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A SIMPLER APPROXIMATION TO QX

BY

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ABSTRACT. McDuff's construction $C^{\pm}(M)$ of a space of positive and negative particles is modified to a space $C^{\pm}(R^{\infty}, X)$, which is weakly homotopy equivalent to $\Omega^{\infty}\Sigma^{\infty}X$, for a locally equi-connected, nondegenerately based space X.

Introduction. Over the past few years, attempts have been made to approximate function spaces with more well-behaved combinatorial models. The greatest immediate success occurred in the case of the n-fold loop space $\Omega^n\Sigma^n X$, where X is connected and nondegenerately based. Here J. P. May **[6]**, G. Segal **[8]**, and others found relatively simple constructions which were homotopy equivalent to the desired space. These are all equivalent to the space $C(R^n, X)$ of finite point sets in R^n parametrized by X. It was considered a drawback that the connectivity is actually necessary; for nonconnected X (e.g., S^0) $\pi_0(C(R^n, X))$ is not even a group anymore. However, in **[8]** Segal managed to prove that if one adjoins "homotopy inverses" to $C(R^n, X)$, one does indeed obtain the homotopy type of $\Omega^n\Sigma^n X$. In what was apparently an effort to do so on the space level, D. McDuff studied a space of "positive and negative particles" with cancellation. She found instead that the resulting space has the wrong homotopy type, and presents her findings about this space and $C(M)$ in **[7]**.

In 1978, the present author together with S. Waner began to try making up an actual homotopical approximation to $\Omega^n S^n$, and came up with a functor $\tilde{C}_n$ and in particular a space $\tilde{C}_n(S^0)$ which may be described as the space of "signed subcubes of R^n modulo mergings along a single coordinate". (I am grateful to Nigel Ray for this concise description.) This does have the proper homotopy type, but both the construction and the proof are far more delicate than we had suspected at first, and the result in **[1]** is a little hard to use.

However, while examining the possibility of a homotopical splitting of $\tilde{C}_n X$ along the lines of **[2]**, the present author noticed that the case $n = \infty$ of $C^{\pm}(R^n)$ actually does yield $\Omega^{\infty}S^{\infty}$. Since this is a basically simpler construction than $\tilde{C}_{\infty}(S^0)$, it was desirable to know if this could be extended to approximate $\Omega^{\infty}\Sigma^{\infty}X$ for nonconnected X. This is the subject of the present work.

1. Construction of $C^{\pm}(M, X)$. In general, we have a nondegenerately based space and an unbased space M, and topologize a "space of configurations of positive and negative particles in M, parametrized by X" so that two particles may annihilate

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each other if they have the same "X-parameter" and opposite signs. Precisely, let

$$C(M, X) = \Big(\coprod_{j>0} F(M,j) \times_{\Sigma_j} X^j\Big)/\sim$$

where $F(M,j)$ is the subspace of M^j consisting of j-tuples of distinct points in M, Σ_j acts by permuting factors in X^j and points in $F(M,j)$, and the equivalence relation $\sim$ is the one generated by

$$(\langle m_1, \ldots, m_j\rangle, x_1, \ldots, x_j) \sim (\langle m_1, \ldots, m_{j-1}\rangle, x_1, \ldots, x_{j-1})$$

whenever x_j is the basepoint $*$ of X. (See [1], [6].)

Denote a typical point of $C(M, X)$ by (A, f), where A is a finite subset of M and $f\colon A \to X$ is a function. Then let $C(M, X)_2$ be the subspace of $C(M, X) \times C(M, X)$ consisting of pairs $((A, f), (B, g))$ such that $f|A \cap B = g|A \cap B$, and define

$$C^{\pm}(M, X) = C(M, X)_2/\approx$$

where $((A, f), (B, g)) \approx ((A', f'), (B', g'))$ if $A - B = A' - B'$, $f|A - B = f'|A' - B'$, $B - A = B' - A'$, and $g|B - A = g'|B' - A'$. We intend to show that $C^{\pm}(R^{\infty}, X)$ is weakly homotopy equivalent to $QX = \Omega^{\infty}\Sigma^{\infty}X$.

The following proposition is what one might expect to be true and is technically significant. Recall that a space is said to be *locally equi-connected* (LEC) if the diagonal map $\Delta\colon X \to X \times X$ is a cofibration (see [3], [4]).

PROPOSITION 1. *Suppose M is LEC, and X and X' are nondegenerately based LEC spaces. Then if $f\colon X \to X'$ is a weak homotopy equivalence, so is $C^{\pm}(\mathrm{id}_M, f)$.*

The proof is deferred to the end.

2. The homotopical approximation. We start off with the necessary lemmas from basic homotopy theory.

LEMMA 2. *If $\{A_n \xrightarrow{f_n} A_{n+1}\}$ and $\{B_n \xrightarrow{g_n} B_{n+1}\}$ are systems of closed inclusions and $\{\alpha_n\colon A_n \to B_n\}$ is a system of homotopy equivalences such that the following diagram is homotopy-commutative*:

$$\begin{array}{ccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \to \cdots & \to A_n & \to \cdots \\
\alpha_1\downarrow & & \alpha_2\downarrow & & \alpha_n\downarrow & \\
B_1 & \xrightarrow{g_1} & B_2 & \to \cdots & \to B_n & \to \cdots
\end{array}$$

then the direct limits $\varinjlim A_n$ and $\varinjlim B_n$ are weakly equivalent.

