Tensor Products of Principal Series for the DeSitter Group

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Abstract. The decomposition of the tensor product of two principal series representations is determined for the simply connected double covering, $G = \text{Spin}(4, 1)$, of the DeSitter group. The main result is that this decomposition consists of two pieces, $T_c$ and $T_d$, where $T_c$ is a continuous direct sum with respect to Plancherel measure on $G$ of representations from the principal series only and $T_d$ is a discrete sum of representations from the discrete series of $G$. The multiplicities of representations occurring in $T_c$ and $T_d$ are all finite.

Introduction. Let $G = \text{Spin}(4, 1)$ be the simply connected double covering of the DeSitter group, $G = KAN$ an Iwasawa decomposition of $G$, $M$ the centralizer of $A$ in $K$, and $P = MAN$ the associated minimal parabolic subgroup of $G$. For $\sigma \in \hat{M}$ and $\tau \in \hat{A}$, $\sigma \times \tau$ is a representation of $P$ via $\sigma \times \tau(m \rho) = \sigma(m)\tau(\rho)$ and a representation of the form $\pi(\sigma, \tau) = \text{Ind}_G^P \sigma \times \tau$ is called a principal series representation of $G$. The main goal of this paper is to determine the decomposition of the tensor product of two principal series representations of $G$ into irreducibles.

It was shown in [7], by using Mackey's tensor product theorem and the Mackey-Anh reciprocity theorem, that this problem reduces to knowing how to decompose the restriction to $MA$ of almost every principal series representation of $G$ and each discrete series representation of $G$. For a representation $\pi$ belonging to the principal series of $G$, the restriction of $\pi$ to $MA$, $(\pi)_{MA}$, was determined by using Mackey's subgroup theorem. However, in that paper, we were not able to determine explicitly $(\pi)_{MA}$ for a representation $\pi$ belonging to the discrete series of $G$. This we do in §3 of this paper by using Lie algebraic methods and the realizations of these representations given by Dixmier in [2].

This paper is organized as follows. In §§1 and 2 we summarize the main results concerning the structure and representation theory of $G$ that we shall use. In §3 we determine $(\pi)_{MA}$ when $\pi$ is a discrete series representation of $G$. We also include the results of [7] concerning the decomposition of $(\pi)_{MA}$ when $\pi$ is a principal series representation of $G$. In §4 we show how to decompose the tensor product of two principal series representations of $G$. The main results are contained in Theorem 4.

The basic methodology used in this paper to decompose principal series tensor products originates in the works of G. Mackey [6], N. Anh [1], and F. Williams [11].
1. The structure of \( G \). In this section we summarize the main results concerning the structure of the DeSitter group and its two-fold covering \( \text{Spin}(4, 1) \) that we shall use in this paper. Further details may be found in [8].

Let \( O(4, 1) \) denote the group of linear transformations of \( \mathbb{R}^5 \) which preserve the quadratic form \(-x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 \). If \( J \) is the diagonal matrix \( J = \begin{bmatrix} -1 & 1 & & & \\ 1 & -1 & & & \\ & & -1 & & \\ & & 1 & -1 & \\ & & & & -1 \end{bmatrix} \), then \( O(4, 1) \) may be identified with \{ \( g \in \text{GL}(5, \mathbb{R}) : g^T J g = J \) \}. The connected component of the identity is the group

\[
G' = \text{SO}_e(4, 1) = \{ g \in O(4, 1) : \det(g) = 1, g_{oo} > 1 \}
\]

which is commonly referred to as the DeSitter group. \( G' \) is a connected semisimple real-rank one Lie group with trivial center. We let \( G = \text{Spin}(4, 1) \) denote the simply-connected double covering of \( G' \). As indicated in [8], we may realize \( G \) as a certain collection of two-by-two matrices over the quaternions. If

\[
F = \{ x, + i x_2 + j x_3 + k x_4 : x, \in \mathbb{R}, j^2 = k^2 = -1, x_0 = -x_4 \}
\]

denotes the quaternions, \( \bar{x} = x_1 - i x_2 - j x_3 - k x_4 \), and \(|x| = \sqrt{x \bar{x}}\), then \( G \) is isomorphic to the group

\[
\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in F, \bar{a} b = c d, |a|^2 - |c|^2 = 1, |d|^2 - |b|^2 = 1 \right\}.
\]

The group \( U = \{ x \in F : |x| = 1 \} \) of unit quaternions is easily seen to be isomorphic to \( SU(2) \) via the mapping

\[
\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad a = x_1 + i x_2, \quad b = x_3 + i x_4,
\]

for example, and if we let

\[
K = \left\{ \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} : u, v \in U \right\} \approx SU(2) \times SU(2) \approx \text{Spin}(4),
\]

\[
A = \left\{ \begin{pmatrix} \text{ch} \frac{1}{2} t & \text{sh} \frac{1}{2} t \\ \text{sh} \frac{1}{2} t & \text{ch} \frac{1}{2} t \end{pmatrix} = a_t : t \in \mathbb{R} \right\} \approx \mathbb{R}^+, \quad a_t = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},
\]

and

\[
N = \left\{ \begin{pmatrix} 1 - x & x \\ -x & 1 + x \end{pmatrix} = n_x : x = \frac{1}{2} (x_2 i + x_3 j + x_4 k) \right\} \approx \mathbb{R}^3,
\]

then \( G = K A N \) is an Iwasawa decomposition of \( G \). If \( M \) denotes the centralizer of \( A \) in \( K \), then

\[
M = \left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} = m_u : u \in U \right\} \approx \text{Spin}(3)
\]

and \( P = M A N \) is a (minimal parabolic) subgroup of \( G \) which contains \( N \) as a normal subgroup. One easily computes that the actions of \( M \) and \( A \) on \( N \) are given by:

\[
m_u n_x m_u^{-1} = m_u \cdot n_x = n_{ux}, \quad a_n x a_n^{-1} = a_n \cdot n_x = n_{nx}, \quad \text{i.e., } M \text{ acts by rotations and } A \text{ acts by dilations. If } M \text{ denotes the normalizer of } A \text{ in } K, \text{ then the Weyl group } W = \tilde{M} / M \text{ has order 2 and we may take } e = (0, 0) \text{ and } w = (0, -1) \text{ as representatives of the cosets of } W. \text{ In addition to the Iwasawa decomposition of } G, \text{ one has the } K A K \text{ decomposition (see [8, p. 366]) and the Bruhat decomposition}
$G = P e P \cup P w P$ (and so there are only two $P : P$ double cosets in $G$ with only one of positive Haar measure). Setting

$$V = \left\{ \begin{pmatrix} 1 - x & -x \\ x & 1 + x \end{pmatrix} : x = \frac{1}{2} (i x_2 + j x_3 + k x_4) \right\}$$

and using the relations $w^{-1} A w = A$, $w^{-1} M w = M$, the latter decomposition can be expressed as $G = P w^{-1} \cup PV$ and so up to a manifold of lower dimension (and so a set of Haar measure zero) we see that $G = PV$.

