A CORRECTION AND SOME ADDITIONS TO
"REPARAMETRIZATION OF n-FLOWS OF ZERO ENTROPY"
BY
J. FELDMAN AND D. NADLER

Abstract. In addition to correcting an error in the previously mentioned paper, we show that if \( v \rightarrow \psi_v \) and \( w \rightarrow \psi_w \) on \( X \) and \( Y \) are \( n \)- and \( m \)-flows, respectively, then the \((n + m)\)-flow \((v, w) \rightarrow \psi_v \times \psi_w \) on \( X \times Y \) is "loosely Kronecker" if and only if \( \varphi \) and \( \psi \) are.

There is a silly and easily correctable mistake in our paper [4]. Recall that an \( n \)-flow is a free, ergodic, probability-preserving action of \( \mathbb{R}^n \). We constructed in [4] an action \( \varphi \) of \( \mathbb{R}^n \) as follows: \( t \rightarrow T_t \) was defined as a suspension over the non-LB, ergodic, zero-entropy transformation of [2]. Then, for an \((n - 1)\)-vector \( u \), \( \varphi_{(t, u)} \) was defined as \( T_t \times \theta_u \). Although \( \varphi \) is indeed ergodic and probability-preserving, it is not free, so of course it is not an \( n \)-flow.

The purpose of the construction was to produce a zero-entropy \( n \)-flow which is not LK in the sense of [4]. First, we would like to change terminology, and use the term "standard" (as in Katok [5]) rather than "LK". The object, then, is to construct a nonstandard \( n \)-flow of zero entropy. One way would be to fix up the prior example as follows: let the above flow \( T \) act on \((Y, \nu)\), and let \( \theta \) be any \((n - 1)\)-flow on a space \((Z, \rho)\). Then \( \varphi_{(t, u)} = T_t \times \theta_u \) will be a nonstandard \( n \)-flow of zero entropy. That it is nonstandard may be seen as in the argument given at the end of [4] and it is easy to see that it has zero entropy. However, we now give a sketch of a more enlightening approach to the matter.

First, we point out

Lemma 1. A standard \( n \)-flow has entropy zero.

The easiest way to see this is to use the ideas of \( r \)-entropy, from [3]: to say \( \varphi \) is standard is to say that for large \( N \), most \( C_N \) names for \((\varphi, \mathcal{F})\) are \( f_N \)-close. The Lebesgue continuity theorem then may be used to get an exponentially small bound on the number of sets of \( d_N \) diameter \( r \) which are required to cover most of the space on which \( \varphi \) acts.

Hereafter, let \( \psi \) be an \( l \)-flow on \((Y, \nu)\) and \( \theta \) an \( m \)-flow on \((Z, \rho)\). If \( \varphi_{(t, u)} = \psi_t \times \theta_u \), then \( \varphi \) is an \((l + m)\)-flow on \((Y \times Z, \nu \times \rho)\).

Lemma 2. \( \varphi \) as above necessarily has entropy zero.

Indication of Proof. By using partitions of the form \( \mathcal{R} \times \mathbb{S} \) where \( h(\psi, \mathcal{R}) \) and \( h(\theta, \mathbb{S}) \) are finite, we may reduce to the case where \( \psi \) and \( \theta \) have finite

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entropy. But now the result follows directly from the definition of entropy, essentially because $(l + m)N / N^{l+m} \to 0$ as $N \to \infty$. See [1] for discussions of this type in the discrete case.

**Lemma 3.** $\varphi$ as above is standard if and only if both $\psi$ and $\theta$ are.

**Proof.** If both $\psi$ and $\theta$ are standard, then for any partition of the form $\mathcal{R} \times \mathcal{S}$, the process $(\psi \times \theta, \mathcal{R} \times \mathcal{S})$ may be seen to be standard by doing $f$-matching for $(\psi, \mathcal{R})$ and $(\theta, \mathcal{S})$ separately, and then combining.

To go in the other direction, one may use a similar argument to that at the end of [4] to make a reduction of dimension. Here are the details.

