RANDOM ERGODIC SEQUENCES ON LCA GROUPS

BY

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ABSTRACT. Let \( \{X(t, \omega)\}_{t \in \mathbb{R}^+} \) be a stochastic process on a locally compact abelian group \( G \), which has independent stationary increments. We show that under mild restrictions on \( G \) and \( \{X(t, \omega)\} \) the random families of probability measures

\[
\mu_T(\cdot, \omega) = B_T^{-1} \int_0^T f(t)X(t, \omega) \, dt \quad \text{for } T > 0,
\]

where \( f(t) \) is a function from \( \mathbb{R}^+ \) to \( \mathbb{R}^+ \) of polynomial growth and \( B_T = \int_0^T f(t) \, dt \), converge weakly to Haar measure of the Bohr compactification of \( G \). As a consequence we obtain mean and individual ergodic theorems and asymptotic occupancy times for these processes.

0. Summary. Let \( G \) be an LCA group of the form \( \mathbb{R}^n \times \mathbb{Z}^m \times \mathbb{K} \) where \( \mathbb{K} \) is a closed subgroup of \( \mathbb{U}^\infty \), the countable product of the unit circle. Let \( \{X(t, \omega)\}_{t \in \mathbb{R}^+} \) be a stochastic process on a probability space \( (\Omega, \mathcal{F}, P) \) with independent, stationary increments and state space \( G \).

For \( \gamma \in \hat{G} \) let \( \phi_\gamma \) be the characteristic function of the \( X(t)'s \). Call a function \( f: \mathbb{R}^+ \to \mathbb{R}^+ \) a weight function if it has polynomial growth, i.e., if there exist positive constants \( C, \tilde{C} \) and a nonnegative \( p \) such that \( Ct^p < f(t) < \tilde{C}t^p \). In this paper we show that for every weight function \( f \) there exists a set \( \Omega_f \subset \Omega \) with \( P(\Omega_f) = 1 \) such that for \( \omega \in \Omega_f \),

\[
\lim_{T \to \infty} B_T^{-1} \int_0^T f(t)X(t, \omega), \gamma \, dt = 0 \quad (1)
\]

for all \( \gamma \in \hat{G} - \{0\} \), where \( B_T = \int_0^T f(t) \, dt \).

If for a given weight function \( f \) we define the random families of probability measures on \( G \) as

\[
\mu_T(dx, \omega) = B_T^{-1} \int_0^T f(t)X(dx, \omega)(X(t, \omega)) \, dt, \quad (2)
\]

then (1) says that for \( \omega \in \Omega_f \) the Fourier transforms \( \tilde{\mu}_T(\gamma, \omega) \) satisfy

\[
\lim_{T \to \infty} |\tilde{\mu}_T(\gamma, \omega)| = 0 \quad \text{for } \gamma \in \hat{G} - \{0\}. \quad (3)
\]

As a consequence we obtain mean ergodic theorems for unitary representations of \( G \) and weighted occupancy times for \( \{X(t, \omega)\} \).

1. Preliminaries. Let \( G \) be an LCA-group of the form \( \mathbb{R}^n \times \mathbb{Z}^m \times \mathbb{K} \) with dual \( \hat{G} = \mathbb{R}^n \times \mathbb{U}^m \times \mathbb{K} \). Since \( \mathbb{K} \) is a closed subgroup of \( \mathbb{U}^\infty \), \( \hat{G} \) is countable. Let \( \hat{G} \) be the Bohr compactification of \( G \) and \( m \) Haar measure on \( \hat{G} \). For details see [4].

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We say that a family \( \{ \mu_T \} \) of probability measures on \( G \) is ergodic if
\[
\lim_{T \to \infty} \hat{\mu}_T(\gamma) = 0 \quad \text{for } \gamma \in \widehat{G} - \{0\}.
\]
If we consider \( \mu_T \) as measures on \( \widehat{G} \) this is equivalent to saying that weak
\[
\lim_{T \to \infty} \mu_T = m.
\]
As shown in [2] ergodic families of measures provide mean ergodic theorems for
unitary representations of \( G \) on a Hilbert space.

A measurable subset \( I \) of \( G \) is called a \( p \)-set if there exists \( p \in [0, 1] \) such that for
every ergodic family (or sequence) \( \{ \mu_T \} \),
\[
\lim_{T \to \infty} \mu_T(I) = p.
\]
If \( B \) is a continuity set in \( \widehat{G} \), i.e., its boundary has measure zero, then, by the Paul Lévy continuity
theorem, \( B = \overline{B} \cap G \) is a \( p \)-set with \( p = m(\overline{B}) \).