If $\phi: G \to G'$ denotes the homomorphism on p. 366 of [8] and we let $K' = \phi(K)$, $A' = \phi(A)$, $N' = \phi(N)$, $M' = \phi(M)$, and $V' = \phi(V)$, then

$$\ker(\phi) = \text{the center of } G = Z(G) = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\} \subseteq K$$

and so we have the following isomorphisms:

$$K / Z(G) \approx K' \approx SO(4), \quad M / Z(G) \approx M' \approx SO(3),$$

$$A \approx A' \approx \mathbb{R}^+, \quad N \approx N' \approx V \approx V' \approx \mathbb{R}^3.$$ 

It is easily seen that

$$K' = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} : k \in SO(4) \right\},$$

$$A' = \left\{ \begin{pmatrix} \text{cha} & \text{sha} & 0 \\ \text{sha} & \text{cha} & 0 \\ 0 & 0 & 1 \end{pmatrix} : a \in \mathbb{R} \right\},$$

$$M' = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & u \end{pmatrix} : u \in SO(3) \right\}.$$ 

The Lie algebra $g$ of $G$ (and of $G'$) is the collection of matrices of the form

$$M(a, b, \ldots, k) = \begin{pmatrix} 0 & a & b & c & d \\ a & 0 & e & f & g \\ \text{b} & -\text{e} & 0 & \text{h} & \text{j} \\ \text{c} & -\text{f} & -\text{h} & 0 & \text{k} \\ \text{d} & -\text{g} & -\text{j} & -\text{k} & 0 \end{pmatrix}$$

where $a, b, \ldots, k \in \mathbb{R}$.

Letting $A = M(1, 0, \ldots, 0)$, $B = M(0, 1, 0, \ldots, 0)$, $C = M(0, \ldots, 0, 1)$, we see that $A, B, \ldots, K$ forms a basis for $g$. If $\mathfrak{f} \subseteq g$ denotes the set of matrices of the form $aE + bF + cG + dH + eJ + fK$ and $\mathfrak{p} \subseteq g$ denotes the set of matrices of the form $aA + bB + cC + dD$, then $\mathfrak{g} = \mathfrak{f} + \mathfrak{p}$ is a Cartan decomposition of $g$ and the associated Cartan involution $\theta$ is just the negative transpose. The subspace $\mathfrak{a} = RA$ is a maximal abelian subalgebra of $\mathfrak{p}$. If $a \in \text{Hom}_{\mathbb{R}}(a, \mathbb{R})$ is given by $\alpha(aA) = a a$ and we define

$$\mathfrak{g}_\alpha = \left\{ X \in \mathfrak{g} : [H, X] = \alpha(H) X, \text{ for all } H \in \mathfrak{a} \right\},$$

then the nonzero weight spaces are

$$\mathfrak{g}_0 = RA + RH + RJ + RK,$$

$$\mathfrak{g}_1 = R(B + E) + R(C + F) + R(D + G) \quad \text{and}$$

$$\mathfrak{g}_{-1} = R(-B + E) + R(-C + F) + R(-D + G).$$
We let $n = g_1$, $v = g_{-1}$ and $m = RH + RJ + RK$. It is easy to check that the Lie algebras of $K, M, A, N,$ and $V$ are, respectively, $f, m, a, n,$ and $v$.

We denote by $U$ the universal enveloping algebra of $g$ and by $gc$ the complexification ($\approx \text{so}(5, \mathbb{C})$) of $g$. Basis elements and a multiplication table for $gc$ may be found in [2].

2. The representation theory of $G$. We shall use the classification of the irreducible representations of $G$ given by Dixmier in [2]. Dixmier’s techniques are very algebraic and rely on the one-to-one correspondence between equivalence classes of irreducible unitary representations of $G$ and certain infinitesimal equivalence classes of algebraically !-finite representations of $g$ which are infinitesimally unitary (see [2] or [10, Vol. I, p. 330]). A more global treatment using induced representations may be found in [8].

We begin by looking at the representation theories of $M, K, A, and C = MA$. $M = \text{Spin}(3) \approx SU(2)$ and it is well known that $\hat{M} = \{\sigma^k : k = 0, \frac{1}{2}, 1, \ldots \}$ where each $\sigma^k$ acts on a space $V_k$ of dimension $2k + 1$ (see p. 110 of [9]). For $k = 0, 1, \ldots$, $\sigma^k$ is a single-valued representation of $M' \approx SO(3)$ while each $\sigma^k$, $k = \frac{1}{2}, 3/2, \ldots$, is a double-valued representation of $M'$.

Since $K = \text{Spin}(3) \times \text{Spin}(3) \approx \text{Spin}(4)$, we have that $\hat{K} = \{\sigma^{k, k'} : k, k' = 0, \frac{1}{2}, 1, \ldots \}$ where each $\sigma^{k, k'}$ acts on $V_{k, k'} = V_k \times V_{k'}$. It is also known that the restriction of $\sigma^{k, k'}$ to $M, (\sigma^{k, k'})_M$, decomposes as

$$\sum_{j = |k - k'|}^{k + k'} \sigma^j \quad [[9, p. 175]].$$

The representation $\sigma^{k, k'}$ is a single- or double-valued representation of $K'$ according to whether $k + k'$ is integral or not.

The irreducible representations (quasi-characters) of $A$ are given by $\lambda^s(a) = e^{is}$ for $s \in \mathbb{C}$. These are unitary precisely when $\text{Re}(s) = 0$ and so we shall view $\hat{A} = \{\lambda^y : y \in \mathbb{R}\}$. Since the group $C = MA$ is a direct product, $\hat{C} = \hat{M} \times \hat{A}$ with Plancherel measure $\mu_C$ on $\hat{C}$ being the product of the Plancherel measures on $\hat{M}$ and $\hat{A}$.

If $\rho \in \hat{G}$ acting on $H_\rho$, then Dixmier [2] shows that the restriction of $\rho$ to $K, (\rho)_K$, contains a given $\sigma^{k, k'}$ with multiplicity 0 or 1. Letting $\Gamma$ be the set of pairs $(k, k')$ which occur in $(\rho)_K$, we then have that $(\rho)_K \simeq \bigoplus_\Gamma \sigma^{k, k'}$ and $H_\rho = \bigoplus_\Gamma V^{k, k'}$ (the Hilbert space direct sum). Furthermore, if

$$\Gamma_0 = \{(k, k') : k, k' = 0, \frac{1}{2}, 1, \ldots \},$$

$$\Gamma_1 = \{(k, k') \in \Gamma_0 : k + k' \equiv 0 \, (\text{mod } 1)\},$$

$$\Gamma_2 = \{(k, k') \in \Gamma_0 : k + k' \equiv \frac{1}{2} \, (\text{mod } 1)\},$$

then $\Gamma_0 \cap \Gamma_1 \cap \Gamma_2$, $\Gamma \subseteq \Gamma_1$ or $\Gamma \subseteq \Gamma_2$, and $\Gamma$ must consist of all points in $\Gamma_1$ or $\Gamma_2$ which lie on and within a region of the following form:
Using the same letters to denote the corresponding representations on the Lie algebras of $G$ and $K$, we have that $\rho$, as a representation of $\mathfrak{g}$, acts on the space $H_p = \sum_{k, k'} V^{k,k'}$ (the algebraic sum) and that the restriction of $\rho$ to $\mathfrak{f}$ acts on $V^{k,k'}$ as $\sigma^{k,k'}$. Dixmier then shows that by starting with a $\Gamma$ of the above form and a basis for each $V^{k,k'}$, $(k, k') \in \Gamma$, it is possible to use a "closest neighbors" technique to extend the above action of $\rho$ on $\mathfrak{f}$ in a compatible way to all of $\mathfrak{g}$ (in possibly many ways in the case of Figure 1) so as to obtain an irreducible representation of $\mathfrak{g}$. Thus in Dixmier's classification of $\mathcal{G}$, the irreducibles, other than the trivial representation, fall into four categories.