Suppose $\varphi$ is standard. Choose a partition $\mathcal{P}$ of $Y$. Then $\mathcal{R} = \{ P \times Z : P \in \mathcal{P} \}$ is a partition of $Y \times Z$. So, referring to the definitions in §3 of [4], we see that for any $\epsilon > 0$ there is some $M > 0$ such that if $M < N$ there is a set $E_N \subset Y \times Z$ with $\nu \times \rho(E_N) > 1 - \epsilon$ and $f^{\mathcal{R}}_N(x, x') < \epsilon$ whenever $x, x' \in E_N$. There is thus some $z \in Z$ so that if we set $F_N = \{ y : (y, z) \in E_N \}$ then $\nu(F_N) > 1 - \epsilon$. Now, if $t \in R^l$ and $u \in R^m$, then $\mathcal{R}(y, z)(t, u) = \mathcal{P}(y)(t)$, independent of $z$. So $f^{\mathcal{P}}_N((y, t), (y', t')) < \epsilon$ provided $y, y' \in F_N$. So for any such $y, y'$, and any $z, z'$, there is some $h \in D_{C^\infty}$ such that

$$\frac{1}{|C^l_{n+m}|} \int_{C^l_{n+m}} \delta(\mathcal{R}(y, z)(h(t, u)), \mathcal{R}(y', z')(t, u)) \, dt \, du < \epsilon.$$ 

(Since there are different dimensions to worry about, we now denote the $N$-cube in $R^p$ by $C^p_N$.) Rewriting, and writing $h(t, u)$ as $(j(t, u), k(t, u))$, where $j : R^{l+m} \to R^l$ and $k : R^{l+m} \to R^m$, we have

$$\frac{1}{|C^l_N|} \frac{1}{|C^m_N|} \int_{C^l_N} \int_{C^m_N} \delta(\mathcal{P}(y)(j(t, u)), \mathcal{R}(y')(t)) \, dt \, du < \epsilon;$$

so for some $u_0$ we have

$$\frac{1}{|C^l_N|} \int_{C^l_N} \delta(\mathcal{P}(y)(j(t, u_0)), \mathcal{R}(y')(t)) \, dt < \epsilon.$$

Set $i(t) = j(t, u_0)$. $i$ is a differentiable function from $C^l_N$ to $C^l_N$ leaving fixed a neighborhood of the boundary. Furthermore $\|i' - I_R\|_\infty < \|h' - I_R^{l+m}\|_\infty < \epsilon$. Finally, assuming $\epsilon < 1$, we have $\|i'(y) - I_R^l\| < 1$ for each $y$, so $i$ is locally invertible (by the Inverse Function Theorem), so--since $C^l_N$ is simply connected--$i$ is globally invertible, i.e. $i \in D_C$. Thus $f^{\mathcal{P}}_N(y, y') < 2\epsilon$ for all $y, y' \in F_N$. But $\epsilon$ was arbitrary, so we are done.

It is now easy to produce, for $n > 2$, examples of nonstandard $n$-flows of zero entropy: just take $\psi$ to be a 1-flow of positive entropy, and $\theta$ any $(n - 1)$-flow whatsoever. Then by Lemma 2, $\varphi$ will have zero entropy. It cannot be standard, because if it were, then by Lemma 3, $\psi$ would also be standard, and therefore by Lemma 1 would have to have entropy zero.

Alternatively, one could, as in [4], take $\psi$ to be some nonstandard 1-flow of zero entropy. Such examples are provided by proving the following fact:

**Lemma 4.** A flow is standard in the present sense if and only if it is LB in the sense of [2] and of zero entropy, or, equivalently, standard in the sense of [5].
The proof is a fairly routine application of the definitions. This construction is in principle more difficult, in that it already needs the existence of non-LB flows in one dimension. However, it may be useful in constructing uncountably many different equivalence classes.

The first construction, setting \( q_{(t,u)} = \psi_t \times \theta_u \) with \( \psi \) of positive entropy, raises the interesting possibility of exhibiting some "natural" equivalence classes other than the standard class, among the entropy zero \( n \)-flows, \( n \geq 2 \).

**References**


**Department of Mathematics, University of California, Berkeley, California 94720**

**Department of Mathematics, University of California, Los Angeles, California 90024**