The proof is trivial by consideration of the "mapping telescope" construction (see [5]) on $\{f_n\}$ and $\{g_n\}$; namely one has a homotopy equivalence $\underrightarrow{\mathrm{Tel}}\, \alpha_n$: $\underrightarrow{\mathrm{Tel}}\, A_n \to \underrightarrow{\mathrm{Tel}}\, B_n$, and these spaces are weakly equivalent to $\varinjlim A_n$ and $\varinjlim B_n$, respectively. □

LEMMA 3. *Suppose we have a commutative diagram*

$$\begin{tikzcd}[column sep=small] A_1 \arrow[rr, "f_1"] \arrow[dr, "\alpha_1"'] & & A_2 \arrow[rr, "f_2"] \arrow[dr, "\alpha_2"'] & & A_3 \arrow[rr] \arrow[dr, "\alpha_3"'] & & \cdots \\ & B_1 \arrow[ur, "\beta_1"'] \arrow[rr, "g_1"'] & & B_2 \arrow[ur, "\beta_2"'] \arrow[rr, "g_2"'] & & B_3 \arrow[r] & \cdots \end{tikzcd}$$

in which the f_i's and g_i's are closed inclusions. Then $\varinjlim A_n$ *is weakly equivalent to* $\varinjlim B_n$.

PROOF. Consider the homotopy commutative diagram:

$$\begin{tikzcd} A_1 \arrow[rr, "f_1"] \arrow[d, "\mathrm{id}"'] & & A_2 \arrow[rr, "f_2"] \arrow[d, "\mathrm{id}"'] & & A_3 \arrow[r] \arrow[d, "\mathrm{id}"'] & \cdots \\ A_1 \arrow[r, "\alpha_1"] & B_1 \arrow[r, "\beta_1"] \arrow[d, "\mathrm{id}"'] & A_2 \arrow[r, "\alpha_2"] & B_2 \arrow[r, "\beta_2"] \arrow[d, "\mathrm{id}"'] & A_3 \arrow[r] & \cdots \\ & B_1 \arrow[rr, "g_1"] & & B_2 \arrow[rr, "g_2"] & & \cdots \end{tikzcd}$$

Lemma 2 shows that $\varinjlim A_n$ is weakly equivalent to $\varinjlim \beta_i\alpha_i$. But $\varinjlim \beta_i\alpha_i = \varinjlim \alpha_{i+1}\beta_i$, which is weakly equivalent to $\varinjlim B_n$. $\square$

We are now ready to prove the main theorem.

THEOREM 4. *If X is LEC and nondegenerately based, then $C^{\pm}(R^{\infty}, X)$ is weakly homotopy equivalent to QX.*

PROOF. In particular, this is true for the case where X has the homotopy type of a CW-complex (see **[4]**).

In **[7]**, McDuff claims that $C^{\pm}(R^n, X)$ is weakly equivalent to

$$\Omega^n(\Sigma^n X \times \Sigma^n X/\Delta\Sigma^n X),$$

where $\Delta\Sigma^n X$ is the diagonal $\{(x, x) | x \in \Sigma^n X\}$ in $\Sigma^n X \times \Sigma^n X$. Denote this loop space by $\Gamma^n X$. The suspension map E: $\Omega^n\Sigma^n X \to \Omega^{n+1}\Sigma^{n+1}X$ induces a map ε: $\Gamma^n X \to \Gamma^{n+1}X$, and the inclusion $R^n \hookrightarrow R^{n+1}$ induces a map $C^{\pm}(R^n, X) \hookrightarrow C^{\pm}(R^{n+1}, X)$. McDuff's approximation is sufficiently natural that the diagram

$$\begin{tikzcd} C^{\pm}(R^n, X) \arrow[r, "\simeq"] \arrow[d, hook] & \Gamma^n X \arrow[d, hook] \\ C^{\pm}(R^{n+1}, X) \arrow[r, "\simeq"] & \Gamma^{n+1} X \end{tikzcd}$$

commutes. Hence $C^{\pm}(R^{\infty}, X) \simeq \varinjlim \Gamma^n X$.