Reich constructed in [3] large classes of \( p \)-sets; the simplest construction can be
obtained as follows: let \( \gamma \in \widehat{G} \) be of infinite order and \( I \) an interval in \( \mathbb{R} \). Then
\[
\{ g \in \widehat{G} | \langle g, \gamma \rangle \in I \}
\]
is a continuity set of measure \( |I| \) and therefore \( \{ g \in \widehat{G} | \langle g, \gamma \rangle \in I \} \) is a \( p \)-set with \( p = |I| \).

2. The main results. Let \( X(t, \omega) = (X_1(t, \omega), \ldots, X_{n+m+1}(t, \omega)) \), i.e., the \( j \)th
coordinate \( X_j \) has state space \( \mathbb{R}, \mathbb{Z}, \mathbb{N} \) for \( 1 < j < n, n + 1 < j < n + m, j = n + m + 1 \) respectively.

By a well-known argument, using stationarity and independence of the increments, we can show that
\[
|\phi_j(\gamma)| = |\phi_j(\gamma)|'
\]

Theorem 1. If \( |\phi_j(\gamma)| < 1 \) for \( \gamma \in \widehat{G} - \{0\} \) and \( E|X_j(t, \omega)| = O(t) \) for \( t > 0 \) and
\( j = 1, 2, \ldots, n + m \), then for every weight function \( f \) of polynomial growth, there
exists a set \( \Omega \subset \mathbb{C} \) with \( P(\Omega) = 1 \) such that for \( \omega \in \Omega \),
\[
\lim_{T \to \infty} |\hat{\mu}_T(\gamma, \omega)| = 0 \quad \text{for all } \gamma \in \widehat{G} - \{0\}.
\]

Remark. Note that \( |\phi_j(\gamma)| < 1 \) for \( \gamma \neq 0 \) is merely a condition to ensure that
\( X(t, \omega) \) is not distributed on a proper closed subgroup of \( G \).

3. Some lemmas. The first two lemmas are from [3].

Lemma 1. Let \( \lambda \) be a positive integer and \( \delta_j = \pm 1, j = 1, 2, \ldots, 2\lambda \), such that
\[
\sum_{j=1}^{2\lambda} \delta_j = 0.
\]
Define \( k_j = -\sum_{j=1}^{2\lambda} \delta_j \delta_{j-1} \) for \( j = 1, 2, \ldots, 2\lambda - 1 \). Then for indeterminates 
\( x_1, \ldots, x_{2\lambda} \),
\[
\sum_{j=1}^{2\lambda} \delta_j x_j = \sum_{j=1}^{2\lambda-1} k_j (x_{j+1} - x_j).
\]
Furthermore, \( |k_j| < 1 \) for all \( j \) and \( k_{2j-1} \neq 0 \) for \( j = 1, \ldots, \lambda \).

The proof is obvious.

Lemma 2. Let \( g \) be a continuous function from \( \mathbb{R}^n \times \mathbb{G}^m \) into the complex plane. Suppose \( K \) is a cube in \( \mathbb{R}^n \times \mathbb{G}^m \), i.e., \( K = [I_j]_{j=1}^{2^n+m} \) where the \( I_j \)'s are intervals in \( \mathbb{R} \),
respectively. Suppose \( \max_{j=1, \ldots, n+m} |\partial g(\alpha)/\partial \alpha_j| \leq C \) for all \( \alpha \); then for any \( \alpha, \beta \in K \),

\[
|g(\alpha)| \leq |g(\beta)| + C \sum_{j=1}^{n+m} |I_j|.
\]

**Proof.** By induction on \( n + m \), the case \( n + m = 1 \) follows from the mean value theorem applied to the real and imaginary part of \( f \).

**Lemma 3.** Let \( L \) be a positive integer, \( f \) a weight function of polynomial growth, \( 0 < r < 1 \),

\[
S = \left\{ (t_1, \ldots, t_{2l}) \in [0, T]^{2l} | 0 < t_1 < t_2 < \cdots < t_{2l} < T \right\}
\]

and \( dt_2 \) Lebesgue measure on \( \mathbb{R}^{2l} \); then

\[
B_T^{-2l} \int_S \prod_{j=1}^{2l} f(t_j) \prod_{j=1}^{l} r_2^{t_{j+1} - t_j} dt_2 \leq C |\ln(r)|^{-l} T^{-l},
\]

where \( C \) only depends on \( f \) and \( L \).

