p\langle T(u_i), \Omega^*(l)\rangle = \langle pT(u_i), \Omega^*(l)\rangle = \langle T(X_i)^p, \Omega^*(l)\rangle = 0.$$

The second equality follows from 5:2. The third equality follows from 3:10 of [**23**]. For $T(X_i)^p$ is indecomposable while $\Omega^*(l)$ is primitive. □

PROOF OF 5:9. By 2:6, $Q_n = Q_0\mathcal{P}^{\Delta_n}Q_0$ maps onto Q^{even} in dimensions $\geqslant 2p^n + 2$. By 2:2, $\mathcal{P}^{\Delta_n}$ is trivial when restricted to Q^{even}. Thus $Q_0\mathcal{P}^{\Delta_n}$ maps onto Q^{even} in dimensions $\geqslant 2p^n + 2$. Since $Q_0 = \beta_p$, 5:9 follows. □

We conclude this section by using our results for $BP_*(\Omega X)$ to deduce results about the algebra structures of $k(n)_*(\Omega X)$ and $K(n)_*(\Omega X)$. We can obtain $k(n)_*(\Omega X)$ from $BP_*(\Omega X)$ by factoring out the ideal $I_n = (p, v_1, \ldots, v_{n-1}, v_{n+1}, \ldots)$. And we can obtain $K(n)_*(\Omega X)$ from $k(n)_*(\Omega X)$ by localizing with respect to v_n. Thus any element of $BP_*(\Omega X)$ defines unique elements in $k(n)_*(\Omega X)$ and in $K(n)_*(\Omega X)$. We will use the same symbol to denote these elements. From 5:1 it follows that:

PROPOSITION 5:10. *For $h = k(n)$ or $K(n)$, $h_*(\Omega X)$ is isomorphic, as an algebra, to $h_*(\text{pt})[\hat{\chi}]/J$ where $\hat{\chi}$ and J are as in* 5:1.

The relations $\{R_X\}$ generating J must be of the following form:

$$\begin{aligned} &\text{if } |X^p| < 2p^n \text{ then } R_X = X^p, \\ &\text{if } |X^p| \geqslant 2p^n \text{ then } R_X = X^p - p\overline{X} \text{ for some } \overline{X} \in h_*(\Omega X). \end{aligned} \tag{5:11}$$

Since we have a commutative diagram

$$\begin{array}{ccccc} BP_*(\Omega X) & \to & k(n)_*(\Omega X) & \to & H_*(\Omega X; \mathbf{Z}/p) \\ \downarrow & & \downarrow & & \downarrow \\ Q\big(BP_*(\Omega X)\big) & \to & Q\big(k(n)_*(\Omega X)\big) & \to & Q\big(H_*(\Omega X; \mathbf{Z}/p)\big) \end{array}$$

it follows from 5:6 and 5:7 that

PROPOSITION 5:12. *For $h = k(n)$ the elements $\{\overline{X}\}$ from* 5:11 *project to a linearly independent set in $Q(H_*(\Omega X; \mathbf{Z}/p))$.*

6. Eilenberg-Moore spectral sequences. In this section we will use the results of §§4 and 5 to complete the proof of Theorem 1:1. By 3:4 and 3:5 we need only show

$$\operatorname{rank} K(n)_*(X) \geqslant N \quad \text{where } N = 2^a p^b. \tag{6:1}$$

Our main tool in proving 6:1 will be Eilenberg-Moore type spectral sequences. For each of the theories $h = H\mathbf{Z}/p$, $k(n)$, and $K(n)$, there is a 1st and 4th quadrant spectral sequence $\{E^r(h)\}$ of $h_*(\text{pt})$ modules satisfying

$$E^2(h) = \operatorname{Tor}^{h_*(\Omega X)}\big(h_*(\text{pt}); h_*(\text{pt})\big), \tag{6:2}$$

$$E^\infty(h) = E^0\big(h_*(X)\big), \tag{6:3}$$

$$d^r \text{ is of bidegree } (-r, r-1). \tag{6:4}$$

In each case the spectral sequence arises from the fact that X has the same homotopy type as $B_{\Omega X}$, the classifying space of ΩX. The space $B_{\Omega X}$ is filtered by an increasing sequence $\{B_n\}$ where B_n is the n-fold projective space of ΩX. This induces a fltration of $h_*(B_{\Omega X})$ and, hence, a spectral sequence $\{E^r(h)\}$. Properties 6:3 and 6:4 follow from the general properties of the construction. Regarding property 6:2, it follows if h is a commutative, associative, multiplicative theory, satisfies a Kunneth formula and $h_*(\Omega X)$ is torsion free (see §4 of **[21]**). By the discussion at the beginning of §5, property 6:2 holds for $H\mathbf{Z}/p$ and $K(n)$. It also holds for $k(n)$. For we can set up the spectral sequence for BP and, by the above reasoning, 6:2 will hold for this spectral sequence. Then we can reduce from BP to $k(n)$. The natural maps $H\mathbf{Z}/p \leftarrow k(n) \rightarrow K(n)$ induce maps $\{E^r(H\mathbf{Z}/p)\} \leftarrow \{E^r(k(n))\} \rightarrow \{E^r(K(n))\}$ between the spectral sequences. The second map is simply localization with respect to v_n. These maps play a key role in our proof of 6:1. We want to study the spectral sequence $\{E^r(K(n))\}$. The spectral sequence $\{E^r(H\mathbf{Z}/p)\}$ has been studied in detail (see **[7]** or **[13]**). Using the above maps, this information is used to determine the differentials in $\{E^r(k(n))\}$ and then in $\{E^r(K(n))\}$.

