CHARACTERIZATIONS OF THE FISCHER GROUPS. I, II, III

BY

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ABSTRACT. B. Fischer, in his work on finite groups which contain a conjugacy class of 3-transpositions, discovered three new sporadic finite simple groups, usually denoted $M(22)$, $M(23)$ and $M(24)$. In Part I two of these groups, $M(22)$ and $M(23)$, are characterized by the structure of the centralizer of a central involution. In addition, the simple groups $U_6(2)$ (often denoted by $M(21)$) and $PSp(7, 3)$, both of which are closely connected with Fischer’s groups, are characterized by the same method.

The largest of the three Fischer groups $M(24)$ is not simple but contains a simple subgroup $M(24)'$ of index two. In Part II we give a similar characterization by the centralizer of a central involution of $M(24)$ and also a partial characterization of the simple group $M(24)'$.

The purpose of Part III is to complete the characterization of $M(24)'$ by showing that our abstract group $G$ is isomorphic to $M(24)'$. We first prove that $G$ contains a subgroup $X = M(23)$ and then we construct a graph (on the cosets of $X$) which is shown to be isomorphic to the graph for $M(24)$.

PART I

The results proved are:

**Theorem A.** Let $G$ be a finite group, $z$ an involution in $G$ and $H = C_G(z)$. Suppose $J = O_2(H)$ is extra-special of order $2^9$ and $C_H(J) \subseteq J$. Then

(i) if $H/J \cong PSp_4(3)$, $G \cong U_6(2)$ or $G = H \cdot O(G)$;

(ii) if $H/J \cong \text{Aut} PSp_4(3)$, $G$ contains a subgroup $G_0$ of index two, $G_0 \cong U_6(2)$ and $G \subseteq \text{Aut} G_0$ or $G = H \cdot O(G)$.

**Theorem B.** Let $G$ be a finite group, $z$ an involution in $G$ and $H = C_G(z)$. Suppose that $J = O_2(H)$ is the direct product of a group of order two with an extra-special group of order $2^9$ and that $J' = \langle z \rangle$. If $C_H(J) \subseteq J$ and $H/J \cong \text{Aut} PSp_4(3)$, then one of the following holds:

(i) $G = H \cdot O(G)$;

(ii) there is an involution $z_1 \in Z(J) - \langle z \rangle$ with $\langle z_1 \rangle \vartriangleleft G$, $G$ contains a subgroup $G_0$ of index two with $G_0/\langle z_1 \rangle \cong U_6(2)$ and $G \subseteq \text{Aut} G_0$;

(iii) $G \cong M(22)$.

**Theorem C.** Let $G$ be a finite group which possesses an involution $z$ such that $H = C_G(z)$ satisfies:

(i) $J = O_2(H)$ is the direct product of a four group and an extra-special group of order $2^9$, and $C_H(J) \subseteq J$;

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(ii) $H$ contains an element $c$ of order three with $H/J \langle c \rangle \cong \operatorname{Aut} \operatorname{PSp}_4(3)$ and $C_H(c)/\langle c, z \rangle \cong \operatorname{PSp}_4(3)$.

Then either $G = H \cdot O(G)$ or $G \cong M(23)$ or $G$ contains a normal four group $\langle z_1, z_2 \rangle$ with $G/\langle z_1, z_2 \rangle \cong \operatorname{Aut} \operatorname{U}_6(2)$.

Previously, D. Hunt [11], [12] has characterized the groups $M(22)$ and $M(23)$ by the centralizer of an involution which is a 3-transposition. The proofs of Theorems B and C rely on his results as well as Theorem A. In Part II these results are used by the author to give a characterization of the larger group $M(24)$.

By using some of the lemmas in the proofs of Theorems B and C we are able to give a characterization of the simple group $P\Omega(7, 3) = B_3(3)$. (A certain element of order three in $M(23)$ has centralizer isomorphic to $Z_3 \times P\Omega(7, 3)$.)

**Theorem D.** Let $G$ be a finite group, $z$ an involution in $G$ and suppose $H = C_G(z)$ satisfies:

(i) $H$ contains a normal subgroup $K_1 \ast K_2 \times K_3$ of index four with $K_1 \cong K_2 \cong \operatorname{SL}(2, 3)$ and $K_3 \cong \mathfrak{A}_4$;

(ii) $H = (K_1 \ast K_2 \times K_3) \langle t, u \rangle$ where $\langle t, u \rangle \cong E_4$, $K_1 = K_2$, $[K_3, t] = 1$ and $ut$ acts fixed-point-free on a Sylow 3-subgroup of $H$.

Then either $G = H \cdot O(G)$; $O_2(K_3) = \langle z_1, z_2 \rangle \triangleleft G$ with $G/\langle z_1, z_2 \rangle \cong Z_3 \cdot \operatorname{Aut} \operatorname{PSp}_4(3)$; or $G \cong P\Omega(7, 3)$.

1. Notation and preliminary results. As far as notation is concerned, we will in general follow Gorenstein [3]. In addition, we will use:

- $X \ast Y$: the central product of the groups $X, Y$;
- $x \sim_X y$: $x$ is conjugate to $y$ in $X$;
- $x^X = \{ y^{-1}xy \mid y \in X \}$ = conjugacy class of $x$ in $X$;
- $Z_n, D_n$: the cyclic, dihedral groups of order $n$, respectively;
- $E_p^n$: the elementary abelian group of order $p^n$ ($p$ prime);
- $Q_8$: the quaternion group of order 8;
- $\Sigma_n, \mathfrak{A}_n$: the symmetric, alternating groups of degree $n$.

**Proposition 1 (Suzuki [3, pp. 328, 105]).** If $x$ is an involution in the finite group $G$ and $x \not\in O_2(G)$, then $x$ inverts an element of odd order in $G^\#$.

**Proposition 2 (Glauberman [2]).** Let $x$ be an involution in a finite group $G$. If $G \neq C_G(x) \cdot O(G)$ then $x$ is conjugate (in $G$) to an involution in $C_G(x) - \langle x \rangle$.

**Proposition 3 (Thompson [10, Corollary 1]).** Let $S$ be a Sylow 2-subgroup of the finite group $G$, $S_0$ a maximal subgroup of $S$ and $x$ an involution in $S - S_0$. If $x$ is not conjugate (in $G$) to any involution in $S_0$ then $G$ contains a subgroup $G_0$ of index two with $x \in G - G_0$.

**Proposition 4.** Let $G$ be a finite group, $z$ an involution in $G$ and $H = C_G(z)$. Suppose that $P \neq 1$ is a $p$-subgroup of $H$ ($p$ an odd prime) which satisfies $(\ast)$ if $g^{-1}P_g \subseteq H$, $g \in G$, then there exists $h \in H$ so that $g^{-1}P_g = h^{-1}P_h$.

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1 In the case $G = O^2(G)$, Theorem D is a special case of a result of Olsson [21] and Solomon [22].
Then for any involution $t \in C_H(P)$, $t \sim_G z$ if and only if $t \sim_{N(P)} z$. If, in addition, $N_G(P) = N_H(P) \cdot C_G(P)$ then $t \sim_G z$ if and only if $t \sim_C(P) z$.

**Proposition 5.** Let $H$ be a finite group with $J = O_2(H)$ extra-special (i.e. $J \cong D_8 \ast \ldots \ast D_8$ or $D_8 \ast \ldots \ast D_8 \ast Q_8$). If $P \neq 1$ is a $p$-subgroup of $H$ ($p$ an odd prime) then each of $C_J(P)$, $[P, J]$ is extra-special or equal to $Z(J)$.

**Proof.** This follows immediately from $C_J(P)[P, J] = J$ [3, Theorem 5.3.5] and the Three Subgroups Lemma [3, Theorem 2.2.3].

The next result is certainly a consequence of Gorenstein and Harada's main theorem [6]. However we will give a short proof which essentially relies on the original characterizations of the groups involved.

**Proposition 6.** Let $G$ be a finite group which contains an involution $z$. Suppose that $H = C_G(z)$ satisfies:

(i) $O_{2,i}(H) = L_1 \ast L_2$, $L_i \cong SL(2, 3)$, $i = 1, 2$;

(ii) $H/O_{2,i}(H) \cong E_4$;

(iii) $L_i \triangleleft H$, $i = 1, 2$.

Then one of the following holds:

(a) $G = H \cdot O(G)$ or

(b) $G \cong L_4(3)$, $U_4(3)$, Aut $PSp_4(3)$ or Aut $G_2(3)$.

**Proof.** Let $L_i = \langle \alpha_i, \beta_i, \sigma_i \rangle$ where $\langle \alpha_i, \beta_i \rangle \cong Q_8$ and $\sigma_i$ is of order three for $i = 1, 2$, and let $V = \langle u, v, z \rangle$ be a Sylow 2-subgroup of $N_H(\langle \sigma_1, \sigma_2 \rangle)$. Note that $V \cdot L_1 \cdot L_2 = H$ and $V \cap O_2(H) = \langle z \rangle$. From the assumptions listed above we can choose the generators of $H$ in such a way that the following relations hold:

\[
\begin{align*}
\sigma_i^w &= \sigma_2, & \alpha_i^w &= \alpha_2, & \beta_i^w &= \beta_2; \\
\sigma_i^v &= \sigma_i^{-1}, & \alpha_i^v &= \alpha_i^{-1}, & \beta_i^v &= \alpha_i \beta_i, & i = 1, 2.
\end{align*}
\]

Suppose now that (a) does not hold. By Proposition 2, $z \sim_G h$ for some $h \in H - \langle z \rangle$. As $O_2(H) - \langle z \rangle$ has only one class of involutions in $H$ with representative $\alpha_1 \alpha_2$ say, and as $C(\alpha_1 \alpha_2) \cap O_2(H) \cong Z_2 \times D_8$, Sylow's theorem yields that $z \sim_G h$, for some $h \in H - O_2(H)$. Thus there are involutions in $V - \langle z \rangle$ by Proposition 1 so $V \cong E_8$, $D_8$ or $Z_4 \times Z_2$.

We first consider the case when $G$ does not contain a subgroup of index two.

If $V \cong E_8$ then $G \cong L_4(3)$ by Phan's result [17]. In this case $G$ has two classes of involutions with $z \sim_G v \sim_G u \sim_G uv$ say and $\alpha_1 \alpha_2 \sim_G uz \sim_G uvz \sim_G z$.

If $V \cong D_8$ and $v$ is of order four then $G \cong U_4(3)$ by another result of Phan [18].

We note that $U_4(3)$ has only one class of involutions. Suppose that $V \cong D_8$ and $v^2 = 1$. Without loss we take $(uv)^2 = z$. As $O_2(C_H(v)) = \langle v, \alpha_1 \alpha_2, z \rangle$, $z \sim_G v$ forces $z \sim_G \alpha_1 \alpha_2$, since all involutions in $vO_2(H)$ are conjugate in $H$. Let $S = \langle u, \alpha_1 \alpha_2, \beta_1 \beta_2, z \rangle$, a Sylow 2-subgroup of $C_H(u)$. Clearly $S$ is the unique (normal) elementary abelian subgroup of order 16 in $V \cdot O_2(H)$, a Sylow 2-subgroup of $H$ (and hence of $G$). Since $N_H(S)/S \cong \Sigma_4$, $C_G(S) = S$ and $z$ is conjugate to an involution in $S - \langle z \rangle$ in $G$, we have $N_G(S)/S \cong L_2(7)$ or $\mathbb{G}_8$. In either case $z \sim_G \alpha_1 \alpha_2$ and $vS \sim_{N(E)} \alpha_1 \beta_2 S$ so that $z \sim_G v$ also. Finally, Proposition 3 yields
$z \sim_G u \sim_G uz$ and $G$ has one class of involutions. As $C_H(\sigma_1)$, $C_H(\sigma_1\sigma_2^{-1})$ and $C_H(\sigma_2)$ have quaternion, cyclic and elementary abelian Sylow 2-subgroups, respectively, it follows that $\langle \sigma_1 \rangle \sim_G \langle \sigma_1, \sigma_2 \rangle \sim_G \langle \sigma_1, \sigma_2^{-1} \rangle$. Proposition 4 now yields that $z \sim_{G(\sigma_1\sigma_2)} u$. We have $C_H(\sigma_1/\sigma_2) \langle \sigma_1, \sigma_2 \rangle \approx D_{12}$ whence Gorenstein and Walter's result [3, Theorem 15.2.1] gives $G(\sigma_1/\sigma_2) \langle \sigma_1, \sigma_2 \rangle \approx L_2(11)$ or $L_2(13)$. It follows that either $5 \mid |G(v)|$ or $7 \mid |G(v)|$ as $v \in N_H(\langle \sigma_1, \sigma_2 \rangle)$. This contradicts $|G(v)| = |H|$, so this case cannot occur.

In the case when $V = \mathbb{Z}_4 \times \mathbb{Z}_2$ we show that $G$ has a subgroup of index two. If $v^2 = 1$ then $\langle v, O_2(H) \rangle$ contains two classes of elements of order four in $H$, with representatives $\alpha_1, \nu \alpha_1 \alpha_2$. Obviously $u \sim_G \alpha_1$, and as $C_H(u)$ has Sylow 2-subgroup isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ while $C_H(\alpha_1, \alpha_2) \approx \mathbb{Z}_4 \times \mathbb{Z}_4$, $u \sim_G \alpha_1 \alpha_2$ either. By Harada's transfer lemma [10], $G$ has a subgroup of index two. If $u^2 = 1$, $\langle u \rangle \cdot O_2(H)$ has two classes of elements of order four in $H$ with representatives $\alpha_1, \nu \alpha_1$. Obviously $v \sim_G \alpha_1$ and as $C_H(v) \approx \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ while $C_H(\nu \alpha_1) \approx \mathbb{Z}_4 \times \mathbb{Z}_4$, we also have $v \sim_G \nu \alpha_1$. Harada's lemma again yields that $G$ has a subgroup of index two. (Clearly, when $(\nu u)^2 = 1$, an identical argument gives the same result.)

Finally we consider the case when $G$ has a subgroup $G_0$ of index two. Without loss we may assume $G_0 \cap H = \langle u \rangle O_2(H)$ or $\langle v \rangle O_2(H)$. In the first case, $G_0 \cong PSp_4(3)$ by Janko [14], and in the second case $G_0 \cong G_2(3)$ by another result of Janko [15]. It follows that $G \cong \text{Aut} PSp_4(3)$, in which case $V \cong E_6$, or $G \cong \text{Aut} G_2(3)$ (and $V \cong D_8$).

We conclude this section by listing various properties of the groups $PSp_4(3)$ and $\text{Aut} PSp_4(3)$ which will be needed in the proofs of the theorems.

Some properties of $PSp_4(3)$. The group $PSp_4(3)$ is a simple group of order $2^6 \cdot 3^4 \cdot 5$. A Sylow 2-subgroup of $PSp_4(3)$ has centre of order two, and if $t$ is an involution in the centre of a Sylow 2-subgroup, $C(t) = L_1 \cdot L_2\langle u \rangle$, where $u^2 = 1$, $L_1 \cong \text{SL}(2, 3)$ and $L_2 \cong L_2(2, 3)$. If we take $L_i = \langle \alpha_i, \beta_i, \sigma_i \rangle$ (as above) we have the following relations for $C(t)$: $\alpha_i^u = \alpha_i, \alpha_i^u = \alpha_i, \beta_i^u = \beta_i$.

Note that $O_2(C(t)) = \langle \alpha_i, \beta_i \mid i = 1, 2 \rangle \cong Q_8 \cdot Q_8$ has one noncentral class of involutions in $C(t)$ with representative $\alpha_i \beta_i$ say.

In $PSp_4(3)$ there are two classes of involutions with representatives $t, ut$ where $t \sim u$ and $ut \sim \alpha_1 \alpha_2$ (the coset $uO_2(C(t))$ contains two classes of involutions in $C(t)$ with representatives $u, ut$). The Sylow 2-subgroup $\langle u, O_2(C(t)) \rangle$ contains precisely one elementary abelian subgroup of order 16, namely $S = \langle u, \alpha_1 \alpha_2, \beta_1 \beta_2, t \rangle$. We note that $C(S) = S$, $N(S)/S \cong \mathbb{Z}_4$ and that $C(u t) \subset N(S)$ is isomorphic to the centralizer of a noncentral involution in $\mathbb{Z}_8$.

We have $\langle \sigma_1, \sigma_2 \rangle$ is a Sylow 3-subgroup of $C(t)$ and $C(\langle \sigma_1, \sigma_2 \rangle) = M(\langle t \rangle)$ where $M$ is elementary of order 27, and $N(M)/M \cong \Sigma_4$. Let $\langle u, t, \beta \rangle$ be a Sylow 2-subgroup of $N(M)$ with $\langle u, t, \beta \rangle \subset N(\langle \sigma_1, \sigma_2 \rangle)$ and $Z(\langle u, t, \beta \rangle) = \langle \nu \rangle$. $PSp_4(3)$ has four classes of elements of order three; $\langle \sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1} \rangle$, and two classes of elements of order nine with cube conjugate to $\sigma_1$ or $\sigma_1^{-1}$. Further, $C(\sigma_1 \sigma_2^{-1}) = M(\langle t \rangle)$ and $N(\langle \sigma_1, \sigma_2^{-1} \rangle) = M(\langle u, t \rangle)$; $C(\sigma_1 \sigma_2) = M(\langle u, t \rangle)$, $N(\langle \sigma_1, \sigma_2 \rangle) = M(\langle u, t, \beta \rangle)$; $C(\sigma_1) = N(\langle \sigma_1 \rangle)$ is a split extension of a nonabelian group $O_2(C(\sigma_1))$ of order 27 and of exponent three, by $L_2(2, 3)$. If $x \in O_2(C(\sigma_i)) \approx \langle \sigma_1 \rangle$ then $x \sim \sigma_1 \sigma_2^{-1}$. A Sylow 3-subgroup of $PSp_4(3)$ is isomorphic to $Z_3 \vee Z_3$. 

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The subgroup $C(\sigma_1)$ is maximal in $PSp_4(3)$ and $O_{3,2}(C(\sigma_1))$ lies in precisely one maximal subgroup of $PSp_4(3)$, namely $C(\sigma_1)$.

Finally, if $B$ is a Sylow 5-subgroup of $PSp_4(3)$ then $N(B)$ is a Frobenius group of order 20.

Some properties of $Aut PSp_4(3)$. It is well known that $Aut PSp_4(3)$ contains a subgroup of index two isomorphic to $PSp_4(3)$. Put $Aut PSp_4(3) = PSp_4(3)\langle v \rangle$ where $v$ is an involution and $v \in C(t)$. We may choose $v$ so that we have the following relations for $C(t)$:

$$[u, v] = 1, \quad \sigma_i^v = \sigma_i^{-1}, \quad \alpha_i^v = \alpha_i^{-1}, \quad \beta_i^v = \alpha_i \beta_i \quad (i = 1, 2).$$

The Sylow 2-subgroup $\langle u, v, O_2(C(t)) \rangle$ now contains four elementary abelian subgroups of order 16: $\langle u, v, t, a_1^{a_2} \rangle \sim \langle v a_1, u \beta_1 \beta_2, t, a_1 a_2 \rangle$, and $\langle u w, a_1 a_2, a_1 \beta_1 \beta_2, t, \rangle$, $S$ which are both normal. In $Aut PSp_4(3)$, $N(S)/S \approx \Sigma_5$. Further, $C(v) \approx Z_2 \times Z_2 \times \Sigma_4$, and, taking $v \sim w t$, $C(u w) \approx Z_2 \times \Sigma_6$. (In $C(t)$, $v O_2(C(t))$ contains one class of involutions, and $u v O_2(C(t))$ has two classes with representatives $u w, u t$.)

From the relations in $C(t)$ we see that $\sigma_1, \sigma_1^{-1}$ are conjugate in $Aut PSp_4(3)$ and so $N(\langle \sigma_1 \rangle) = C(\sigma_1)\langle v \rangle$, $C(\sigma_1)$ clearly being the same as in $PSp_4(3)$. Also we have $C(\sigma_2) = M\langle t, u \rangle$ and $N(\langle \sigma_2 \rangle) = M\langle t, u, v \rangle$ while $C(\sigma_1) = M\langle t, u, v \beta \rangle$ and $N(\langle \sigma_1 \sigma_2 \rangle) = M\langle u \rangle \times \langle t, u, v \rangle$. Note that $N(M)/M \approx Z_2 \times \Sigma_4$ and $C(\sigma_2)/M \approx D_8$. (Most of the properties of $PSp_4(3)$ listed above may be found in Janko [14].)

For the remainder of Part I we will adopt the following notation: if $x$ was used to denote an element of $PSp_4(3)$ or $Aut PSp_4(3)$ above then $x$ will denote an element of $H$ such that $x J$ satisfies the same properties in $H/J$ as $x$ did in $Aut PSp_4(3)$. In addition $t$ will belong to $C_H(\langle \sigma_1, \sigma_2 \rangle) (\langle \sigma_1, \sigma_2 \rangle = E_9)$, $u, v$ will normalize $\langle \sigma_1, \sigma_2 \rangle$, but we do not know if $t$ is an involution in $H$ or if $[u, t] = 1$. (We only have $t^2, [u, t] \in J$.)

2. A nonsimple case in Theorems A, B.

**Theorem 1.** Suppose that $G, H, z$ satisfy the hypotheses of Theorems A or B. If, in addition, we have $C_j(\sigma_1) = C_j(\sigma_2)$ then $G = H : O(G)$.

The proof will be given in a series of lemmas which, taken together, will yield $z \sim _G h$ for any involution $h \in H - \langle z \rangle$. The theorem then follows immediately from Proposition 2. Much of the proof is independent of the different assumptions about $H$. However it will be necessary at times to refer to the three possibilities:

**Case (i):** $H/J \cong PSp_4(3)$, $Z(J) = \langle z \rangle$;

**Case (ii):** $H/J \cong Aut PSp_4(3)$, $Z(J) = \langle z \rangle$;

**Case (iii):** $H/J \cong Aut PSp_4(3)$, $Z(J) = \langle z, z_1 \rangle \cong E_4$.

As $H$ contains an element $y$ with $y^3 = \sigma_1$, Proposition 5 yields $[\sigma_1, J] \cong Q_8 \ast Q_8 \ast Q_8$ and $J_0 = C_j(\sigma_1) = Z(J) \ast D$ where $D \cong D_8$ or $Q_8$. We suppose that $J_0 = C_j(\sigma_2)$ also. Since $\langle J, N_H(\langle \sigma_1 \rangle) \rangle, C_j(\sigma_2) = H$ (see §1) it follows that $J_0 \leq H$. Note that if $U$ is a Sylow 3-subgroup of $O_{2,3}(C_H(\sigma_1))$ then $U - \langle \sigma_1 \rangle$ only contains conjugates of $\sigma_1^{-1}$. It follows therefore that $C_j(\sigma_1^{-1}) \cap [\sigma_1, J] \cong Q_8$. Thus
$C_J(\sigma_1 \sigma_2) \cap [\sigma_1, J] \cong Q_8 \rtimes Q_8$ and $O_{2,3}(C_H(\sigma_1 \sigma_2)) \cap C(J_0) = \langle \sigma_1 \sigma_2 \rangle \times Z(J) \rtimes L_3 \rtimes L_4$ where $L_3 \cong L_4 = \text{SL}(2, 3)$. This yields that $J_1$ contains one class of involutions in $H$ in cases (i), (ii) and two classes in case (iii) (with representatives $j_1, j_2, j_1 \in [\sigma_1, J]$). In particular any involution in $J$ is conjugate to an involution in $C_J(B)$ ($C_J(B) \cap [\sigma_1, J] \cong D_8$).

**Lemma 1.1.** We have $z \sim_G j$ for any involution $j \in J - \langle z \rangle$.

**Proof.** Since $|H : C_H(Z(J))| < 2$, $O_2(C_G(Z(J))) = J$. whence $N_H(Z(J)) = N_G(Z(J))$. It follows that $z^G \cap Z(J) = \{z\}$ and that a Sylow 2-subgroup of $H$ is a Sylow 2-subgroup of $G$. Let $j$ be any involution in $J - J_1$, with $K$ a Sylow 2-subgroup of $C_H(j)$. By Lemma 6 of [16] we have $O_1(Z(K)) \subseteq \langle j, Z(J) \rangle$. Clearly $[H, H'] \cap J \subseteq J$, which means $[K, K'] \cap O_1(Z(K)) \subseteq Z(J)$. On the other hand, $K' \cap J \not\subseteq Z(K)$ so $\langle z \rangle \subset [K, K']$. It follows that $\langle z \rangle \subset N_G(K)$ whence $K$ is a Sylow 2-subgroup of $C_G(j)$; i.e. $z \sim_G j$.

Now let $j \in J_1 - Z(J)$ with $z \sim_G j$. From our remarks above we may suppose $j \in C_J(B)$ whence Proposition 4 implies $z \sim j$ in $C_G(B)$. Let $L$ be a Sylow 2-subgroup of $C(j) \cap C_H(B)$. Since $|L : C_J(j) \cap L| < 2$, Proposition 7 of [16] yields that we must be in cases (ii) or (iii) with $L \cap J \cong Z_2 \times D_8$ or $Z_2 \times Z_3 \times D_9$, respectively. In addition, $L - J$ must contain involutions. However, an involution $x \in C_J(B) - C_J(J_1)$ must centralize $[B, j_1] \cong D_8 \times Q_8$. As $[x, j_1] \neq 1$, $x$ acts as an outer automorphism on $C(B) \cap [\sigma_1, J]$ ($\cong D_8$) whence $C(x) \cap C_J(B) = Z(J)$.

It follows that $z \sim_G j$ and the lemma is proved.