**Proof.** From \( C_1^p < f(t) < C_T^p \) we obtain

\[
C_T^{p+1}/(p + 1) < B_T < C_T^{p+1}/(p + 1).
\]

Now by induction on \( l \), let \( l = 1 \) and \( p > 0 \). Then

\[
\int_0^T \int_0^T f(t_1)f(t_2) r_2^{t_2 - t_1} dt_2 dt_1 \leq \int_0^T t_1^{-p} \int_0^T t_2^{-p} r_2^{t_2 - t_1} dt_2 dt_1
\]

\[
= \int_0^T t_1^{-p} \left[ \int_0^T t_2^{-p} r_2^{t_2 - t_1} dt_2 \right] dt_1
\]

\[
< \int_0^T t_1^{-p} \frac{t_1^{p+1} + T^p}{|\ln(r)|} dt_1 < 2C_T^{2p+1} |\ln(r)|^{-1}.
\]

Now divide both sides by the lower bound in (1) to obtain the inequality.

For the case \( p = 0 \) we can compute the iterated integral directly.

Now assume true for \( l \), to prove the inequality for \( l + 1 \). Write \( \int_S \ldots dt_2 \) as an iterated integral, split off the two innermost integrals which are handled as for \( l = 1 \), then apply the induction hypothesis.

**Lemma 4.**

\[
E \left( \sup_{\gamma \in \hat{G}} \max_{j=1, \ldots, n+m} \left| \frac{\partial}{\partial \gamma_j} \hat{\mu}_T(\gamma, \omega) \right| \right) = O(T).
\]

**Proof.** By hypothesis there is some positive \( C \) such that

\[
\max_{j=1, \ldots, n+m} E|X_j(t, \omega)| < C \cdot t.
\]

For \( \gamma \in \hat{G}, \gamma = (\gamma_1, \ldots, \gamma_{n+m}, \gamma_{n+m+1}) \), hence

\[
\langle X(t, \omega), \gamma \rangle = \prod_{j=1}^{n+m+1} \langle X_j(t, \omega), \gamma_j \rangle
\]
and, therefore,
\[ \frac{\partial}{\partial \gamma_j} \langle X(t, \omega), \gamma \rangle = iX_j(t, \omega) \langle X(t, \omega), \gamma \rangle \quad \text{for} \ j = 1, \ldots, n + m. \]

From the last equation it follows that
\[ \left| \frac{\partial}{\partial \gamma_j} \mu_T(\gamma, \omega) \right| = \left| B_T^{-1} \int_0^T f(t) \frac{\partial}{\partial \gamma_j} \langle X(t, \omega), \gamma \rangle \, dt \right| < CB_T^{-1} \int_0^T |f(t)| X(t, \omega) \, dt. \]

Taking expectations on both sides, using (1) and the fact that \( f \) has polynomial growth finishes the proof.

**Lemma 5.** Let \( l \) be a positive integer and \( \gamma \in \hat{G} \) such that \( k \gamma \neq 0 \) for \( 1 < |k| < l \). Then
\[ E \left| \mu_T(\gamma, \omega) \right|^{2l} < C \cdot \left( \ln \left( \max_{1 < |k| < l} |\phi_k(k \gamma)| \right) \right)^{-l} T^{-l} \]
where \( C \) is independent of \( T \) and \( \gamma \).

**Proof.**
\[
|\mu_T(\gamma, \omega)|^{2l} = \prod_{j=1}^l B_T^{-1} \int_0^T f(t_j) \langle X(t_j, \omega), \gamma \rangle \, dt_j \times \prod_{j=l+1}^{2l} B_T^{-1} \int_0^T f(t_j) \langle X(t_j, \omega), \gamma \rangle \, dt_j
\]
\[ = B_T^{-2l} \int_{[0,T]^{2l}} \prod_{j=1}^{2l} f(t_j) \prod_{j=1}^{2l} \langle \delta_j X(t_j, \omega), \gamma \rangle \, dt^{2l} \]
\[ = B_T^{-2l} \int_{[0,T]^{2l}} \prod_{j=1}^{2l} f(t_j) \left( \sum_{j=1}^{2l} \delta_j X(t_j, \omega), \gamma \right) \, dt^{2l} \]
where
\[ \delta_j = \begin{cases} 1 & \text{for} \ j = 1, \ldots, l, \\ -1 & \text{for} \ j = l + 1, \ldots, 2l. \end{cases} \]