As in **[7]**, one notes that if e: $\Sigma^n X \times \Sigma^n X \to \Omega\Sigma^{n+1}X$ is given by $e(a, b) = Ea + (-Eb)$, then e is null-homotopic when restricted to $\Delta = \Delta\Sigma^n X$. Hence it induces a map

$$\tilde{e}\colon C(\Delta) = (\Sigma^n X \times \Sigma^n X) \cup_i C\Delta \to \Omega\Sigma^{n+1}X$$

where $C\Delta$ is the cone on Δ, and the attaching map i identifies the base of the cone with the subspace Δ. But as $\Sigma^n X$ is a CW-complex, it is LEC (by the result of **[4]**) and the map $C(\Delta) \to \Sigma^n X \times \Sigma^n X/\Delta$ induced by collapsing $C\Delta$ to a point is a homotopy equivalence. Thus there is a unique factorization (up to homotopy) of $\Omega^n e$ as

$$\Omega^n(\Sigma^n X \times \Sigma^n X) \to \Omega^n(\Sigma^n X \times \Sigma^n X/\Delta) \xrightarrow{\gamma_n} \Omega^{n+1}\Sigma^{n+1}X.$$

Let ι_n: $\Omega^n\Sigma^n X \to \Gamma^n X$ be given by $\iota_n(a) = (a, 0)$. Then as in [7], $\gamma_n \circ \iota_n \simeq E$. Further, the composite

$$\Omega^n(\Sigma^n X \times \Sigma^n X) \xrightarrow{i} \Omega^n C(\Delta) \xrightarrow{-\varepsilon + (\iota_{n+1} \circ \gamma_n)} \Gamma^{n+1}X$$

sends (a, b) to $(-Ea + (Ea - Eb), -Eb + 0)$, and this is deformable to $(-Eb, -Eb)$ $= *$ in $\Gamma^{n+1}X$; hence the map $-\varepsilon + \iota_{n+1} \circ \gamma_n$ is null-homotopic, so $\iota_{n+1} \circ \gamma_n \simeq \varepsilon$ and the following diagram commutes:

$$\begin{tikzcd}
\Omega^n\Sigma^n X \arrow[rr, "E"] \arrow[dr, "\iota_n"'] & & \Omega^{n+1}\Sigma^{n+1}X \arrow[rr] \arrow[dr, "\iota_{n+1}"'] & & \cdots \\
& \Gamma^n X \arrow[ur, "\gamma_n"] \arrow[rr, "\varepsilon"'] & & \Gamma^{n+1}X \arrow[r] & \cdots
\end{tikzcd}$$

In view of Lemma 3, the result follows. □

Thus we are done modulo the proof of our first technical result, in the next section.

3. The weak homotopy invariance of $C^{\pm}(M, -)$.

Proof of Proposition 1. Let $F_{i,j} = F_{i,j}C^{\pm}(M, X)$ denote the subspace of points $((A, f), (B, g))$ such that A has cardinality $\leqslant i$ and B has cardinality $\leqslant j$. Let $D_{i,j,k,l}$ refer to the subspace of $F(M, i) \times F(M, j) \times_{(\Sigma_i \times \Sigma_j)} X^i \times X^j$ of pairs $((\langle m_1, \ldots, m_i\rangle, \langle m'_1, \ldots, m'_j\rangle), x_1, \ldots, x_i; x'_1, \ldots, x'_j)$ where $m_k = m'_l$ and $x_k = x'_l$, where $1 \leqslant k \leqslant i$ and $1 \leqslant l \leqslant j$. Finally let σX^m denote the subspace of points $(x_1, \ldots, x_m)$ in X^m such that $x_k = *$ for some k, and let $D_{i,j}$ denote the union of the spaces $D_{i,j,k,l}$ over all k and l.

Then if $i, j > 0$, the following is a pushout diagram:

$$\begin{array}{ccc}
D_{i,j} \cup F(M, i) \times F(M, j) \times_{\Sigma_i \times \Sigma_j} \sigma(X^i \times X^j) & \to & F_{i,j-1} \cup F_{i-1,j} \\
\downarrow & & \downarrow \\
F(M, i) \times F(M, j) \times_{\Sigma_i \times \Sigma_j} (X^i \times X^j) & \to & F_{i,j}
\end{array} \qquad (1)$$

where the vertical maps are inclusions and the horizontal maps result under the identifications from the definition of $C^{\pm}(M, X)$.

Since M and X are LEC, the inclusion of $D_{i,j}$ into the product on the lower left is a cofibration, and since X is nondegenerately based, $\sigma(X^i \times X^j) \hookrightarrow (X^i \times X^j)$ is a $\Sigma_i \times \Sigma_j$-equivariant cofibration; hence both vertical inclusions are cofibrations.

The proof now proceeds by induction. Let

$$F_k C^{\pm}(M, X) = \bigcup_{\min(i,j) < k} F_{i,j}$$

and note that $F_0 = (\bigvee_i F_{i,0}) \vee (\bigvee_j F_{0,j})$; these are just wedges of filtrations of $C(M, X)$, hence if F'_k denotes $F_k C^{\pm}(M, X')$ and f: $X \to X'$ is a weak homotopy equivalence, then so is f_*: $F_0 \to F'_0$. But each F_{k+1} and F'_{k+1} is built up by pushouts of the form (1) from F_k and F'_k, where the map of pushouts is already a weak equivalence on the three corners which form the pushout, by the induction hypothesis. Hence the homotopy invariance of weak equivalences under pushouts by cofibrations [5] allows us to conclude that $F_{i,j} \to F'_{i,j}$ is a weak equivalence. By passage to limits, the proposition follows. $\square$

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