We begin by considering the E^2 terms in these spectral sequences. From now on we will use $\mathrm{Tor}(h)$ to denote $\mathrm{Tor}^{h_*(\Omega X)}(h_*(\mathrm{pt}), h_*(\mathrm{pt}))$. It follows from 4:6 and 5:10 that:

PROPOSITION 6:5. *For $h = H\mathbf{Z}/p$, $k(n)$, or $K(n)$, $\mathrm{Tor}(h)$ is isomorphic to the homology of the complex $E(s\chi) \otimes \Gamma(t\chi_1) \otimes h_*(\mathrm{pt})$ where sX has bidegree $(1, |X|)$, tX has bidegree $(2, 2p|X|)$, and the differential d acts by the rule $dsX = 0$, $d\gamma_s(tX) = \gamma_{s-1}(tX)Q_X$. Here Q_X is determined from R_X of 5:1 by the rule $Q_X = \sum \lambda_i sX_i$.*

(See §7 of **[15]** as well as §§3 and 8 of **[24]** for a justification of this theorem. In **[15]** we showed that $\mathrm{Tor}^{BP_*(\Omega X)}(BP_*(\mathrm{pt}); BP_*(\mathrm{pt}))$ was the homology of the complex $E(s\chi) \otimes \Gamma(t\chi_1)$ with $BP_*(\mathrm{pt})$ coefficients. So now we are simply changing our coefficient rings.) The results of §5 give strong restrictions on the elements $\{R_X\}$. We now use this information to impose further restrictions on $\mathrm{Tor}(h)$. The elements $s\chi$ define an isomorphism

$$s\colon Q\big(h_*(\Omega X)\big) \simeq \mathrm{Tor}_{1,*}(h). \tag{6:6}$$

Furthermore, the isomorphisms 6:6, in the cases $h = k(n)$ and $h = H\mathbf{Z}/p$, are related by the following commutative diagram:

$$\begin{array}{ccc} Q\big(k(n)_*(\Omega X)\big) & \simeq & \mathrm{Tor}_{1,*}(k(n)) \\ \downarrow & & \downarrow \\ Q\big(H_*(\Omega X; \mathbf{Z}/p)\big) & \simeq & \mathrm{Tor}_{1,*}(H\mathbf{Z}/p) \end{array}$$

Consider the map $k(n)_*(\Omega X) \rightarrow Q(k(n)_*(\Omega X)) \simeq \mathrm{Tor}_{1,*}(k(n))$. (The first map is the quotient map while the second comes from 6:6.) Under this map R_X is mapped to Q_X. Thus the restrictions imposed on $\{R_X\}$ by 5:11 and 5:12 imply that we can rewrite the elements of $s\chi$ so that

LEMMA 6:7. $\mathrm{Tor}(k(n)) =$ *the homology of the complex* $A \otimes B \otimes C \otimes \mathbf{Z}/p[v_n]$ *where*

$$A = \otimes E(x_i), \qquad dx_i = 0,$$
$$B = \otimes \Gamma(y_j), \qquad dy_j = 0, \qquad |y_j| < 2p^n - 2,$$
$$C = \otimes E(x_k) \otimes \Gamma(y_k), \quad d\gamma_s(y_k) = v_n \gamma_{s-1}(y_k) x_k, \quad |y_k| \geqslant 2p^n - 2.$$

REMARK. The generators $S = \{x_i\} \cup \{x_k\} \cup \{y_j\} \cup \{y_k\}$ are obtained from the set $s\chi \cup t\chi_1$ by rewriting elements.

REMARK. We are using $|x|$ to denote total dimension in the bigraded module $\mathrm{Tor}(k(n))$. We will follow the same convention for the rest of this section.

We also have the identities

LEMMA 6:8. $\mathrm{Tor}(K(n)) = A \otimes B \otimes \mathbf{Z}/p[v_n, v_n^{-1}]$.

LEMMA 6:9. $\mathrm{Tor}(H\mathbf{Z}/p) = A \otimes B \otimes C$.

Both of these results are corollaries of 6:7. As our last result on the E^2 terms of the spectral sequences we observe that, by the usual homological algebra arguments, the short exact sequence $0 \to k(n)_*(\Omega X) \xrightarrow{v_n} k(n)_*(\Omega X) \to H_*(\Omega X; \mathbf{Z}/p) \to 0$ induces the exact couple:

$$\begin{tikzcd} \mathrm{Tor}(k(n)) \arrow[rr, "v_n"] & & \mathrm{Tor}(k(n)) \\ & \mathrm{Tor}(H\mathbf{Z}/p) \arrow[ul, "\Delta_n"] \arrow[ur, "\rho_n"'] & \end{tikzcd} \tag{6:10}$$

We now consider how differentials act in the Eilenberg-Moore spectral sequences. We begin with $\{E^n(H\mathbf{Z}/p)\}$. It is analogous to the spectral sequence considered in §4 and satisfies analogous properties. It is a spectral sequence of bicommutative, biassociative Hopf algebras. Moreover $E^2 = \mathrm{Tor}(H\mathbf{Z}/p)$ as Hopf algebras and $E^\infty = H_*(X; \mathbf{Z}/p)$ as coalgebras. (See [7]. For the last property see the proof of 2:8 of [13].)