Let \( \varphi_{2l} \) be the permutations of \( \{1, 2, \ldots, 2l\} \) and for \( \sigma \in \varphi_{2l} \) define
\[ S_\sigma = \{(t_1, \ldots, t_{2l}) \in [0, T]^{2l} | t_{\sigma(1)} < t_{\sigma(2)} < \cdots < t_{\sigma(2l)}\}. \]

Then \( \{S_\sigma\}_{\sigma \in \varphi_{2l}} \) is an up to measure zero disjoint partition of \([0, T]^{2l}\) and therefore
\[ E \left| \mu_T(\gamma, \omega) \right|^{2l} = \sum_{\sigma \in \varphi_{2l}} B_T^{-2l} E \int_{S_\sigma} \prod_{j=1}^{2l} f(t_j) \left( \sum_{j=1}^{2l} \delta_j X(t_j, \omega), \delta \right) \, dt^{2l} \]
\[ = \sum_{\sigma \in \varphi_{2l}} B_T^{-2l} \int_{S_\sigma} \prod_{j=1}^{2l} f(t_{\sigma(j)}) \left( \sum_{j=1}^{2l} \delta_{\sigma(j)} X(t_{\sigma(j)}, \omega), \gamma \right) \, dt^{2l}. \quad (1) \]

From the definition of the \( \delta_j \)'s and \( \delta_{\sigma(j)} \)'s it follows that they satisfy the hypothesis of Lemma 1; therefore for each \( \sigma \) we can find integers \( k_j, j = 1, 2, \ldots, 2l - 1, \)
such that in the last equality
\[
B_T^{-2l}E \left| \int_{S_o} \ldots \ dt^{2l} \right|
\]
\[
= B_T^{-2l}E \int_{S_o} \sum_{j=1}^{2l} f(t_{a(j)}) \left( \sum_{j=1}^{2l-1} k_j [X(t_{a(j+1)}, \omega) - X(T_{a(j)}, \omega)] \right) \gamma \ dt^{2l}
\]
\[
= B_T^{-2l} \int_{S_o} \prod_{j=1}^{2l} f(t_{a(j)}) E \prod_{j=1}^{2l-1} \langle k_j [X(t_{a(j+1)}, \omega) - X(t_{a(j)}, \omega)] \rangle \ dt^{2l}
\]
\[
= B_T^{-2l} \int_{S_o} \prod_{j=1}^{2l} f(t_{a(j)}) E \prod_{j=1}^{2l-1} \langle X(t_{a(j+1)}, \omega) - X(t_{a(j)}, \omega), k_j \gamma \rangle \ dt^{2l}
\]
\[
\leq B_T^{-2l} \int_{S_o} \prod_{j=1}^{2l} f(t_{a(j)}) \prod_{j=1}^{2l-1} E \langle X(t_{a(j+1)}, \omega) - X(t_{a(j)}, \omega), k_j \gamma \rangle \ dt^{2l}
\]
\[
= B_T^{-2l} \int_{S_o} \prod_{j=1}^{2l} f(t_{a(j)}) \prod_{j=1}^{2l-1} |\phi_1(k_j \gamma)|^{t_{\omega_j} - t_{\omega_j - 1}} \ dt^{2l}
\]
\[
\leq B_T^{-2l} \int_{S_o} \prod_{j=1}^{2l} f(t_{a(j)}) \prod_{j=1}^{2l} \left( \max_{1 \leq |k| \leq l} |\phi_1(k \gamma)| \right)^{t_{\omega_j} - t_{\omega_j - 1}} \ dt^{2l}
\]
\[
\leq C \left| \ln \left( \max_{1 \leq |k| \leq l} |\phi_1(k \gamma)| \right) \right|^{-l} \cdot T^{-l}.
\]

The first inequality follows from the fact that on $S_o$, $t_{a(1)} < t_{a(2)} < \ldots < t_{a(2l)}$ and independent increments of $X(t, \omega)$. The second and third inequalities follow from Lemma 1 since $|k_j| < l$ for all $j$ and $k_2j - 1 \neq 0$ for $j = 1, 2, \ldots, l$. For the last inequality apply Lemma 3.