Let $T_i$, $i = 1, 2, 3$, denote Sylow 2-subgroups of $C_H(\sigma_1)$, $C_H(\sigma_1 \sigma_2^{-1})$, $C_H(\sigma_1 \sigma_2)$, respectively. As $T_1 \cap J = J_0$ and $C_G(J_0)$ covers $T_1 / J_0 (\cong Q_8)$, we must have $T_1 = J_0 \times T_1^*$, $T_1^* \cong Q_8 \times Q_8$ has trivial multiplier [13, p. 643]); without loss we take $\langle t \rangle = \Omega_1(T_1^*)$. Clearly $[t, \langle \sigma_1, \sigma_2 \rangle] = 1$ from which it follows that $[t, C_J(\sigma_1 \sigma_2^{-1})] = 1$ and that $t$ interchanges the two quaternion subgroups in $[\sigma_1 \sigma_2^{-1}, J]$. Finally an easy calculation yields that any involution in $tJ$ is conjugate (in $H$) to an involution in $tC_J(\sigma_1 \sigma_2^{-1})$.

**Lemma 1.2.** We have $\langle \sigma_1 \rangle \sim_G \langle \sigma_1 \sigma_2^{-1} \rangle \sim_G \langle \sigma_1 \sigma_2 \rangle$.

**Proof.** From Lemma 1.1 and the fact that $\langle z \rangle \subset T_2 \subset T_2 \cap J$ we see that $T_2$ is a Sylow 2-subgroup of $C_G(\sigma_1 \sigma_2^{-1})$. As $|T_2| > 2^9$ and $|T_2| < 2^8$, we have $\sigma_1 \sigma_2^{-1} \sim_G \sigma_1 \sigma_2$. If $\langle \sigma_1 \rangle \sim_G \langle \sigma_1 \sigma_2^{-1} \rangle$ then there exists $g \in G - H$ with $T_1^g \subseteq T_2^g$ by Sylow's theorem. However $z \in T_1^g$ implies $z^g \in T_2^g \subset T_2 \cap J$. Thus $z^g = z$ by Lemma 1.1, a contradiction.

**Lemma 1.3.** For any involution $x \in tJ$ we have $x \sim_G z$.

**Proof.** Let $x$ be an involution in $tJ$. By the remarks preceding Lemma 1.2 we may suppose $x \in tC_J(\sigma_1 \sigma_2^{-1})$. Since $[t, C_J(\sigma_1 \sigma_2^{-1})] = 1$, $z \in (C(x) \cap T_2) \cap J$. It follows from Lemma 1.1 that $C(x) \cap T_2$ is a Sylow 2-subgroup of $C(x) \cap C_G(\sigma_1 \sigma_2^{-1})$ and that $z \sim x$ in $C_G(\sigma_1 \sigma_2^{-1})$. Proposition 4 and Lemma 1.2 now yield $z \sim_G x$.

**Lemma 1.4.** No involution in $tJ$ is conjugate to $z$ in $G$. 

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Proof. In the notation of §1 we have $ut \in C_H(\sigma_1 \sigma_2)$ and that $ut$ inverts $\sigma_1 \sigma_2^{-1}$. Thus $C(ut) \cap [u, j] \cong Z_2 \times Q_8$ whence $C_j(x)$ contains a subgroup $K \cong Z_4 \ast Q_8$ for any (involution) $x \in utJ$. If $z \sim_G x$ then $C_G(x)$ contains a 2-subgroup $Y$ with $|Y : X| = 2$, $X$ a Sylow 2-subgroup of $C_H(x)$. Let $y \in Y - X$ and consider $K^y(\subset X)$. Since $z^G \cap J = \{z\}$ (Lemma 1.1), $K^y \cap J = 1$ so $K^y J / J \cong K \cong Z_4 \ast Q_8$. This is a contradiction, however, as $C_H(ut) J / J$ cannot contain a subgroup isomorphic to $K$ (see §1).

We remark that the proof of Theorem 1 is now complete if we are in case (i). For the remaining two lemmas we assume therefore that we are in cases (ii) or (iii).

Lemma 1.5. If $x$ is an involution in $uvJ$ then $z \sim_G x$.

Proof. As we may assume $uv \in C_H(B)$, $[uv, [B, J]] = 1$ forces $C_j(x)$ to be nonabelian for any (involution) $x$ in $uvJ$. Hence if $X$ is a Sylow 2-subgroup of $C_H(x)$, we have $z \in X'$. Since $z^G \cap H^* = \{z\}$ by the lemmas already proved, $X$ is a Sylow 2-subgroup of $C_G(x)$ and $x \sim_G z$.

Lemma 1.6. There are no involutions in $uvJ$ conjugate to $z$ in $G$.

Proof. Suppose $x$ is conjugate to an involution in $uvJ$. Then $z \sim_G x$ for some involution $x$ in $uvJ$ (see §1). From the proof of Lemma 1.5 we see that $C_j(x)$ must be elementary abelian. Hence $uvJ$ has at most two classes of involutions in $\langle uv, J \rangle$ so we may assume $x \in C_H(\sigma_1 \sigma_2^{-1})$. By Lemma 1.2 and Proposition 4 we must have $z \sim x$ in $C_G(\sigma_1 \sigma_2^{-1})$. Now $T_2 = \langle x, t, C_j(\sigma_1 \sigma_2^{-1}) \rangle$ is a Sylow 2-subgroup of $C_H(x)$ and $[t, C_j(\sigma_1 \sigma_2^{-1})] = 1$, $K = C(x) \cap T_2$ must be elementary abelian.

If all involutions in $x C_j(\sigma_1 \sigma_2^{-1})$ are conjugate to $x$ in $T_2$ then $\langle z^G \cap K \rangle = \langle x, C_j(\sigma_1 \sigma_2^{-1}) \rangle$, whence $\{y | y \in C_j(\sigma_1 \sigma_2^{-1}) \setminus \langle z \rangle\} \leq N(K)$. Thus $\langle z \rangle \leq N(K)$ which implies $K$ is a Sylow 2-subgroup of $C(x) \cap C_G(\sigma_1 \sigma_2^{-1})$ and $x \sim_G z$. It follows therefore that we are in case (iii), $K \cong E_{32}$ and $z$ has 5 conjugates in $N(K) \cap C_G(\sigma_1 \sigma_2^{-1})$. However $C_j(\sigma_1 \sigma_2^{-1}) K / K (\cong E_4)$ is a Sylow 2-subgroup of $N(K)/K$ (of order 20). This contradicts the structure of $GL(5, 2)$ which completes the proof of the lemma.

3. The proof of Theorem A. In this section we will complete the proof of Theorem A by proving the following two results.

Theorem 2. Suppose $G$ satisfies the assumptions of Theorem A(i) (i.e. $H / J \cong PSp_4(3)$). If, in addition, $G \neq H \cdot O(G)$ and $C_j(\sigma_1) \neq C_j(\sigma_2)$, then $G \cong U_6(2)$.

Theorem 3. Suppose $G$ satisfies the assumptions of Theorem A(ii) (i.e. $H / J \cong Aut PSp_4(3)$). If, in addition, $G \neq H \cdot O(G)$ and $C_j(\sigma_1) \neq C_j(\sigma_2)$, then $G$ contains a subgroup $G_0$ of index two, $G_0 \cong U_6(2)$ and $G \subseteq Aut G_0$.

We begin with some remarks on the structure of $H$ which apply to both cases. Since $\sigma_2$ acts nontrivially on $C_j(\sigma_1)$, Proposition 5 gives $C_j(\sigma_1) \cong Q_8$ and $J \cong Q_8 \ast Q_8 \ast Q_8 \ast Q_8$ (i.e. $J$ is of type $+$). If $r \in C_j(\sigma_1) - \langle z \rangle$ then $r$ has order four.
and $C_H(r)$ covers $O_{2,3}(C_H(\sigma_1))/C_j(\sigma_1)$, whence $C_H(r) \subset C_H(\sigma_1)J$. Thus $r$ has at least 240 conjugates in $H$, so all elements of order four in $J$ are conjugate in $H$. This yields $C_j(B) = \langle z \rangle$.

As any 3-element of $O_{2,3}(C_H(\sigma_1)) - \langle \sigma_1 \rangle$ is conjugate to $\sigma_1 \sigma_2^{-1}$, we have (again using Proposition 5) $J = C_j(\sigma_1) \cdot C_j(\sigma_2) \cdot C_j(\sigma_1 \sigma_2^{-1})$; i.e. $O_{2,3}(C_H(\sigma_1 \sigma_2^{-1})) \cong Z_3 \times SL(2, 3) \times SL(2, 3)$. Finally we see that $C_j(\sigma_1 \sigma_2) = \langle z \rangle$ and that $J$ has only one class of noncentral involutions in $H$.

Let $T_1$ be a Sylow 2-subgroup of $C_H(\sigma_1)$. As $T_1 \cap J \cong T_j / T_1 \cap J \cong Q_8$ and $C_H(T_1 \cap J) = \langle t, z \rangle$ so that $[t, \sigma_2] = 1$ also. Let $T_2, T_3$ be Sylow 2-subgroups of $C_H(\sigma_1, \sigma_2^{-1})$ and $C_H(\sigma_1 \sigma_2)$, respectively, with $t \in T_i, i = 2, 3$. Let $M$ be a Sylow 3-subgroup of $C_H(\langle \sigma_1, \sigma_2 \rangle)$ so that $M \cong E_{29}$ and $C_H(\langle \sigma_1, \sigma_2 \rangle) = M \cdot \langle t, z \rangle$. In the notation of §1 take $\langle u, t, z \rangle$ as a Sylow 2-subgroup of $N_H(\langle \sigma_1, \sigma_2 \rangle)$ in case (i) and $\langle v, u, t, z \rangle$ in case (ii). Without loss, assume that $t \sim_{N(M)} u \sim_{N(M)} u\langle z \rangle$—see §1. Thus $\langle u, t, z \rangle \simeq E_8$ or $D_8$ depending on whether $(ut)^2 = 1$ or $z$.

Since $\sigma_1^2 = \sigma_2$ we have $C_j(\sigma_1)^{st} = C_j(\sigma_2)$ and $[C_j(\sigma_1 \sigma_2^{-1}), u] = 1$. Conversely, $[t, C_j(\sigma_1) \cdot C_j(\sigma_2)] = 1$ while $t$ interchanges the two quaternion groups in $C_j(\sigma_1 \sigma_2^{-1})$, so that $C_j(t) = E_4 \times Q_8 \times Q_8$. Thus $tJ$ contains 3 classes of involutions in $\langle t, \sigma_1, \sigma_2, J \rangle$ with representatives $t, tz, ij$ where $t, tz$ each have four conjugates, $ij$ has 72 and $j$ is any involution in $C_j(t) - Z(C_j(t))$. Let $F = C_j(u t) = C_j(u) \cap C_j(t)$ so that $F \cong E_{32}$. Hence $utJ$ contains only 2 classes of elements with square in $\langle z \rangle$; we take $ut, utz$ as representatives of the two classes.

Let $T$ be a Sylow 2-subgroup of $H$ with $T = J \cdot O_2(C_H(\langle t \rangle)) \langle u \rangle$ in case (i) and $T = JO_2(C_H(\langle t \rangle)) \langle u, v \rangle$ in case (ii). Observe that $T$ is also a Sylow 2-subgroup of $G$ because $C_G(J) = Z(J) = \langle z \rangle$. As $\langle u \rangle \cdot \langle \sigma_1, \sigma_2 \rangle$ acts irreducibly on $C_j(t)/Z(C_j(t)) \cong E_{16}$, it follows that $F < T\langle \sigma_1, \sigma_2 \rangle$. Clearly $F \lhd C_H(\sigma_1 \sigma_2^{-1})$ so the structure of $PSp_4(3)$ yields that $N_H(F)/J \cong E_16 \cdot D_8$ (case (i)), $E_16 \cdot \Sigma_3$ (case (ii)). Take $E = C_H(\sigma_1 \sigma_2^{-1})$ and observe $u, t \in E$. Thus $E$ covers $O_2(N_H(F))/J$ which yields $|E| = 2^9$ and $N_H(E)/E \cong N_H(F)/J$. Finally we have $E \cong E_{29}$ if $[u, t] = 1$ or $E \cong E_{16} \times D_8 \times Q_8$ if $[u, t] = z$.

Proof of Theorem 2. This will be carried out in a series of lemmas.

Lemma 2.1. The subgroup $\langle \sigma_1 \sigma_2^{-1} \rangle$ is not conjugate to either $\langle \sigma_1 \rangle$ or $\langle \sigma_1 \sigma_2 \rangle$ in $G$. Further we have $C_G(\langle \sigma_1 \sigma_2^{-1} \rangle) \langle \sigma_1 \sigma_2^{-1} \rangle \cong PSp_4(3)$ or $C_G(\sigma_1 \sigma_2^{-1}) = O(C_G(\langle \sigma_1 \sigma_2^{-1} \rangle))C_H(\sigma_1 \sigma_2^{-1})$.

Proof. From the remarks above we see that $C_H(\langle \sigma_1 \sigma_2^{-1} \rangle)/\langle \sigma_1 \sigma_2^{-1} \rangle$ is isomorphic to the centralizer of a (central) involution in $PSp_4(3)$. The second statement now follows from Proposition 6. In particular, $T_2$ is a Sylow 2-subgroup of $C_H(\sigma_1 \sigma_2^{-1})$.

Clearly $\langle \sigma_1 \rangle \not\simeq_G \langle \sigma_1 \sigma_2 \rangle$ as $T_1 \cong Q_8 \times Q_8 \not\cong T_2$. If $T_2 \subset T_2$ for some $g \in G$ then $|T_2| = 8$ implies $|C(z) \cap T_2| = 8$. This is not possible in $T_2$ so we have $\langle \sigma_1, \sigma_2^{-1} \rangle \not\simeq_G \langle \sigma_1, \sigma_2 \rangle$, as required.

Lemma 2.2. For any involution $j \in J - \langle z \rangle$ we have $z \sim_G j$. In addition we may suppose $z \sim_G tz$. 

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Proof. Let \( j \in T_2 \cap J - \langle z \rangle \) and recall \( t \in T_2 \). By Proposition 4 and Lemma 2.1 we have that \( z \) is conjugate to \( j \), \( t \) or \( tz \) in \( G \) only if it is conjugate in \( C_G(\sigma_1 \sigma_2) \). If \( C_G(\sigma_1 \sigma_2) / \langle \sigma_1 \sigma_2 \rangle \approx \PSp(3) \) then we may assume \( z \sim t \sim j \sim tz \) in \( C_G(\sigma_1 \sigma_2) \) (see §1). In the other case \( \langle z \rangle \) is weakly closed in \( T_2 \) with respect to \( C_G(\sigma_1 \sigma_2) \) and so \( z \) is not conjugate to any of \( j \), \( t \), \( tz \). The proof is completed by recalling that all involutions in \( J - \langle z \rangle \) are conjugate in \( H \).

**Lemma 2.3.** The subgroup \( E \) is elementary abelian of order \( 2^8 \) and is weakly closed in \( T \) with respect to \( G \).

Proof. If \( E \) is not elementary abelian then \( [u, t] = z \) whence \( z \sim_G t \). By Proposition 2.2, we may have \( z \sim_G tj_1 \), \( j_1 \) an involution in \( C_j(t) = Z(C_j(t)) \). It follows that \( \langle z^G \cap T \rangle = EJ \). Hence if \( K \) is a Sylow 2-subgroup of \( C_H(tj_1) \) we have \( z \in \langle K \cap Z(G) \rangle \subset J \cap K \); i.e. \( \langle z \rangle \not\sim_G N_G(K) \). Sylow's theorem immediately yields \( z \sim_G tj_1 \) and we have proved that \( E \) is elementary abelian.

Let \( g \in G \) with \( E^g \subset T \). As \( |E^g \cap J| \leq 32 \) and \( |E^g \cdot J / J| < 16 \), we have \( E^g \cap J \approx E_{32} \) and \( E^g \subset EJ \). Thus \( E^g \) contains all \( (32) \) involutions in \( uJ \) so \( E^g \cap J = F = C_J(ut) \). It follows immediately that \( E^g = E \) as \( E = C_G(F) \).

**Lemma 2.4.** The group \( G \) has precisely three classes of involutions; namely we have \( z \sim_G t \), \( j \sim_G tz \), \( ut \) and \( tj \sim_G utz \), where \( j \) is an involution in \( C_j(t) = Z(C_j(t)) \). In addition, \( N_G(E) / E \cong L_3(4) \).

Proof. As all classes of involutions in \( H \) are represented in \( E \), we must have \( z \sim_G e \), \( e \in E - \langle z \rangle \), by Proposition 2.1. It follows from Lemma 2.3 and Sylow's theorem that \( z \sim_{N(E)} e \); i.e. \( N_H(E) \neq N_G(E) \). We will now show \( N_G(E) / E \cong L_3(4) \).

If \( O(N_G(E) / E) \neq 1 \) then there exists a subgroup \( J_1 \) of index two in \( J \) with \( J_1E / E \) centralizing a subgroup of odd order in \( O(N_G(E) / E) \). This is not possible however as \( \langle z \rangle \subset Z(JE) \subset J \cap E \). Hence we have \( O(N_G(E) / E) = 1 \). As \( \sigma_1 \sigma_2 \) acts fixed-point-free on \( JE / E \), \( T / E \) is of type \( L_3(4) \) (see [6, Lemma 2.6, pp. 79–80]). It follows from a result of Gorenstein and Harada [4, Theorem C] that \( N_G(E) / E \) is isomorphic to a subgroup of \( PGL(3, 4) \). Let \( B \) be a Sylow 5-subgroup of \( N_H(E) \). Then \( C_{E}(B) = \langle z \rangle \) which implies that \( N_G(E)' = N_G(E) \). Thus \( N_G(E) / E \cong L_3(4) \) as \( N_G(E) \supset N_H(E) \).

We observe that \( |N_G(E) : N_H(E)| = 21 \) so \( z \) has 21 conjugates in \( E \). Hence \( z \sim_G t \) and \( tz \sim_G j \) by Lemma 2.1. Let \( \langle \sigma, \sigma_1, \sigma_2 \rangle \) be a Sylow 3-subgroup of \( N_G(E) \). Clearly \( z, t, u \) are conjugate under the action of \( \sigma \) on \( C_E(\sigma_1 \sigma_2) = \langle u, t, z \rangle \); it follows therefore that \( tz, uz, ut \) are conjugate under \( \sigma \) and \( C(\sigma) \cap C_E(\sigma_1 \sigma_2) = \langle utz \rangle \). Thus \( j \) has at least 210 conjugates in \( N_G(E) \). Since \( utz, tj \) have 160, 120 conjugates in \( N_H(E) \), respectively, the order of \( L_3(4) \) and the fact that a Sylow 7-normalizer of \( L_3(4) \) is a Frobenius group of order 21 gives \( j \sim_{N(E)} utz \sim_{N(E)} tj \). The lemma follows from the observation once again that two involutions of \( E \) are conjugate in \( G \) if and only if they are conjugate in \( N_G(E) \).

**Lemma 2.5.** We have \( C_G(tj) \subset N_G(E) \) and \( C_G(tj) / E \cong U_3(2) \cong E_9 \cdot Q_8 \).
Proof. As $|C(t) \cap N(E)/E| = 2^3 \cdot 3^2$ and $N_G(E)/E \cong L_3(4)$, we have $C_G(t) \cap N(E)/E \cong E_3 \cdot Q_8$ (a Sylow 3-normalizer in $L_3(4)$). Let $R$ be a Sylow $2$-subgroup of $C(t) \cap N_G(E)$ and $V$ a Sylow $2$-subgroup of $N(R) \cap C(t) \cap N_G(E)$. From $V \cap E = \langle t \rangle$ we conclude $V \cong Z_2 \times Q_8$. Since $C(t) = E_3 \times D_8$ there is an involution $j_0 \in C(t) - E$. By Proposition 1 we may assume $j_0 \in V$ (so $\Omega_1(V) = \langle j_0, j \rangle$) and by Sylow's theorem we may assume $V \subset T$.

Let $Y = EV = T \cap C(t) \cap N_G(E)$ and let $X = C_G(j_0)$. As $E$ char $Y$, $Y$ must be a Sylow $2$-subgroup of $X$. Further, we compute that $j_0E$ contains two classes of involutions in $Y$ with representatives $j_0, j_0tj$ and that $C_E(Y) = \langle t, z \rangle$. Thus $\{z\} = Z^G \cap Z(C_Y(j_0))$ whence Sylow's theorem yields $C_Y(j_0)$ is a Sylow $2$-subgroup of $C_Y(j_0)$. In particular no involution in $Y - E$ is conjugate to an involution in $E$ in $X$. An application of Grün's first theorem [3, Theorem 7.4.2] gives $Y \cap X' \subseteq \Omega_1(Y)$. Hence $X$ contains a normal subgroup $X_1$ of index four with $X_1 \cap Y = \Omega_1(Y) = \langle E, j_0 \rangle$. Proposition 3 applied to $X_1$ yields that $X_1$ has a subgroup of index two which does not contain $j_0$. It follows that $X_1$ possesses a subgroup $X_2$ of index two with $N(E) \cap X_2 = E \cdot R$. We conclude that $N_X(Y) = VC_X(Y)$, which implies $N_X(R) \cap X_2 = C_X(R)$.

If $C_X(R) = R \times \langle t \rangle$, two applications of Burnside's transfer theorem [3, Theorem 7.4.3] yield $X_2 = O(X_2) \cdot E \cdot R$. The lemma will follow if we can show $C_X(r) \subseteq R \cdot E$, $r \in R^\#$, and $O(X) = 1$. To do this, we use the fact that $utz \sim N(t)$ and show $C(s_1s_2) \cap C_G(utz) \subset N_G(E)$ and $O(C_G(utz)) = 1$.

As $u, t, z$ are all conjugate in $C_G(utz) \cap N_G(E)$, each of $u, t, z$ acts fixed-point-free on $O(C_G(utz))$. Thus $O(C_G(utz)) = 1$. Recall that
$$C(s_1s_2) \cap C_H(utz) = \langle s_1s_2, u, t, z \rangle$$
$$= \langle s_1s_2 \rangle \times \langle u, t \rangle \times \langle u, t \rangle \approx Z_3 \times Z_2 \times Z_4.$$
Clearly, $C_G(utz) \cap C_H(utz) = C_H(utz)$, so we conclude that $C(utz) \cap C(s_1s_2) \cap C_G(utz)$ is abelian. A result of Suzuki (see [19]; or see [3, pp. 420–423]) yields $C(s_1s_2) \cap C_G(utz) \approx \tilde{E}_4$ or $\hat{E}_5$. The second case is not possible as $R \sim N(E) \langle s, s_1s_2 \rangle$ and $N_X(R) = C_X(R) \cdot V$. Thus $C(s_1s_2) \cap C_G(utz) \subset N_G(E)$ as required.

**Lemma 2.6.** If $W = \langle tz \rangle \times Z(C_T(tz))$ then
$$C_G(W) = O_2(C_H(tz)) \cdot \langle s_1s_2^{-1}, u \rangle,$$
$$N_G(W) = C_G(W) \cdot (N_G(W) \cap C_G(s_1s_2^{-1}))$$
and $N_G(W)/C_G(W) \cong \tilde{E}_4$. Further, we have $C_G(tz) \subset N_G(W)$ so that $C_G(tz) = C_G(W) \langle s_1s_2, l \rangle$ where $\langle s_1s_2, l \rangle \approx \Sigma_3, z^l = t$ and $l \in N_G(E)$.

**Proof.** We have $W = \langle tz \rangle \times Z(C_T(tz)) \approx E_{16}, W \subseteq E \cap C_H(s_1s_2^{-1})$ and $W \triangleleft C_H(tz)$. By Lemmas 2.1 and 2.4, $C_G(s_1s_2^{-1})/\langle s_1s_2^{-1} \rangle \cong PSp_4(3)$. Since $u$ inverts $s_1s_2^{-1}$ and centralizes a complement to $\langle s_1s_2^{-1} \rangle$ in $C_H(s_1s_2^{-1})$, it follows that $N_G(\langle s_1s_2^{-1} \rangle) = \langle s_1s_2^{-1}, u \rangle \times P$ where $P \cong PSp_4(3)$. Now $N_P(W)/W \cong \hat{E}_5$ (see §1) and as $z, tz$ have precisely 5, 10 conjugates in $W$, it follows (from the structure of $\tilde{E}_4$) that $N_G(W) = N_P(W)C_G(W)$. In particular,
$$C_G(W) = O_2(C_H(tz)) \cdot \langle s_1s_2^{-1}, u \rangle \quad (C_H(tz) = C_G(W)\langle s_1s_2 \rangle).$$
Since \( tz \) has 10 conjugates in \( N_p(W) \), \( C_G(tz) \cap N_p(W) = W \langle \sigma_1 \sigma_2, l \rangle \) where \( \langle \sigma_1 \sigma_2, l \rangle \cong S_3 \). Clearly \( l \not\in H \) whence \( z' = t \) as \( H_{\sigma_1 \sigma_2} = \langle t, z \rangle \). Finally \( [l, u] = 1 \) so \( l \) normalizes \( O_2(C_H(tz)) \langle u \rangle \supseteq E \) and hence \( l \in N_G(E) \).

It remains to show that \( C_G(tz) \subseteq N_G(W) \). Firstly we note that

\[
C_G(tz) \cap N_G(E) = O_2(C_H(tz)) \cdot \langle \langle u \rangle \times \langle \sigma_1 \sigma_2, l \rangle \rangle \subseteq N_G(W),
\]
as \( tz \) has 210 conjugates in \( E \). Clearly \( L = \langle O_2(C_H(tz)), u, l \rangle \) is a Sylow 2-subgroup of \( C_G(tz) \), \( E \not\subset L \) and \( z \) has only 2 conjugates in \( N(E) \cap C_G(tz) \), namely \( t, z \). As \( z^G \cap L \subseteq E \), Proposition 2 applied to \( C_G(tz)/\langle tz \rangle \) yields

\[
C_G(tz) = O_2(C_G(tz)) \cdot N_G(\langle t, z \rangle).
\]

Since \( tj \sim_G tjz \) and \( C_G(tj) \subseteq N_G(E) \) (Lemma 2.5), each of \( z, tj \) and \( tjz \) must act fixed-point-free on \( O_2(C_G(tz)) \). Thus \( O_2(C_G(tz)) = 1 \) and \( C_G(tz) = N_G(\langle t, z \rangle) \) whence \( C_G(tz) \subseteq N_G(W) \). The lemma is proved.