(A). The representations $\nu_{p, a}$. These are the representations arising from Figure 1. To each half-integer $p$ one gets a one parameter family of irreducibles indexed by $a$. The range of $a$ is as follows: if $p = 0$, $a > -2$; if $p = 1, 2, 3, \ldots$, $a > 0$; while if $p = \frac{1}{2}, 3/2, \ldots$, $a > \frac{1}{2}$. These representations arise quite naturally within the framework of induced representations. They constitute what are usually called the principal series (essentially for $a > \frac{1}{2}$) and complementary series of irreducible unitary representations of $G$. If $\sigma^p \in \hat{M}$ and $\lambda^s$ is a quasi-character of $A$, we let $(\sigma^p \times \lambda^s)'$ denote the representation of $P$ given by $(\sigma^p \times \lambda^s)'(\text{man}) = \sigma^p(m)\lambda^s(a)$ and set $\pi(p, s) = \text{Ind}^{G}(\sigma^p \times \lambda^s)'$. When $\text{Re}(s) = 0$, the representation $\pi(p, s)$ is unitary and it is known that $\pi(p, s)$ is reducible iff $s = 0$ and $p = \frac{1}{2}, 3/2, \ldots$ (see [4]). For $p = \frac{1}{2}, 3/2, \ldots$, $\pi(p, 0)$ splits into the direct sum of 2 irreducibles which we shall denote by $\pi_{p, 1/2}^\pm$ [8, pp. 390, 421]. It is well known that $\pi(p, iy) \simeq \pi(p, -iy)$ and that $\pi(p, iy) \simeq \pi(q, ix)$ if $p \neq q$ or $|y| \neq |x|$. The collection $\hat{G}_p = \{\pi(p, iy) : y > 0, p = 0, \frac{1}{2}, \ldots\}$ is called the principal series of $G$. We will write

$$\hat{G}_i = \{\pi(p, iy) : \pi(p, iy) \in \hat{G}_p ; y \neq 0 \text{ if } p = \frac{1}{2}, \frac{3}{2}, \ldots\}$$
for the irreducible principal series and

\[ \hat{G}_r = \{ \pi_{p,1/2}^\pm : p = \frac{1}{2}, \frac{3}{2}, \ldots \} \]

for the collection of irreducibles arising from the reducible principal series represen-
tations. Finally, the representation \( \pi(p, i\nu) \in \hat{G}_p \) corresponds to \( \nu_{p,\sigma} \) in Dixmier’s classification with \( \sigma = \frac{1}{4} + \nu^2 \).

When \( \text{Re}(\nu) \neq 0 \), the representation \( \pi(p, s) \) is not unitary. However, the results of [4] show that when \( p = 0, 1, 2, \ldots \), it is possible to define a new inner product on the Hilbert space in question for certain real values of \( s \) in a “critical interval” \( 0 < s < c_p \) (for \( p = 0, c_p = \frac{3}{2} \), while for \( p = 1, 2, \ldots, c_p = \frac{3}{2} \)). With respect to this new inner product, \( \pi(p, s) \) is unitary. The collection of unitary representations

\[ \hat{G}_c = \{ \pi(p, s) : 0 < s < 3/2 \text{ if } p = 0, 0 < s < \frac{1}{2} \text{ if } p = 1, 2, \ldots \} \]

obtained in this fashion are all irreducible and pairwise inequivalent. They con-
stitute the representations in the complementary series of \( G \). The results of [8, p. 391] show that \( \pi(p, s) \simeq \nu_{p,\sigma} \) for \( \sigma = \frac{1}{4} - \nu^2 \).

Representations of the form \( \nu_{0,\sigma} \) are called class one representations since they con-
tain the trivial representation when restricted to \( K \).

(B). The representations \( \pi_{p,q}^+, p = \frac{1}{2}, 1, \ldots; q = p, p - 1, \ldots, 1 \) or \( \frac{1}{2} \). These are the represen-
tations arising from Figure 2.

(C). The representations \( \pi_{p,q}^-, p = \frac{1}{2}, 1, \ldots; q = p, p - 1, \ldots, 1 \) or \( \frac{1}{2} \). These are the represen-
tations arising from Figure 3.

The collection \( \hat{G}_d = \{ \pi_{p,q}^\pm, q \neq \frac{1}{2} \} \) is called the discrete series of \( G \). The representa-
tions in \( \hat{G}_d \) are the only irreducible unitary representations of \( G \) which are square integrable (in fact they are integrable for \( q > \frac{3}{2} \)). Global realizations of these representations are given in [8].

(D). The representations \( \pi_{p,0}, p = 1, 2, \ldots \). These are the representations arising from Figure 4. We denote this collection by \( \hat{G}_e \).

The above representations are single- or double-valued representations of \( G \) according to whether \( p = 0, 1, 2, \ldots \) or not.

Thus we may write \( \hat{G} = \hat{G}_i \cup \hat{G}_r \cup \hat{G}_c \cup \hat{G}_d \cup \hat{G}_e \cup \{ I \} \) where \( I \) is the trivial representa-
tion of \( G \). Plancherel measure \( \mu_G \) on \( \hat{G} \) is concentrated in \( \hat{G}_i \cup \hat{G}_r \cup \hat{G}_d \) (see [10, Vol. II]). On \( \hat{G}_i \cup \hat{G}_r \), it is a continuous measure (see [4]) while \( \mu_G(\pi) > 0 \) for each \( \pi \in \hat{G}_d \).

3. Restrictions to \( MA \). In this section we determine the restriction of each
\( \pi \in \hat{G}_p \cup \hat{G}_d \) to the subgroup \( C = MA \). For \( \pi \in \hat{G}_p \), \( (\pi)_C \) was determined in [7] by using Mackey’s subgroup theorem. We summarize these results in Theorem 1. For \( \pi \in \hat{G}_d \), we find \( (\pi)_C \) by working on the Lie algebra \( g \) of \( G \) and using Dixmier’s realizations of these representations. Throughout this paper we denote by \( ^AR \) the regular representation of \( A \) acting on \( L^2(A) \).
Theorem 1. Let $\rho = \pi(n, iy) \in \hat{G}_p$. Then

$$(\rho)_C \simeq \begin{cases} 
\{\sigma^0 \oplus 3\sigma^1 \oplus \cdots \oplus (2n + 1)(\sigma^n \oplus \sigma^{n+1} \oplus \cdots)\} \times A^R & \text{if } n = 0, 1, \ldots, \\
\{2^{1/2} \sigma^{3/2} \oplus \cdots \oplus (2n + 1)(\sigma^n \oplus \sigma^{n+1} \oplus \cdots)\} \times A^R & \text{if } n = \frac{1}{2}, \frac{3}{2}, \ldots.
\end{cases}$$