To finish the proof combine (1) and (2) to conclude that
\[
E| \hat{\mu}_T(\gamma, \omega)|^{2l} \leq (2l)! CT^{-l} \left| \ln \left( \max_{1 \leq |k| \leq l} |\phi_1(k \gamma)| \right) \right|^{-l}.
\]

4. Proof of Theorem 1. Let $K = \prod_{j=1}^{n+m} I_j \times \{\alpha\}$, where the $I_j$'s are closed intervals in $\mathbb{R}$, $\alpha$ for $1 \leq j \leq n, n + 1 \leq j \leq n + m$, respectively, and $\alpha \in \mathcal{K}$; we will call a set of this form a cube.

Fix $l = 3(n + m) + 4$ and suppose for $\gamma \in K$, $k \gamma \neq 0$ for $1 \leq |k| < l$, i.e., $K$ contains no roots of unity of order $< l$.

Define
\[
r = \max_{1 \leq |k| \leq l} \sup_{\gamma \in K} |\phi_1(k \gamma)|.
\]

Then
\[
r < 1. \tag{1}
\]

This follows from the assumption $|\phi_1(\gamma)| < 1$ for $\gamma \neq 0$ and the fact that $|\phi_1(\gamma)|$ is continuous and $K$ is compact and contains no roots of unity of order $< l$. 

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For a positive integer $N$, divide $K$ into $[N^{3/2}]^{n+m} = \bar{N}$ subcubes $\{K_j\}_{j=1}^{\bar{N}}$ of equal measure, which are disjoint up to measure zero ($[\ ]$ denotes the greatest integer part), i.e., divide each $I_j$ into $[N^{3/2}]$ subintervals and take product sets. In each $K_j$ fix a point $y_j$ and let

$$A_N = \left\{ \max_{j=1, \ldots, N} |\hat{\mu}_N(y_j, \omega)| < N^{-1/4} \right\}.$$  

Then by Chebychev's inequality, Lemma 5 and (1),

$$P(A_N^c) \leq \sum_{j=1}^{\bar{N}} N^{21/4} E|\hat{\mu}_N(y_j, \omega)|^2 < CN^{-1/2} |\ln(r)|^{-1} N^{-1} < CN^{-1/2} N^{3/2(n+m)} |\ln(r)|^{-1} < CN^{-2} |\ln(r)|^{-1}. \quad (2)$$

The constant $C$ only depends on $f$ and $l$ by Lemma 5.

Let

$$B_N = \left\{ \max_{j=1, \ldots, n+m} \left| \frac{\partial}{\partial y_j} \hat{\mu}_N(y, \omega) \right| < N^{5/4} \right\}.$$  

Then by Lemma 4 and Chebychev's inequality,

$$P(B_N^c) < \sum_{j=1}^{n+m} N^{-5/4}O(N) = O(N^{-1/4}). \quad (3)$$

Hence by (2) and (3),

$$\sum_{N=1}^{\infty} P((A_N^c \cap B_N^c)^c) < \infty,$$

which by the Borel-Cantelli lemma implies that

$$P\{\omega | \omega \text{ is outside of at most finitely many of the } A_N^c \cap B_N^c\text{'s}\} = 1. \quad (4)$$

If $\omega \in A_N^c \cap B_N^c$, then for $\gamma \in K$ there is a subcube $K_j$ such that $\gamma \in K_j$. Therefore by Lemma 2, Lemma 4 and the fact that to obtain the $K_j$'s we divided each $I_j$ into $[(N^{8})^{3/2}]$ subintervals of equal length, we get

$$|\hat{\mu}_N^*(\gamma, \omega)| < |\hat{\mu}_N^*(y_j, \omega)| + \sum_{k=1}^{n+m} N^{10} \cdot |I_j| \cdot \left[ N^{12} \right]^{-1} < N^{-2} + (n + m) \left( \max_{j=1, \ldots, n+m} |I_j| \right) 2N^{-2} = O(N^{-2}).$$

Since this inequality does not depend on $\gamma$, we get for $\omega \in A_N^c \cap B_N^c$,  

$$\sup_{\gamma \in K} |\hat{\mu}_N^*(\gamma, \omega)| < O(N^{-2}). \quad (5)$$

Therefore, by (4) and (5),

$$\lim_{N \to \infty} \sup_{\gamma \in K} |\hat{\mu}_N^*(\gamma, \omega)| = 0 \text{ with probability one.}$$