**Lemma 2.7.** The order of \( G \) is \( 2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11 \).

**Proof.** Thompson's order formula and Lemmas 2.4–2.6 give

\[
|G| = 2^{15} \cdot 3^4 \cdot 5 \cdot a(tj) + 2^{13} \cdot 3^2 \cdot a(z) + 2^{13} \cdot 3^4 \cdot 5 \cdot a(tz)
\]

where, for any involution \( g \in G \),

\[
a(g) = |\{(x, y) | (xy)^n = g \text{ for some positive integer } n, \text{ with } x \sim_G z, \ y \sim_G tj\}|.
\]

In the computations for \( a(tj) \) and \( a(tz) \) we will use the notation introduced in Lemmas 2.5 and 2.6. Recall that \( |z^G \cap N_G(E)| = |z^G \cap E| = 21 \) and that \( z^G \cap H = \{z\} \cup \nu \). For the rest of the proof we will assume \( x \sim_G z, y \sim_G tj \).

\[
a(tj) = 9. \text{ Since } [R, E] \times \langle tj \rangle = E, \text{ and } [R, E] \not\subset C_G(tj), \text{ we may assume } e \in E \text{ if } (xy)^n = tj \text{. In } C_G(tj), \ z \text{ has 9 conjugates, while in } C_G(utz), \ z \text{ has 12 conjugates. The result follows because } tjz \sim_G tj \text{ while } ut \sim_G tz.
\]

\[
a(z) = 0. \text{ If } (xy)^n = z \text{ we easily verify that } x \neq z \text{ and if } x = t, \ y \in C_H(t). \text{ Suppose } x = t \text{ and } y \in H - C_H(t). \text{ If } [t, y] \in J, \text{ we only have to consider } y = u_{j2}, j_2 \in C_f(u) - C_f(t). \text{ However } (tu_{j2})^3 = (j_2)^3 \neq z \text{ as } z \not\in [1, J]. \text{ Suppose next that } tJ \sim_H yJ, \text{ so that } (ty)^m \text{ lies in a conjugate of } tJ \text{ or } utJ, \text{ for some } m. \text{ Now } F = \langle z \rangle \times [ut, J] \text{ and } [u, T] \cap J = \emptyset \text{ (see §1) so } (ty)^m J \sim_H tJ \text{ and } (ty)^q \sim_H \sigma_1 \sigma_2^{-1} \text{ for some integer } q. \text{ Similarly if } tJ \sim_H yJ, \text{ the product is conjugate to } \sigma_1 \sigma_2^{-1}. \text{ Finally we check that there are no } x, y \in N_H(\langle \sigma_1 \sigma_2^{-1} \rangle) \text{ with } (xy)^n = z \text{ for any } n.
\]

\[
a(tz) = 2^4 \cdot 3^2 \cdot 19. \text{ Recall that } N_G(\langle \sigma_1 \sigma_2^{-1} \rangle) = \langle \sigma_1 \sigma_2^{-1}, u \rangle \times P, \text{ } P \cong PSp_4(3), \text{ and}
\]

\[
C_G(tz) = O_2(C_H(tz)) \cdot \langle \langle \sigma_1 \sigma_2^{-1}, u \rangle \times \langle \sigma_1 \sigma_2, l \rangle \rangle.
\]

Since \( P \) has two classes of involutions with representatives \( z, tz \), it follows that \( l \sim_G tz \) and \( ul \sim_G utz \sim_G tj \). We put \( D = O_2(C_G(tz)) = O_2(C_H(tz)) \) and compute that \( ul \) has \( 2^6 \cdot 3^2 \) conjugates in \( C_G(tz) \) and all involutions in \( ulD \) are conjugate to \( ul \). Further, all involutions in \( lD \) are conjugate to \( l \) while in \( uD, u \) has 16 conjugates,
$u_j (\sim G J)$ has 96 ($j_0 \in Z(C_f(t)))$, $utzt$ has 64 and all other involutions (there are 80) in $uD$ are conjugate to $tz$ in $G$. (All involutions in $uD$ lie in $E$.) Finally we have $|z^G \cap D| = 5$ while $|z^G \cap C_G(tz) - D| = |u^{C_{tz}}| = 48.$

Suppose $(xy)^n = tz$ and firstly that $x \in D$. If $x = z$ we get no pairs if $y \in D$ but $2^6 \cdot 3^2$ pairs if $y \sim u$ in $C_G(tz)$. When $x \neq z$ in $C_G(tz)$ we take $x = z$ in $C_G(tz)$ and easily show there are no pairs in this case. Hence there are $2^7 \cdot 3^2$ pairs if $x \in D$.

Now suppose $x = u$ and observe that $u \cdot utzt = tz$ so we get one pair if $y \in C_G(u)$. Since $[\langle u \rangle, D] \subseteq E - tz(E \cap J)$ we get zero pairs if $y \in D$. If $y \in u_1 D$ where $l_1 D \sim lD$ then $(u \cdot u_1 d)^2 = (l_1 d)^2 = [u, d]$ for those $d \in D$ with $ul_1 d$ an involution. There are zero pairs therefore in this case. Next suppose $y \in u_1 l_1 D$, $uD \neq u_1 l_1 D \sim uD,$ $l_1 D \sim lD$ in $C_G(tz)$. Then $(w)^3 \in l_1 D$ and $\langle u, y \rangle D = \langle (u, \sigma_1 \sigma_2^{-1}) \rangle \cdot D$. As $C_G(u) \supseteq C_G(\sigma_1 \sigma_2^{-1}) \cap \langle l_1, D \rangle$, $(u)^3$ is an involution, so we again get zero pairs. Finally if $y \in u_1 D$ for $uD \neq u_1 D \sim uD$ then $\langle u, y \rangle D = \langle \sigma_2^{-1}, u \rangle D$. Since $utzt \sim G J$ the structure of $C(\sigma_1 \sigma_2^{-1}) \cap C_G(tz)$ shows that we get two pairs for each conjugate of $\langle \sigma_1 \sigma_2^{-1} \rangle$ normalized by $u$. There are 16 such conjugates. We therefore get $48 \times 33$ pairs if $x \notin D$. The lemma follows immediately.

**Lemma 2.8.** There exists $k \in G$ such that $G = H' \cup HlH \cup HkH$ where $|H'| \cap H| = 2^{13} \cdot 3^2$ and $|H^k \cap H| = 2^6 \cdot 3^4 \cdot 5$. Thus $G$ acts as a rank three permutation group on the cosets of $H$.

**Proof.** As $H' = C_G(t)$, $H' \cap H = C_H(t)$ whence $|H' \cap H| = 2^{13} \cdot 3^2$. Since $N_G(\langle \sigma_1 \sigma_2^{-1} \rangle) = \langle u, \sigma_1 \sigma_2^{-1} \rangle \times P$, there exists $u_1 \in N_G(\langle \sigma_1 \sigma_2^{-1} \rangle), u_1 \sim G u$ such that

$$C_G(u_1) \cap C_G(u) \cap N(\langle \sigma_1 \sigma_2^{-1} \rangle) = P.$$ 

Hence there exists $k \in G$ with $C_H(z^k) \cap J = H$, as $u \sim G z$. Finally, as $z^k \notin H$ we have $C_H(z^k) \cap J = 1$; i.e., $C_H(z^k) \cong PSp_3(3)$. The lemma follows from Lemma 2.7.

**Lemma 2.9.** The group $G$ is simple and is isomorphic to $U_6(2)$.

**Proof.** From Lemma 2.6, $O(C_G(tz)) = 1$ whence $z, t, tz$ all act fixed-point-free on $O(G)$. It follows that $O(G) = 1$. If $1 \neq N \subseteq G, z \in N$ because $Z(T) = \langle z \rangle$ and $O(G) = 1$. We observe that $\langle z^G \cap H \rangle = H$ so $H \subseteq N$. Thus $N = G$ as $N_G(T) = T (N_G(T) \subseteq H)$, and $G$ is a simple group.

In the notation of Lemma 2.8, we have $(zz^l)^2 = (zl)^2 = 1$ and $(z \cdot z^k)^3 = 1$. (Note that $k$ was chosen in Lemma 2.8 so that

$$N_G(\langle z \rangle) = \langle z \rangle, z^k \rangle \times C_H(z^k) \sim G N_G(\langle \sigma_1 \sigma_2^{-1} \rangle) \cong \Sigma_3 \times PSp_4(3).$$

Thus $z^G$ is a class of 3-transpositions. We conclude that $G \cong U_6(2)$ because of Fischer's result [1].

**Proof of Theorem 3.** We now suppose $G, H, z$ satisfy the assumptions of Theorem 3. To begin the proof we make some remarks on the structure of $H$ in this case.

Recall that $\langle v, t, u, z \rangle$ is a Sylow 2-subgroup of $N_H(\langle \sigma_1, \sigma_2 \rangle)$, with $\sigma_i^v = \sigma_i^{-1}$, $i = 1, 2$. Take $\langle v, t, u, \beta, z \rangle$ to be a Sylow 2-subgroup of $N_G(M)$, where $\langle t, u, \beta, z \rangle \sim D_8$ and note that $N_H(M)/C_H(M) \cong Z_2 \times \Sigma_4$. As $vJ \sim H$ utzJ
and \( C_M(uv) = \langle \sigma_1 \sigma_2 \rangle \), we have \( C_M(vt) = 1 \) whence \( vt \in C_H(M) \). Since \( v \in N_H(T_1) \) so that \([v, t] = 1\) also (recall \( T_1 \cong Q_8 \times Q_8 \), \( \mathcal{O}_1(T_1) = \langle t, z \rangle \)). It follows immediately that \([\langle vt \rangle, \langle u, t \rangle] = 1\). In addition we will take \( T_2 = \langle vu, t \rangle \cdot C_2(\sigma_1 \sigma_2) \) and \( T_3 = \langle t, u, z, v \beta \rangle \) (note that \( T_3/\langle z \rangle \cong D_8 \)).

Since \( vt \) inverts \( \langle \sigma_1, \sigma_2 \rangle \) and \([vt, u \rangle = 1\), we easily compute \( C_2(vt) = C_2(v) = E_{16} \), \( C_2(vu) = E_{32} \). Further if \( h \in H - H' \) with \( h^2 \in \langle z \rangle \) then \( h \) is conjugate to one of \( v, u, uz \) in \( H \).

**Lemma 3.1.** The elements \( \sigma_1, \sigma_1 \sigma_2^{-1}, \sigma_1 \sigma_2 \) lie in distinct conjugate classes in \( G \).

**Proof.** As in the proof of Lemma 2.1, \( T_2 \) is a Sylow 2-subgroup of \( C_G(\sigma_1 \sigma_2) \). If there exists \( g \in G \) with \( T^g \subset T_2 \) then \( z^g \in T^g_2 \subset T_2 \cap J \). However \( \Omega_8(T_1) = \langle t, z \rangle \) while \( C_2(z^g) \cap T_2 = Z_2 \times D_8 \). We conclude that \( \sigma_1 \sim \sigma_1 \sigma_2 \). If \( T^g \subset T_2 \) for some \( g \in G \) then \( |C(z^g) \cap T_2| = 16 \), whence there exists \( h \in H \) with \( h \in C_H(\sigma_1 \sigma_2) \) and \( z^gh = z^gvt \). However \( C(vu) \cap T_2 \) must be abelian in this case as \( z \in \langle t, C_2(\sigma_1 \sigma_2) \rangle \) and \( |C_2(v) \cap T_2| = 4 \). Clearly this contradicts \( T_3/\langle z \rangle \cong D_8 \). It remains to show \( \sigma_1 \sim \sigma_1 \sigma_2 \).

Suppose that \( T_1 \) is not a Sylow 2-subgroup of \( C_G(\sigma_1) \). By Sylow’s theorem there is a subgroup \( T^*_1 \) of \( C_G(\sigma_1) \) with \( |T^*_1 : T_1| = 2 \). It follows that \( T^*_1 \cong Q_8 \) wr \( Z_2 \) and \( \langle z, t \rangle \) char \( T^*_1 \). Thus \( T^*_1 \) is a Sylow 2-subgroup of \( C_G(\sigma_1) \). In either case it is immediate that neither \( T^*_1 \) nor \( T_1 \) contains a subgroup isomorphic to \( T_2 \). We conclude that \( \sigma_1 \sim \sigma_1 \sigma_2 \).

**Lemma 3.2.** Either \( C_G(\sigma_1 \sigma_2) = Aut \, \text{PSp}_4(3) \) or \( L_4(3) \), or \( C_G(\sigma_1 \sigma_2) = O(C_G(\sigma_1 \sigma_2)) \cdot C_H(\sigma_1 \sigma_2) \). In addition, \( z \sim j \) for any involution \( j \in J - \langle z \rangle \) and we may assume \( z \sim tz \).

**Proof.** By Proposition 4 and Lemma 3.1, \( z \sim tz \) if and only if \( z \sim tz \) in \( C_G(\sigma_1) \). Recall \( T_1 = T_{11} \times T_{12}, T_{11} \approx T_{12} = Q_8 \) and assume \( T^*_1 = \langle z \rangle, T^*_{12} = \langle t \rangle \). It follows immediately from the last part of the proof of Lemma 3.1 and Burnside’s lemma [3, Theorem 7.1.1] that \( z \sim tz \).

Suppose that \( C_G(\sigma_1 \sigma_2) \neq C_H(\sigma_1 \sigma_2) \). By Proposition 6, \( C_G(\sigma_1 \sigma_2) = L_4(3), U_4(3), Aut \, \text{PSp}_4(3) \) or \( Aut \, \text{G}_2(3) \). Since \( z \sim tz \) and \( U_4(3) \) has one class of involutions, \( U_4(3) \) is not a possibility. We show that \( C_G(\sigma_1 \sigma_2) = \langle \sigma_1 \sigma_2 \rangle \). If \( C_G(\sigma_1 \sigma_2) \approx \langle \sigma_1 \sigma_2 \rangle \) then \( \langle vu, t \rangle \approx D_8 \), hence \( \langle u, t \rangle \approx D_8 \). Let \( K \) be a subgroup of index two in \( C_G(\sigma_1 \sigma_2) \) with \( K/\langle \sigma_1 \sigma_2 \rangle \approx G_2(3) \). Then \( K \cap T_2 = \langle vu, t \rangle \approx \langle \sigma_1 \sigma_2 \rangle \). It follows immediately that \( \sigma_1 \sigma_2 \) inverts \( M, \) a Sylow 3-subgroup of \( C_H(\sigma_1 \sigma_2) \), and \( u \) centralizes a complement to \( \langle \sigma_1 \sigma_2 \rangle \) in \( O_2(3,C_H(\sigma_1 \sigma_2) \langle v \rangle \). The structure of \( G_2(3) \) (see [15]) yields that either \( u \) or \( uz \) centralizes \( C_G(\sigma_1 \sigma_2) \). Hence \( N_G(\langle \sigma_1 \sigma_2 \rangle \langle u \rangle) \cong \Sigma_3 \times G_2(3) \). Hence \( \langle [t, u] = 1 \), a contradiction. The lemma is proved.

**Lemma 3.3.** The subgroup \( E \) is elementary abelian of order \( 2^9 \) and weakly closed in \( T \) with respect to \( G \).
PROOF. If \( C_G(\sigma_1\sigma_2^{-1})/\langle \sigma_1\sigma_2^{-1} \rangle \cong L_4(3) \) or \( \text{Aut } PSp_4(3) \) then \( \langle uu, t, z \rangle \cong E_8 \) and hence \( \langle u, t, z \rangle \cong E_8 \); i.e. \( E \cong E_2 \). For the case when \( \langle z \rangle \) is weakly closed in \( C_H(\sigma_1\sigma_2^{-1}) \) with respect to \( C_G(\sigma_1\sigma_2^{-1}) \), the proof of the first part of Lemma 2.3 may be repeated to yield \( E \cong E_2 \).

In order to prove \( E \) is weakly closed in \( T \), observe that \( E^g \subset T \) (\( g \in G \)) implies \( E^g \cap eJ \neq \emptyset \) for some \( e \in E \) with \( eJ \sim_H uJ \) (see §1 for a list of subgroups of \( \text{Aut } PSp_4(3) \) isomorphic to \( E_{16} \)). As \( C_J(e) = F \) and \( E = C_G(F) \), we have \( E^g = E \) as required.

**Lemma 3.4.** The subgroup \( N_G(E) \) contains a subgroup \( K \) of index two with \( K/E \cong L_3(4) \) and \( K \cap H = N_G(E) \cap H' \).

**Proof.** By Lemma 3.2, \( z \sim_G e \) for some \( e \in E - \langle z \rangle \). It follows therefore from Lemma 3.3 that \( z \sim e \) in \( N_G(E) \) whence \( N_G(E) \neq N_H(E) \). If \( N_G(E) \) contains a subgroup \( K \) with \( |N_G(E)/E : K/E| = 2 \) then \( K/E \cap N_H(E)/E \cong E_{16} \delta_3 \) and \( K/E \cong L_3(4) \) as in the proof of Lemma 2.4.

For \( j_1 \in J - E \), \( \langle z \rangle \subset \langle j_1 \rangle \subset J \cap \langle z \rangle \) whence \( C_{N(E)}(j_1) = T^h/E \), \( h \in N_H(E) \). If we assume that \( N_G(E) \) contains no subgroup of index two then \( N_G(E)/E \) is “fusion-simple” (in the sense of [5]). Further, \( T/E \), a Sylow 2-subgroup of \( N_G(E)/E \) is of type \( \tilde{A}_8 \) (see [6, Lemma 2.6, pp. 79–80]). However the conditions contradict a result of Gorenstein and Harada [5, Theorem A]. Thus \( N_G(E)/E \) must possess a subgroup of index two and the lemma is proved.

**Lemma 3.5.** There are 3 classes of involutions in \( E \) in \( N_G(E) \) with representatives \( z, tz, utz \). These classes are not fused in \( G \). Further, \( C_G(\sigma_1\sigma_2^{-1})/\langle \sigma_1\sigma_2^{-1} \rangle \cong \text{Aut } PSp_4(3) \) and for any involution \( h \in H - H' \), we have \( z \sim_G h \).

**Proof.** We have \( z \sim t \sim j \sim tz \sim ut \sim utz \sim tj \) in \( N_G(E) \) as in Lemma 2.4 (\( j \) an involution in \( C_J(t) - Z(C_J(t)) \)). Further, these classes are not fused in \( G \) because of Lemma 3.3. As \( z \sim_G t \), Proposition 4 and Lemmas 3.1, 3.2 yield \( C_G(\sigma_1\sigma_2^{-1}) \neq O(C_G(\sigma_1\sigma_2^{-1}))C_H(\sigma_1\sigma_2^{-1}) \).

Suppose that \( C_G(\sigma_1\sigma_2^{-1})/\langle \sigma_1\sigma_2^{-1} \rangle \cong L_4(3) \). Since \( \langle u, z \rangle \) centralizes \( C_H(\sigma_1\sigma_2^{-1})/\langle \sigma_1\sigma_2^{-1} \rangle \), the structure of \( L_4(3) \) (see [17]) and \( u \sim_G z \) yield \( N_G(\langle \sigma_1\sigma_2^{-1} \rangle) = \langle \sigma_1\sigma_2^{-1}, uz \rangle \times L \), \( L \equiv L_4(3) \). However \( t \sim_G z \) forces \( t \cdot uz \sim z \cdot uz = u \) in \( N_G(\langle \sigma_1\sigma_2^{-1} \rangle) \), clearly a contradiction. Thus \( C_G(\sigma_1\sigma_2^{-1})/\langle \sigma_1\sigma_2^{-1} \rangle \cong \text{Aut } PSp_4(3) \) and, as \( z \sim t \) in \( C_G(\langle \sigma_1\sigma_2^{-1} \rangle) \), we have \( z \) is not conjugate to any involution in \( T_2 - \langle t, C_J(\sigma_1\sigma_2^{-1}) \rangle \) in \( C_G(\sigma_1\sigma_2^{-1}) \).

**Lemma 3.6.** The group \( G \) contains a subgroup \( G_0 \) of index two with \( G_0 \cong U_6(2) \).

**Proof.** Recall that \( C_J( uv ) = E_{32} \) and \( C_H( uv )/ C_J( uv ) \cong Z_2 \times S_8 \) (see §1). Let \( Y \) denote a Sylow 2-subgroup of \( C_J( uv ) \). Clearly \( \langle z \rangle \subset Z( Y ) \subset \langle j, uv \rangle \) whence \( z^G \cap Z( Y ) = \{ z \} \) by Lemmas 3.5 and 3.2. Thus \( Y \) is a Sylow 2-subgroup of \( C_J( uv ) \), and as \( | Y | = 2^{10} \), \( u \sim_G tz \) and \( uv \sim_G tj \); i.e. \( uv \sim_G x \) for any involution \( x \in H' \). It follows immediately from Proposition 3 that \( G \) contains a subgroup \( G_0 \) of index two with \( uv \in G - G_0 \). Thus \( G_0 \cap H = H' \) whence \( G_0 \cong U_6(2) \) by Theorem 2. The lemma is proved.
As $G \subseteq \text{Aut } G_0$, the proof of Theorem 3 and hence that of Theorem A has been completed.

4. The proof of Theorem B. Throughout this section we will work under the following assumptions.

**Hypothesis 1.** Let $G$ be a finite group, $z$ an involution in $G$ and $H = C_G(z)$.
Suppose that $G \neq H \cdot O(G)$ and that $H$ satisfies:

(I) $J = O_2(H)$ is the direct product of a group of order two and an extra-special subgroup of order $2^9$, $J' = \langle z \rangle$ and $C_H(J) \subseteq J$;

(II) $H/J = \text{Aut } PSp_4(3)$;

(III) $C_J(x) \neq C_J(y)$.

We will put $Z(J) = \langle z, z_1 \rangle$ and otherwise use the same notation as in the proof of Theorem 3 in §3.

As $T_1 \cap J = Z_2 \times Q_8$ and $T_1/J \cap T_1 = Q_8$, $T_1/Z(J) = E_4 \times Q_8$ and $T_1 = T_1(T_1 \cap J)$, $T_1 = Q_8$ (although $T_1$ is not necessarily a direct product this time).

We take $\Omega_1(T_1) = Z(T_1) = \langle t, z, z_1 \rangle$. Further, $T_2 = \langle z_1 \rangle \times [M, C_4(x_1)] \cdot \langle t, u, v \rangle$ and $T_3 = \langle z_1, z \rangle \cdot \langle t, u, v \rangle$ with $T_3/Z(J) = D_8$. Note that $Z(T_2) \subseteq Z(J)$ and $Z(T_3) \subseteq \langle Z(J), u \rangle$.

In the same way as in §3 we compute that $J - Z(J)$ has two classes of involutions with representatives $j, jz_1$—choose $j$ in $[M, C_4(x_1)]$. (To see $j \sim_H jz_1$ use the fact that $j \sim_H jz_1$ is in $C_H(x_1)$.) Further $t, tz, tz_1, tz_2$ each have 4 conjugates and $t_1, t_1z_1$ each have 72 conjugates in $\langle t, x_1, x_2 \rangle$. (Here $j_1$ is an involution in $C_4(t) - Z(C_4(t))$.) Finally, if $x^2 \in \langle z \rangle$ and $xJ \sim_H uJ$, then $x$ is conjugate to (at least) one of $ut, utz, utz_1, utz_2$; if $x^2 \in \langle z \rangle$ and $xJ \sim_H uJ$ then $x \sim_H v$ or $x \sim_H vz_1$; or if $x^2 \in \langle z \rangle$ and $xJ \sim_H uvJ$ then $x$ is conjugate to one of $uv, uz, uz_1, uz_2$.

We begin the proof of Theorem B by considering the case when $Z(J) = Z(H)$.

**Theorem 4.** Suppose Hypothesis 1 holds and, in addition, $Z(J) = Z(H)$. Then

\[ Z(G) = \langle z_1 \rangle \quad \text{or} \quad \langle zz_1 \rangle \quad \text{and} \quad G \text{ contains a subgroup } G_0 \text{ of index two with } G_0/Z(G) = U_6(2) \quad \text{and} \quad G \subseteq \text{Aut } G_0. \]

As usual, the proof will be carried out in a series of lemmas.

**Lemma 4.1.** The involutions $z, z_1, zz_1$ lie in distinct conjugacy classes in $G$. Hence a Sylow 2-subgroup of $H$ is a Sylow 2-subgroup of $G$.

**Proof.** As $H = C_G(\langle z \rangle)$, $J = O_2(G(\langle z \rangle))$, whence $\langle z \rangle = J' \cap N_G(\langle z \rangle)$. If $T$ is a Sylow 2-subgroup of $H$ then $Z(T) = \langle z, z_1 \rangle$, so $\langle z \rangle \subseteq N_G(T)$. Thus $T = N_G(T)$ and $T$ is a Sylow 2-subgroup of $G$ (by Sylow’s theorem) and $z, z_1, zz_1$ lie in distinct conjugacy classes by Burnside’s lemma [3, Theorem 7.1.1].

**Lemma 4.2.** There is an element $g$ in either $C_G(z_1)$ or $C_G(zz_1)$ such that $z^g \in H - \langle z \rangle$. 

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Proof. Suppose that $z^g \in H$, $g \in C_G(z_1)$ or $C_G(zz_1)$, implies $z^g = z$. We will show, in a number of steps below, that $\langle z \rangle$ is weakly closed in $H$—this will contradict Proposition 2 as we have assumed $G \neq H \cdot O(G)$.