Starting out from the E^2 term given by 6:9 and using the arguments of [7], it follows that d^r acts trivially unless $r = 2p^s - 1$ for some $s \geqslant 1$. Furthermore, $E^{2p^s-1}(H\mathbf{Z}/p)$ is isomorphic, as a differential Hopf algebra, to a tensor product of differential Hopf algebras where each factor is either a differential Hopf algebra on which $d = 0$ or a differential Hopf algebra of the form

$$M_s = E(x) \otimes \Gamma(y), \qquad d\gamma_{p^s}(y) = x.$$

Since $E^\infty(H\mathbf{Z}/p)$ is a finite $\mathbf{Z}/p$ module, it follows that

$$E^\infty(H\mathbf{Z}/p) = E(T) \otimes \left[\bigotimes_{i=1}^{l} H(M_{s_i})\right] \text{ for some set } T \text{ and some integers } s_1, \ldots, s_l \tag{6:11}$$

(that is any divided polynomial algebra $\Gamma(y) \subset E^2(H\mathbf{Z}/p)$ will eventually appear in a factor of type M_s). The homology of M_s, $H(M_s)$, is a divided polynomial algebra truncated at height p^s. Therefore, by dualizing 6:11, we have a Borel

decomposition of $H^*(X; \mathbf{Z}/p)$. By the definition of $N = 2^a p^b$ it follows that

$a = T^\#$

$b =$ the number of nontrivial divided pth powers which appear in the various $H(M_{s_i})$ and have total dimension $< 2p^n$.

Next, consider the spectral sequence $E^n(K(n))$. Again this is a spectral sequence of bicommutative, biassociative Hopf algebras (see **[26]** for the necessary multiplicative structure on $K(n)$ to ensure this Hopf algebra structure). Starting out from $E^2(K(n))$ as given by 6:8, and using the arguments of [7], it follows that $d^n = 0$ unless $r = 2p^s - 1$ for some $s \geqslant 1$. Also, $E^{2p^s-1}(K(n))$ can be written as a tensor product of differential Hopf algebras on which $d = 0$ or a differential Hopf algebra of the form $M_s \otimes \mathbf{Z}/p[v_n, v_n^{-1}]$. Since $E^\infty(K(n))$ is a finitely generated $\mathbf{Z}/p[v_n, v_n^{-1}]$ module it follows that

$$E^\infty(K(n)) = E(T') \otimes \left[\bigotimes_{j=1}^{m} H(M_{s_j}) \right] \otimes \mathbf{Z}/p[v_n, v_n^{-1}] \quad \text{for some set } T' \tag{6:12}$$

and some integers $s_1, \ldots, s_m$.

Thus rank $E^\infty(K(n))$ (as a $\mathbf{Z}/p[v_n, v_n^{-1}]$ module) $= 2^c p^d$ where

$c = T'^\#$,

$d =$ the number of nontrivial divided pth powers which appear in the various $H(M_{s_j})$.

To prove 6:1 we want to show that rank $E^\infty(K(n)) \geqslant 2^a p^b$. Thus it suffices to show that $c \geqslant a$ and $d \geqslant b$.

LEMMA 6:13. $c = a$.

PROOF. This follows from a counting argument. We can think of the spectral sequences $\{E^r(H\mathbf{Z}/p)\}$ and $\{E^r(K(n))\}$ as having an E^1 term given by 6:7. Thus $E^1(H\mathbf{Z}/p) = A \otimes B \otimes C$ and $E^1(K(n)) = A \otimes B \otimes C \otimes \mathbf{Z}/p[v_n, v_n^{-1}]$. Hence both E^1 terms have the same number, α, of odd-dimensional generators and the same number, β, of even-dimensional generators. Next, in both spectral sequences, the number of odd-dimensional generators which disappear equals the number of even-dimensional generators. For, in $\{E^r(H\mathbf{Z}/p)\}$, odd-dimensional generators disappear only by appearing in a factor M_s while, in $\{E^r(K(n))\}$, they disappear only by appearing in a factor of C or in a factor $M_s \otimes \mathbf{Z}/p[v_n, v_n^{-1}]$. Also, as we have already argued, the number of such factors, in each case, equals the number of even-dimensional generators. We conclude that $a = \alpha - \beta = c$. $\square$

LEMMA 6:14. $b \leqslant d$.

PROOF. Let

$e =$ the number of divided pth powers of $E^2(H\mathbf{Z}/p) = A \otimes B \otimes C$ which have total dimension $< 2p^n$ and survive nontrivially to $E^\infty(H\mathbf{Z}/p)$.

$f =$ the number of divided pth powers of $E^2(K(n)) = A \otimes B \otimes \mathbf{Z}/p[v_n, v_n^{-1}]$ which survive nontrivially to $E^\infty(K(n))$.

We will prove 6:14 in three steps. We wll show $b = e \leqslant f \leqslant d$.

(i) $b = e$. By the arguments of [7], we can rewrite the elements of $E^2(H\mathbf{Z}/p) = A \otimes B \otimes C$ such that the factors $E(x)$ and $\Gamma(y)$ of each M_s can be identified with

factors $E(x_i)$, $\Gamma(y_j)$, $E(x_k)$, or $\Gamma(y_k)$ of $E^2(H\mathbf{Z}/p)$ which have survived to $E^{2p^s-1}(H\mathbf{Z}/p)$. Moreover, this rewriting does not disturb any of the previous structure theorems. In particular, 6:7, 6:8 and 6:9 are still valid. For rewriting the elements of A or B or C only involves other elements of A or B or C, respectively. First of all, the element x in M_s has total dimension $\equiv -1 \pmod{2p}$ while the elements $\{x_k\}$ in C have total dimension $\equiv 3 \pmod{2p}$. Therefore x comes from the elements of A. By rewriting the elements of A we can identify x with one of the $\{x_i\}$ in A. Secondly, the element y in M_s comes from an element of B if $|y| < 2p^n$. Rewriting the elements of B, we can identify y with one of the $\{y_j\}$. Thirdly, if $|y| > 2p^n$ we can rewrite the elements of C to identify y with one of the $\{y_k\}$. This canonical form we have given to the spectral sequence implies equality (i).