And since $B_T$ grows geometrically with $T$ by a well-known argument, we can conclude

$$\lim_{T \to \infty} \sup_{\gamma \in K} |\hat{\mu}_T(\gamma, \omega)| = 0 \text{ almost surely.}$$
From the structure of $\hat{G}$ we see that $\hat{G}$-{roots of unity of order $< l$} is a countable union of such cubes $K$ and that there are at most countably many roots of unity of order $< l$. If $\gamma$ is a root of unity of order $< l$ and $\gamma \neq 0$, then letting

$$A_N = \{ \omega \mid \hat{\mu}_N(\gamma, \omega) < N^{-1/4} \},$$

it follows from Lemma 5 with $l = 1$ that

$$P(A_N) < N^{1/2}E|\hat{\mu}_N(\gamma, \omega)|^2 < CN^{-1/2}$$

and therefore $\sum_{N=1}^{\infty} P(A_N) < \infty$. Now by an argument as above using the Borel-Cantelli lemma,

$$\lim_{T \to \infty} |\hat{\mu}_T(\gamma, \omega)| = 0 \text{ almost surely.}$$

Taking the intersection of this countable collection of sets of probability one, gives us the desired result.

**5. Some examples.** Let $X_1(t, \omega), \ldots, X_n(t, \omega)$ be Brownian motions on $\mathbb{R}$ such that:

(i) the random variables $X_1(1, \omega), \ldots, X_n(1, \omega)$ are linearly independent, i.e., $P(\sum_{j=1}^{n} r_j X_j(1, \omega) = 0) = 1$ iff $r_1 = \cdots = r_n = 0$; and

(ii) for $0 < r < s < t$, $X_j(t, \omega) - X_j(s, \omega)$ is independent of $X_k(r, \omega)$ for all $j, k$.

Then the process $X(t, \omega) = (X_1(t, \omega), \ldots, X_n(t, \omega))$ on $\mathbb{R}^n$ has independent stationary increments by (ii) and the characteristic function satisfies the hypothesis of Theorem 1 by (i). In particular, (ii) is satisfied if the processes $X_j$ are independent. Similarly, using Poisson processes, we can construct a process on $\mathbb{Z}^m$, which satisfies the conditions of Theorem 1. Combining these processes we obtain a process on $\mathbb{R}^n \times \mathbb{Z}^m$ with the desired properties.

**6. Applications to unitary representations.** Let $\{ U_g \}_{g \in G}$ be a weakly continuous unitary representation of $G$ on a Hilbert space $\mathcal{H}$. Denote by $P_{\mathcal{H}}$ the orthogonal projection onto the closed subspace $\overline{\mathcal{H}}$ of invariant elements under $\{ U_g \}$.

**Theorem 2.** Let $\{ X(t, \omega) \}, f, \Omega_f$ be as in Theorem 1, and $\{ U_g \}_{g \in G}$ any weakly continuous unitary representation of $G$ on a Hilbert space. Then for $\omega \in \Omega_f$,

$$\lim_{T \to \infty} \left\| B_T^{-1} \int_0^T f(t)(U_{X(t, \omega)}h) \, dt - P_{\mathcal{H}}h \right\| = 0$$

for all $h \in \mathcal{H}$.

**Proof.** Since

$$B_T^{-1} \int_0^T f(t)(U_{X(t, \omega)}h) \, dt = \int_G (U_g h) \mu_t(dg, \omega)$$

and $\hat{\mu}_T(\gamma, \omega) \to 0$ for $\gamma \in \hat{G} - \{0\}$, the result follows from a theorem in [2].

**Theorem 3.** Let $\{ X(t, \omega) \}, f, \Omega_f$ be as in Theorem 1. Let $\{ U_g \}_{g \in G}$ be a weakly continuous representation on some $L^2$ space. Then there exists a dense set $\mathcal{D} \subset L^2$ such that for $\omega \in \Omega_f$,

$$\lim_{N \to \infty} B_{N^4}^{-1} \int_0^{N^8} f(t)(U_{X(t, \omega)}h(y)) \, dt = P_{\mathcal{D}}h$$

for almost every $y$ and all $h \in \mathcal{D}$. 