(a) If $W \supseteq Z(J)$, $W \cong E_8$, then $N_G(W) \subseteq H$. For $g \in N_G(W)$, we have $\langle z, z_1 \rangle^g \cap \langle z, z_1 \rangle \neq 1$. If $g \notin H$, $z \neq z^g \in W \subset H$ and $g \in C_G(z_1)$ or $C_G(zz_1)$, as $z_1 \sim_G zz_1$.

(b) $z \sim_G j$ for any $j \in J - \langle z \rangle$. (If $j \in J - Z(J)$ and $K$ is a Sylow 2-subgroup of $C_H(j)$, then $Z(K) = \langle z, z_1, j \rangle$. Thus $N_G(K) \subseteq H$ by (a) and $z \sim_G j$. We already have $z_1 \sim_G z \sim_G zz_1$—Lemma 4.1.)

(c) $z \sim_G x$ for any involution $x$ in $tJ$. (Using the structure of $Aut PSp_4(3)$ and the remarks at the beginning of this section we easily compute that if $x$ is an involution in $tJ$ and $X$ is a Sylow 2-subgroup of $C_H(x)$, then $Z(X) \subseteq \langle t, J \rangle$. Suppose there is a 2-subgroup $Y$ of $C_G(x)$ with $|Y : X| = 2$. For $y \in Y - X$, $\langle z, z_1 \rangle^y \cap J \neq 1$ so $z \neq z^y \in Z(X)$ with $y \in C_G(z_1)$ or $y \in C_G(zz_1)$ by (b) and Lemma 4.1. We conclude that $z \sim_G X$.)

(d) $\sigma_1, \sigma_1\sigma_2^{-1}$ and $\sigma_1 \sigma_2$ lie in distinct conjugacy classes in $G$. (As $Z(T_2) = Z(J)$, $Z(T_1) = \langle t, z, z_1 \rangle, Z(J) \subseteq Z(T_3) \subseteq \langle ut, z, z_1 \rangle$, (a) and (b) yield that $T_1, T_2, T_3$ are Sylow 2-subgroups of $C_G(\sigma_1), C_G(\sigma_1\sigma_2^{-1}), C_G(\sigma_1\sigma_2)$, respectively. The result follows as the $T_i$ have different orders.)

(e) $z \sim_G x$ for any involution $x \in utJ$. (If $z \sim_G x$, $x$ would be conjugate to an involution in $Z(T_3) = \langle ut, z, z_1 \rangle$. The result follows from Proposition 4, (d), (a) and Burnside’s lemma.)

(f) $z \sim_G x$ for any involution $x \in wuJ$. (If $z \sim_G x$ then $z$ is conjugate to an involution in $\langle ux, z, z_1 \rangle$. However $\langle ux, z, z_1 \rangle$ is a Sylow 2-subgroup of $C_H(B)$ where $B$ is a Sylow 5-subgroup of $H$. We get the required contradiction in the same way as in (e).)

(g) $z \sim_G v$ and $z \sim_G vz_1$. (If $z$ is conjugate to $v$ in $G$ then $z$ must be conjugate to an involution $x$ in $C_G(\sigma_1\sigma_2^{-1}), x \in wuJ \cap C_H(\sigma_1\sigma_2^{-1})$, by (d) and Proposition 4. If $X$ is a Sylow 2-subgroup of $C(x) \cap T_2$ then $X$ is abelian as $[t, J] \cap Z(J) = 1$ and $|C_G(x) \cap T_2| = 8$. We can apply the same argument as given in the second paragraph of the proof of Lemma 1.6 to $\Omega_4(X)$ to show $V$ is a Sylow 2-subgroup of $C(x) \cap C_G(\sigma_1\sigma_2^{-1})$. Thus $v \sim_G z$. A similar argument yields $z \sim_G vz_1$ also.)

This completes the proof of the lemma.

Let $F = C_J(\langle u, \ i \rangle)$ and $E = C_H(F)$. In this case, $F \cong E_6$. $|E| = 2^{10}$ and (using the same arguments as in §3) $N_H(E)/E \cong \Sigma_5$.

Lemma 4.3. We may assume that $C_G(z_1)$ contains a subgroup $U$ of index two with $C_G(z_1) = U\langle v \rangle$ and $U/\langle z_1 \rangle \cong U_6(2)$. Further, $E$ is weakly closed in $T$ with respect to $G$ and $N_G(E)/E \cong L_3(4)$.

Proof. By Lemma 4.2 we may assume that there exists $g \in C_G(z_1)$ with $z^g \in H - \langle z \rangle$. Thus $C_G(z_1) \neq O(C_G(z_1)) \cdot H$ by Proposition 2. It follows therefore from Theorem 3 that $C_G(z_1)$ contains a subgroup $U$ with the properties stated. The proof that $E$ is weakly closed in $T$ is the same as that given in Lemma 3.3.

Lemma 4.4. We have $N_G(E) \subseteq C_G(z_1)$ and $z_1$ is not conjugate (in $G$) to any involution in $H'\langle z_1 \rangle - \langle z_1 \rangle$. 

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Proof. If \( z \sim_G j \) for some involution \( j \in J - Z(J) \) and \( X \) is a Sylow 2-subgroup of \( C_H(j) \), there exists a 2-group \( Y \subseteq C_G(j) \) with \(|Y:X| = 2\). For \( y \in Y - K \), \( z^y \in \langle j, z, z_1 \rangle - \langle z \rangle \) and \( y \in C_G(z_1) \) or \( C_G(zz_1) \). However this is not possible—see Lemma 3.2. We conclude \( z \sim_G j \) for any \( j \in J - \langle z \rangle \). It follows by the argument of Lemma 3.4 that \( N_G(E) \) contains a subgroup \( K \) of index two with \( N_G(E) = K \langle v \rangle \), and \( K/E \cong L_3(4) \). Thus \( N_G(E) \subseteq C_G(z_1) \) by Lemma 4.3. Finally, recall that each involution in \( H' \langle z_1 \rangle \) is conjugate to an involution in \( E \). Hence as \( E \) is weakly closed in \( T \) and \( C_G(z_1) \supseteq N_G(E) \), it follows immediately that \( z_1^G \cap H' \langle z_1 \rangle = \{z_1\} \).

**Lemma 4.5.** We have \( G = C_G(z_1) \).

Proof. We claim that \( G \) contains a subgroup \( G_0 \) of index two. In order to use the proof of Lemma 3.6 we only have to show \( z \sim_G x, x \in uvJ \). (We have already shown \( z^G \cap J = \{z\} \) in the proof of Lemma 4.4.) As in the proof of Lemma 4.2(f) let \( \langle uv, z, z_1 \rangle \) be a Sylow 2-subgroup of \( C_H(B) \), \(|B| = 5\). By Proposition 4, \( z \) is conjugate to an involution in \( uvJ \) if and only if \( z \) is conjugate to an involution in \( uvZ(J) \) in \( C_G(B) \). It follows that \( z \) must be conjugate to an involution in \( uvZ(J) \) in \( C_G(z_1) \) or \( C_G(zz_1) \), which is not the case. We conclude, as in Lemma 3.6, that \( G \) contains a subgroup \( G_0 \) with \( G = G_0uv \).

If \( z_1 \notin G_0 \) then we conclude from Theorem 3 that \( G_0 \) contains a subgroup \( G_1 \) of index two with \( G_1 \cong U_4(2) \). In this case \( G \subseteq C_G(z_1) \) and the lemma is proved. Therefore we will assume \( z_1 \in G_0 \). It follows that \( G_0 \cap H = H' \langle z_1 \rangle \) whence \( z_1^G \cap H = \{z_1\} \). Proposition 2 yields \( G = C_G(z_1) \cdot O(G) \) and it remains to show that \( O(G) = 1 \).

Since we may assume \( z \sim_G t \), it follows that \( [tz, O(G)] = 1 \). However \( tz \sim_G j_0 \) for some \( j_0 \in J \) (Lemma 3.5) and \( j_0 \sim_H j_0^z \). Thus \( [j_0, O(G)] = [j_0^z, O(G)] = 1 \) whence \( O(G) \subseteq H \) so \( O(G) = 1 \). The lemma is proved.

It is clear that Theorem 4 follows immediately from Lemmas 4.5 and 4.3. To complete the proof of Theorem B it only remains to consider the case \( Z(J) \neq Z(H) \).

**Theorem 5.** Suppose Hypothesis 1 holds and, in addition, \( Z(J) \neq Z(H) \). Then \( G \cong M(22) \).

We proceed (as usual) to give the proof in a series of lemmas, after making the obvious remark that \( C_H(Z(J))/J \cong PSp_4(3) \), and if \( h \in H - C_H(Z(J)) \) then \( z_1^h = z_1^z \).

**Lemma 5.1.** The cosets \( uvJ \) and \( uvJ \) each contain precisely one class of involutions in \( H \).

Proof. Let \( X = \langle uv, \sigma_1\sigma_2, J \rangle \) so that \( X/J \cong \Sigma_3 \). By our assumptions above, \( \langle uv, Z(J) \rangle \) is a Sylow 2-subgroup of \( N_X(\langle \sigma_1\sigma_2 \rangle) \). As \( \langle uv, Z(J) \rangle \) is nonabelian, \( \langle uv, Z(J) \rangle \cong D_8 \) so \( N_X(\langle \sigma_1\sigma_2 \rangle) \) contains precisely one class of involutions which are not in \( J \). The result follows for the coset \( uvJ \) from Proposition 1, and exactly the same argument applied to \( \langle v, \sigma_1\sigma_2, J \rangle \) yields the result for \( uvJ \).

**Lemma 5.2.** The involution \( z_1 \) is not conjugate (in \( G \)) to any involution in \( J - \{z_1, z_1z\} \).
Proof. If \( Y \) is a Sylow 2-subgroup of \( C_H(z_1) \) then clearly \( Z(Y) = Z(J) = \langle z, z_1 \rangle \). As \( J = O_2(C_G(\langle z, z_1 \rangle)) \) and \( J' = \langle z \rangle \), \( Y \) must be a Sylow 2-subgroup of \( C_G(z_1) \) whence \( z_1 \not\sim_J z \). Recall that \( J - Z(J) \) has two classes of involutions in \( H \) with representatives \( j, jz_1 \) (and each has 270 conjugates in \( H \)). If \( K \) is a Sylow 2-subgroup of \( C_H(j) \) then \( K \leq C_H(Z(J)) \) so \( Z(K) = \langle z, j \rangle \). If \( j \) is conjugate to \( z_1 \) in \( G \), then there exists \( g \in G \) with \( \langle z, j \rangle^g = \langle z, z_1 \rangle \). Thus \( z^g = z \) and \( g \in H \) which is a contradiction. The same argument shows \( z_1 \not\sim_G jz_1 \) and the lemma follows.

Lemma 5.3. We have \( \langle \sigma_1 \rangle \not\sim_G \langle \sigma_1 \sigma_2^{-1} \rangle \sim_G \langle \sigma_1 \sigma_2 \rangle \).

Proof. Note that \( |T_2| = 2^8 \) and \( Z(T_2) = \langle z \rangle \), so that \( T_2 \) is a Sylow 2-subgroup of \( C_G(\langle \sigma_1 \rangle) \). Now \( |T_1| = 2^7 \) and \( Z(T_1) \approx E_8 \) whence if \( T_2^g \subseteq T_2 \) for some \( g \in G \), \( |Z(T_2)| > 4 \). We conclude that \( \langle \sigma_1 \rangle \not\sim_G \langle \sigma_1 \sigma_2^{-1} \rangle \). If \( T_2^g \subseteq T_2 \) for some \( g \in G \) then \( z \in T_3 \) implies \( z^g \in T_2 \subseteq T_2 \cap J \). However \( |T_3| = 2^5 \) while \( |C(z^g) \cap T_2| > 2^7 \), clearly a contradiction. The lemma is proved.

Lemma 5.4. Either \( C_G(\langle \sigma_1 \rangle) = O(C_G(\langle \sigma_1 \rangle)) \cdot C_H(\langle \sigma_1 \rangle) \) or \( C_G(\langle \sigma_1 \rangle) \) contains a subgroup \( U \) of index two with \( U \langle z_1 \rangle = C_G(\langle \sigma_1 \rangle) \) and \( U/\langle \sigma_1 \rangle \approx U_4(3) \). Further, if \( z \) is conjugate to some involution in \( T_2 - \langle z \rangle \) in \( G \) then, with appropriate choice of elements, we have the following fusion in \( C_G(\langle \sigma_1 \sigma_2 \rangle) \):

\[
\begin{align*}
z & \sim j \sim t \sim tz \sim uw \sim uvz_1 \sim jz_1 \sim tz_1z \sim z_1 \sim tz_1.
\end{align*}
\]

Proof. If \( z \) is not conjugate to any involution in \( T_2 - \langle z \rangle \) then \( C_G(\langle \sigma_1 \rangle) = O(C_G(\langle \sigma_1 \rangle)) \cdot C_H(\langle \sigma_1 \rangle) \) by Proposition 2. Hence we will suppose \( z \sim_G x \) for some \( x \in T_2 - \langle z \rangle \). By Proposition 4 and Lemma 5.3, \( z \sim x \) in \( C_G(\langle \sigma_1 \rangle) \). Let \( X = C_G(\langle \sigma_1 \rangle) \) and \( Y = C_H(\langle \sigma_1 \rangle) \). We take \( uw \) to be an involution (in \( N_Y(\langle \sigma_1 \rangle) \)) and \( C_Y(uw) \cap \langle t, z, z_1 \rangle = \langle z, tz_1 \rangle \) so that \( t^uw = tz_1 \). \( \langle t, z, z_1 \rangle = C_Y(\langle \sigma_1 \rangle) \cap T_2 \). As \( t, uw \) interchange the two quaternion subgroups in \( O_2,3(X) = [M, Y \cap J] \), we have \( C_Y(t) \cap Y = E_8, C_Y(uw) \cap Y = E_8 \) and \( C_Y(uwz_1) \cap Y = Z_4 \times Z_2 \). Thus \( Y \) contains 9 classes of involutions with representatives \( z, z_1, j, jz_1, t, tz_1, tz_1z, uw, uwz_1 \). Further, \( T_2 \) contains precisely one elementary abelian subgroup of order 32, namely \( W = \langle t \rangle \times (C_Y(t) \cap T_2) \).

Suppose at first that \( z \) is not conjugate (in \( X \)) to any involution in \( W - \langle z \rangle \). Thus \( z \sim_X uw \) or \( z \sim_X uwz_1 \). If \( K \) is a Sylow 2-subgroup of \( C_Y(uw) \) then \( Z(K) = \langle uw, z, j' \rangle \) (for some \( j' \in J \cap Y - Z(J) \)). However \( \langle j', j'z \rangle \not\sim_N K(X) \) (by Lemma 5.1) whence \( \langle z \rangle \not\sim_X K(X) \) so \( K \) is a Sylow 2-subgroup of \( C_Y(uw) \) and \( z \not\sim_X uw \). If \( L \) is a Sylow 2-subgroup of \( C_Y(uwz_1) \) then \( \langle z \rangle \subseteq \mathfrak{S}^1(L) \subseteq L \cap J, \) so \( \langle z \rangle \not\sim_X L \) and \( z \not\sim_X uwz_1 \) either. We have proved that \( z \sim_X w \) for some \( w \in W - \langle z \rangle \) and thus \( z \sim w \) in \( N_X(W) \) (as \( W \) is normal in any Sylow 2-subgroup which contains it). Thus \( N_X(W) \neq N_Y(W) \).

We have \( C_X(W) = W \times \langle \sigma_1 \sigma_2^{-1} \rangle, N_X(W)/C(W) = \mathfrak{S}_4, T_2/W = D_8 \) and \( T_2 \) is a Sylow 2-subgroup of \( N_X(W) \). If \( xC(W) \) centralizes \( j_2C(W) \) in \( N_X(W)/C_X(W) \), where \( \langle j_2 \rangle C(W)/C(W) = Z(T_2C(W)/C(W)) \) then \( x \) normalizes \( C_{\mathfrak{S}_4}(j_2) = \langle z, z_1, j' \rangle \) for some \( j' \in W \cap J \) and therefore \( x \) normalizes \( \langle z_1z, z_1 \rangle \) by Lemma 5.2; i.e. \( x \in N_Y(W) \). It now follows from a result of Gorenstein and Walter [7, Theorem 1] that \( N_X(W)/C_X(W) \approx \mathfrak{S}_5, L_2(7) \) or \( \mathfrak{S}_6 \).
In \(N_x(W)\), \(z\) has 1 conjugate, \(z_1\) has 2 conjugates, \(j, jz_1\) each have 6, \(t\) has 8, and \(tz_1, tz_1z\) each have 4 conjugates. If \(N_x(W) / C_x(W) \cong \Sigma_5\), as \(z_1\) must have 10 conjugates we may assume (without loss) that \(j\) still has only 6 conjugates in \(N_x(W)\). This forces \(z \in \langle j^{N(W)} \rangle \subset J \cap W\) which is not possible as \(z\) has 5 conjugates in \(N_x(W)\) (recall \(j \sim jz\) in \(T_2 \cap J\)). If \(N_x(W) / C_x(W) \cong L_2(7)\), \(z, z_1\) must have 7, 14 conjugates, respectively, and we again may assume that \(j\) has only 6 conjugates in \(N_x(W)\). This forces \(J \cap W = \langle z^{N(W)}, j^{N(W)} \rangle \leq N_x(W)\), a contradiction. We have proved therefore that \(N_x(W) / C_x(W) \cong \mathfrak{S}_6\) and (as above) determine that \(z \sim j \sim t\) (\(z \sim j \sim tz\) forces \(z \sim j\) as \([\langle \sigma_1 \sigma_2 \rangle, W \rangle \subset [M, J]\)), \(z_1 \sim tz_1\) and \(jz_1 \sim tz_1z\) (as we may interchange \(t, tz_1, tz\) if necessary). Since \(\mathfrak{S}_6\) has one class of involutions, it follows that all involutions in \(N_x(W) - W\) are conjugate either to \(z\) or \(jz_1\) (as \(jzW\) contains two classes in \(Y\) with representatives \(j_2, jz_1\)), for some \(j_2 \in (J \cap T_2) - W\). It follows from the uniqueness of \(W\) (in \(T_2\)) that \(z_1\) is not conjugate to any involution in \(\langle t, uw \rangle \cdot [M, Y \cap J]\) and so \(X\) has a subgroup \(X_1\) of index two with \(z_1 \in X - X_1\). Clearly \(z \in X_1 \cap W\) whence \(X_1 \cap W = \langle z^{N(W)} \rangle \cong E_{16}\).

We see that \(X_1\) has no subgroup of index two and that \(X_1 \cap Y\) satisfies the assumptions for the centralizer of an involution in Proposition 6. Thus we conclude that \(X_1 / \langle \sigma_1 \sigma_2^{-1} \rangle \cong U_4(3)\) as \(\langle t, uw \rangle \cong D_8\). This completes the proof of the lemma.

As above we take \(F = C_J(\langle u, t \rangle) \cong E_{64}\) and \(E = C_H(F)\). Thus \(|E| = 2^{10}\) and \(N_H(E) / E \cong \Sigma_5\). Let \(T\) be a Sylow 2-subgroup of \(N_H(E)\) (whence \(T\) is a Sylow 2-subgroup of \(G\)).

**Lemma 5.5.** The subgroup \(E\) is elementary abelian (of order \(2^{10}\)) and weakly closed in \(T\) with respect to \(G\).

**Proof.** As we chose \(T_3 = \langle u, t, v \beta \rangle Z(J)\), \(u\) normalizes \(\langle t, Z(J) \rangle\) whence \([u, t] = 1\) or \((ut)^2 = z\) which implies \(tz_1^u = tz_1z\). Thus if \((ut)^2 = z\), \(z\) cannot be fused to any involution in \(T_2 - \langle z \rangle\) by Lemma 5.4 and so \(z^G \cap H \subset \{z\} \cup z_1^H \cup z_1z^H\). This leads to a contradiction as in the proof of Lemma 2.3. Hence \([t, u] = 1\) and \(E\) is elementary abelian. The proof that \(E\) is weakly closed in \(T\) follows as in the proof of Lemma 3.3.

**Lemma 5.6.** We have \(N_G(E) / E \cong M_{22}\) (the Mathieu group on 22 letters) and \(C_G(\sigma_1 \sigma_2^{-1}) = U \langle z_1 \rangle\) where \(U / \langle \sigma_1 \sigma_2^{-1} \rangle \cong U_4(3)\).

**Proof.** By Lemma 5.4 we have that \(z \sim_G e\) for some \(e \in E - \langle z \rangle\). Hence Lemma 5.5 forces \(z \sim e\) in \(N_G(E)\) and \(N_G(E) \neq N_H(E)\). We first show that for any \(j_0 \in J - E\), \(O(C_{N(E) / E}(j_0E)) = 1\). Suppose \(gE \in O(C(j_0E))\). Then as \(JE/E \subseteq C(j_0E)\) it follows that \([g, K] \subseteq E\) for \(K / E\) a subgroup of index two in \(JE / E\). As \(\langle z, z_1 \rangle \subset Z(K) \subset J \cap E\) and \(g\) normalizes \(Z(K)\) it follows by Lemma 5.2 that \(g\) normalizes \(\{z_1, z_1z\}\). Hence \(g \in H\) so \(g \in E\) and \(O(C(j_0E)) = 1\). This forces, in addition, that \(O(N_G(E) / E) = 1\) also.

If \(N_G(E) / E\) possesses a subgroup \(L / E\) of index two then \(L / E \cong L_2(4)\) as in the proof of Lemma 2.4. However this means \(z\) must have 21 conjugates in \(N_G(E)\) so \(z \sim_G tz_1\) or \(tz_1z\), against Lemma 5.4. Thus \(N_G(E) / E\) does not contain a subgroup of index two. If \(j_0 \in J - E\) then \(|C_{N(E) / E}(j_0E)| < 2^7 \cdot 15\) as \(z \in [j_0, E]\) of order at
most 16, and \( N_G(\mathcal{E}) \subseteq H \). Thus as \( T/E \) is of type \( \hat{A}_8 \) (see Lemma 3.4), the combined work of Gorenstein and Harada [5] and Phan [18] yields \( N_G(\mathcal{E})/E \cong M_{22} \) or \( U_d(3) \). Since \( N_G(\mathcal{E})/E \cong U_d(3) \) implies \( |N_G(\mathcal{E})| > |E| \), we have \( N_G(\mathcal{E})/E \cong M_{22} \), and \( z \) has 231 conjugates in \( N_G(\mathcal{E}) \). It follows that \( z \sim j \sim t \sim ut \) say, where \([ut, v\beta] = 1 \) in \( T_3 \), \( z_1 \sim tz_1 \), and all other involutions in \( E \) (there are 770) are conjugate to \( jz_1 \) in \( N_G(\mathcal{E}) \). (This last fact follows from some simple computations and the structure of \( M_{22} \) — however it is not needed in this work.)

**Lemma 5.7.** We have \( C_G(z_1)/\langle z_1 \rangle \cong U_d(2) \) and \( G \cong M(22) \).

**Proof.** As \( \langle z \rangle \) is not weakly closed in \( C_H(z_1) \) with respect to \( C_G(z_1) \) (use either Lemma 5.4 or 5.6), we have \( C_G(z_1)/\langle z_1 \rangle \cong U_d(2) \) by Theorem 2. We next show that \( z_1 \in C_G(z_1)' \) so that \( C_G(z_1) \cong \hat{U}_d(2) \) — a two-fold covering group of \( U_d(2) \).

Recall that \( Z(T_1) = \langle t, z, z_1 \rangle \) and the Sylow 2-subgroup \( \langle v, T_1 \rangle \) of \( N_G(\langle \sigma_1 \rangle) \) has centre \( \langle tz_1, z \rangle \). As \( z^G \cap Z(T_1) = \{z, t, tz\} \), \( z_1^G \cap Z(T_1) = \{z_1, z_1z, tz_1\} \), it follows immediately from Sylow's theorem that \( T_1 \) is a Sylow 2-subgroup of \( C_G(\sigma_1) \).

Now \( T_1 \) contains \( \langle u, t, z, z_1 \rangle \cong E_{16} \) so \( \sigma_1 \cong \sigma_1 \sigma_2 \) whence Lemma 5.3 and Proposition 4 imply \( z \sim t \sim tz \) in \( C_G(\sigma_1) \). Thus \( z \sim t \sim tz \) in \( N(T_1) \cap C_G(\sigma_1) \) by Burnside's lemma. If \( z_1 \in T_1 \) then \( T_1 \cong Z_2 \times Q_8 \times Q_8 \) (see §3, introductory remarks) with \( \Phi_1(T_1) = \langle t, z \rangle \). However it is now impossible for \( z, t, tz \) to be conjugate in \( N(T_1) \cap C_G(\sigma_1) \) so we have proved \( z_1 \in T_1 \) whence \( z_1 \in C_G(z_1)' \).

It follows from a result of Griess [8] that \( C_G(z_1) \) is uniquely determined. Thus D. Hunt's result [11] yields \( G \cong M(22) \) once we have observed that \( G \) is simple. As \( z \sim G t \sim G tz \), we have \( O(G) = 1 \) immediately, so if \( 1 \neq N \triangleleft G \) then \( z \in N \). Hence, as \( \langle z^G \cap H \rangle = H \) we have \( H \subseteq N \), so \( N = G \) as \( Z(T) = \langle z \rangle \). This completes the proofs of the lemma and Theorem 5.