\textbf{Proof.} One easily computes that the actions of $M$ and $A$ on $V$ are given by:

$m_{u,v_x}m_{u}^{-1} = v_{ux}$ and $a_{x}v_{x}a_{x}^{-1} = v_{e_x}$, i.e., $MA$ acts on $V$ by rotations and dilations. Thus there are just two orbits in $V$ under this action—the zero orbit and everything else. Taking $v_x$ with $x = i$ as a representative of the nonzero orbit, one easily computes that the stability group corresponding to this orbit is

$$M_0 = \left\{ \left( \begin{array}{cc} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{array} \right) : \theta \in \mathbb{R} \right\}.$$

Writing $\lambda_n(t_\theta) = (e^{i\theta})^{2n}$, we have that $M_0 = \{\lambda_n: n = 0, \pm \frac{1}{2}, \pm 1, \ldots\}$. One sees easily, using the isomorphism $M \simeq SU(2)$ and the realization of $\sigma^k \in \hat{M}$ given on p. 110 of [9], that

$$(\sigma^k)_{M_0} \simeq \lambda_{-k} \oplus \lambda_{-k+1} \oplus \cdots \oplus \lambda_k \quad \text{for } k = 0, \pm \frac{1}{2}, \pm 1, \ldots.$$

According to the results of [7], $S = \{v_0, v_1\}$ is up to a set of measure zero, a cross-section for $V/MA$ as well as for $P \setminus G/MA$ and $(\rho)_C \simeq \text{Ind}_{M_0}(\sigma^n)_{M_0} \simeq \text{Ind}_{M_0}(\sigma^n)_{M_0} \times A^R$. Theorem 1 now follows by using compact reciprocity and the fact that $(\sigma^n)_{M_0} \simeq \lambda_{-n} \oplus \cdots \oplus \lambda_n$.

We now turn our attention to $\hat{G}_d$ (we will also include the representations in $\hat{G}_r$ ($ q = \frac{1}{2}$ since our arguments hold for these representations as well). We first note that for $\rho \in \hat{G}_d$, the results of [7], combined with our knowledge of Plancherel measure on $\hat{C} = \hat{M} \times \hat{A}$, imply that

$$(\rho)_C \simeq \bigoplus n(\sigma, \rho) \sigma \times \int_\hat{M} \tau d\mu_A(\tau) \simeq \bigoplus n(\sigma, \rho) \sigma \times A^R$$

and so the fact that $A^R$ always occurs in $(\rho)_C$ for $\rho \in \hat{G}_d$ will come as no surprise in our next theorem. The main task will be to determine the multiplicities $n(\sigma, \rho) \in \{0, 1, \ldots, \infty\}$.

Theorem 2. For $n = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and $q = n, n - 1, \ldots, 1 \text{ or } \frac{1}{2}$,

$$\left(\pi_{n,q}^\pm\right)_C \simeq \{\sigma^q \oplus 2\sigma^{q+1} \oplus \cdots \oplus (n - q + 1)(\sigma^n \oplus \sigma^{n+1} \oplus \cdots)\} \times A^R.$$ 

\textbf{Proof.} We will consider the case of $\pi_{n,q}^-$. The case $\pi_{n,q}^+$ is similar. If we let $\rho = \pi_{n,q}$, then $H_\rho = \bigoplus_\Gamma V^{k,k}$ (the Hilbert space direct sum) where $\Gamma$ has the form indicated in Figure 3, §2. We will use the realization of $\rho$ as a representation of $q$ given in [2] acting on $H = \Sigma_\Gamma V^{k,k'}$ (the algebraic sum). We begin by using our knowledge of $(\sigma^{k,k'})_M$ to decompose each $V^{k,k'}$ into $M$-invariant subspaces. Since $(\sigma^{k,k'})_M \simeq \sigma^k \otimes \sigma^{k'} \simeq \bigoplus_{|k-k'|} \sigma^j$ we know that, under the change of basis (using the Clebsch-Gordan coefficients) described in [9, pp. 174–194], $V^{k,k'} = V^{k-k'} \oplus \cdots \oplus V^{k+k'}$ where $(\rho)_M$ acts on each $V^j$ as $\sigma^j$. Fortunately we do not need to
describe this change of basis (via the Clebsch-Gordan coefficients on each $V^{k,k'}$) explicitly. We only need to know that such a change exists and that these new basis elements can be chosen so as to transform under $(\rho)_n$ according to the formulas on p. 13 of [2]—in particular, each new basis element in $V^j$ is an eigenvector of $\rho(X)$ for $X = \frac{i}{2}(L - L')$, the eigenvalues running from $i(-j), i(-j + 1), \ldots, i(j)$.

Next we rearrange the spaces $V^{k,k'}$ appearing in Dixmier's description of $(\rho)_K$ into $n - q + 1$ columns $C_q, C_{q+1}, \ldots, C_n$ where $C_j = \{V^{k,k'}: k - k' = j\}$ for $j = q, q + 1, \ldots, n$. So in our new basis, we may view $H$ as being the algebraic sum of the following spaces (a box with a $j$ in it denotes a copy of $V^j$):

We note that under this arrangement, the "closest neighbors" of $V^{k,k'}$ are the four (or less in certain cases) horizontal and vertical neighbors of $V^{k,k'}$ with $V^{k+1/2,k'+1/2}, V^{k-1/2,k'-1/2}$ lying above and below $V^{k,k'}$, respectively, and $V^{k+1/2,k'-1/2}, V^{k-1/2,k'+1/2}$ lying to the right and left of $V^{k,k'}$, respectively. We will denote by $[p]^{j,k}$ the copy of $V^p$ in column $j$ at level $k$. We note that $[p]$'s occur in the first $p - q + 1$ columns for $q < p < n$, in each of the $n - q + 1$ columns for

We note that under this arrangement, the “closest neighbors” of $V^{k,k'}$ are the four (or less in certain cases) horizontal and vertical neighbors of $V^{k,k'}$ with $V^{k+1/2,k'+1/2}, V^{k-1/2,k'-1/2}$ lying above and below $V^{k,k'}$, respectively, and $V^{k+1/2,k'-1/2}, V^{k-1/2,k'+1/2}$ lying to the right and left of $V^{k,k'}$, respectively. We will denote by $[p]^{j,k}$ the copy of $V^p$ in column $j$ at level $k$. We note that $[p]$'s occur in the first $p - q + 1$ columns for $q < p < n$, in each of the $n - q + 1$ columns for

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\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5}
\caption{Figure 5}
\end{figure}
$p > n$, and that there are no $[p]$'s for $p < q$ or $p - q \equiv \frac{1}{2} \pmod{1}$. This is essentially where the multiplicities alluded to in our theorem come from.