(ii) $e \leqslant f$. Pick a divided pth power $\gamma_{p^s}(y_t)$ of $E^2(H\mathbf{Z}/p)$ which has total dimension $< 2p^n$. By the relation between the E^2 terms given by 6:7–6:9, it follows that there are corresponding elements $\gamma_{p^s}(y_t)$ in $E^2(k(n))$ and in $E^2(K(n))$. To prove (ii) it suffices to show

(*) $d^r\gamma_{p^s}(y_t) = 0$ for all r in $\{E^2(H\mathbf{Z}/p)\}$ implies $d^r\gamma_{p^s}(y_t) = 0$ for all r in $\{E^r(K(n))\}$.

So suppose $d^r\gamma_{p^s}(y_t) = 0$ for all r in $\{E^r(H\mathbf{Z}/p)\}$. Consider $\gamma_{p^s}(y_t) \in E^2(k(n))$. By the exact triangle 6:10 it follows that $E^2(k(n)) \simeq E^2(H\mathbf{Z}/p)$ in total dimension $< 2p^n$. Since d^r lowers dimension by 1 we can prove that, for all r, $E^r(k(n)) \simeq E^r(H\mathbf{Z}/p)$ in total dimension $< 2p^n$. We work by double induction, first on r and secondly on total dimension. Thus $d^r\gamma_{p^s}(y_t) = 0$ in $E^r(H\mathbf{Z}/p)$ implies $d^r\gamma_{p^s}(y_t) = 0$ in $E^r(k(n))$. Since $E^r(K(n))$ is obtained from $E^r(k(n))$ by localizing it follows that $d^r\gamma_{p^s}(y_t) = 0$ in $E^r(K(n))$.

(iii) $f \leqslant d$. This inequality is trivial. $\square$

7. *BP* theories. The last three sections of this paper are devoted to proving Theorem 1:5. We will work with cohomology rather than homology throughout the discussion. In §§7 and 8 we will describe some useful machinery. In §9 we will prove the theorem. In this section we describe some spectral sequences associated to various *BP* theories.

First of all, dual to the homology spectral sequence described in §3 is a cohomology spectral sequence $\{B_r, d_r\}$ induced by the exact couple

$$\begin{tikzcd} k(n)^*(X) \arrow[rr, "v_n"] & & k(n)^*(X) \arrow[dl, "\rho_n"] \\ & H^*(X; \mathbf{Z}/p) \arrow[ul, "\Delta_n"] & \end{tikzcd} \tag{7:1}$$

Moreover properties 3:1–3:3 are still valid when we replace homology by cohomology. By letting $k(0)^*(X) = H^*(X)$ and $v_0 = p$ we can include the case $n = 0$ in the above as well.

For $1 \leqslant n < \infty$ let $BP\langle n\rangle^*(X)$ be the *BP* theory where $BP\langle n\rangle^*(\mathrm{pt}) = \mathbf{Z}_{(p)}[v_1, v_2, \ldots, v_n]$. Also let $BP\langle 0\rangle^*(X)$ denote $H^*(X)$, $BP\langle -1\rangle^*(X)$ denote $H^*(X; \mathbf{Z}/p)$, and $BP\langle \infty\rangle^*(X)$ denote $BP^*(X)$. Given integers $-1 \leqslant t < s \leqslant \infty$ there is a canonical map $\rho(t, s)$: $BP\langle s\rangle^*(X) \to BP\langle t\rangle^*(X)$. For $n \geqslant 0$ there is an

exact couple

$$\begin{array}{ccc} BP\langle n\rangle^*(X) & \xrightarrow{v_n} & BP\langle n\rangle^*(X) \\ & \Delta_n \nwarrow \quad \swarrow \rho(n-1, n) & \\ & BP\langle n-1\rangle^*(X) & \end{array} \tag{7:2}$$

where v_n is multiplication by v_n ($v_0 = p$). We can derive a Bockstein spectral sequence $\{\hat{B}_r, \hat{d}_r\}$ from this exact couple. This spectral sequence satisfies the following properties:

(7:3) $\hat{B}_1 = BP\langle n-1\rangle^*(X)$.

(7:4) $\hat{B}_\infty = BP\langle n\rangle^*(X)/T_n + v_n BP\langle n\rangle^*(X)$ where T_n is the v_n torsion submodule of $BP(n)^*(X)$.

(7:5) Image $\hat{d}_s = \rho(n-1, n)T_n(s)$ where $T_n(s) = \{x \in T_n | v_n^s x = 0\}$.

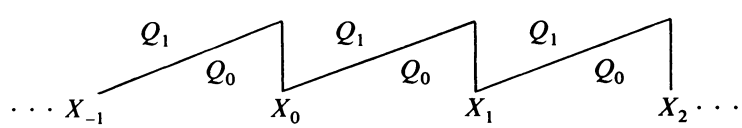
(7:6) For $n \geqslant 1$, Image $\hat{d}_s$ is represented in $\hat{B}_s$ by v_{n-1} torsion elements of $\hat{B}_1$ which have survived to $\hat{B}_s$.