5. The proof of Theorem C. Throughout this section, \( G \) will denote a finite group which satisfies the assumptions of Theorem C. In addition we will assume that \( G \neq H \cdot O(G) \) so that \( z \sim_G h \), for some \( h \in H \) — \( \langle z \rangle \) by Proposition 2. From our assumptions on the structure of \( H \) we have \( C_H(c) = \langle c \rangle \times P \) where \( P/\langle z \rangle \cong PSp_4(3) \). Thus \( P \cong Z_2 \times PSp_4(3) \) or \( PSp_4(3) \) (the covering group for \( PSp_4(3) \)). Further, \( N_H(\langle c \rangle) = \langle c \rangle \times P \langle v \rangle \), and we will take all the elements used to describe the structure of \( \text{Aut} \ PSp_4(3) \) in §1 to lie in \( \langle c \rangle \langle v \rangle \). In particular we see that \( t^2 = 1 \).

As \( |J'| = 2 \), we have \( J' = \langle z \rangle \), and if \( \langle z_1, z_2 \rangle = [\langle c \rangle, Z(J)], \) then \( Z(J) = \langle z, z_1, z_2 \rangle \) and \( J = \langle z_1, z_2 \rangle \times J_0, \) \( J_0 \cong Q_8 \times Q_8 \times Q_8 \). Clearly \( H' = C_H(c) \cdot J \) and \( H'' = P \cdot J = C_H(Z(J)) \). If \( T = \langle j, t, \alpha_1, \alpha_2, \beta_1, \beta_2, u, v \rangle \) then \( T \) is a Sylow 2-subgroup of \( H \) and we take \( z_2^v = z_1 \) so \( Z(T) = \langle z, z_1 \rangle \). As usual, \( T_2 = \langle C_J(\sigma_1 \sigma_2), uv, t \rangle \) is a Sylow 2-subgroup of \( C_H(\sigma_1 \sigma_2) \) and \( Z(T_2) = \langle z, z_1 \rangle \).

Repeating the arguments of previous sections we see that \( J \) has 5 classes of involutions with representatives \( z(1), z(3), z(xz(3), j(270), jz_1(810)) \) where \( j \in [C_J(\sigma_1 \sigma_2), M] \). (The numbers in brackets of course denote the number of conjugates of each of the involutions.) In \( tJ, t(4), tz_1(4), tz_1z_1(12), tz_1z(12), j_1(72), j_1z_1(216) \) (\( j_1 \in C_J(t) - Z(C_J(t)) \)) are representatives of the classes of involutions, where \( t \sim_H tz \) and \( tz_1 \sim_H tz_1z \) only if \( P \cong PSp_4(3) \).
In $v J$, $u_{16}$, $u_{16}$, $u_{16}$, $u_{16}$, $u_{16}$, $u_{16}$ represent the classes of elements with square in $<z>$, while if $x^{2} \in <z>$, $x \in v J$ implies $x \sim_{H} v$, and $x \in \mu J$ implies $x \sim_{H} w v$ or $x \sim_{H} w v z$. In addition, $C_{J}(v) = E_{32}$ while $C_{J}(w v) = E_{64}$, $C_{J}(u t) = E_{128}$ and $C_{J}(t) = E_{16} \times Q_{8} \star Q_{8}$.

Finally we consider the elements of order three. If $y^{3} = 1$, $y \in H$, then $y$ is conjugate to an element in $<c, \sigma_{1}, \sigma_{2}^{*}>$, $C_{H}(<c, \sigma_{1}, \sigma_{2}^{*}>) = (M \times <e>)<t, z>$ and $N_{H}(<c, \sigma_{1}, \sigma_{2}^{*}>) = C_{H}(<c, \sigma_{1}, \sigma_{2}^{*}>) \times <u, v>$. We see that there are 9 conjugacy classes of elements of order 3 in $H$. In the table below $Y$ denotes a Sylow 2-subgroup of $C_{H}(y)$.

### Conjugacy classes of elements of order three in $H$

<table>
<thead>
<tr>
<th>$y$</th>
<th>$Y \cap J = C_{J}(y)$</th>
<th>$Y / Y \cap J$</th>
<th>$Z(Y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{1}$</td>
<td>$E_{4} \times Q_{8}$</td>
<td>$Q_{8}$</td>
<td>$&lt;t, Z(J)&gt;$</td>
</tr>
<tr>
<td>$\sigma_{2}^{1}$</td>
<td>$E_{4} \times Q_{8} \star Q_{8}$</td>
<td>$E_{4}$</td>
<td>$&lt;z_{1}, z&gt;$</td>
</tr>
<tr>
<td>$\sigma_{2}$</td>
<td>$Z(J)$</td>
<td>$D_{8}$</td>
<td>$&lt;ut, z_{1}, z&gt;$</td>
</tr>
<tr>
<td>$c$</td>
<td>$&lt;z&gt;$</td>
<td>type $PSp_{4}(3)$</td>
<td>$&lt;z, t&gt;$</td>
</tr>
<tr>
<td>$c \sigma_{1}^{-1}$</td>
<td>$Q_{8} \star Q_{8} \star Q_{8}$</td>
<td>$Q_{8}$</td>
<td>$&lt;z&gt;$</td>
</tr>
<tr>
<td>$c \sigma_{1}$</td>
<td>$&lt;z&gt;$</td>
<td>$Q_{8}$</td>
<td>$&lt;t, z&gt;$</td>
</tr>
<tr>
<td>$c \sigma_{2}$</td>
<td>$Q_{8}$</td>
<td>$Z_{2}$</td>
<td>$&lt;t, z&gt;$</td>
</tr>
<tr>
<td>$c \sigma_{1} \sigma_{2}^{-1}$</td>
<td>$Q_{8} \star Q_{8}$</td>
<td>$E_{4}$</td>
<td>$&lt;u, z&gt;$</td>
</tr>
<tr>
<td>$c \sigma_{1} \sigma_{2}$</td>
<td>$Q_{8} \star Q_{8}$</td>
<td>$E_{4}$</td>
<td>$&lt;t, z&gt;$</td>
</tr>
</tbody>
</table>

**Lemma 6.1.** The involutions $z, z_{1}, z_{1} z$ lie in distinct conjugacy classes in $G$. Further, a Sylow 2-subgroup $T$ of $H$ is a Sylow 2-subgroup of $G$.

**Proof.** As $J = O_{2}(C_{G}(Z(T)))$ and $<z> = J'$, $N_{G}(Z(T)) \subseteq H$ whence $N_{G}(T) = T$ and the result follows from the theorems of Sylow and Burnside.

**Lemma 6.2.** If $<\sigma_{1} \sigma_{2}^{*}> \subseteq H$ for some $g \in G$ then there exists $h \in H$ with $<\sigma_{1} \sigma_{2}^{*}>^{h} = <\sigma_{1} \sigma_{2}^{*}>^{g}$.

**Proof.** Let $y^{3} = 1$, $y \in H$ but $y \sim_{H} \sigma_{1} \sigma_{2}^{-1}$. We have to show $y \sim_{G} \sigma_{1} \sigma_{2}^{-1}$. As $Z(T_{2}) = <z, z_{1}>$ we see that $T_{2}$ is a Sylow 2-subgroup of $C_{G}(\sigma_{1} \sigma_{2}^{-1})$ by Lemma 6.1. As $|T_{2}| = 2^{9}$ it is immediate that $\sigma_{1} \sigma_{2}^{-1} \sim_{G} c \sigma_{1}$. Further, if $\sigma_{1} \sigma_{2}^{-1} \sim_{G} y$, then if $|Y| < 2^{9}$, we must have $|Z(Y)| > 8$. Hence it only remains to consider $y = \sigma_{1}$ or $\sigma_{1} \sigma_{2}$. We have $\sigma_{1} \sim_{G} \sigma_{1} \sigma_{2}^{-1}$ by the argument of Lemma 3.1. If $T_{3} \subseteq T_{2}$ for some $g \in G$ ($T_{3}$ a Sylow 2-subgroup of $C_{H}(\sigma_{1} \sigma_{2})$, as usual), then since $|T_{3}| = 2^{6}$, $z^{8} \in \mu J \cap T_{2}$ ($|C(\mu J) \cap T_{2}| = 2^{5}$). However in this case $Z(T_{3}) = <ut, z, z_{1}> = E_{8}$ while $C(z^{8}) \cap T_{2}$ has centre of order 16. We conclude that $\sigma_{1} \sigma_{2}^{-1} \sim_{G} \sigma_{1} \sigma_{2}$ and the lemma is proved.

As in the proof of Lemma 5.4, let $W = <t> \times (C_{J}(t) \cap T_{2})$, the unique elementary abelian subgroup of order 64 in $T_{2}$.

**Lemma 6.3.** If $X = C_{G}(\sigma_{1} \sigma_{2}^{-1})$ and $Y = C_{H}(\sigma_{1} \sigma_{2}^{-1})$ then one of the following holds:

(i) $X = O(X) \cdot Y$ and $z^{G} \cap Y = \{z\}$;
(ii) $N_X(W)/C_X(W) = G$ and with appropriate choice of $t, uv$ we have the following fusion of involutions in $X$:

$$z \sim t \sim tz_1z \sim jz_1 \sim uv \sim uvt \sim z_1 \sim uvz \sim z_1z \sim tz;$$

(iii) $N_X(W)/C_X(W) \cong Z_3^3 \cdot \Sigma_5$, $\langle z_1, z_2 \rangle \triangleleft N_X(W)$ and $|Z_G \cap W| = 5$ with $z \sim G t$.

Proof. As usual, Proposition 4 and Lemma 6.2 imply that for any involution $y \in Y, z \sim y$ if and only if $z \sim X y$. Assume $X \neq O(X) \cdot Y$ so that $z \sim y$ for some $y \in Y - \langle z \rangle$ by Proposition 2. We note that $Y = Y_0 \cdot Y_1$ where $Y_0 \cap Y_1 = 1$, $Y_0 = \langle z_1, z_2, c \rangle \cong \mathbb{Z}_4$, $Y_1 = \langle uv, t, M \rangle[O, Y \cap J]$ with $Y_1/\langle \sigma_1 \sigma_2 \rangle$ satisfying the assumptions of the centralizer of an involution in Proposition 6.

Suppose at first that $z \sim X w$ for any $w \in W - \langle z \rangle$. Then it follows, as in the proof of Lemma 1.6, that $z \sim uv$ and if $z \sim uv$, then $C_Y(uv) = \langle \sigma_1 \sigma_2 \rangle \times L$ where $L = E_{32}$. In this case we have $N_Y(L)/C_Y(L) \cong E_8$ whence $z$ has 9 conjugates in $N_X(L)$. Thus $N_X(L)$ contains a normal subgroup $L_1$ with $|L_1 : L| = 2$. However $L_1 \subset T_2$ and $(z) = Z_G \cap Z(L_1)$ which yields $\langle z \rangle \triangleleft N_X(L)$, a contradiction. We conclude that $z \sim X w$ for some $w \in W - \langle z \rangle$ and hence $z \sim w$ in $N_X(W)$.

For the rest of the proof of this lemma, we will use the “bar-convention” for $N_X(W)/C_X(W)$. (Note that $C_X(W) = W \times \langle \sigma_1 \sigma_2 \rangle$.) We have $N_Y(W) = Z_3^2 \cdot \Sigma_4$ and $\langle \sigma_1 \sigma_2 \rangle = O_3(N_Y(W))$. If $\langle \sigma_1 \sigma_2 \rangle \not\subset O(N_X(W))$ then $N_X(W)/O(N_X(W)) \not\cong \mathbb{Z}_7$ by Gorenstein and Walter’s result [7]. It follows immediately that $N_X(W) = \mathbb{Z}_7$ and that $z$ has 35 conjugates in $N_X(W)$. Since $\sigma_1 \sigma_2 \sim \sigma_1 \sigma_3$ in $N_X(W)$, $j \sim z_1$ and $z_1, z_1 z$ have 21, 7 conjugates, respectively, in $N_X(W)$. The rest of the fusion in (ii) follows from an appropriate choice of $t, uv$ and the fact that $\mathbb{Z}_7$ has one class of involutions.

Suppose now that $\langle \sigma_1 \sigma_2 \rangle \subset O(N_X(W))$. If $j_0 \in T_2 \cap J - W$ then $[j_0, W] = \langle z, j \rangle$ for some $j \in J \cap W$. Thus $|C_{N_X(W)}(j_0)| = 2^3 \cdot 3$ or $2^3 \cdot 3^2$ whence $O(N_X(W)) = \langle \sigma_1 \sigma_2 \rangle$ or $O(N_X(W)) = 4^3$ (recall that $j_0 \in O_2(N_Y(W))$). In the latter case, $|C_{N_X(W)}(j_0)| = 16$, $\langle \sigma_1 \sigma_2 \rangle \subset Z(O(N_X(W)))$. Thus $N_Y(W)$ acts faithfully on $Z(O(N_X(W)))$ which forces $|Z(O(N_X(W)))| > 27$. Hence $O(N_X(W))$ must be abelian, which is a contradiction. It follows that $O(N_X(W)) = \langle \sigma_1 \sigma_2 \rangle$ and, therefore, $\langle z_1, z_2 \rangle = [\sigma_1 \sigma_2, W] \triangleleft N_X(W)$ and $\langle z, t, j_1, j_2 \rangle = C_W(\sigma_1 \sigma_2) \triangleleft N_X(W)$, where $\langle j_1, j_2 \rangle \times Z(J) = J \cap W$. It follows from Gorenstein and Walter’s result [7] and the fact that $N_X(W)$ has a subgroup of index two that $N_X(W) \cong Z_3^3 \cdot \Sigma_5$.

Finally we see that $z$ has only 5 conjugates in $N_X(W)$, whence $|Z_G \cap X| = 5$ also, and take $z \sim G t$ (replacing $t$ by $iz$ if necessary). The lemma is proved.

As in previous sections let $E = C_H(F)$, where $F = C_Y(\langle u, t \rangle)$ ($\cong E_2$). In this case $|E| = 2^{11}$ and $N_H(E)/E \cong Z_3^3 \cdot \Sigma_5$.

**Lemma 6.4.** The subgroup $E$ is elementary abelian of order $2^{11}$ and is weakly closed in $T$ with respect to $G$. In addition, $N_G(E)/E \cong M_{23}$ (the Mathieu group on 23 letters) or $N_G(E)/E$ contains a subgroup of index two which is isomorphic to $PGL(3, 4)$. 

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Proof. If $E$ is nonabelian then $(ut)^2 = z$ and $t \sim_H tz$. Hence we must be in case (i) of Lemma 6.3, so $z^G \cap H \subseteq \{z\} \cup \{ut^H \cup utz^H\}. This leads to a contradiction, as in the proof of Lemma 2.3. The proof that $E$ is weakly closed in $T$ follows as in Lemma 3.3.

Lemma 6.3 implies that there exists $e \in E - \langle z \rangle$ with $z \sim e$ in $N_G(E)$; i.e. $N_G(E) \neq N_H(E)$. The argument in Lemma 5.6 combined with $z^G \cap J = \{z\}$ in cases (i), (iii) of Lemma 6.3 and $\langle zz^J \cap J \rangle = Z(J)$ in case (ii) yield that $O(C_{N(E)}/E(jE)) = 1$ (where $j \in J - E$). In particular we have $O(N_G(E)/E) = 1$.

If $N_G(E)/E$ has a subgroup of index two then this subgroup is isomorphic to $PGL(3, 4)$ by Gorenstein’s and Harada’s result [4]. If $N_G(E)/E$ has no subgroup of index two then $N_G(E)/E \cong M_{22}, M_{23}$, or $U_4(3)$ (by [5], [18]) as $j_0E$ has centralizer of order $< 2^7 \cdot 45$ in $N_G(E)/E$ and $N_G(EE) \subseteq H$. It now follows that $N_G(E)/E \cong M_{23}$, as neither $M_{22}$ nor $U_4(3)$ contains a subgroup isomorphic to $E_{16}(Z_3 \cdot \Sigma_3)$.

Lemma 6.5. If $N_G(E)/E \cong PGL(3, 4) \cdot Z_2$ then $\langle z_1, z_2 \rangle = G$ with $G/\langle z_1, z_2 \rangle = Aut U_6(2)$.

Proof. In this case, $z$ has 21 conjugates in $N_G(E)$ so we are in case (iii) of Lemma 6.3. As $N_X(W)/W \leq Z_3 \cdot \Sigma_5$ and $\langle z_1, z_2 \rangle \leq N_X(W)$, it follows from Theorem 2 that $C_G(\langle z_1, z_2 \rangle)/\langle z_1, z_2 \rangle \cong U_6(2)$. In addition, $C_G(z_1) \subseteq N_G(\langle z_1, z_2 \rangle)$ by Theorem 3. Further, $N_G(\langle z_1, z_2 \rangle)/\langle z_1, z_2 \rangle \cong Aut U_6(2)$, and the structure of $Aut U_6(2)$ yields $N_G(E) \subseteq N_G(\langle z_1, z_2 \rangle)$. Thus $z^G \cap H = \{z\} \cup t^H$ and, in particular, $z^G \cap J = \{z\}$ (by Lemmas 6.3 and 6.4).

As $t \sim_G tz$, $[ut, t] = 1$, and as $z \sim x t$, we have $(ut)^2 = 1$. If $z \sim_G uv$ then $z \sim x uv$ by Proposition 4 and Lemma 6.2. If $K$ is a Sylow 2-subgroup of $C_G(ut)$ then $Z(K) = \langle z, z_1, j, u \rangle$ for some $j \in J \cap T_2$, and we can repeat the argument in Lemma 1.6 to yield $\langle z \rangle \leq N_X(K)$ whence $z \sim_G uv$. Following the proof of Lemma 3.6 we have that $G$ contains a subgroup $G_0$ of index two with $uv \in G - G_0$. If $N = N_G(\langle z_1, z_2 \rangle) \cap G_0$ then $N/\langle z_1, z_2 \rangle \cong U_6(2) \cdot Z_3$ and $N \cap C_G(\langle z_1, z_2 \rangle)$.

We will prove that $C(x) \cap G_0 \subseteq N$ for any involution $x \in N$. Note that $N$ has 5 classes of involutions with representatives $z_1, z, z_1z, t, utz, tzzz, utzzz$ (which are not fused in $G$). Firstly, we observe that $O(C_G(x)) = 1$ for any involution $x \in N$ as $\langle z_1, z_2 \rangle \subseteq C_G(x)$ and $C_G(\langle x \rangle) \subseteq C_G(\langle z_1, z_2 \rangle)$. If $C_N(x) \subseteq C_G(\langle z_1, z_2 \rangle)$ then $C_N(x) \cap G_0 \subseteq N$ by Proposition 2. Thus as $H \cap G_0 \subseteq N$ and the argument of Lemma 2.6 may be repeated to yield $C_G(tz) \subseteq C_G(\langle t, z \rangle) \subseteq N \cap \langle uv \rangle$, it only remains to prove that $C(utz) \cap G_0 \subseteq N$.

Let $C = C(utz) \cap G_0$ and recall $C \cap N \subseteq N(E)$ with $C \cap N/E \cong E_9 \cdot SL(2, 3)$ (see Lemma 2.5). Take $\langle \sigma_1, \sigma_2, d \rangle$ to be a Sylow 3-subgroup of $C \cap N$ with $R = \langle \sigma_1, \sigma_2, d \rangle \subseteq C_G(\langle z_1, z_2 \rangle)$. From the structure of $PGL(3, 4)$, we see there are 3 classes of subgroups of order three in $N_G(E)$ with representatives $\langle \sigma_1, \sigma_2 \rangle$, $\langle c \rangle$, $\langle cd \rangle$. Further, we have

$C_E(\sigma_1 \sigma_2) = \langle z_1, z_2, z, u, t \rangle \cong E_{32}$ and $C(cd) \cap N_G(E) \cong Z_3 \times E_9 \cdot F_{21}$ (F21, a Frobenius group of order 21). It follows that $C_E(\sigma_1 \sigma_2)$ is a Sylow 2-subgroup of $C(\sigma_1 \sigma_2) \cap G_0$ and hence $\langle \sigma_1 \sigma_2 \rangle \sim_G \langle c \rangle$. If $\langle cd \rangle \sim_G \langle \sigma_1 \sigma_2 \rangle$ then $C(cd) \cap G_0$
would contain a Sylow 2-subgroup of order 32 which would be normalized by a nonabelian group of order 21. Since \(7 \mid |N_G(C_e(\sigma_1, \sigma_2))|\) we see that \(\langle cd \rangle \cong G \langle \sigma_1, \sigma_2 \rangle\).

We have proved that \(\langle \sigma_1, \sigma_2, d, c \rangle \) is a Sylow 3-subgroup of \(C\) by proving that \(N_c(R) \subseteq C \cap N\). Since \(N_c(R) \cap N\) has Sylow 2-subgroup \(\langle z_1, z_2, utz \rangle\) it follows that \(N_c(R) = C_c(R) \cdot (C \cap N)\). Further,

\[
N(\langle z_1, z_2, utz \rangle) \cap C_c(R) = R \times \langle z_1, z_2, utz \rangle
\]

whence \(O(C_c(R)) = R\), and Burnside's transfer theorem [3, Theorem 7.4.3] yields \(C_c(R) \subseteq N\).

We have shown therefore that \(\langle \sigma_1, \sigma_2, d, c \rangle\) is a Sylow 3-subgroup of \(C\). As \(N_e(\langle \sigma_1, \sigma_2, d, c \rangle)/\langle R, utz \rangle = Z_6\), Grün's transfer theorem [3, Theorem 7.4.2] yields that \(C\) has a normal subgroup \(C_0\) of index three with \(C_0 \cap N = G_0(\langle z_1, z_2 \rangle)\). It follows from Proposition 2 that \(C \subset N\) as required. This completes the proof that \(N\) contains the centralizer (in \(G_0\)) of each of its involutions. By [3, Theorem 9.2.1], it follows that \(G_0 = N\) or that \(N\) contains one class of involutions. Thus \(N = G_0\) and the lemma is proved.

**Lemma 6.6.** If \(N_G(E)/E \cong M_{23}\) then \(G \cong M(23)\).

**Proof.** If \(N_G(E)/E \cong M_{23}\), Lemma 6.3(ii) holds and \(z, z_1, zz_1\) have 1771, 253, 23 conjugates, respectively, in \(N_G(E)\). Further, as \(M_{23}\) has one class of involutions, \(G\) has precisely 3 classes of involutions. In addition (with appropriate choice of \(ut\)), we have \(z_1 \sim j \sim tz_1, z \sim ut, zz_1 \sim tz\) and if \(e \in E^0\) is not conjugate to one of these involutions in \(N_H(E)\) then \(e \sim z\) in \(N_G(E)\).

Since \(C_H(zz_1)/\langle zz_1 \rangle\) satisfies the hypotheses of Theorem 5, we have \(C_G(zz_1)/\langle zz_1 \rangle = M(22)\). As \(z_1 = [z_2, v], zz_1 \in C_G(zz_1)\), so \(C_G(zz_1)\) is isomorphic to the double cover of \(M(22)\) (see [9]). It follows that \(G \cong M(23)\) (see [12]) once we have proved that \(G\) is simple.

As \(z \sim_{\varphi} jz_1 \sim_{\varphi} jz_1, O(G) = 1\). Thus if \(N \lhd G\), \(N\) contains one of \(z, z_1, zz_1\). However, \(z^G \cap H = z_1^G \cap H = H\), while \(zz_1^G \cap H = H'\), so we conclude \(H \subset N\), whence \(N = G = N_G(T) = T\). This completes the proof of Lemma 6.6 and Theorem C.

**6. The proof of Theorem D.** In this section we will use the notation in the statement of Theorem D, and in addition we put \(J = O_2(H) = E_4 \times O_7 \times Q_6; T = J(u, t), \) a Sylow 2-subgroup of \(H; W = \langle t \rangle \times C_7(t), \) the unique elementary abelian subgroup of order 64 in \(T\); and let \(\langle \sigma_i \rangle\) be a Sylow 3-subgroup of \(K_i\), \(i \in \{1, 2, 3\}.\) Let \([u, z_2] = z_1\) so that \(Z(T) = \langle z, z_1 \rangle\) (recall \(O_2(K_3) = \langle z_1, z_2 \rangle \cap H\)). We suppose \(G \neq H \cdot O(G)\) and give the proof in a series of lemmas.

**Lemma 7.1.** The elements \(z, z_1, zz_1\) lie in distinct conjugacy classes in \(G\) and \(T\) is a Sylow 2-subgroup of \(G\).

**Proof.** As \(Z(T) = \langle z, z_1 \rangle\) and \(\varphi(T') = \langle z \rangle\), the lemma follows from Sylow's theorem and Burnside's lemma.
Lemma 7.2. Either $N_G(W)/W \cong \mathfrak{S}_7$ and $G$ has precisely 3 conjugacy classes of involutions with
\[ z \sim t \sim tz_1z \sim jz_1 \sim u \sim ut \sim z_1 \sim j \sim tz_1 \sim uz \sim z_1z \sim tz; \]
or $N_G(W)/W \cong Z_3 \cdot \Sigma_5$ with $\langle z_1, z_2 \rangle \triangleleft N_G(W)$, $|z^G \cap W| = 5$ and
\[ z \sim t \sim z_1 \sim zz_1 \sim jz \sim jz_1 \sim tzz_1. \]

Proof. Lemma 6.3.

Lemma 7.3. If $N_G(W)/W \cong Z_3 \cdot \Sigma_5$ then $\langle z_1, z_2 \rangle \triangleleft G$ with $G/C_G(\langle z_1, z_2 \rangle) \cong \Sigma_3$ and $C_G(\langle z_1, z_2 \rangle)/\langle z_1, z_2 \rangle \cong PSp_4(3)$.