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If, in addition, the $U_g$'s are uniformly bounded on $L^\infty$ and the set of eigenvalues does not have any limit points, then we can find a dense $\mathcal{D} \subset L^2$ such that
\[
\lim_{T \to \infty} B_T^{-1} \int_0^T f(t)(U_x(t, \omega)h(y)) \, dt = P_\omega h
\]
for almost every $y$ and all $h \in \mathcal{D}$.

Remark. Note that the two statements of the theorem hold for all $\omega \in \Omega_f$, i.e., the set of probability one does not depend on the unitary representation nor the particular function selected from $\mathcal{D}$.

Proof. Let $E(\cdot)$ denote the resolution of the identity for $\{U_g\}$ on $\hat{G}$. Let $h \in L^2$ and $\{\gamma_j\}$ be the nonzero eigenvalues such that $E(\gamma_j)h = h\gamma_j \neq 0$. Assume first
\[
h = \sum_{j=1}^{\infty} h\gamma_j + P_\omega h. \tag{1}
\]
Then for $\varepsilon > 0$ and $N$ sufficiently large,
\[
\tilde{h} = \sum_{j=1}^{N} h\gamma_j + P_\omega h \text{ is } \varepsilon\text{-closed to } h. \tag{2}
\]
For $\tilde{h}$ we get for $\omega \in \Omega_f$,
\[
\lim_{T \to \infty} \int_G U_g \tilde{h} \mu_T(dg, \omega) = \lim_{T \to \infty} \sum_{j=1}^{N} \tilde{\mu}_T(\gamma_j, \omega)h\gamma_j + P_\omega h = P_\omega h
\]
since the $\gamma_j$'s are nonzero.

Assume now that $h \in L^2$ such that
\[
E(\gamma)h = 0 \text{ for all } \gamma \in \hat{G}. \tag{3}
\]
This implies the Borel measure $(E(dy)h, h)$ is continuous on $\hat{G}$. Therefore, for $\varepsilon > 0$ by the $\sigma$-compactness of $\hat{G}$ we can find a compact cube $\tilde{K}$ such that
\[
\|E(\tilde{K})h - h\|_2 < \varepsilon/2. \tag{4}
\]
From the structure of $\hat{G}$ one sees that a compact cube $K$ only can contain finitely many roots of unity of order $< l$. Deleting sufficiently small cubical open neighborhoods around each root of order $< l$ from $\tilde{K}$ gives us a compact set $K$ such that
\[
(i) \quad \|E(K)h - E(\tilde{K})h\|_2 < \varepsilon/2;
(ii) \quad K = \bigcup_{j=1}^{M} K_j, \tag{5}
\]
the $K_j$'s are disjoint and each $K_j$ is of the form $\prod_{j=1}^{\alpha} I_j \times \{\alpha\}$ where the $I_j$'s are intervals (not necessarily closed) and $\alpha \in \hat{G}$. Also note that the closure of $K_j$ does not contain any roots of order $< l$.

Since $E(K)h = \sum_{j=1}^{M} E(K_j)h$, it is sufficient to prove pointwise convergence for each function $E(K_j)h$.

From (5) in the proof of Theorem 1 it follows that for $\omega \in \Omega_f$ and $N$ sufficiently large
\[
\sup_{\gamma \in K_j} |\tilde{\mu}_N(\gamma, \omega)| < O(N^{-2}). \tag{6}
\]
Therefore for $\lambda > 0$, letting

$$F_N = \left\{ y \mid \left| \int_G U_y \left[ E(K_j) h \right](y) \mu_{N^2}(dg, \omega) \right| < \lambda \right\},$$

we obtain the estimate

$$|F_N| < \lambda^{-2} \left\| \int_G U_y \left[ E(K_j) h \right] \mu_{N^2}(dg, \omega) \right\|^2_2$$

$$\leq \lambda^{-2} \int_{K_j} |\hat{\mu}_{N^2}(\gamma, \omega)|^2 (E(d\gamma) h, h) < \lambda^{-2} N^{-4} \|h\|_2^2.$$  \(7\)

The last inequality follows from (6). From (7) and the Borel-Cantelli lemma it follows that except for a set of measure zero all $y$'s are at most in finitely many of the $F_N$'s; since $\lambda$ can be made arbitrarily small, we deduce pointwise convergence a.e. to 0 for $E(K_j) h$ and therefore also for $E(K) h$. Finally, each function in $L^2$ is a sum of two functions of the form given in (1) and (3).