We now investigate the action of $H = \frac{1}{2}(X_\gamma + X_{-\gamma})$, as given on p. 15 of [2], on these basis vectors. In light of the $KAK$ decomposition, the fact that $\rho$ is an irreducible representation of $G$ (recall that $A = \{\exp(\alpha t) : \alpha \in \mathbb{R}\}$), and the fact that $K$ leaves each connected set of squares invariant, we expect that the $H$-action will be somewhat complicated. We do note the following:

1. If $f \in H$ is a (finite) linear combination of basis elements of the form $f_{\kappa'}^{k,k'}$ (in the "old" basis) and we select a nonzero component of $f$ of the form $cf_{\kappa'}^{k,k'}$ with $k - k'$ minimal and then $k + k'$ maximal (i.e., as far left and up on Figure 5 as possible), then $\rho(H)(cf_{\kappa'}^{k,k'})$ will always have a nonzero component in $V^{k+1/2,k'+1/2}$ which cannot be cancelled out by any of the other components of $\rho(H)f$. So $\rho(H)f \neq 0$ whenever $0 \neq f \in H$ and hence the operator $\rho(H)$ will always be 1-1 on $H$.

2. If we let $\rho^+(H)f_{\kappa'}^{k,k'}$ denote the components of $\rho(H)f_{\kappa'}^{k,k'}$ which lie in $V^{k+1/2,k'+1/2} \oplus V^{k+1/2,k'-1/2}$ and $\rho^-(H)f_{\kappa'}^{k,k'}$ denote the components of $\rho(H)f_{\kappa'}^{k,k'}$ which lie in $V^{k-1/2,k'+1/2} \oplus V^{k-1/2,k'-1/2}$, then $\rho(H) = \rho^+(H) + \rho^-(H)$. The same argument as in (1) shows that $\rho^+(H)$ is also 1-1 on $H$. One may have $\rho^-(H)f = 0$ for $0 \neq f \in H$ (as we shall see).

3. For $X \in m$, $[X, H] = 0$ and so $\rho(X)\rho(H) = \rho(H)\rho(X)$ on $H$. From this one easily sees that $\rho(X)\rho^\pm(H)f_{\kappa'}^{k,k'} = \rho^\pm(H)\rho(X)f_{\kappa'}^{k,k'}$ for each $f_{\kappa'}^{k,k'} \in H$ and so both $\rho^+(H)$ and $\rho^-(H)$ commute with the action of $(\rho)_m$ as well.

4. If $W \subseteq H$ is a direct sum of a finite number of distinct copies of $[p]$ for a fixed $p \in \{q, q + 1, \ldots \}$ and $\tilde{W} = \rho(H)W = \{\rho(H)w : w \in W\}$, then $\tilde{W}$ is invariant under $(\rho)_m$ from (3) and in fact the action of $(\rho)_m$ on $\tilde{W}$ is equivalent to that of $(\rho)_m$ on $W [\rho(H)$ is the bijection from $W$ to $\tilde{W}$ that intertwines them]. Similarly, the action of $(\rho)_m$ on such a $W$ is equivalent to that of $(\rho)_m$ on $\rho^+(H)W$. The same is true for such a $W$ when the action of $\rho^-(H)$ is 1-1.

5. Applying (3) with $X = j(L - L') \in m$, we see that if $v \in H$ is an eigenvector of $\rho(X)$ with eigenvalue $c$, then $\rho(H)v$, $\rho^+(H)v$, and $\rho^-(H)v$ (whenever $\rho^-(H)v \neq 0$) will also be eigenvectors of $\rho(X)$ with eigenvalue $c$.

6. For $p \in \{q, q + 1, \ldots \}$ we know that $\rho(H)[p]^j,k \subseteq V^{j,k+1} \oplus V^{j+1,k} \oplus V^{j-1,k} \oplus V^{j,k-1}$. From (4) we know that $(\rho)_m$ acts on $\rho(H)[p]^j,k$ as $\sigma^p$, and since $(\rho)_m$ also leaves each of the spaces $V^{j,k+1}, V^{j+1,k}, V^{j-1,k}, V^{j,k-1}$ invariant, we must have

$$\rho(H)[p]^j,k \subseteq [p]^{j+1,k} \oplus [p]^{j+1,k} \oplus [p]^{-1,k} \oplus [p]^{j-1,k} \oplus [p]^{j,k-1},$$

i.e., for each $f \in [p]^j,k$, $\rho(H)f$ is a linear combination of vectors in the closest neighboring $[p]$'s of $[p]^j,k$. From (5) we also know that if $f \in [p]^j,k$ is an eigenvector of $\frac{1}{2}(L - L')$ with eigenvalue $c$, then $\rho(H)f$ will be a linear combination of eigenvectors in each of the closest neighboring $[p]$'s of $[p]^j,k$ with the same eigenvalue $c$. Similarly, $\rho^+(H)[p]^j,k$ is contained in $[p]^{j,k+1} \oplus [p]^{j+1,k}$, and if $f \in [p]^j,k$ is an eigenvector of $\frac{1}{2}(L - L')$ with eigenvalue $c$, each of the components of $\rho^+(H)f$ in $[p]^{j,k+1}, [p]^{j+1,k}$ will also be an eigenvector of $\frac{1}{2}(L - L')$ with eigenvalue $c$. 

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(7) Since \( \rho(H) \) is hermitian, we have that \( \langle \rho(H)f, g \rangle = -\langle f, \rho(H)g \rangle \) for \( f, g \in H \). A routine calculation (first on basis vectors of the form \( f_{\alpha_1, \alpha_2}^{k_1, k_2} \) and then by linearity to all of \( H \)) shows that \( \langle \rho^+(H)f, g \rangle = -\langle f, \rho^-(H)g \rangle \) for \( f, g \in H \).

(8) If for a fixed \( f \in H \) we let \( e_n = \rho^n(H)f \) for \( n = 0, 1, \ldots \), \( (e_n) = \mathcal{C}e_n^* \), \( n = 0, 1, \ldots \), and \( H_f = \sum_{n=0}^{\infty} (e_n) \), then \( H_f \) is invariant under the action of \( \alpha \) and \( \rho(H)e_n = e_{n+1} \). We claim that the action of \( (\rho)_n \) on \( H_f \) is equivalent to the regular representation of \( \alpha \). To see this we first note that, according to the results of [7, p. 202], the restriction to \( A \) of the discrete series representation \( D_1 \) of \( \text{SL}(2, \mathbb{R}) \) is unitarily equivalent to \( \mathcal{A}R \). The action of \( D_1 \) as a representation of the Lie algebra of \( \text{SL}(2, \mathbb{R}) \) is described in [5, p. 119] \( (D_1 \simeq d\pi_1) \). \( D_1 \) acts on \( H' = \sum_{n \geq 2, n \equiv 0 (\text{mod} 2)} (\phi_n), \ H = \frac{1}{2} (E^+ + E^-), \ \alpha = \{\exp(iH); \ t \in \mathbb{R}\}, \) and \( D_1(H)\phi_n = \frac{1}{2} \{- (n - 2)\phi_{n-2} + (n + 2)\phi_{n+2}\} \). Let \( \tilde{e}_n = D_1^*(H)\phi_2 \) for \( n = 0, 1, \ldots \). Then \( H' = \sum_{n=0}^{\infty} (\tilde{e}_n) \) and \( D_1(H)\tilde{e}_n = \tilde{e}_{n+1} \). Certainly the action of \( (D_1)_n \) on \( H' \) (which is equivalent to the regular representation of \( \alpha \)) is equivalent to the action of \( (\rho)_n \) on \( H_f \) and so our claim follows.