See **[11]** for $BP(n)$ and all of the above properties. Also, since there is a natural map from 7:2 to 7:1 it follows that:

(7:7) There is a natural map $\{\hat{B}_r, \hat{d}_r\} \to \{B_r, d_r\}$ which agrees on the B_1 terms with the map $\rho(-1, n-1)$: $BP\langle n-1\rangle^*(X) \to H^*(X; \mathbf{Z}/p)$.

For $n = 0$ the above two spectral sequences agree and are identical with the classical Bockstein spectral sequence (see **[3]**).

8. $E(Q_0, Q_1)$ modules. In §9 we will also need to be able to describe how Q_0 and Q_1 act on $H^*(X; \mathbf{Z}/p)$. Since $Q_0^2 = Q_1^2 = 0$ and $Q_0Q_1 = -Q_1Q_0$ we can consider $H^*(X; \mathbf{Z}/p)$ as a graded module over the exterior algebra $E(Q_0, Q_1)$. It is very convenient to describe the action of Q_0 and Q_1 in terms of such a module structure. In particular we can use the machinery from Part III of **[2]** where Adams classified finite-dimensional $E(Q_0, Q_1)$ modules. The key to his work is the "lightning flash" module. Let x_i have dimension $2i(p-1)$ ($i \in \mathbf{Z}$). Let L = the $E(Q_0, Q_1)$ module generated by $\{x_i\}$ with relations $Q_1(x_i) = Q_0(x_{i+1})$. We can display L schematically as:



Two $E(Q_0, Q_1)$ modules M and N are stably isomorphic if there exist free modules F and G such that $M \oplus F \cong N \oplus G$. Adams showed in **[2]** that the stable class of any finite-dimensional $E(Q_0, Q_1)$ module is a direct sum of stable classes of subquotient modules of L (see Theorem 16:11 of Part III of **[2]**). In the situation dealt with in §9 we will only need the following modules derived from L. For each $t \in \mathbf{Z} \cup \{\infty\}$ define the module $L(t)$ as follows.

$t < 0$: $L(t)$ = the $E(Q_0, Q_1)$ submodule of L generated by $\{x_i | t \leqslant i < 0\}$.

$t \geqslant 0$: $L(t) =$ the $E(Q_0, Q_1)$ quotient module of L formed by factoring out the submodule generated by $\{x_i | i < 0 \text{ or } i > t\}$.

$t = \infty$: $L(\infty) =$ the $E(Q_0, Q_1)$ quotient module of L formed by factoring out the submodule generated by $\{x_i | i < 0\}$.

For a schematic representation of $L(t)$ take the display of L and truncate it at both ends in the appropriate manner. The $L(t)$ modules are interrelated in the following manner. Given $q, r \in \mathbf{Z}$ then

$$L(q) \otimes L(r) \cong L(q + r) \oplus F \quad \text{where } F \text{ is a free } E(Q_0, Q_1) \text{ module.} \tag{8:1}$$

PROOF. We will only do the case $q \geqslant 0$, $r \geqslant 0$. The other cases are similar. Define a map γ: $L(q + r) \to L(q) \otimes L(r)$.

$$\gamma(x_s) = \sum_{i+j=s;\, 0 \leqslant i \leqslant q;\, 0 \leqslant j \leqslant r} x_i \otimes x_j.$$

It is easy to verify that γ is an injective map of $E(Q_0, Q_1)$ modules. If we pass to Q_0 or Q_1 homology then, in each case, γ is an isomorphism. It follows from 16:3 of Part III of [2] that $L(q + r)$ and $L(q) \otimes L(r)$ are stably isomorphic. More exactly $L(q + r) \oplus F \cong L(q) \otimes L(r) \oplus G$, where $F =$ cokernel γ and $G =$ kernel γ are free. Since γ is injective 8:1 follows. $\square$

The following is a simple corollary of 8:1. Let m be a positive integer. Then

$$L(1)^m \cong L(m) \oplus F \quad \text{where } F \text{ is a free } E(Q_0, Q_1) \text{ module.} \tag{8:2}$$

9. Proof of Theorem 1:5. In this section we will prove Theorem 1:5. Fix an integer $m \geqslant 1$. Our study of $BP\langle 1\rangle^*(\prod_{i=1}^m X)$ will depend on a knowledge of the structure of $H^*(\prod_{i=1}^m X; \mathbf{Z}/p)$ as an $E(Q_0, Q_1)$ module. First of all:

(9:1) $H^*(X; \mathbf{Z}/p) = E(a) \otimes E(b) \otimes T(c)$ where $|a| = 3$, $|b| = 2p + 1$, $|c| = 2p + 2$ and the action of the elements $\{Q_s\}$ on $H^*(X; \mathbf{Z}/p)$ is determined by $Q_0(b) = Q_1(a) = c$.

(For $p = 2$ we have the further relation $c = a^2$ which we ignore.) Let Σ be the $E(Q_0, Q_1)$ module with $\mathbf{Z}/p$ in dimension 1 and 0 in all other dimensions. For any integer $s \geqslant 1$ and any $E(Q_0, Q_1)$ module M define the s-fold suspension $\Sigma^s M$ to be M tensored with s copies of Σ. Then

$$H^*(X; \mathbf{Z}/p) = L(0) \oplus \Sigma^3 L(1) \oplus \Sigma^{2p^2+2p-2} L(-1) \oplus \Sigma^{2(p^2+p+1)} L(0) \oplus F$$
$$\text{where } F \text{ is a free } E(Q_0, Q_1) \text{ module.} \tag{9:2}$$

We obtain this decomposition by writing down elements of $H^*(X; \mathbf{Z}/p)$ in order of dimension.

$$(a, b, c)(ab, ac, bc, c^2) \cdots (abc^{p-2}, ac^{p-1}, bc^{p-1})(abc^{p-1}). \tag{1}$$

The brackets indicate the decomposition of $H^*(X; \mathbf{Z}/p)$ into a direct summand of $E(Q_0, Q_1)$ modules. In particular, $L(0)$ and $\Sigma^3 L(1)$ are the first two summands while $\Sigma^{2p^2+2p-2} L(-1)$ and $\Sigma^{2(p^2+p+1)}$ are the last two summands. All other summands are free.