For the second part, for $h \in L^2 \cap L^\infty$ and $\epsilon > 0$ find first a compact cube $\tilde{K}$ such that

$$\|E(\tilde{K}) h - h\|_2 < \epsilon/2.$$  \(8\)

Then as before delete sufficiently small neighborhoods around all roots of order $< \lambda$ and all eigenvalues in $\tilde{K}$ to obtain a compact set $K$ such that

$$\left\| E(\tilde{K}) h - \left( E(K) h + \sum_{\gamma \in \hat{K}} E(\{\gamma\}) h \right) \right\|_2 < \epsilon/2.$$  \(9\)

From the assumption that the $e$-values have no limit points we conclude that there are only finitely many $e$-values in $\tilde{K}$ and therefore

$$\sum_{\gamma \in \hat{K}} E(\{\gamma\}) h$$

is a finite sum. \(10\)

Let $\Theta$ be an open cover of $K$ which has compact closure such that all roots of unity of order $< \lambda$ and all $e$-values are in the interior of $\Theta^c$ and let $\sigma$ be a finite measure on $G$ such that

(i) $0 < \sigma(\gamma) < 1$, \hspace{1cm} $\gamma \in \hat{G}$,

(ii) $\sigma(\gamma) = \begin{cases} 1 & \text{for } \gamma \in K, \\ 0 & \text{for } \gamma \in \Theta^c. \end{cases}$  \(11\)

We define

$$h^* = \int_G U_y h\sigma(dg).$$

From the assumption of uniform boundedness of $\{U_y\}$ on $L^\infty$ it follows that

$$h^* \in L^\infty \cap L^2.$$  \(12\)

Finally, define

$$h_\epsilon = h^* + \sum_{\gamma \in \hat{K}} E(\{\gamma\}) h.$$  \(13\)
From (8) and (9) conclude that $h_\varepsilon$ is $\varepsilon$-closed to $h$, and from (10) we see that $\sum_{\gamma \in \hat{G}} E(\gamma)h$ converges pointwise.

For $h^*$ we obtain

\begin{equation}
\left\| \int_G U_\varepsilon h^* \mu_{N^*}(dg, \omega) \right\|_2^2 = \int_G |\hat{\sigma}(\gamma)|^2 |\hat{\mu}_{N^*}(\gamma, \omega)|^2 (E(\ddot{\gamma})h, h) \leq \sup_{\gamma \in \Theta} |\hat{\mu}_{N^*}(\gamma, \omega)|^2 \|h\|^2 \leq N^{-4}\|h\|^2 \tag{14}\end{equation}

for all $\omega \in \Omega_f$. The last inequality follows as in (6).

Now we argue as in (7) to obtain

$$\lim_{N \to \infty} B_{N^*}^{-1} \int_0^N f(t) U_{X(t, \omega)} h^* \, dt = 0 \quad \text{a.e.}$$

Then

$$\lim_{T \to \infty} B_T^{-1} \int_0^T f(t) U_{X(t, \omega)} h^* \, dt = 0 \quad \text{a.e.}$$

follows from the fact that $h^* \in L^\infty$, $\{U_\varepsilon\}$ is uniformly bounded on $L^\infty$ and the $B_T$'s grow geometrically.

7. $p$-occupancy. Let $\{X(t, \omega)\}$ be a process as in Theorem 1; then for $\omega \in \Omega_f$, $\{\mu_T(dg, \omega)\}$ is an ergodic family of measures on $G$ (as defined in §1). Hence for $I_p$ a $p$-set,

$$\lim_{T \to \infty} \mu_T(I_p, \omega) = p \quad \text{for all } \omega \in \Omega_f; \tag{1}$$

in particular, if $\gamma \in \hat{G}$ of infinite order and $I$ an interval in $\mathcal{G}$,

$$\lim_{T \to \infty} \frac{1}{B_T} \int_0^T f(t) \chi_{\{\varepsilon, \gamma \in I\}}(X(t, \omega)) \, dt = |I| \tag{2}$$

for all $\omega \in \Omega_f$.

It should be noted that for $f \equiv 1$, (1) and (2) are the limit of the average amount of time the process spends in the given set up to time $T$; this case is a generalization of a result on random walks in [1].

REFERENCES


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