We now fix \( p \in \{q, q + 1, \ldots \} \). Then \( [p] \) will appear for the first time in column \( q \) at level

\[
s = \begin{cases} 
1 & \text{for } q < p < n, \\
p - n + 1 & \text{for } p > n.
\end{cases}
\]

We set \( D_1 = [p]^{q, q} \). We let \( D_2 \) denote the direct sum of all \([p]\)'s which appear in the main diagonal above \( D_1 \), \( D_3 \) denote the direct sum of all \([p]\)'s which appear in the main diagonal above \( D_2 \), etc. So for \( p > n \),

\[
D_2 = [p]^{q, q-n+2} \oplus [p]^{q+1, q-n+1},
\]

\[
D_3 = [p]^{q, q-n+3} \oplus [p]^{q+1, q-n+2} \oplus [p]^{q+2, q-n+1}, \quad \text{for } k < n - q + 1,
\]

\[
D_k = [p]^{q, q-n+k} \oplus [p]^{q+1, q-n+k-1} \oplus \cdots \oplus [p]^{q+k-1, q-n+1},
\]

\[
D_k = [p]^{q, q-n+k} \oplus [p]^{q+1, q-n+k-1} \oplus \cdots \oplus [p]^{q+k-1, q-n+1},
\]

for \( k > n - q + 1 \).

For \( q < p < n \), \([p]\)'s appear only in the first \( p - q + 1 \) columns and so for \( k < p - q + 1 \),

\[
D_k = [p]^{q, q-k} \oplus [p]^{q+1, q-k-1} \oplus \cdots \oplus [p]^{q+k-1, 1},
\]

while for \( k > p - q + 1 \),

\[
D_k = [p]^{q, q-k} \oplus \cdots \oplus [p]^{q, 1} \oplus [p]^{q, k-(p-q)}.
\]

We note that for \( q < p < n \), \( D_1 \) has one copy of \([p]\), \( D_2 \) has (possibly) two copies of \([p]\), and so forth until we get to \( D_{p-q+1} \) which has \( p - q + 1 \) copies of \([p]\); for \( k > p - q + 1 \), \( D_k \) will always have \( p - q + 1 \) \((n - q + 1)\) copies of \([p]\). For \( p > n \), \( D_1 \) starts with one copy of \([p]\), \( D_2 \) has two copies, and so forth until we get to \( D_{n-q+1} \) which has \( n - q + 1 \) copies of \([p]\); for \( k > n - q + 1 \), \( D_k \) will always
have \( n - q + 1 \) copies of \([p]\). So for \( q < p < n \), we have

\[
\dim(D_{k+1}) = \dim(D_k) + (2p + 1) \quad \text{for } k < p - q + 1 \text{ and } \\
\dim(D_{k+1}) = \dim(D_k) \quad \text{for } k \geq p - q + 1.
\]

while for \( p \geq n \), we have

\[
\dim(D_{k+1}) = \dim(D_k) + (2p + 1) \quad \text{for } k < n - q + 1 \text{ and } \\
\dim(D_{k+1}) = \dim(D_k) \quad \text{for } k \geq n - q + 1.
\]

We also note that for \( k > 2 \), \( \rho(H)D_k \subseteq D_{k-1} \oplus D_k \) with \( \rho^{-}(H)D_k \subseteq D_{k-1} \) and \( \rho^{+}(H)D_k \subseteq D_{k+1} \) while \( \rho(H)D_1 = \rho^{+}(H)D_1 \subseteq D_2 \).

We now establish the inductive step of the procedure we shall use for obtaining a copy of \( m_p \sigma^p \times A^R \) in \((\rho)_c\) where

\[
m_p = \begin{cases} 
p - q + 1 & \text{for } q < p < n, \\
n - q + 1 & \text{for } p \geq n.
\end{cases}
\]

**Claim 1.** If \( D_k = W^1_k \oplus \cdots \oplus W^j_k \) where each \( W^i_k \), \( i = 1, \ldots, j \), is an invariant subspace of \( D_k \) on which \((\rho)_m\) acts as \( \sigma^p \) and we set \( W^i_{k+1} = \rho^+(H)W^i_k \) for \( i = 1, \ldots, j \), then:

(i) If \( \dim(D_{k+1}) = \dim(D_k) \), we have that \( D_{k+1} = W^1_{k+1} \oplus \cdots \oplus W^j_{k+1} \) where \((\rho)_m\) acts on each \( W^i_{k+1} \) as \( \sigma^p \).

(ii) If \( \dim(D_{k+1}) = \dim(D_k) + (2p + 1) \), we have that \( D_{k+1} = W^1_{k+1} \oplus \cdots \oplus W^i_{k+1} \oplus W^{i+1}_{k+1} \) where \( W^{i+1}_{k+1} \) is the orthogonal complement in \( D_{k+1} \) of \( \rho^+(H)D_k \) and \((\rho)_m\) acts on each \( W^i_{k+1} \) as \( \sigma^p \) for \( i = 1, \ldots, j + 1 \).

**Proof of Claim 1.** (i) We know from note (2) that \( \rho^+(H)D_k \) will be a subspace of \( D_{k+1} \) with the same dimension as \( D_k \) and so, in this case, \( \rho^+(H)D_k = D_{k+1} \). From note (3), we have that \( \rho^+(H) \) commutes with the action of \( m \) and so each \( W^i_{k+1} \) will be invariant under the action of \((\rho)_m\) and this action will be equivalent to that of \( \sigma^p \).

(ii) As in (i), we have \( \rho^+(H)D_k = W^1_{k+1} \oplus \cdots \oplus W^j_{k+1} \), but in this case, \( \rho^+(H)D_k \) is properly contained in \( D_{k+1} \). Certainly \( D_{k+1} = \rho^+(H)D_k \oplus W^{j+1}_{k+1} \) and the action of \((\rho)_m\) on \( W^i_{k+1} \) is also equivalent to \( \sigma^p \).

**Claim 2.** (i) If \( \dim(D_{k+1}) = \dim(D_k) \) and \( D_k = W^1_k \oplus \cdots \oplus W^j_k \), \( D_{k+1} = W^1_{k+1} \oplus \cdots \oplus W^j_k \) are as in Claim 1, then for each \( i = 1, \ldots, j \), \( \rho^{-}(H)W^i_{k+1} = W^i_k \) while \( \rho^-(H)W^i_{k+1} = 0 \).

(ii) As in (i), \( \rho^{-}(H)W^i_{k+1} = W^i_k \) for \( i = 1, \ldots, j \). Now let \( g \in W^j_{k+1} \) and \( f \in D_k \). From note (7), we have \( \langle f, \rho^{-}(H)g \rangle = -\langle f, \rho^{+}(H)g \rangle \) and so \( \rho^{-}(H)g \neq 0 \) whenever \( f \neq 0 \) in \( W^j_{k+1} \) and \( \rho^{-}(H)W^j_{k+1} = W^j_k \).

**Proof of Claim 2.** (i) Let \( g \in W^j_{k+1} \) with \( g = \rho^{+}(H)f \) for \( f \in W^j_k \). Then by note (7), \( \langle g, g \rangle = \langle \rho^{+}(H)f, g \rangle = -\langle f, \rho^{-}(H)g \rangle \) and so \( \rho^{-}(H)g \neq 0 \) whenever \( f \neq 0 \) in \( W^j_{k+1} \) and \( \rho^{-}(H)W^j_{k+1} = W^j_k \).