We can extend 9:2 to a decomposition of $H^*(\prod_{i=1}^m X; \mathbf{Z}/p) = \bigotimes_{i=1}^m H^*(X; \mathbf{Z}/p)$ by using 8:1. In particular $H^*(\prod_{i=1}^m X; \mathbf{Z}/p)$ contains the summand $\Sigma^{3m}L(1)^m = \Sigma^{3m}L(m) \oplus F$. It is this summand which will produce the $BP\langle 1\rangle$ torsion described in 1:5. We now demonstrate this fact.

(A) *The case* $\tilde{H}^*(Y; \mathbf{Z}/p) = \Sigma^s L(m)$ *for some* $s \geqslant 0$. We will show that the torsion submodule of $BP\langle 1\rangle^*(Y)$ is generated by elements $\{z_1, \ldots, z_m\}$ which satisfy the relations

$$pz_m = 0, \quad v_1 z_1 = 0, \quad pz_i = v_1 z_{i+1} \qquad (1 \leqslant i < m). \tag{9:3}$$

Moreover, any relation satisfied by the elements $\{z_1, \ldots, z_m\}$ is a consequence of 9:3. In particular, we have the relations

$$v_1^{i-1} z_i \neq 0 \quad \text{while } v_1^i z_i = 0. \tag{9:4}$$

To see that we have generators $\{z_i\}$ satisfying 9:3 and 9:4 we will use Bockstein spectral sequence arguments. We will use the spectral sequences described in §7 in order to pass from $H^*(Y; \mathbf{Z}/p)$ to $H^*(Y)$ and then to $BP\langle 1\rangle^*(Y)$. First we pass from $H^*(Y; \mathbf{Z}/p)$ to $H^*(Y)$ using $\{\hat{B}_r\}$ for the case $n = 0$. From the fact that $\tilde{H}^*(Y; \mathbf{Z}/p) \cong L(m)$ we can easily calculate that

$$\tilde{H}^*(Y) = \mathbf{Z}_{(p)} \oplus \mathbf{Z}/p \oplus \cdots \oplus \mathbf{Z}/p \quad (m \text{ copies of } \mathbf{Z}/p). \tag{9:5}$$

Choose generators $\{y, y_1, \ldots, y_m\}$ of $\tilde{H}^*(Y)$ where y generates the free summand while $\{y_1, \ldots, y_m\}$ generate the torsion summands. Under the map $\tilde{H}^*(Y) \to \tilde{H}^*(Y; \mathbf{Z}/p)$, y maps to x_0 while y_i maps to $Q_0(x_i)$. Observe that

$$|y_i| = |y| + 2i(p-1) + 1. \tag{9:6}$$

Next we pass from $H^*(Y)$ to $BP\langle 1\rangle^*(Y)$ using $\{\hat{B}_r\}$ for the case $n = 1$. The $\hat{B}_1$ term is given by 9:5. The elements $\{y_1, \ldots, y_m\}$ lift to elements $\{z_1, \ldots, z_m\}$ in $BP\langle 1\rangle^*(Y)$. For since $\hat{d}_r$ changes degree by $2r(p-1) + 1$ we can use 9:6 plus dimension arguments to show that the elements $\{y_1, \ldots, y_m\}$ are permanent cycles in the spectral sequence.

The relations 9:3. We now show that the elements $\{z_i\}$ can be chosen to satisfy 9:3. First of all, since $BP\langle 1\rangle^i(Y) \cong H^i(Y)$ in dimension $\geqslant |y_m|$ it follows that

$$pz_m = 0. \tag{9:7}$$

Secondly, we can choose z_1 to satisfy

$$v_1 z_1 = 0, \tag{9:8}$$

while for any choice of $\{z_i\}$ we have

$$v_1 z_i \neq 0 \quad \text{for } 2 \leqslant i \leqslant m. \tag{9:9}$$

To prove these two facts it suffices, by 7:5, to show that $y_1 \in \text{Image } \hat{d}_1$ while $y_i \notin \text{Image } \hat{d}_1$ if $2 \leqslant i \leqslant m$. Map from $\{\hat{B}_r\}$ to $\{B_r\}$ using 7:7. Then the elements $\{y_i\}$ map to $\{Q_0(x_i)\}$. However, only $Q_0(x_1)$ is hit (under Q_1) by an element which lifts to $H^*(X)$.

Thirdly, for each $1 \leqslant i < m$, we have the relation

$$v_1 z_{i+1} = p\bar{z} \quad \text{for some } \bar{z} \in BP\langle 1\rangle^*(Y). \tag{9:10}$$

For using the spectral sequence $\{B_r\}$ for $n = 1$ and the fact that $\tilde{H}^*(Y; \mathbf{Z}/p) \cong L(m)$, we can easily deduce that $k(1)^*(Y)$ has no higher v_1 torsion. The exact sequence.

$$BP\langle 1\rangle^*(Y) \xrightarrow{xp} BP\langle 1\rangle^*(Y) \to k(1)^*(Y)$$

then yields 9:10.