Proof. By Proposition 6, $C_G(\langle z_1, z_2 \rangle)/\langle z_1, z_2 \rangle \cong PSp_4(3)$. As in the proof of Lemma 6.5, $C_p(u)$ is a Sylow 2-subgroup of $C_G(u)$. Thus $G$ contains a subgroup $G_0$ of index two (Proposition 3) with $G_0 \cap H = K_1 \cdot K_2 \cdot K_3(\langle t \rangle)$. Since $\langle z_1, z_2 \rangle$ is strongly closed in $T \cap G_0$, Proposition 2 yields
\[ C_G(z_1) = O(C_G(z_1)) \cdot (N_G(\langle z_1, z_2 \rangle) \cap C_G(z_1)). \]
As $z \sim t$ and $t_2 \sim j \sim jz$ in $C_G(z_1)$, we have $[t, O(C_G(z_1))] = 1$ whence
\[ [j, O(C_G(z_1))] = [jz, O(C_G(z_1))]. \]

It follows that $\langle z_1, z_2 \rangle \triangleleft G$ by [3, Theorem 9.2.1].

For the rest of this section we will assume $N_G(W)/W \cong \mathfrak{S}_7$.

Lemma 7.4. We have $C_G(zz_1) = U(\langle z_2 \rangle)$, $U/\langle zz_2 \rangle \cong U_4(3)$ ($U$ is a covering group of $U_4(3)$) and $\langle z_2, zz_1 \rangle \triangleleft C_G(z_2)$ with $C_G(z_2)/\langle z_2, zz_1 \rangle \cong \text{Aut PSp}_4(3)$ (and $C_G(\langle z_2, zz_1 \rangle) \cong E_4 \times PSp_4(3)$).

Proof. The structure of $C_G(zz_1)$ follows from Lemma 5.4. As $\langle z, jz_1 \rangle$ normalizes $O(C_G(z_2))$ we have $O(C_G(z_2)) = 1$. In $C(z_2) \cap N_G(W)$ ($\cong E_{64} \cdot \Sigma_3$), $z_2$ has 5 conjugates while $z_1z$ has only two, namely $z_1z, zz_1z$. It follows therefore from Proposition 2 that $\langle z_1z, z_2 \rangle \triangleleft C_G(z_2)$. Thus $C_G(z_2)/\langle z_1z, z_2 \rangle \cong \text{Aut PSp}_4(3)$ by Proposition 6 (as $C_G(z_2)$ contains a subgroup of index two, namely $C_G(\langle z_1, z_2 \rangle)$).

Lemma 7.5. The order of $G$ is $2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$.

Proof. Let $a(g) = |(x, y)(xy)^n = g$ for some positive integer $n$ where $x \sim g z, y \sim g zz_1|$ for any involution $g \in G$. We compute that $a(z) = 2^2 \cdot 3^3, a(z_1) = 3^6 \cdot 5$ and $a(z_1z) = 0$. The lemma follows from Thompson's order formula (see Lemma 2.7).

Lemma 7.6. We have $K = O_3(C_G(\sigma_1\sigma_2^{-1})) \cong E_{35}, C_G(\sigma_1\sigma_2^{-1})/K \cong \Sigma_6$ and
\[ C_G(\sigma_1\sigma_2^{-1}) = \langle \sigma_1\sigma_2^{-1} \rangle \times (C_G(\langle tz \rangle) \cap C_G(\sigma_1\sigma_2^{-1})) \]
with $\langle tz, \sigma_1\sigma_2^{-1} \rangle \cong \Sigma_3$.

Proof. Put $S = \langle z_1, z_2, z_1, u \rangle$, a Sylow 2-subgroup of $C_G(\sigma_1\sigma_2^{-1}) \cap C_G(z_1)$ (recall $z_1 \sim g z_2$). As $S' = \langle z_1 \rangle, S$ is a Sylow 2-subgroup of $C_G(z_1)$. Further, $\langle z_1, z_2, uz \rangle \cong D_8$ with $z_1 \sim z_2 \sim z_1z_2$ in $C_H(\sigma_1\sigma_2^{-1})$ and $z_1 \sim uz \sim uzz_1$ in $C_G(\sigma_1\sigma_2^{-1}) \cap C_G(z_1)$.
Since $S = \langle z_1, z_2, uz \rangle$ contains 3 conjugates each of $z$ and $zz_1$, Proposition 3 yields that $C_G(\sigma_1\sigma_2^{-1})$ has a subgroup of index two with Sylow 2-subgroup isomorphic to $D_8$. It now follows from Gorenstein and Walter [7] that $C_G(\sigma_1\sigma_2^{-1})/K \cong \Sigma_6$ (we know that $C_H(\sigma_1\sigma_2^{-1}) \cong E_9 \cdot (Z_2 \times \Sigma_4)$. $C_G(\sigma_1\sigma_2^{-1}) \cap C_G(zz_1) \cong E_3 \cdot (Z_2 \times \Sigma_4)$). We get $|K| = 3^5$ by applying the Brauer-Wielandt formula [20] to $K\langle z_1, z_2 \rangle$.

Recall that $tz$ inverts $\sigma_1\sigma_2^{-1}$ and that $[t, S] = 1$. It follows that $C_G(tz)$ covers $N_G(\langle \sigma_1\sigma_2^{-1} \rangle)/K$ because $tS$ contains precisely one conjugate of $zz_1$, namely $tz$.

Finally, $C_K(z) = \langle \sigma_1, \sigma_2 \rangle$ so $C_K(tz) \neq 1$. It follows that $C_K(tz) \cong E_8$ and the lemma is proved.

**Lemma 7.7.** The group $G$ is a rank three extension of $C_G(zz_1)$, and $zz_1^G$ is a class of 3-transpositions.

**Proof.** We have $|C_G(z_2) \cap C_G(zz_1)| = 2^8 \cdot 3^4 \cdot 5$ by Lemma 7.4, and by Lemma 7.6 there exists $g \in G$ with $|C_G(zz_1) \cap C_G(zz_f)| = 2^4 \cdot 3^6 \cdot 5$ (since $zz_1 \sim_G tz$). The lemma follows from Lemma 7.5 and $\langle tz, \sigma_1\sigma_2^{-1} \rangle \cong \Sigma_3$.

**Lemma 7.8.** We have $G \cong P \Omega(7, 3) = B_3(3)$.

**Proof.** Suppose $1 \neq N \triangle G$. Then $N \cap Z(T) \neq 1$ as each of $z_1, z_1z_2, z_2$ acts fixed-point free on $O(G)$. This implies that $W \subseteq N$ and hence $N_G(W) \subseteq N$. Thus $T \subset N$ and so $N = G$ as $N_G(T) = T$. It follows now from Fischer’s result [1] that $G \cong P \Omega(7, 3)$.

**PART II**

The following result is proved.

**Theorem.** Let $G$ be a finite group, $z$ an involution in $G$ and suppose $H = C_G(z)$ satisfies:

(i) $J = O_2(H)$ is extra-special of order $2^{13}$ with $C_H(J) \subseteq J$;

(ii) $H$ contains a normal subgroup $K$ with $K/O_{2,3}(H) \cong U_4(3)$ and $O_{2,3}(H)/J \cong Z_3$.

Then one of the following holds:

(a) $G = H \cdot O(G)$;

(b) $H/K \cong Z_2$ and $G$ is a simple group with $|G| = |M(24)|$;

(c) $H/K \cong Z_2 \times Z_2$ and $G \cong M(24)$.

The notation used will be as in Part I, and, in addition, $P_{27}$ will denote a nonabelian group of order 27 and exponent 3.

1. Some properties of $U_4(3)$. Much of the information given below comes from Phan [18]. We will, in general, follow his notation. The group $U_4(3)$ is a simple group of order $2^7 \cdot 3^6 \cdot 5 \cdot 7$ and has only one class of involutions. For an involution $i$ in $U_4(3)$ we have $C(i) = (\overline{L}_i \bullet \overline{L}_j)(\overline{u}, \overline{v})$ where $\overline{L}_i = \langle \overline{a}_i, \overline{b}_i \rangle \langle \overline{a}_i \rangle \cong SL(2, 3)$, $\langle \overline{a}_i, \overline{b}_j \rangle \cong Q_8$, $\langle \overline{a}_i \rangle \cong Z_3$ for $i = 1, 2$, and the relations involving $\overline{u}, \overline{v}$ are as follows:

\[
\overline{u}^2 = (\overline{uv})^2 = 1, \quad \overline{v}^2 = \overline{i} \quad (i.e. \langle \overline{u}, \overline{v} \rangle \cong D_8);
\]
\[
\overline{a}_1\overline{a}_2 = \overline{a}_2, \quad \overline{b}_1 = \overline{b}_2, \quad \overline{\sigma}_1 = \overline{\sigma}_2;
\]
\[
\overline{a}_1\overline{a}_1 = \overline{a}_i^{-1}, \quad \overline{b}_1\overline{b}_2 = \overline{a}_i\overline{b}_2, \quad \overline{a}_1 = \overline{a}_i^{-1}, \quad \overline{b}_2 = \overline{a}_i\overline{b}_2, \quad \overline{a}_1 = \overline{a}_i^{-1}, \quad i = 1, 2.
\]
The Sylow 2-subgroup $O_2(C(i))\langle \bar{u}, \bar{v} \rangle$ contains precisely two elementary abelian subgroups of order 16:

$$F_1 = \langle i, \bar{a}_1\bar{a}_2, \bar{b}_1\bar{b}_2, \bar{u} \rangle, \quad F_2 = \langle i, \bar{a}_1\bar{a}_2, \bar{b}_2\bar{a}_2\bar{b}_1, \bar{u}\bar{v} \rangle$$

with $N(F_i)/F_i \cong \mathbb{Z}_6$, $i = 1, 2$.

There are four classes of elements of order 3 in $U_4(3)$ with representatives \(\bar{a}_1, \bar{a}_1\bar{a}_2, \bar{a}_1\bar{a}_2^{-1}, \bar{a}_3\). For the Sylow 3-subgroup \(\langle \bar{a}_1, \bar{a}_2 \rangle\) of \(C(i)\) we have $N(\langle \bar{a}_1, \bar{a}_2 \rangle) = M \langle \bar{u}, \bar{v} \rangle$ where $M \cong E_{21}$ and $N(M)/M \cong \mathbb{Z}_6$. Further, $C(\bar{a}_1\bar{a}_2^{-1})/M \cong C(\bar{a}_1)/M \cong \mathbb{Z}_4$ and $N(\langle \bar{a}_1, \bar{a}_2 \rangle)/M \cong N(\langle \bar{a}_1, \bar{a}_2 \rangle)/M \cong \Sigma_4$. For \(\bar{a}_1\), we have $C(\bar{a}_1) = U_1 \cdot U_2$ where $U_1 \cong P_{27}$, $U_2 \cong P_{27}$, $N(\bar{a}_1) = C(\bar{a}_1)$ and $L_2\langle \bar{v} \rangle$ acts irreducibly on $U_1/\langle \bar{a}_1 \rangle$. Clearly $U_1\langle \bar{a}_1 \rangle$ is a Sylow 3-subgroup of $U_4(3)$ with centre $\langle \bar{a}_1 \rangle$. Finally, we may assume $C(\bar{a}_1) \subset U_1$, $N(\langle \bar{a}_1 \rangle) = C(\bar{a}_1)$ with $C(\bar{a}_1) \cong \mathbb{Z}_3 \times P_{27}$, and we note that $U_1(M)$ contains 2 (20) conjugates of $\bar{a}_1$, 48 (30) conjugates of each of $\bar{a}_1\bar{a}_2^{-1}, \bar{a}_1\bar{a}_2$ and 144 (0) conjugates of $\bar{a}_3$.

The Sylow 5- and Sylow 7-normalizers in $U_4(3)$ are Frobenius groups of orders 20, 21, respectively.

We now consider $U_4(3)\langle \bar{\pi}_1, \bar{\pi}_2 \rangle$ where $\bar{\pi}_1, \bar{\pi}_2$ are involutory (outer) automorphisms of $U_4(3)$ which satisfy the following relations:

$$(\bar{\pi}_1) = i \quad \text{where} \quad \bar{\pi}_3 = \bar{\pi}_1\bar{\pi}_2 \quad \text{and} \quad [\bar{\pi}_1, \bar{\pi}_2] = [\bar{\pi}_3, \bar{u}] = 1, \quad [\bar{\pi}_1, \bar{v}] = [\bar{\pi}_2, \bar{u}] = [\bar{\pi}_2, \bar{v}] = [\bar{\pi}_3, \bar{u}] = i. \quad \text{For} \quad i = 1, 2, \quad \langle \bar{\pi}_i, F_i \rangle \cong E_{32}, \quad \text{while if} \quad j \neq i \quad \text{and} \quad j \in \{1, 2, 3\}, \quad \langle \bar{\pi}_j, N(F_i) \rangle/F_i \cong \Sigma_6.$$

There are 6 classes of involutions outside $U_4(3)$ with representatives $\bar{\pi}_i, \bar{\pi}_i\bar{a}_1\bar{a}_2$ ($i = 1, 2$), $\bar{\pi}_3\bar{u}, \bar{\pi}_3\bar{a}_1$, which have centralizers in $U_4(3)$ isomorphic to $PSp_4(3)$, $E_{16} \cdot E_9 \cdot Z_4$, $\Sigma_6$ and $U_3(3)$, respectively. In addition, we note that we may assume

$$\bar{\pi}_1 \sim \bar{\pi}_1\bar{u}, \quad \bar{\pi}_1\bar{a}_1\bar{a}_2 \sim \bar{\pi}_1\bar{u} \sim \bar{\pi}_1\bar{v}, \quad \bar{\pi}_2 \sim \bar{\pi}_2\bar{u}\bar{v}i,$$

$$\bar{\pi}_2\bar{a}_1\bar{a}_2 \sim \bar{\pi}_2\bar{u}\bar{v} \sim \bar{\pi}_2\bar{v} \quad \text{and} \quad \bar{\pi}_3\bar{u} \sim \bar{\pi}_3\bar{v} \sim \bar{\pi}_3\bar{u}\bar{v}.$$

With regard to the elements of order 3, $\bar{a}_1$ centralizes $\bar{\pi}_1, \bar{\pi}_2$ and $\bar{\pi}_3\bar{a}_1$, $\bar{a}_1\bar{a}_2^{-1}$ centralizes $\bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_2\bar{u}\bar{v}$ and $\bar{\pi}_3\bar{u}\bar{v}$; $\bar{a}_1\bar{a}_2$ centralizes $\bar{\pi}_1, \bar{\pi}_1\bar{u}, \bar{\pi}_2, \bar{\pi}_3\bar{u}$ while $\bar{a}_3$ centralizes conjugates of $\bar{\pi}_1\bar{a}_1\bar{a}_2$ ($i = 1, 2$) and $\bar{\pi}_3\bar{a}_1$.

Finally we remark that $U_4(3)\langle \bar{\pi}_1, \bar{\pi}_2 \rangle$ is the maximal subgroup of $Aut U_4(3)$ ($|Aut U_4(3)| = 2$) in which $\bar{\pi}_1\bar{a}_1\bar{a}_2 \sim \bar{\pi}_1\bar{a}_2$.

2. Some properties of $H$. We will use the notation of the theorem and note that if $T$ is a Sylow 2-subgroup of $H$ then $Z(T) = \langle z \rangle$ so $T$ is a Sylow 2-subgroup of $G$. Let $\langle c \rangle$ denote a Sylow 3-subgroup of $O_{23}(H)$. Thus $C_k(c)/\langle c, C_j(c) \rangle \cong U_4(3)$. If $C_j(c) \neq \langle z \rangle$ then $C_j(c)$ and $\langle c \rangle, J$ are both extra-special groups of order $< 2^{11}$ (see Part I, Proposition 5). It follows from assumption (i) and [13, pp. 356, 357] that at least one of the orthogonal groups $O^+(10, 2)$ or $O^-(10, 2)$ must contain a subgroup isomorphic to $U_4(3)$. This is easily seen to be impossible by considering the Sylow 3-subgroups of these three groups. Therefore we have proved $C_j(c) = \langle z \rangle$. It follows immediately that $J$ is the central product of 6 quaternion groups (or 6 dihedral groups) of order 8.
The “bar convention” will be used for \( C_k(c)/\langle c, z \rangle \) and this group will be identified with \( U_4(3) \) (as described in §1). As each element of order three in \( U_4(3) \) is conjugate to its inverse we may take \( \sigma_i \) (\( i = 1, 2, 3 \)), \( \sigma_1 \sigma_2^{-1}, \sigma_1 \sigma_2 \) to be elements of order three in \( C_k(c) \), and also assume each of these elements is conjugate to its inverse in \( C_k(c) \). As \( C(\sigma_i) \) acts irreducibly on \( O_3(C(\sigma_i))/\langle \sigma_i \rangle \) and there exists \( \bar{d} \in C(\sigma_i) - O_3(C(\sigma_i)) \) with \( (\bar{d})^3 = \sigma_i \), it follows that \( \sigma_i \) lies in the centre of a Sylow 3-subgroup of \( C_k(c) \). Let \( U_1 \) be a Sylow 3-subgroup of \( O_3(C(\sigma_i) \cap C_k(c)) \) so that \( O_3(C(\sigma_i)) = \bar{U}_1 = U_1 \times \langle \bar{z} \rangle/\langle c, z \rangle \). If \( M \) is a Sylow 3-subgroup of \( O_3(C(\sigma_i) \cap C_k(c)) \) then \( M \cong E_3 \) and \( O_3(C(\sigma_1 \sigma_2)) = \bar{M} = M \times \langle \bar{z} \rangle/\langle c, z \rangle \). Finally we take \( t, u, w \) to be involutions in \( C_k(c) \), so that \( \langle c, \sigma_1, \sigma_2 \rangle \) is a Sylow 3-subgroup of \( C_k(t) \) and \( \langle t, u, v \rangle, \langle c, \sigma_1, \sigma_2 \rangle \) is \( \langle \sigma_1, \sigma_2 \rangle \).

Since we have assumed \( \sigma_1^* = \sigma_1^{-1}, (\sigma_1 c)^* = \sigma_1 c \), it follows that \( C_j(\sigma_1 c) \neq \langle z \rangle \) and \( [\sigma_1, J] \) is the central product of 2, 4 or 6 quaternion groups. As \( \sigma_1 \in U_1 \) and \( \bar{U}_1 \langle \bar{c} \rangle \) acts irreducibly on \( \bar{U}_1 \langle \bar{c} \rangle \), it follows that \( [\sigma_1, J] \) admits a group of automorphisms of order \( \geq 3^6 \). Hence \( [\sigma_1, J] = J(3^6 | |O_3^1(2)) \), \( C_j(\sigma_1) = \langle z \rangle \) and \( C_j(\sigma_1 c) \cong C_j(\sigma_1 c) \cong Q_8 \cdot Q_8 \cdot Q_8 \). Let \( J_2 = C_j(\sigma_1 c), J_3 = C_j(\sigma_1 c) \) so \( J_2 = J_2 \cdot J_3 \) and put \( U_i = C_k(U_i) \cap U_1, i = 2, 3 \).

As \( U_2 \cap U_3 \cap Z(U_i) = 1 \), it follows that \( U_1 = U_2 \times U_3 \) and \( U_i \cong P_{27}, i = 2, 3 \). In particular, \( c \in U_i \) so \( c \in C_k(c) \). Further, \( C(U_i) \cap C_k(c) = U_i \times \langle \bar{z} \rangle \) and \( C_k(c)/\langle U_i, z \rangle \) is isomorphic to a Sylow 3-normalizer in \( PS_p(3) (= P_{27} : SL(2, 3)) \), \( i = 2, 3 \). By the Frattini argument, \( N_H(\langle \sigma_1, c \rangle) \) covers \( H/K \). If \( x \in N_H(\langle \sigma_1, c \rangle) - K \), replacing \( x \) by \( xv \) if necessary, we may assume \( x \) normalizes both \( \langle \sigma_1, c \rangle \) and \( \langle \sigma_1 c \rangle \) and also, therefore, \( J_i, U_i, i = 2, 3 \). Either \( [x, J_2] \neq 1 \) or \( [x, J_3] \neq 1 \) so for \( i = 2, 3 \) (or perhaps both), \( \langle x, C_k(c) \rangle = \langle U_i, z \rangle \) is isomorphic to a subgroup of \( \text{Aut} \, PS_p(3) \cong O_3^- (2) \). It follows immediately that \( x^2 \in C_k(\sigma_1) \) and \( H = K \) or \( H/K \cong Z_2 \) or \( Z_2 \times Z_2 \). We collect all these results in the following lemma.

**Lemma 1.** The subgroup \( J = O_2(H) \) is the central product of six quaternion (dihedral) groups and \( C_j(c) = C_j(\sigma_1) = \langle c \rangle \). We have \( K/J \cong C_k(c)/\langle c \rangle \) is isomorphic to a non-split extension of a group of order three by \( U_4(3) \) and either \( H = K \) or \( H/K \cong Z_2 \) or \( Z_2 \times Z_2 \). Finally if \( J_2 = C_j(\sigma_1 c), J_3 = C_j(\sigma_1 c) \) and \( U_1 = O_3(C_k(c)) \), then \( J_i \cong Q_8 \cdot Q_8 \cdot Q_8, U_i = C_k(U_i) \cap U_1 \cong P_{27}, i = 2, 3 \) and \( U_1 = U_2 \times U_3 \).

If \( \mu, \nu \) are elements of order 5, 7, respectively, in \( C_k(c) \), then from Lemma 1 it follows that \( \langle \mu \rangle, \langle \nu \rangle \) are Sylow 5-, 7-subgroups, respectively, in \( H \) and \( C_k(\mu) \cong Q_8 \cdot Q_8 \). The 48 elements of order three in \( U_i - Z(U_i), i = 2, 3 \), are all conjugate in \( N_k(\langle \sigma_1 \rangle) \) and are therefore conjugate to \( \sigma_1 \sigma_2 \) or \( \sigma_1 \sigma_2 \) in \( K \). We will assume that \( U_2 - \langle \sigma_1 c \rangle \) (only) contains conjugates of \( \sigma_1 \sigma_2 \). It follows immediately therefore that

\[
C_j(\sigma_1 \sigma_2^{-1}) = Q_8 \cdot Q_8 \cdot Q_8
\]

and we take \( C_k(\sigma_1 \sigma_2^{-1}) = C_j(\sigma_1 \sigma_2^{-1}) \cdot M \cdot \langle uv, t \rangle \cdot \langle \sigma_3 \rangle \).

Also, \( \sigma_1 \sigma_2 \sim_H \sigma_1 \sigma_2 \cdot c^i \) and \( C_j(\sigma_1 \sigma_2 \cdot c^i) \cong Q_8, i = \pm 1 \).
With this choice of elements we also have \( \sigma_1 \sigma_2 \sim_K \sigma_1 \sigma_2 c^i, \ i = \pm 1, \)
\[
C_j(\sigma_1 \sigma_2) = [\langle \sigma_1 \sigma_2^{-1} \rangle, J] \cong Q_8 \ast Q_8
\]
and
\[
C_K(\sigma_1 \sigma_2) = C_j(\sigma_1 \sigma_2) \cdot M \cdot \langle u, t \rangle.
\]
In the same way, \( \sigma_3 \sim_K \sigma_3 c^i, \ i = \pm 1, \)
\[C_j(\sigma_3) \cong Q_8 \ast Q_8 \ast Q_8 \quad \text{and} \quad C_K(\sigma_3) = C_j(\sigma_3) \cdot (C_K(\sigma_3) \cap U_1).\]
We have proved

**Lemma 2.** The group \( H \) has 7 classes of subgroups of order three with representatives \( \langle c \rangle, \langle \sigma_1 \rangle, \langle \sigma_1 c \rangle, \langle \sigma_1 \sigma_2^{-1} \rangle, \langle \sigma_1 \sigma_2^{-1} c \rangle, \langle \sigma_1 \sigma_2 \rangle, \langle \sigma_3 \rangle, \) where

\[C_j(\sigma_1 \sigma_2^{-1}) \cong Q_8 \ast Q_8 \ast Q_8, \ C_j(\sigma_1 \sigma_2^{-1} c) \cong Q_8 \]

and
\[C_j(\sigma_1 \sigma_2) \cong C_j(\sigma_3) \cong Q_8 \ast Q_8.\]

Further, we have \( C_j(\mu) \cong Q_8 \ast Q_8, \ C_j(\nu) = \langle z \rangle, \) where \( \langle \mu \rangle, \langle \nu \rangle \) are, respectively, Sylow 5-, 3-subgroups of \( H. \)

As \( t \) centralizes the subgroup \( \langle c, \sigma_1 \sigma_2^{-1} \rangle \) which acts faithfully on \( C_j(\sigma_1 \sigma_2) \) we have \( [t, C_j(\sigma_1 \sigma_2)] = 1. \) If \( M_1 = [\langle t, w u \rangle, J] \times \langle c \rangle \cong E_{81} \) then \( M = M_1 \times \langle \sigma_1 \sigma_2^{-1} \rangle \) and \( C_H(t) \cap M_1 = \langle c, \sigma_1 \sigma_2 \rangle. \) Let \( \tau_1 \in M_1 \) with

\[
C_j(\tau_1) \cap C_j(\sigma_1 \sigma_2^{-1}) \cong Q_8 \ast Q_8 \ast Q_8
\]
(as \( C_H(\sigma_1 \sigma_2^{-1})/O_{2,3}(C_H(\sigma_1 \sigma_2^{-1})) \cong \hat{\mathbb{Z}}_4, \) \( M_1 \) acts faithfully on \( C_j(\sigma_1 \sigma_2^{-1}) \)). Now \( \tau_1 \) has 4 conjugates in \( \langle w u, t \rangle \cdot M_1 \) whence \( C_j(\sigma_1 \sigma_2^{-1}) \cdot M_1 \) is the central product of four subgroups each isomorphic to \( \text{SL}(2, 3). \) These four subgroups are permuted by \( C_H(\sigma_1 \sigma_2^{-1})/C_j(\sigma_1 \sigma_2^{-1}) \cdot M \) (as \( \hat{\mathbb{Z}}_4 \)) acting in the natural way. Hence \( C(t) \cap C_j(\sigma_1 \sigma_2^{-1}) \cong E_{32} \) and therefore \( C_j(t) \cong E_{16} \times Q_8 \ast Q_8. \) It follows that \( J \) contains 3 classes of involutions in \( K \) with representatives \( t, tz, tj \) (\( j \) an involution in \( C_j(\sigma_1 \sigma_2) \) and \( t \neq K tz \) as \( (ut)^2 = 1 \) and \( v^2 \in \langle z \rangle \)).