(ii) As in (i), \( \rho^{-}(H)W^j_{k+1} = W^j_k \) for \( i = 1, \ldots, j \). Now let \( g \in W^j_{k+1} \) and \( f \in D_k \). From note (7), we have \( \langle f, \rho^{-}(H)g \rangle = -\langle \rho^{+}(H)f, g \rangle = 0 \) by the definition of \( W^j_{k+1} \).

Now for \( p = q, q + 1, \ldots \) we begin with \( D_1 = W^1_1 \) and use induction to decompose each \( D_k \) into subspaces having the properties stated in our claims. So
for \( q < p < n \),
\[
D_k = W_k^1 \oplus \cdots \oplus W_k^k \quad \text{for } k < p - q + 1,
\]
\[
D_k = W_k^1 \oplus \cdots \oplus W_k^{p-q+1} \quad \text{for } k > p - q + 1.
\]

For \( p > n \),
\[
D_k = W_k^1 \oplus \cdots \oplus W_k^k \quad \text{for } k < n - q + 1
\]
\[
D_k = W_k^1 \oplus \cdots \oplus W_k^{n-q+1} \quad \text{for } k > n - q + 1.
\]

For \( q < p < n \), we let
\[
H_p^1 = \sum_{k=1}^{\infty} W_k^1, \quad H_p^2 = \sum_{k=2}^{\infty} W_k^2, \ldots, \quad H_p^{p-q+1} = \sum_{k=p-q+1}^{\infty} W_k^{p-q+1}.
\]

Each of these \( p-q+1 \) subspaces of \( H \) is invariant under the action of \((\rho)_{m\oplus a}\), and by (8), \((\rho)_{m\oplus a}\) acting on each of these spaces is equivalent to \(\sigma^p \times A^R\). So for \( q < p < n \), we see that \(\sigma^p \times A^R\) will occur in \((\rho)_{m\oplus a}\) with multiplicity \( p-q+1 \). For \( p > n \), we let
\[
H_p^1 = \sum_{k=1}^{\infty} W_k^1, \ldots, \quad H_p^{n-q+1} = \sum_{k=n-q+1}^{\infty} W_k^{n-q+1}.
\]

Each of these \( n-q+1 \) subspaces of \( H \) is invariant under the action of \((\rho)_{m\oplus a}\) and equivalent to \(\sigma^p \times A^R\). So for \( p > n \), we see that \(\sigma^p \times A^R\) will occur in \((\rho)_{m\oplus a}\) with multiplicity \( n-q+1 \) and Theorem 2 has been proven.

4. Tensor products of principal series. In this section we combine the results of the previous section with those of [7] to determine the decomposition of the tensor product of two principal series representations of \( G \). If \( \sigma^m, \sigma^n \in \hat{\mathcal{M}} \) with \( m \geq n \), \( \lambda^\mu, \lambda^\nu \in \hat{A} \), then (as described in [7]) we have, after a routine application of Mackey’s tensor product theorem, that \( \pi(m, is) \otimes \pi(n, iy) \simeq \text{Ind}_{\mathcal{L}}^G(\sigma^m \otimes \sigma^n) \times \lambda^{i(s+y)} \). Thus the problem of decomposing the tensor product of two principal series representations reduces to knowing the decomposition \( \sigma^m \otimes \sigma^n \simeq \bigoplus_{k=-n}^{m-n} \sigma^k \) and then the decomposition of \( \text{Ind}_{\mathcal{L}}^G \) for all \( L \in \hat{\mathcal{L}} \). Since inducing from \( C \) is independent of the character on \( A \) (see [7, p. 188]), the latter decomposition is known once one knows how to decompose \( \text{Ind}_{\mathcal{L}}^G \) for \( \mu_C \)-almost all \( L \in \hat{\mathcal{L}} \). By the Mackey-Anh reciprocity theorem, the problem of finding \( \text{Ind}_{\mathcal{L}}^G \) for \( \mu_C \)-almost all \( L \in \hat{\mathcal{L}} \) is equivalent to finding \( (\pi)_C \) for \( \mu_C \)-almost all \( \pi \in \hat{G} \), i.e., for almost every principal series representation and every discrete series representation of \( G \).

**Theorem 3.** Let \( \sigma^n \in \hat{\mathcal{M}} \) and \( \tau \in \hat{A} \). Then \( \text{Ind}_{\mathcal{L}}^G \sigma^n \otimes \tau \simeq T_c \oplus T_d \) where \( T_c \) is a continuous direct integral with respect to Plancherel measure on \( \hat{G} \) of representations \( \pi(k, s) \) from the principal series of \( G \) with \( k+n \equiv 0 \pmod{1} \) and \( T_d \) is a discrete direct sum of discrete series representations \( \pi_{k,q}^+ \) with \( k+n \equiv 0 \pmod{1} \) and 1 or \( \frac{1}{2} < q < \min\{k, n\} \) (and so \( T_d = 0 \) for \( n = 0, \frac{1}{2} \)). For 0 or \( \frac{1}{2} < k < n \), the representation \( \pi(k, s) \) (\( s > 0 \) for \( k = 0, 1, \ldots \) and \( s > 0 \) for \( k = \frac{1}{2}, \frac{3}{2}, \ldots \)) occurs in \( T_c \) with multiplicity \( 2k+1 \) while the representation \( \pi_{k,q}^+ \oplus \pi_{k,q}^- \) or \( 3/2 \) or \( 1 < q < k \), occurs in \( T_d \) with multiplicity \( k+q+1 \). For \( k > n \), the representation \( \pi(k, s) \) occurs in \( T_c \) with multiplicity \( 2n+1 \) while the representation \( \pi_{k,q}^+ \oplus \pi_{k,q}^- \) or \( 3/2 < q < n \), occurs in \( T_d \) with multiplicity \( n+q+1 \). For \( k > n \) and \( n < q < k \), the multiplicity of \( \pi_{k,q}^+ \oplus \pi_{k,q}^- \) in \( T_d \) is zero.
Proof. This is immediate from the Mackey-Anh reciprocity theorem in conjunction with Theorems 1 and 2.