Lastly,

$$\bar{z} = z_i. \tag{9:11}$$

To prove 9:11 we need only show that $\bar{z}$ is not divisible by v_1. For it then follows from the exact sequence

$$BP\langle 1\rangle^*(Y) \xrightarrow{v_1} BP\langle 1\rangle^*(Y) \to H^*(Y)$$

that $\bar{z}$ maps nontrivially to y_i and, hence, can be chosen to be z_i. Suppose $\bar{z}$ is divisible by v_1. Let $\bar{z} = v_1 w$. By 9:10 we have $v_1(z_{i+1} - pw) = 0$. But replacing z_{i+1} by $z_{i+1} - pw$ we then contradict 9:9.

Relations 9:3 now follow from 9:8–9:10.

The relations 9:4. Relations 9:3 have the following consequences:

$$p^j z_i = v_1^j z_{i+j}. \tag{9:12}$$

(We employ the convention that $z_s = 0$ when $s < 0$ or $s > m$.) In particular, $p^i z_1 = v_1^i z_i = 0$. Thus to prove 9:4 it suffices to show that $v_1^{i-1} z_i \neq 0$. We will use the spectral sequences $\{\hat{B}_r\}$ for $n = 1$. In particular, we will use the relation between v_1 torsion and differentials given by 7:5. As already noted the $\hat{B}_1$ term is given by 9:5 and the torsion generators $\{y_1, \ldots, y_m\}$ of $\hat{B}_1$ are permanent cycles. Thus differentials can only act nontrivially on elements from the free summand of $\hat{B}_1$ generated by y. To determine exactly how differentials do act observe that the elements $\{y_1, \ldots, y_m\}$ are killed by differentials in the spectral sequence. This follows from 7:5 and the fact that $v_1^i z_i = 0$ for $1 \leqslant i \leqslant m$. It now follows from dimension arguments (see 9:6) that there is only one way in which differentials can act. Namely,

$$\begin{gathered} p^{i-1}y \text{ and } \{y_i, y_{i+1}, \ldots, y_m\} \text{ survive to } \hat{B}_i \text{ and we have} \\ \hat{d}_i(p^{i-1}y) = y_i \text{ for } 1 \leqslant i \leqslant m. \end{gathered} \tag{9:13}$$

In particular, since y_i survives nontrivially to $\hat{B}_i$ it follows from 7:5 that $v_1^{i-1} z_i \neq 0$.

(B) *The case* $\tilde{H}^*(Y; \mathbf{Z}/p) = \Sigma^s L(\infty)$ *for some* $s \geqslant 0$. The argument in part (A) extends in an obvious way to show that $BP\langle 1\rangle^*(Y)$ contains an infinite collection of elements $\{z_1, z_2, \ldots\}$ satisfying 9:3 and 9:4 (except of course for the relation $pz_m = 0$).

(C) *The case free* $E(Q_0, Q_1)$ *modules in* $H^*(Y; \mathbf{Z}/p)$. Spectral sequence arguments as in part (A) show that each free $E(Q_0, Q_1)$ summand of $H^*(Y; \mathbf{Z}/p)$ produces two $\mathbf{Z}/p$ summands in $H^*(Y)$ and one $BP\langle 1\rangle^*(\mathrm{pt})/(p, v_1)$ $(= \mathbf{Z}/p)$ summand in $BP\langle 1\rangle^*(Y)$. In particular, suppose that $\tilde{H}^*(Y; \mathbf{Z}/p) = \Sigma^s L(t) \oplus F$ where F is a free $E(Q_0, Q_1)$ module. Then when we calculate the various Bockstein spectral sequences as in part (A) the action of the differentials produced by the free

summands $E(Q_0, Q_1)$ is completely independent of the action produced by the summand $L(t)$. This shows that the results obtained in (A) and (B) still hold if $H^*(Y; \mathbf{Z}/p) = \Sigma^s L(t) \oplus F$ where F is a free $E(Q_0, Q_1)$ module.

(D) *Proof of Theorem* 1:5. To calculate $BP\langle 1\rangle^*(\prod_{i=1}^m X)$ we can always replace $\prod_{i=1}^m X$ by its stable homotopy type. When we stabilize $\prod_{i=1}^m X$ we obtain the direct summand $X(m) = X \wedge \cdots \wedge X$ (m copies). For convenience we will only look at $BP\langle 1\rangle^*(X(m))$. By 8:2 and 9:2, $H^*(X(m); \mathbf{Z}/p)$ contains the summand $\Sigma^{3m}L(m)$. Let $Y = X^{2p+2}$, the $2p+2$ skeleton of X. By 9:2, $\tilde{H}^*(Y; \mathbf{Z}/p) \cong \Sigma^3 L(1)$. Let $Y(m) = Y \wedge \cdots \wedge Y$ (m copies). Then, by 8:2, $\tilde{H}^*(Y(m); \mathbf{Z}/p) \cong \Sigma^{3m}L(m) \oplus F$. Let $f\colon Y(m) \to X(m)$ be the natural inclusion. Then

$$\begin{gathered}\text{the map } f\colon Y(m) \to X(m) \text{ induces an isomorphism } \alpha\colon \\ \Sigma^{3m}L(m) \subset H^*(X(m); \mathbf{Z}/p) \to H^*(Y(m); \mathbf{Z}/p) \to \Sigma^{3m}L(m).\end{gathered} \tag{9:14}$$