In the same way as above we see that \( [wu, C_j(\sigma_1 \sigma_2)] = 1 \) and \( C_j(\sigma_1 \sigma_2) \cap C_j(\sigma_1 \sigma_2^{-1}) \cong E_{32} \). Further, \( Z = Z(C_j(\langle w, t \rangle J)) \cong E_{81}, \) all involutions in \( Z \) are conjugate in \( \langle J, c \rangle \) and therefore \( C_H(Z) \) covers \( F_2J/J \) where \( F_2 = F_2J, z/J\langle c, z \rangle \cong E_{16}. \) Similarly \( C_H(Z) \) covers \( N_K(F_2)J/J \) so if \( j_2 \in Z \) then \( C_K(j_2)/C_j(j_2) \cong E_{16} \ast \hat{\mathbb{Z}}_4 \) (as this subgroup is maximal in \( U_d(3) \)).

Let \( j_1 \) be an involution in \( C_j(\sigma_1 c) = \langle z \rangle. \) Without loss we may take \( U_2(\sigma_2 c) \) to be a Sylow 3-subgroup of \( C_j(\sigma_1 c) \) and \( U_2(\sigma_2 c) \) is a Sylow 3-subgroup of \( C_H(j_1). \) It now follows by counting the number of involutions in \( J \) that \( |C_K(j_1)/C_j(j_1)| = 2^5 \cdot 3^4 \cdot 5 \) or \( 2^6 \cdot 3^4 \cdot 5. \) The structure of \( U_d(3) \) immediately yields \( C_K(j_1)/C_j(j_1) \cong PSp_4(3). \) We have therefore determined all conjugacy classes of involutions in \( K. \)

**Lemma 3.** There are precisely two conjugacy classes of involutions in \( J \) with representatives \( j_1, j_2 \) where \( C_K(j_1)/C_j(j_1) \cong PSp_4(3) \) and \( C_K(j_2)/C_j(j_2) \cong E_{16} \ast \hat{\mathbb{Z}}_4. \) In \( K - J \) there are precisely three classes of involutions with representatives \( t, tz, tj \) where \( j \in C_j(\sigma_1 \sigma_2) = \langle z \rangle. \)
If \( H \neq K \), it follows from our assumptions and the lemmas above that \( H = K\langle \pi_1 \rangle \) or \( H = K\langle \pi_1, \pi_2 \rangle \) where \( \pi_i \in N_H(\langle c \rangle) \) and \( \pi_i(c, z)/\langle c, z \rangle = \tilde{\pi}_i, \ i = 1, 2, 3 \), correspond to the elements introduced in the previous section. In addition we may assume \( \pi_i^2 \in \langle z \rangle, i = 1, 2, \) and \( \pi_3^2 = \tau \).

Suppose \( [\pi_i, c] = 1 \) for \( i = 1, 2 \) or 3. As we may assume \( [\pi_i, \langle \sigma_1, \sigma_2 \rangle] = 1 \) and as \( C_K(\pi_i) \) covers \( C_K(\sigma_1)/O_{3,2}(C_K(\sigma_1)) \), it follows we may assume \( C(\pi_i) \cap U_1 = \langle U_2, c \rangle, C(\pi_i) \cap U_1 = \langle U_3, c \rangle \). Thus \( \pi_i^* = \pi_i t \) so \( i = 1 \) or 2; i.e. \( \pi_3 \) inverts \( c \). As \( U_2 \) acts irreducibly on \( J_3/\langle z \rangle, [\pi_i, J_3] = 1 \). Further \( \pi_i \) inverts an element \( \lambda \) in \( U_3 - \langle \sigma_1^{-1} c \rangle, \lambda \sim_K \sigma_1 \sigma_2^{-1} \). As \( \langle \lambda \rangle \sim \langle \lambda \sigma_1^{-1} c \rangle \sim \langle \lambda \sigma_1^{-1} c \rangle \) in \( U_3 \) and \( C(\lambda) \cap J_2 = Q_8 \), it follows that \( [\pi_i, C(\lambda) \cap J_2] = 1 \) while \( \pi_i \) swaps the two quaternion groups \( C(\lambda \sigma_1^{-1} c) \cap J_2, C(\lambda \sigma_1 c^{-1}) \cap J_2 \). Thus

\[
C_J(\pi_i) \cong E_4 \times Q_8 \times Q_8 \times Q_8 \times Q_8,
\]

and we suppose \( j_1 \in C(J(\pi_i)) \). Then all involutions in \( Z(C_J(\pi_i)) - \langle z \rangle \) are conjugate to \( j_1 \) (in \( <c, J_3> \)) and \( C(j_1) \) covers \( C(\pi_i)/C_J(\pi_i) \cdot \langle c \rangle \cong PSp_4(3) \).

If \( <\mu> \) is a Sylow 5-subgroup of \( C_K(\pi_i) \) then \( C_5(\mu) \cap C_J(\pi_i) = Z(C_J(\pi_i)) \). Now if \( <\mu'> \) is a Sylow 5-subgroup of \( C_K(\mu_2) \) as \( <\pi_2, F_2, J_3>/J = E_{32} \) we may suppose \( j_2 \in C(J(\pi_2) \cap C(j_2') \). Since \( Z(C_J(\pi_i)) \) only contains conjugates of \( j_1 \) we have proved that \( \pi_2 \) inverts \( c \) also.

Finally, we will show \( [\pi_i, c] = 1 \). From \( \S 1 \), \( \pi_1 u \) centralizes a Sylow 3-subgroup of \( N(\tilde{F}_1) \) in \( C(c)/\langle c, z \rangle \). However a Sylow 3-subgroup of \( N(\tilde{F}_1) \) contains conjugates of \( \tilde{\sigma}_1 \tilde{\sigma}_2, \tilde{\sigma}_3 \) and \( C_K(\sigma_1 \sigma_2) = C_J(\sigma_1 \sigma_2) \cdot M(\mu, t) \). Therefore \( c \in C(\pi_i) \cap C_K(\sigma_1 \sigma_2) \) because \( M \) does not contain conjugates of \( \sigma_2 \); i.e. a Sylow 3-subgroup of \( N_K(F_1J) \) cannot split over \( \langle c \rangle \).

If \( h \in H - K \) with \( h^2 \in \langle z \rangle \) and \( c^h = c^{-1} \), then \( C_J(h) \) is elementary abelian of order 64 or 128 depending whether \( h \sim_H hz \) or not. Since \( 3^3 \mid |GL(5, 2)| \), we must have \( C_J(\pi) \cong C_J(\sigma_1 \sigma_2) \cong E_{128} \). Let \( \langle \sigma_1 \sigma_2^{-1}, \sigma_3 \rangle \) be a Sylow 3-subgroup of \( C_H(\pi_2 \mu \sigma) \), \( \sigma_3 \sim_K \sigma_3 \), and note that \( C_J(\sigma_3) \subset C_J(\sigma_1 \sigma_2^{-1}) \). Thus \( \pi_2 \mu \sigma \) acts fixed-point-free on \( O_{3,2,3}(C_J(\sigma_3))/O_{3,2}(C_H(\sigma_3)) \cong E_9 \) whence \( \pi_2 \mu \sigma \) normalizes the two quaternion subgroups in \( C_J(\sigma_3) \) and so \( \pi_2 \mu \sigma \sim_K \pi_2 \mu \sigma \). Similarly, \( \pi_2 \mu \sigma \) acts fixed-point-free on \( \langle \sigma_1 \sigma_2^{-1}, c \rangle \) and so normalizes the two quaternion subgroups in \( C_J(\sigma_1 \sigma_2) \). Thus \( \pi_2 \mu \sigma \cong C_J(\pi_3 \mu) \cong E_{64} \). These results are collected in the following lemma.

**Lemma 4.** If \( \pi_1 \in H \) then \( [\pi_1, c] = 1 \) and \( C_J(\pi_1) \cong E_4 \times Q_8 \times Q_8 \times Q_8 \times Q_8 \). If \( \pi_2 \in H \) then \( \pi_2 \mu \sigma \) and if \( h \in \pi_2 K \) with \( h^2 \in \langle z \rangle \), \( h \) is conjugate to one of \( \pi_2 \), \( \pi_2 \mu \sigma \), \( \pi_2 \mu \sigma \mu \sigma \cdot C_H(\pi_2) \cong E_{128}, C_J(\pi_2 \mu \sigma) \cong E_{64} \). Finally, if \( \pi_3 \in H \) then \( \pi_3 \mu \sigma \) inverts \( c \) also and any element in \( \pi_3 K \) with square in \( \langle z \rangle \) is conjugate to one of \( \pi_3 \mu \sigma \), \( \pi_3 \mu \sigma \mu \sigma \), \( \pi_3 \mu \sigma \mu \sigma \mu \sigma \) whose \( C_J(\pi_3 \mu \sigma) \cong E_{128} \) and \( C_J(\pi_3 \mu \sigma) \cong E_{64} \).

**3. The proof of case (b).**

**Lemma 5.** Let \( G \) be a finite group which satisfies the assumptions of the theorem. Then \( z \sim_G j_1 \), and if \( z \sim_G j_2 \), then either \( H = K \langle \pi_2 \rangle \) or \( |H : K| = 4 \).

**Proof.** If \( x \) is an involution in \( J - \langle z \rangle \), \( z \sim_G x \) implies that \( O_2(C_H(x))/C_J(x) \) contains a normal elementary abelian subgroup of order 32 by [16, Proposition 7].
It follows from Lemma 3 and the structure of $N(F_2)$ given in §1 that $z \sim_G j_1$ and $z \sim_G j_2$ implies $\pi_2 \in H$. The lemma is proved.

The following simple result will be needed a number of times in this section.

**Proposition.** Let $G$ be a finite group, $z$ an involution in $G$ and $H = C_G(z)$. Suppose that $P \neq 1$ is a $p$-subgroup of $H$ ($p$ an odd prime) which satisfies

$(\ast)$ if $g^{-1}Pg \subset H$, $g \in G$, then there exists $h \in H$ so that $g^{-1}Pg = h^{-1}Ph$.

Then for any involution $x \in C_H(P)$, $x \sim_G z$ if and only if $x \sim_{N(P)} z$. If, in addition, $N_G(P) = N_H(P)C_G(P)$, then $x \sim_G z$ if and only if $x \sim_{C(P)} z$.

For the rest of this section, all results will be proved under the following assumptions.

**Hypothesis 1.** Let $G$ be a finite group which satisfies the assumptions of the theorem. In addition, suppose $G \neq H \cdot O(G)$ and $H = K = K\langle \pi_2 \rangle$ or $H = K\langle \pi_3 \rangle$.

**Lemma 6.** We have $z \sim_{G} \pi_3u$ and $z \sim_{G} j_2$ if and only if $z \sim_{G} \pi_2$ and $j_1 \sim_{G} \pi_2z$.

**Proof.** If $H = K\langle \pi_3 \rangle$ then $z \sim_{G} j_2$ by Lemma 5. If $\langle \mu \rangle$ is a Sylow 5-subgroup of $C_H(\pi_3u)$ then by the proposition, $z \sim_{G} \pi_3u$ if and only if $z \sim \pi_3u$ in $C_G(\mu)$. For some $j' \in C_j(\mu) - \langle \pi_3u, z, j' \rangle$ is a Sylow 2-subgroup of $C_H(\mu) \cap C_G(\pi_3u)$. If $z \sim_{G} \pi_3u$ then $(j', j'z) \subset N_G(\langle \pi_3u, z, j' \rangle)$ whence $\langle \pi_3u, z, j' \rangle$ is a Sylow 2-subgroup of $C_G(\mu)$. Thus $z \sim_{G} \pi_3u$.

Now suppose $H = K\langle \pi_2 \rangle$. Without loss we take $\pi_2 \subset C_H(\mu)$ and observe that if $\pi_2^2 = 1$, $C_H(\mu)/\langle \mu \rangle$ is isomorphic to the centralizer of a central involution in $\vartheta_8$ ($C_j(\pi_2) \cap C_j(\mu) \cong E_8$). Since $C_H(j_2)/O_2(C_H(j_2)) \cong \vartheta_6$, if we assume $j_2 \subset C_j(\mu)$ also we have $C_H(j_2)$ does not cover $N_H(\langle \mu \rangle)/C_H(\mu)$ whence $j_2$ has 12 conjugates in $C_H(\mu)$ (the other 6 are conjugate to $j_1$). If $C_G(\mu)$ has a subgroup $K$ of index two then $K \cap C_H(\mu) = \langle \mu \rangle \times C_j(\mu) \cdot \langle e \rangle$ and $\langle z \rangle$ is weakly closed in $C_H(\mu)$ with respect to $C_G(\mu)$. On the other hand if $C_G(\mu)$ has no subgroup of index two we see easily that $\pi_2^2 = 1$. A result of D. Held [24] yields $C_G(\mu)/\langle \mu \rangle \cong \vartheta_8$, $\vartheta_9$ or Hol($E_8$) = $E_8 \cdot D_2(7)$. The last case cannot occur as $z \sim_{G} j_1$.

Thus we have proved that in $C_G(\mu)$, either $z$ is not conjugate to any involution in $C_H(\mu) - \langle z \rangle$ or $z \sim j_2 \sim \pi_2$, say, and $j_1 \sim \pi_2z$. The corresponding result for $G$ follows from the proposition.

**Lemma 7.** Either $\langle \sigma_1 \sigma_2 \rangle$ satisfies condition $(\ast)$ of the proposition (i.e. $\langle \sigma_1 \sigma_2 \rangle^H = \langle \sigma_1 \sigma_2 \rangle^G \cap H$) or $\sigma_1 \sigma_2 \sim_{G} c$, $H = K\langle \pi_2 \rangle$ and

$z \sim_{G} j_2 \sim_{G} j \sim_{G} j_1 \sim_{G} \pi_2$, $j_1 \sim_{G} tz \sim_{G} \pi_2z$.

**Proof.** We begin by listing some properties of Sylow 2-subgroups of the centralizers of elements of order three in $H$. In the table, $y$ is an element of order 3, $Y$ is a Sylow 2-subgroup of $C_H(y)$, $Y/Y \cap J$ is given for $H = K\langle \pi_2 \rangle$ (and $H = K\langle \pi_3 \rangle$). Recall that $Y \cap J = \langle z \rangle$ or $Y \cap J$ is the central product of quaternion groups.

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In addition we note that when \( H = K \), \( T_i \) \((i = 1, \ldots, 6)\) has order \(2^8, 2^4, 2^{11}, 2^7, 2^{10}, 2^5, 2^5\), respectively.

If \( H = K \) then \( <z> = T_3 \cap Z(T_3) \), while if \( H \neq K \), \( Z(T_3) \subseteq T_3 \subseteq <t, T_3 \cap J> \) whence \( <z> = Z(T_3) \cap [T_3, T_3] \). Thus \( T_3 \) is a Sylow 2-subgroup of \( C_G(\sigma_1\sigma_2) \) so \( \sigma_1\sigma_2^{-1} \sim_G \sigma_1\sigma_2 \sim_G \sigma_1 \), and, in addition, \( c \sim_G \sigma_1\sigma_2 \) if \( H = K \). Clearly \( \Omega_i(T_3) = <z> \) and \( Z(T_3) = <z> \) so \( T_3, T_6 \) are Sylow 2-subgroups of \( C_G(\sigma_1\sigma_2^{-1}) \), \( C_G(\sigma_3) \), respectively. In particular, \( c, \sigma_2^{-1}c \sim_G \sigma_1\sigma_2 \sim_G \sigma_3 \). Since \( Z(T_3) = <t, z> \), if \( x \) is an involution in \( T_3 \) then \( C_G(x) \cap T_3 \) contains an elementary abelian subgroup of order 16.

As \( F, \sigma_1 \), clearly contains no such subgroup, \( \sigma_1 \sim_G \sigma_1\sigma_2 \).

It remains to consider the case \( \sigma_1\sigma_2 \sim_G c \). We may assume here that \( t \sim_G z \) \((Z(T_0) = Z(T_3) = <t, z>)\), whence

\[
C_G(t) \cap C_G(\sigma_1\sigma_2)/<t, \sigma_1\sigma_2> \approx U_4(3).
\]

As \( U_4(3) \) has one class of involutions, \( z \sim \pi_3u \) or \( \pi_3ut \) in \( C_G(\sigma_1\sigma_2) \). This is impossible as \( \pi_3u \sim_H \pi_3ut \) and \( z \not\sim \pi_3u \) by Lemma 6. Thus we have \( H = K(\pi_2) \).

We note that if \( j \in T_3 \cap J \) then \( j \sim_H j_1 \) as \( C_H(j_1) \) only contains conjugates of \( \sigma_1\sigma_2^{-1}, \sigma_3 \). It follows therefore from \( z(<t> \sim j<\langle t \rangle) \) that \( z \sim tj \) and \( j_1 \sim tz \). Also from \( z(<t> \sim \pi_2<\langle t \rangle) \) we may assume \( z \sim \pi_2 \) and \( \pi_2t \sim j_1 \) (thus \( \pi_2t \sim_H \pi_2z \)). This completes the proof of the lemma.

**Lemma 8.** If \( z \sim_G j_2 \) then \( z^G \cap K = \{z\} \). In particular, \( H \neq K \).

**Proof.** If \( z \sim_G j_2 \) then \( <\sigma_1\sigma_2> \) satisfies \((*)\) of the proposition. It follows therefore from \( Z(T_3) = <t, z> \) and Burnside's lemma [13, IV, 2.5] that \( t \not\sim_G z \sim_G tz \). If \( z \sim_G tj \), Lemma 6 implies that \( z^G \cap T_3 \subseteq J \langle t \rangle \cap T_3 \). However

\[
<z^G \cap C(j) \cap T_3> = \langle tj \rangle \times C_j(j) \cap T_3 \approx Z_2 \times Z_2 \times D_8
\]

and so \( C(j) \cap T_3 \) is a Sylow 2-subgroup of \( C(j) \cap C_G(\sigma_1\sigma_2) \). Thus \( z^G \cap K = \{z\} \).

If \( H = K \), Glauberman's theorem [2] yields \( G = H \cdot O(G) \) against Hypothesis 1. The lemma is proved.

**Lemma 9.** We have \( H = K(\pi_2) \).

**Proof.** Suppose \( H = K(\pi_3) \) whence \( z \sim_G j_2 \). We will show \( z \sim_G \pi_3a_1 \), and hence \( z^G \cap H = \{z\} \) by Lemmas 6 and 8. However, this will contradict Glauberman's result [2]; i.e. we will have shown \( H \neq K(\pi_3) \).
We may assume $\pi_3a_1 \in T_1$, where $T_1/\langle z \rangle \cong Q_8 \rtimes Z_4$ in this case. Since $\langle \sigma_2 \rangle$ normalizes $T_1$ but does not centralize $T_1 \cap K \cong Z_2 \times Q_8$, $Z(T_1) = \langle \pi_3, z \rangle$ say with $\pi_3^2 = t$. By Lemma 8, $tz \not\sim_G t \sim_G t$ so $T_1$ is a Sylow 2-subgroup of $C_G(\pi_3)$. It follows immediately that $\langle \sigma_1 \rangle$ satisfies $(*)$ of the proposition as $\Omega^1(T_3) = \langle z \rangle$ whereas $T_1$ contains $t$ or $tz$. In order to complete the proof of the lemma we only need to show $z \sim_G \pi_3a_1$ in $C_G(\sigma_1)$.

Now $C(\pi_3a_1) \cap T_1 = \langle \pi_3a_1, \pi_3, z \rangle$ so $\Omega^1(C(\pi_3a_1) \cap T_1) = \langle t \rangle$. Thus if $\pi_3a_1 \sim z$ in $C_G(\sigma_1)$ there exists $g \in C_G(t) \cap C_G(\pi_3)$ with $z^g \in T_1 - \langle z \rangle$. However $T_1/\langle t \rangle \cong E_{16}$ (because $\pi_3a_1$ is an involution, $a_1^2 = t$ also) whence $z\langle t \rangle$, $z^8\langle t \rangle$ are conjugate in $N(T_1)/\langle t \rangle$ by Burnside's lemma. This is clearly impossible so $H \neq K(\pi_3)$. It follows therefore from Lemma 8 that $H = K(\pi_2)$.

**Lemma 10.** We have $z \sim_G j_2$ if and only if $z \sim_G \pi_2uv$. Therefore we have $z \sim_G j_2$. Further, $c \sim_G \sigma_1\sigma_2$ and the fusion of involutions has been completely determined.

**Proof.** If $\sigma_3 \sim_G \sigma_1$ then there exists $g \in G$ with $T_3^g \subset T_6$. However $T_1 \cong Z_2 \times Z_2 \times Q_8$ so $T_3^g \cap T_6 \cap J$ is abelian of order 16, or isomorphic to $Z_2 \times Q_8$. Both of these cases are impossible ($T_6 \cap J \cong Q_8 \rtimes Q_8$). It follows that $\langle \sigma_3 \rangle$ satisfies condition $(*)$ of the proposition—see the proof of Lemma 7. Since $T_6$ is of type $G_2(3)$, we easily show $z \sim j_2$ in $C_G(\pi_3)$ if and only if $z \sim \pi_2uv$ ($T_6$ has at most three classes of involution in $C_H(\pi_3)$ with representatives $z, j_2, \pi_2uv, j_2 \sim_H j_2$). The first part of the lemma now follows from the proposition.

If $z \sim_G j_2$, it follows from Lemmas 6 and 8 that $z^G \cap H = \{z\}$. We have the same contradiction as in the previous two lemmas. It remains to show therefore that $z \sim_G j_2$ implies $\sigma_1\sigma_2 \sim_G c$.

Suppose to the contrary that $\sigma_1\sigma_2 \sim_G c$. By Lemma 7, $\langle \sigma_1\sigma_2 \rangle$ satisfies $(*)$ of the proposition, whence by Lemma 6 we have $z \sim \pi_2$ in $C_G(\sigma_1\sigma_2)$. Let $X = C_G(\sigma_1\sigma_2)$, $Y = C_H(\sigma_1\sigma_2)$ and $W = \langle \pi_3, t \rangle \times (C_G(\pi_3) \cap T_3) \cong E_{32}$. We easily see that $T_3$ contains only two elementary abelian subgroups of order 32, both of which are normal in $T_3$. Thus $z \sim \pi_2$ in $N_X(W)$. As above, $Z(T_3) = \langle t, z \rangle$ so $t, tz$ lie in distinct conjugacy classes in $X$. We have $C_X(W) = W \times \langle \sigma_1\sigma_2 \rangle$, $N_Y(W)/C_X(W) = \Sigma_4$, and in $N_Y(W)$, $z(1), t(1), tz(1), j(6), tj(6), \pi_2(8), \pi_2z(8)$ ($j \in T_3 \cap J$) represent the classes of involutions in $W$, with the number of conjugates given in brackets. Since $27 | |GL(5, 2)|$, $z$ cannot have 9 conjugates in $N_X(W)$ so $z$ has 15 conjugates in $N_X(W)$: $z \sim \pi_2 \sim tj$. Replacing $t$ by $tz$ if necessary we have $iz \sim j \sim \pi_2z$ (recall $j \sim_H j_1 \sim_G \pi_2x$ by Lemma 6) and $\langle t \rangle \not\subset N_X(W)$. It now follows from Part I, Proposition 6, $z\langle t \rangle \sim j\langle t \rangle$ in $N_X(W)/\langle t \rangle$ and the fact that $\pi_2uv(J \cap T_3)$ does not contain involutions, that $C_X(t)/\langle t, \sigma_1\sigma_2 \rangle \cong U_4(3)$. However, this yields $z\langle t \rangle \sim_{CH} u\langle t \rangle$ so $z \sim x t$ or $z \sim x tz$, a contradiction. This completes the proof of the lemma.

**Lemma 11.** Let $F = C_f(t) \cap C_f(u)$ and $E = C_H(F)$. Then $E \cong E_{24}$ and $N_G(E)/E \cong M_{24}$, the Mathieu group on 24 letters. Further we have $C_G(j_1) \cap N_G(E)/E \cong \text{Aut } M_{22}$.
Proof. As \( u \sim t \sim ut \) in \( N_H(\langle \sigma_1 \sigma_2, c \rangle) \) and \( C_J(\langle u, t \rangle) \cap C_J(\sigma_1 \sigma_2) = E_b \), we have \( F = C_J(t) \cap C_J(u) = E_2 \). In the same way as in Part I, §3, we have \( |C_H(t)|: N_H(F) \cap C_H(t) = |C_H(u): N_H(F) \cap C_H(u)| = 3 \), from which it follows that \( N_H(F)/J \cong E_{16} \cdot Z_3 \cdot \Sigma_6 \). Hence \( E = C_H(F) \) covers \( O_2(N_H(F))/J \) whence \( E = E_{21} \).