Remarks. (1) We have found tables of the following form helpful in generating the multiplicities which appear in Theorem 3. In these tables, the index \( k \) appears between the double lines while the index \( q \) appears in the left-hand column below these double lines. The numbers above the double lines indicate multiplicities of principal series representations occurring in \( T_c \) while those below the double lines indicate multiplicities of the sum \( \pi_{k,q}^+ \oplus \pi_{k,q}^- \) in \( T_d \). So, for example, when \( n = 0, 1, \ldots \) we would write

\[
\begin{array}{cccccccc}
1 & 3 & 5 & 7 & \ldots & 2n-1 & 2n+1 & 2n+1 & \ldots \\
q \\
1 & 2 & 3 & \ldots & n-1 & n & n & \ldots \\
2 & 1 & 2 & \ldots & n-2 & n-1 & n-1 & \ldots \\
3 & 1 & \ldots & n-3 & n-2 & n-2 & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
n-1 & 1 & 2 & 2 & \ldots \\
n & 1 & 1 & \ldots \\
\end{array}
\]

(2) Since \( \pi(n, s) \otimes \pi(0, t) \simeq \text{Ind}_G^{\pi} \sigma^n \times \lambda^{i(s+t)} \), Theorem 3 already provides a decomposition for this tensor product. We note that in contrast to the \( SL(2, \mathbb{R}) \) case (see [7, p. 204]), the tensor product of two principal series representations need not contain discrete series representations in its decomposition (i.e., \( T_d = 0 \))–as in the case for \( n = 0 \) and \( n = \frac{1}{2} \). For \( n = 0 \), each principal series representation \( \pi(k, s) \) for \( k = 0, 1, \ldots \) and \( s > 0 \) will occur with multiplicity 1 while for \( n = \frac{1}{2} \), each \( \pi(k, s) \) for \( k = \frac{1}{2}, \frac{3}{2}, \ldots \) and \( s > 0 \) will occur with multiplicity 2.

Theorem 4. Let \( \pi(m, \cdot) \) and \( \pi(n, \cdot) \) be two principal series representations of \( G = \text{Spin}(4, 1) \) with \( m > n \). Then \( \pi(m, \cdot) \otimes \pi(n, \cdot) \simeq T_c \oplus T_d \) where \( T_c \) is a continuous direct integral with respect to Plancherel measure on \( \hat{G} \) of representations \( \pi(k,s) \) from the principal series of \( G \) with \( k + m + n \equiv 0 \) (mod 1) and \( T_d \) is a discrete direct sum of discrete series representations \( \pi_{k,q}^+ \) with \( k + m + n \equiv 0 \) (mod 1) and \( 1 \) or \( 3/2 < q < \min\{k,m+n\} \) (and so \( T_d = 0 \) for \( m = n = 0 \) or \( m = \frac{1}{2}, n = 0 \)). The representation \( \pi(k,s) \) occurs in \( T_c \) with multiplicity

\[
(2k + 1)(2n + 1) \quad \text{for } 0 \text{ or } \frac{1}{2} < k < m - n,
\]

\[
(2k + 1)(2n + 1) - \sum_{i=1}^{h} 2i \quad \text{for } k = m - n + h, h = 1, 2, \ldots, 2n,
\]

\[
(2m + 1)(2n + 1) \quad \text{for } k > m + n.
\]
The representation \( \pi_{k,q}^+ \oplus \pi_{k,q}^- \) occurs in \( T_d \) with multiplicity

\[
\begin{align*}
(k - q + 1)(2n + 1) & \quad \text{for } 0 \leq k \leq m - n, 1 \text{ or } 3/2 < q < k, \\
(k - q + 1)(2n + 1) - \sum_{i=1}^{h} i & \quad \text{for } k = m - n + h, h = 1, 2, \ldots, 2n, 1 \text{ or } 3/2 < q < m - n, \\
(k - q + 1)(2n + 1 - j) - \sum_{i=1}^{h-j} i & \quad \text{for } k = m - n + h, h = 1, 2, \ldots, 2n, q = m - n + j, j = 1, \ldots, h, \\
(m - q + 1)(2n + 1) & \quad \text{for } k > m + n, 1 \text{ or } 3/2 < q < m - n, \\
(2n - j)^2 - \sum_{i=1}^{2n-j} i & \quad \text{for } k > m + n, q = m - n + j, j = 1, \ldots, 2n, \\
0 & \quad \text{for } k > m + n, m + n < q < k.
\end{align*}
\]

Proof. We have that \( \pi(m, \cdot) \otimes \pi(n, \cdot) \simeq \bigoplus_{m-n} \text{Ind}^d_{\gamma} \sigma^p \times (\cdot) \) where each of the \( 2n + 1 \) representations in the latter sum may be decomposed by using Theorem 3. Note that multiplicities of principal and discrete series representations appearing in a given \( \text{Ind}^d_{\gamma} \sigma^p \times (\cdot) \) increase (by 2 for principal series representations and 1 for discrete series representations) until \( k = p \) and that for \( k > p \) they are “at rest.” So, for \( 0 \leq k \leq m - n, \pi(k, s) \) will occur in the decomposition of \( \pi(m, \cdot) \otimes \pi(n, \cdot) \) with multiplicity \((2n + 1)(2k + 1)\). For \( m - n + 1 < k < m + n, \) the multiplicities of \( \pi(k, s) \) in \( k - (m - n) \) of the representations \( \text{Ind}^d_{\gamma} \sigma^p \times (\cdot), p = m - n, \ldots, m + n, \) are at rest and so the actual multiplicity of \( \pi(k, s) \) in \( T_d \) will be \((2n + 1)(2k + 1) - \Sigma_{i=1}^{2n} 2i = 2n(2n + 1)\) and so the actual multiplicity will be \((2(m + n) + 1)(2n + 1) - 2n(2n + 1) = (2m + 1)(2n + 1)\). The multiplicity of \( \pi(k, s) \) in our decomposition will then be \((2m + 1)(2n + 1)\) for all \( k > m + n \). Similar reasoning applies for the multiplicities of discrete series representations appearing in \( T_d \) except we must also account for the fact that when \( k = m - n + h, h = 1, \ldots, 2n, \) and \( q > m - n \) with \( q = m - n + j, j = 1, \ldots, h, \) there are only \( 2n + 1 - j \) contributions with \( h - j \) of them “at rest.”

Remarks. (1) One can use tables similar to those described in Remark 1 of Theorem 3 to generate the above multiplicities. For example, in the case of \( m + n \equiv 0 \pmod{1} \) we would write the integers 0, 1, \ldots, \( m + n \) in between a pair of lines to denote the \( k \)-index of a representation appearing in our decomposition. Next we would write the integers 1, \ldots, \( m + n \) in a column to the left and below the above row to denote the \( q \)-index of discrete series representations appearing in this decomposition. The multiplicities for the \( \pi(k, s) \)'s for \( 0 < k < m + n \) are then placed above their appropriate \( k \)-indices. They are easy to generate from the theorem. Now if we denote the multiplicity of \( \pi_{k,q}^+ \oplus \pi_{k,q}^- \) in \( T_d \) by \( m_{k,q} \), then \( m_{q,q} = 2n + 1 \) for \( 1 < q < m - n \), while \( m_{q,q} = 2n + 1 - j \) for \( q = m - n + j, j = 1, \ldots, 2n \). These numbers are now placed along the main diagonal in our table. The remaining \( m_{k,q} \)'s for \( k > q \) may now be filled in from our knowledge of
the $m_{q,q}$'s by noting (after checking each of the cases in Theorem 4) that $m_{k+1,q} - m_{k,q} = m_{k+1,k+1}$ for each $k$, and so $m_{q+j,q} = m_{q,q} + m_{q+1,q+1} + \ldots + m_{q+j,q+j}$ for $1 \leq j \leq m + n - q$. The multiplicities $m_{k,q}$ may now be filled in one row at a time.

For example, $\pi(5, \cdot) \otimes \pi(2, \cdot)$ might be exhibited as

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(2) The above decomposition for $m = n = 0$ has also been obtained in [3, p. 202].

**Bibliography**


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