Since we are working stably we can replace $X(m)$ and $Y(m)$ by their suspension spectra and pass to the stable category of CW spectra. Consider the Eilenberg-Mac Lane spectrum $K = K(\mathbf{Z}_{(p)}, 0)$. By using some ideas of Frank Peterson it follows that

$$H^*(K; \mathbf{Z}/p) \cong L(\infty) \oplus F \quad \text{where } F \text{ is a free } E(Q_0, Q_1) \text{ module.} \tag{9:15}$$

PROOF. We can identify $H^*(K; \mathbf{Z}/p)$ as a Steenrod module with $B^* = A^*(p)/A^*(p)Q_0$. When we dualize we have $B_* \subset A_*(p) = E(\tau_0, \tau_1, \ldots) \otimes \mathbf{Z}/p[\xi, \xi_2, \ldots]$ (the dual of the Steenrod algebra). Applying the canonical antiautomorphism χ we have $\chi(B_*) \subset A_*(p)$ where $\chi(B_*)$ denotes the image of B_* under χ. It turns out that the simplest approach is to describe the action of $E(Q_0, Q_1)$ on $\chi(B_*)$. This is mainly because

$$\chi(B_*) = E(\tau_1, \tau_2, \ldots) \otimes \mathbf{Z}/p[\xi_1, \xi_2, \ldots]. \tag{9:15:1}$$

To prove 9:15:1 we will use R and L to denote right and left actions of $A^*(p)$. We have the following exact sequences, each derived from the previous one:

$$A^*(p) \overset{R(Q_0)}{\to} A^*(p) \to B^* \to 0,$$

$$A^*(p) \overset{L(Q_0)}{\leftarrow} A_*(p) \leftarrow B_* \leftarrow 0,$$

$$A_*(p) \overset{R(Q_0)}{\leftarrow} A_*(p) \leftarrow \chi(B_*) \leftarrow 0.$$

The right action of Q_0 is determined by the rules: $\tau_0 Q_0 = 1$, $\tau_s Q_0 = \xi_s Q_0 = 0$ for $s \geqslant 1$. (This can be read off the coalgebra structure of $A_*(p)$. See Theorem 3 of [22].) We can now easily deduce 9:15:1.

We are interested in the left action of $E(Q_0, Q_1)$ on $\chi(B_*)$. This action is determined by the rules: $Q_0\tau_s = \xi_s$ and $Q_1\tau_s = \xi_{s-1}^p$ ($\xi_0 = 1$) for $s \geqslant 1$ (again consider Theorem 3 of [22]). Thus $H(\chi(B_*); Q_0) = \mathbf{Z}/p$ is generated by 1 while $H(\chi(B_*); Q_1) = \{0\}$. The inclusion $\gamma\colon E(\tau_1) \otimes \mathbf{Z}/p[\xi_1] \to \chi(B_*)$ is a map of left $E(Q_0, Q_1)$ modules and induces an isomorphism in Q_0 and Q_1 homology. It follows from 16:3 of Part III of [2] that $E(\tau_1) \otimes \mathbf{Z}/p[\xi_1]$ and $\chi(B_*)$ are stably isomorphic. Since γ is injective it follows that $\chi(B_*) = E(\tau_1) \otimes \mathbf{Z}/p[\xi_1] \oplus F$ where F is a free $E(Q_0, Q_1)$ module. Dualizing we obtain 9:15. $\square$

It follows from 9:5 that $\Sigma^{3m}L(m) \subset H^*(X(m); \mathbf{Z}/p)$ produces a summand $\mathbf{Z}_{(p)} \subset H^{3m}(X(m))$. Let $g\colon X(m) \to \Sigma^{3m}K(\mathbf{Z}_{(p)}, 0)$ be a map classifying a generator of this summand.

$$\text{The map } g\colon X(m) \to \Sigma^{3m}K(\mathbf{Z}_{(p)}, 0) \text{ induces a surjective map } \beta\colon \Sigma^{3m}L(\infty) \subset H^*(\Sigma^{3m}K; \mathbf{Z}/p) \to H^*(X(m); \mathbf{Z}/p) \to \Sigma^{3m}L(m). \tag{9:16}$$

PROOF. The map β is a map of $E(Q_0, Q_1)$ modules. By the definition of g, β induces an isomorphism in dimension $3m$. It is then easy to deduce from the $E(Q_0, Q_1)$ structure of $L(\infty)$ and $L(m)$ that β must be surjective. $\square$

We now prove Theorem 1:5. By 7:5 it suffices to take the Bockstein spectral sequence $\{\hat{B}_r\}$ for $n = 1$ and show that when applied to $X(m)$ we have $\hat{d}_m \neq 0$ in $\hat{B}_m$. If we take $\{\hat{B}_r\}$ for the space $Y(m)$ then it follows from parts (A) and (C) that we can find $x \in \hat{B}_m^{3m}$ where $\hat{d}_m(x) \neq 0$. Similarly if we take $\{\hat{B}_r\}$ for the space $\Sigma^{3m}K$ then it follows from part (B) and (C) that we can find $y \in \hat{B}_m^{3m}$ such that $\hat{d}_m(y) \neq 0$. Moreover, it follows from 9:14 and 9:16 that we can let $x = (gf)^*(y)$. It now follows that if we take $\{\hat{B}_r\}$ for the space $X(m)$ then $\hat{d}_m g^*(y) \neq 0$ in $\hat{B}_m$.

REMARK 9:17. A more detailed analysis would allow us to conclude that the entire situation described in 9:3 occurs in $BP\langle 1\rangle^*(X(m))$. In particular, there is higher p torsion occurring in $BP\langle 1\rangle^*(X(m))$ as well.

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