Let \( T \) be a Sylow 2-subgroup of \( H \) so that \( C_H(t) \) covers \( T/J \). Suppose \( E^g \subseteq T \) for some \( g \in G \). We will show \( E^g = E \). Observe first that \( 2^8 < |E^g \cap J| < 2^7, 2^4 < |E^g \cdot J| \cdot |J| < 2^5 \) and \( E^g \cap J \neq \emptyset \). If \( E^g \subseteq K \cap T \) then we have \( E^g \cap J \neq \emptyset \) or \( E^g \cap uJ \neq \emptyset \). Since \( C_j(t_j) \subseteq C_j(t) \) for any \( t_j \in J \), and as \( C_j(u) \cap C_j(t) \cong E_4 \times Q_8 \times Q_8 \), we have \( E^g \cap J = F \) and so \( E^g = E \). If \( E^g \not\subseteq K \cap T \) then \( E^g \) contains conjugates of \( \tau_2 \nu \), whence \( |E^g \cap J| = 64 \). It follows that \( E^g \cap J \neq \emptyset \) or \( E^g \cap uJ \neq \emptyset \) whence \( E^g \cap J \subseteq F \). However if \( e \in E^g \) and \( eJ \sim_h \tau_2 \nu J \), then as \( e \) inverts \( c, |C_c(e)| = 2^4 \). This contradiction means we have proved that \( E \) is weakly closed in \( T \) with respect to \( G \).

There are 1771 conjugates of \( z \) in \( E \) (the remaining 276 are conjugate to \( j_1 \)) and they are all conjugate to \( z \) in \( N_G(E) \) by the above remarks. Thus \( |N_G(E)/E| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 = |M_{24}|. \) If \( R \) is a Sylow 23-subgroup of \( N_G(E) \) then \( C_E(R) = 1 \) from which it follows that \( N_G(E)/E \) is a simple group. A result of R. Stanton [28] yields \( N_G(E)/E \cong M_{24}. \) The last fact, that \( C_G(j_1) \cap N_G(E)/E \cong Aut M_{22}, \) follows from the structure of \( M_{24} \) (see [23], for example).

Lemma 12. We have \( C_G(j_1)/\langle j_1 \rangle \cong Aut M_{22} \) with \( j_1 \in C_G(j_1)' \).

Proof. Set \( C_j(\sigma_1 \sigma_2) = \langle r_1, s_1 \rangle \ast \langle r_2, s_2 \rangle \) where \( \langle r_1, s_1 \rangle \cong Q_8, i = 1, 2, \) are chosen so that we have

\[
\begin{align*}
r_1^u &= r_2, & s_1^u &= s_2, & r_1^s &= r_2^{-1}, & s_1^s &= r_2 s_2.
\end{align*}
\]

Let \( j = r_1 r_2, z_1 = s_1 s_2 \) and \( j_0 = s_1 r_2 s_2. \) It follows that we have

\[
\langle u, \tau_2 \rangle \subseteq C_H(j), \quad z_1^u = z_1, \quad z_1^\tau_2 = z_1 j, \quad j_0 = j_0 j, \quad j_0^\tau_2 = j_0.
\]

Recall that \( j_1 \sim_h j \sim_h z_1 \sim_h j_0 \) and that \( |O_{23}(C_H(\sigma_1 \sigma_2))| = 3^3 \). If \( D = \langle j, z_1, z \rangle \) then \( D \subseteq C_j(u) \cap C_j(t) = F \) so \( [D, E] = 1 \). Also \( [D, O_{23}(C_H(\sigma_1 \sigma_2))] = 1 \), so from the structure of \( PSp_4(3) \) we have \( D \triangleleft C_H(j) \). Clearly \( C_H(D)/C_J(D) \cong PSp_4(3) \) and \( C_H(D)/j_0 \tau_2 \cong C_H(j) \).

Let \( l \) be an involution in \( C_H(j) - \langle z, j \rangle \) and \( L \) a Sylow 2-subgroup of \( C_H(l) \cap C_H(j) \). Since \( Z(L \cap J) = \langle l, z, j \rangle \), it follows that \( Z(L) = \langle l, j, z \rangle \). If \( S \) is a Sylow 2-subgroup of \( C_H(j_0) \cap C_H(j) \) then as \( j_0 \not\in S' \) we have \( \langle z \rangle \subseteq S' \cap Z(S) \subseteq \langle j, z \rangle \). Thus \( S \) is a Sylow 2-subgroup of \( C_G(j_0) \cap C_G(j) \) (as \( j \sim_h jz \sim_h j_1 \)), and if \( j_0 \sim l \) in \( C_G(j) \) for \( l \in C_G(j) \), then \( j_0 \sim l \) in \( C_H(j) \).

In \( N_G(E) \cap C_G(j) \) we see that \( z \) (and hence \( jz \)) has 231 conjugates and \( z_1 \) has 44. Thus if \( e \in E \) with \( e \sim_G j_1 \) then \( e \) is conjugate to \( j, jz \) or \( z_1 \) in \( N_G(E) \cap C_G(j) \). If \( x \in (N_G(E) \cap C_G(j))' - E \) with \( x \sim_G j_1 \), then \( xe \sim_j' E \) where we may take \( j' \in C_j(u) \cap C_j(j) - E \) (\( M_{22} \) has only one class of involutions). As all involutions in \( uJ \) which are conjugate to \( j_1 \) in \( G \) lie in \( E \), it follows that \( x \) is conjugate to an involution in \( C_G(j) \).

Since \( N_H(E) \cap C_G(j)/E \cong E_{32} \cdot \Sigma_5 \) and \( C_K(\sigma_1 \sigma_2) \cap N_H(E) \cap C_G(j) = E_4(\sigma_1 \sigma_2)(\sigma_1 \sigma_2)(j_0) \), the structure of \( M_{22} \) yields \( j_0 \not\in (N_G(E) \cap C_G(j))' \). Combining
this with our work in the two paragraphs above we see that \( j_0 \) is not conjugate to any involution in \( (N_G(E) \cap C_G(j))^\prime \). Thus Thompson’s transfer theorem [29, Lemma 5.38] yields that \( C_G(j) \) contains a subgroup \( X \) of index two with \( j_0 \notin X \).

It follows that \( C_H(j) \cap X/\langle j \rangle \) satisfies the assumptions of Part I, Theorem B. Thus either \( \langle j, z_1 \rangle \triangleleft X \) with \( X/\langle j, z_1 \rangle \cong U_6(2) \), or \( X/\langle j \rangle \cong M(22) \). The first case is impossible as \( \langle j, z_1 \rangle \triangleleft C_H(j) \), so we conclude \( X/\langle j \rangle \cong M(22) \). Finally, \( j \in C_H(j) \) and \( j_0 \) acts on \( C_H(j) \cap X \) as an outer automorphism whence \( C_G(j)/\langle j \rangle \cong \text{Aut } M(22) \).

**Lemma 13.** The group \( G \) has order \( 2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29 = |M(24)| \).

**Proof.** Thompson’s order formula yields

\[
|G| = 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot a(j_1) + 2^{19} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot a(z)
\]

where, for any involution \( g \) in \( G \),

\[
a(g) = |\{(x, y) | (xy)^n = g \text{ for some positive integer } n, \text{ with } x \sim_G z, y \sim_G j_1\}|
\]

\[
a(z) = 2^2 \cdot 3^3 \cdot 7 \times 2,820,636.\]

In the computation of \( a(z) \), \( x, y \) will be involutions in \( H - \langle z \rangle \) with \( (xy)^n = z \) for some \( n \) and \( x \sim_G z, y \sim_G j_1 \). Thus \( x \) is conjugate in \( H \) to one of

\[
j_2(2 \cdot 3^5 \cdot 7), \quad t(2^4 \cdot 3^4 \cdot 5 \cdot 7), \quad t(j(2^3 \cdot 3^6 \cdot 5 \cdot 7)),
\]

\[
\pi_2(2^3 \cdot 3^3 \cdot 7), \quad \pi_2uv(2^8 \cdot 3^5 \cdot 5 \cdot 7);
\]

while \( y \) is conjugate to either \( j_1(2^2 \cdot 3^3 \cdot 7), tz(2^4 \cdot 3^4 \cdot 5 \cdot 7) \) or \( \pi_2z(2^7 \cdot 3^3 \cdot 7) \), where the number in brackets is the number of conjugates of the involution in \( H \). Note that if either \( x \sim_G xz \) or \( y \sim_G yz \) then \( (xy)^n = z \) implies \( n \) is odd.

Suppose at first that \( y \sim_H j_1 \). A computation yields that there are 1728 conjugates of \( j_2 \) in \( J - C_j(j_1) \) whence there are \( 2^2 \cdot 3^3 \cdot 7 \times 1728 \) pairs \( (x, y) \) with \( x \sim_H j_2 \). By the remark above we see there are no such pairs when \( x \sim_H t \) or \( x \sim_H \pi_1 \). We have \( (jt)^2 = z \) if and only if \( [j, y] = z \) and \( \langle j, y \rangle \subset C_j(t) \). There are 32 such \( y \)'s and so \( 2^{10} \cdot 3^6 \cdot 5 \cdot 7 \) pairs \( (x, y) \) with \( x \sim_H tj \) (\( [tj, J] \) is elementary abelian so \( |tj| \leq 4 \)). For \( \pi_2uv \), \( (\pi_2uv)^2 = z \) implies \( n = 4 \); i.e. \( (\pi_2uv)^2 \) is of order four in \( J \). Now there is precisely one class (in \( H \)) of elements of order four in \( J \) with representative \( r_1 \) say, where \( C_H(r_1)/C_j(r_1) \cong E_{81} \cdot \Sigma_3 \). We compute that \( \pi_2uv \) inverts 64 elements of order four in \( J \) which are all conjugate under the action of \( C_H(\pi_2uv) \). There are 6 conjugates of \( y \) with \( (\pi_2uv)^2 \) of order four so there are \( 2^{15} \cdot 3^6 \cdot 5 \cdot 7 \) pairs \( (x, y) \) with \( x \sim_H \pi_2uv \).

From the remarks above, if \( (xtz)^n = z \) then either \( n = 1 \) and \( x = t \) or \( xtz = az \), \( a \) of odd order (\( \neq 1 \)). (In this latter case, \( xz \cdot tz = a, \langle xz, tz \rangle \) is a dihedral group of order \( 2|\sigma| = 2n \).) If \( y_1, \ldots, y_m \) are representatives of the \( N_H(\langle a \rangle) \)-classes of \( \langle tz \rangle^H \) which invert \( a \), then

\[
I = \sum_{i=1}^{m} ([\sigma] - 1)|C_H(tz)|/|C_H(y_i) \cap N_H(\langle \sigma \rangle)|
\]

gives the number of involutions \( x \in H \) with \( xtz \sim_H az \). If \( a \sim_H a \mu \) then \( I = 2^{11} \cdot 3^2 \); if \( a \sim_H a_1a_2 \), \( I = 2^{10} \); if \( a \sim_H a_1a_2^{-1} \), \( I = 2^8 \); if \( a \sim_H a_3 \), \( I = 2^{11} \cdot 3 \); otherwise
\[ I = 0. \text{ Hence there are} \]
\[ 2^4 \cdot 3^4 \cdot 5 \cdot 7(1 + 2^{11} \cdot 3^2 + 2^{10} + 2^8 + 2^{11} \cdot 3) \]

pairs \((x, y)\) with \(y \sim_H tz\).

Similarly if \(y \sim_H \sigma_2 z\) and \(\sigma \sim_H \epsilon\) then \(I = 2^7\); while if \(\sigma \sim_H \sigma_1 \sigma_2, I = 2^8 \cdot 3 \cdot 5\); otherwise \(I = 0\). Hence there are \(2^7 \cdot 3^3 \cdot 7(1 + 2^7 \cdot 2^8 \cdot 3 \cdot 5)\) pairs \((x, y)\) with \(y \sim_H \sigma_2 z\). Thus

\[ a(z) = 2^2 \cdot 3^3 \cdot 7 \times 2,820,636. \]

\[ a(j_1) = 3^5 \cdot 5 \cdot 7 \cdot 11 \cdot 11 \cdot 13 \times 3,608,577. \]

Again we will take \(x \sim_G z, y \sim_G j_1\) but this time with \((xy)^n = j\) for some \(n\). Clearly \(x, y \in C_G(j)\) and we will use the same notation as in Lemma 12. There are precisely three classes of involutions in \(M(22)\) and another three classes in \(\text{Aut } M(22) - M(22)\). In \(C_G(j) - \langle j \rangle\) we have four classes of involutions with representatives \(z, jz, z_1, j_2\), where \(j_2 \in C_G(j) - C_G(\sigma_1 \sigma_2)\) and \(j_2 \sim j_2 j, z_1 \sim z_1 j \in C_G(j)\). In \(C_G(j) - C_G(j)\) there are two classes of involutions with representatives \(j_0 j_0'\) where \(\langle j_0, z_1 \rangle \times \langle j \rangle = C_G(\mu) \cap C_G(j), \langle \mu \rangle\) a Sylow 5-subgroup of \(C_H(j)\). Further we have \(j_0 \sim j_0 j, j_0' \sim j_0 j\) in \(C_G(j)\). In addition, there is one class of elements of order four in \(C_G(j) - C_G(j)\) with square \(j\); namely \((j_0 \mu z_1)^2 = j\). Since \(C_G(j)\) is isomorphic to the centralizer of a 3-transposition in \(M(23)\), no element of \(C_G(j)\) squares to \(j\) (see [1]).

If \((xy)^n = j, n \text{ odd}\), then by the above remarks, \(x \sim z, y \sim j z\) in \(C_G(j)\). It is enough therefore to determine the number of conjugates of \(z \langle j \rangle\) in \(C_G(j)\) whose product with \(z \langle j \rangle\) has odd order. With the use of the character table for \(M(22)\) [25] and the properties of \(M(22)\) given in Part I, we compute that there are 1, \(2^{14} \cdot 3^3, 2^{10}, 2^{13} \cdot 5\) conjugates of \(z \langle j \rangle\) whose product with \(z \langle j \rangle\) is conjugate to 1, \(\mu, \sigma_1 \sigma_2^{-1}, \sigma_1 \sigma_2\), respectively. Hence if \(n\) is odd we get \(3^5 \cdot 5 \cdot 7 \cdot 11 \cdot 11 \cdot 13 \times 484,353\) pairs.

Now suppose \(n\) is even. By the remarks above we need only consider \(x, y \in N_G(\langle j_0 \mu z_1 \rangle) \cap C_G(j) - C_G(j_0 \mu z_1) \cap C_G(j)\). Let \(w = j_0 \mu z_1\) and we will use the relations given for \(C_H(\sigma_1 \sigma_2)\) in the proof of Lemma 12. In addition, as there exists \(g \in G\) with \(\sigma_1 \sigma_2^g = c\) and \(z^g = t\), we have \([u, v] = [u, \tau_2] = [v, \tau_2] = tz\). A computation yields that \(\langle t, j_0 \pi_2 u v \rangle (\approx E_4) \subseteq C_{E}(w)\) and that \(w_{z_1}\) inverts \(w\). Also, \(\langle C(u) \cap O_3(C_H(\sigma_1 \sigma_2)) = \langle \sigma_1 \sigma_2, \sigma \rangle = E_9\) is a Sylow 3-subgroup of \(C_H(w)\). Without loss we assume \([u, C_j(\sigma_1 \sigma_2 c)\] = 1 so that \(C_j(u) \cap C_j(\sigma_1 \sigma_2 c^{-1}) = E_8\). Thus

\[ J_0 = C_j(u) \cap \langle \sigma_1 \sigma_2 \rangle, J \cong E_4 \times Q_8 \times Q_8 \]

and \(C_j(w) = J_0 \times \langle j \rangle\).

From the structure of \(\text{Aut } M_{22}\), we have \(C(wE)\) in \(N_X(E)/E\) is isomorphic to \(E_8 \cdot GL(3, 2)\). Further, \(C_E(j_0) = C_E(w) \cong E_7\) and \(w\) has 128 conjugates in \(wE, j_0\) has 16, while the remaining 112 involutions are conjugate to \(j_0'\) in \(C_G(j)\) (with \(j_0' \sim_G z\)). It follows that

\[ C_H(w) = \langle w \rangle \times J_0(\sigma_1 \sigma_2, \sigma) \langle t, j_0 \pi_2 u v \rangle. \]

Let \(V = J_0 \langle t, j_0 \pi_2 u v \rangle\) and \(W = V \cap E\). We easily see that \(V \times \langle w \rangle\) is a Sylow 2-subgroup of \(C_G(w)\) and that \(V\) is of type \(\Omega_4(3)\) (see Part I, §6). Further, we can
show $N_G(W) \cap C_G(w)/W \times \langle w \rangle = L_2(7)$ (the argument is similar to Part I, Lemma 6.3). It follows that $C_G(w)/\langle w \rangle$ is simple and, hence, $C_G(w)/\langle w \rangle \cong PSp_6(2)$ by a result of Solomon [27]. Since $u_1$ centralizes $\langle \sigma_1, \sigma_2, \sigma \rangle$, assuming $u_1 \sim G_j$ (rather than $u_1z$) we have $N_G(\langle w \rangle) = \langle u_1, w \rangle \times P$, $P \cong PSp_6(2)$.

The group $P$ has 4 classes of involutions with representatives $z, z_2, z_2z, z_3$ where $z_2 \in Z(J_0) - \langle z \rangle, j_1 \in \langle C_P(\sigma_2) \cap W \rangle - \langle z \rangle$ (so all four involutions lie in $W$). It follows that $u_1z_2$ has 5 classes of involutions with only $u_1z_2$ conjugate to $j_1$ (we assume $z_2$ has only 7 conjugates in $W$ and $C_P(z_2)/W \cong \Sigma_3$). The coset $w \cdot u_1z_2 = j_0P$ also has 5 classes of involutions with only $j_0, j_0z$ conjugate to $j_1$ (recall $j_0E$ contains only 8 conjugates of $j$, and $j_0 \sim j_0z^2$).

Since $xy = w$ implies $x \sim yw$, the remarks above yield that if $xy = w$ then $x \sim u_1z$ and $y \sim j_0z$ or $x \sim u_1z_2$ and $y \sim u_1z_2$ in $N_G(\langle w \rangle)$. From the structure of $P \cong PSp_6(2)$ we compute that $\{z^g|g \in P\text{ and } z_2 \cdot z^g \text{ has odd order}\} = 1 + 2^7$ while $\{z^g|g \in P\text{ and } z_2 \cdot z^g \text{ has odd order}\} = 1 + 2^5$. Also, $u_1z_2$ has $2 \cdot 3^2 \cdot 7$ conjugates in $N_G(\langle w \rangle)$ while $u_1z_2$ has $2 \cdot 3^2 \cdot 7$ conjugates. Hence if $n$ is odd we get

$$2^8 \cdot 3^5 \cdot 5 \cdot 11 \cdot 13 \times (2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 129 + 2 \cdot 3^2 \cdot 7 \cdot 33)$$

$$= 3^5 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \times 3,124,224 \text{ pairs.}$$

It follows that

$$a(j_1) = 3^5 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \times 3,608,577$$

and therefore that $|G| = |M(24)|$.

**Lemma 14.** The group $G$ is simple.

**Proof.** We have $O(G) = 1$ because $z \sim G_j \sim G_jz$ and $O(H) = 1$. If $1 \neq N \lhd G$ then $z \in N$ since if $T$ is a Sylow 2-subgroup of $H$ (and hence of $G$), $Z(T) = \langle z \rangle$. Thus $H = \langle z^G \cap H \rangle \subseteq N$ and so $N = G$ (using $Z(T) = \langle z \rangle$ and the Frattini argument).

This completes the proof of the theorem in case (b).

**4. The proof of case (c).** Throughout this section we will assume:

**Hypothesis 2.** Let $G$ be a finite group which satisfies the assumptions of the theorem. In addition, suppose that $G \neq H \cdot O(G)$ and $H = K\langle \pi_1 \rangle$ or $H = K\langle \pi_1, \pi_2 \rangle$.

**Lemma 15.** The group $G$ has a subgroup $G_0$ of index two with $G_0 \cap H = K\langle \pi_2 \rangle$. Further $G_0$ is a simple group and $|G_0| = |M(24)|$.

**Proof.** Recall that $C_f(\pi_1) = E_4 \times Q_8 \times Q_8 \times Q_8$ and that $C_K(\pi_1)/C_f(\pi_1) = Z_3 \times PSp_4(3)$. Let $Z = Z(C_f(\pi_1)) = E_8$ so that $C_H(Z)$ covers $C_K(\pi_1)/C_f(\pi_1) \cdot \langle c \rangle$. Thus $Z - \langle z \rangle$ contains 6 conjugates of $j_1$. Let $S$ denote a Sylow 2-subgroup of $C_H(\pi_1)$ and note that $Z(S) \subseteq \langle z, \pi_1 \rangle$. Suppose at first that $\pi_1^2 = 1$.

Since $z \in S' \cap Z(S) \subseteq Z$ we see that $S$ is a Sylow 2-subgroup of $C_G(\pi_1)$. As $|S| < 2^{19}$ it follows that $\pi_1$ is not conjugate (in $G$) to any involution in $J$. Now let $T_3(\langle v \rangle)$ be a Sylow 2-subgroup of $N_H(\langle \sigma_1, \sigma_2 \rangle)$ with $u, \pi_1 \in T_3 \subseteq C_H(\sigma_1, \sigma_2)$. As $\langle t \rangle \lhd T_3$ and $[v, \pi_1] \in t\langle z \rangle$, $\langle t, z \rangle \subseteq (T_3 \cap C_H(t))$. Let $j \in C_f(u) \cap T_3$ as usual.
so that \( \langle u, C_j(j) \cap T_3 \rangle = \langle j, z \rangle \). Since \( C_H(j) \cap N_H(\langle \sigma_1 \sigma_2 \rangle) \) covers \( \langle v \rangle T_3 \cap T_3 \cap J \) we have \( (C(j) \cap T_3 \langle v \rangle)' \subseteq \langle j, z \rangle \). Thus if \( l \) is any involution in \( J \) and \( L \) is a Sylow 2-subgroup of \( C_H(l) \) we have \( l \in L' \). It follows immediately that \( l \sim_G \pi_1 \).

If \( \pi_1^2 \neq 1 \) then \( \pi_1^2 = z \), so \( \langle \pi_1 r \rangle^2 = 1 \) where \( r \) is an element of order 4 in \( C_j(\pi_1) \). We compute that \( C_H(\pi_1 r) \) has Sylow 2-subgroup \( S_0 \) of order 214 if \( H = K(\langle \pi_1 \rangle), 2^{15} \) if \( |H : K| = 4. \) In either case \( Z(S_0) \subset \langle \pi_1, r, Z \rangle \) whence \( z \in S_0 \cap \Omega_1(Z(S_0)) \subset Z \). It follows that \( S_0 \) is a Sylow 2-subgroup of \( C_G(\pi_1 r) \) and that \( \pi_1 r \) is not conjugate (in \( G \)) to any involution in \( K \).

We will now show that \( \pi_1 \) (or \( \pi_1 r \)) is not conjugate to any involution in \( K(\langle \pi_1 \rangle) = K \). Let \( l \) be an involution in \( K(\langle \pi_1 \rangle) = K \) and \( L \) a Sylow 2-subgroup of \( C_H(l) \). If \( z \sim_G j_2 \) then there exists \( g \in N_G(\langle z, j_2 \rangle) \) with \( C_j(j_2) \cong \langle j, z \rangle \), \( L' \cap Z(L) = \langle z \rangle \). Thus if \( L \) is not a Sylow 2-subgroup of \( C_G(\langle \pi_1 \rangle) \) then \( l \) is conjugate to an involution in \( J \). Hence \( l \sim_G \pi_1 \) and if \( l \sim_G \pi_1 r \) then \( l \sim_H \pi_2 \) or \( \pi_2 z \) (as \( C_H(\pi_2 \nu v) \) has Sylow 2-subgroup of order 214). However, as \( z \not\subset L \) \( Z(L) = \langle z \rangle \) in this case we must have \( l \sim_H \pi_1 r \), which is impossible.

By Thompson's transfer theorem \([29]\) we have that \( G \) contains a subgroup \( G_0 \) of index two with \( G_0 \cap H = K \), \( K(\langle \pi_1 \rangle) \). The lemma now follows from our results in \( \S 3 \).

**Lemma 16.** We have \( \pi_1^2 = 1 \) and \( C_G(\pi_1) \cong Z_2 \times M(23) \).

**Proof.** From the proof of Lemma 6, \( C_G(\mu) \subset G_0 \langle \mu \rangle \cong \mathbb{S}_6 \) or \( \mathbb{S}_9 \). Since \( \pi_1 \) acts as an outer automorphism on \( C_G(\mu) \), we have \( C_G(\mu) \langle \mu \rangle \cong \mathbb{S}_8 \) or \( \mathbb{S}_9 \). As \( c \sim_G \sigma_1 \sigma_2 \), \( C_G(c) / C_G(\langle \sigma_2 c \rangle) = Z_2 \times U_4(3) \times Z_2 \) whence \( C_G(c) \cap C_G(\mu) \) contains a four group, i.e. \( C_G(\mu) / \langle \mu \rangle \cong \mathbb{S}_9 \). Thus \( \pi_1^2 = 1 \) and \( C_G(\pi_1) \) covers \( H / K \). It follows that \( C_H(\pi_1) \cap G_0 \) satisfies the assumptions of Part I, Theorem C. Replacing \( \pi_1 \) by \( \pi_1 z \) if necessary we have \( C_G(\pi_1) \cap C_G(\mu) \cong Z_2 \times Z_4 \times \mathbb{S}_7 \). Hence \( \langle z \rangle \) is not weakly closed in \( C_G(\pi_1) \) and \( C_G(\pi_1) \) does not contain a normal four group. It follows immediately from Theorem C of Part I that \( C_G(\pi_1) \cong Z_2 \times M(23) \).

**Lemma 17.** The conjugacy class \( \pi_1^G \) is a class of 3-transpositions and \( G \cong M(24) \).

**Proof.** From the character table for \( M(23) \) given in \([26]\) we see that the only proper subgroups of \( M(23) \) of index less than \( |G : C_G(\pi_1)| = 306,936 \) have index 31,671 (= \( 3^4 \cdot 17 \cdot 23 \)), 137,632 (= \( 2^5 \cdot 11 \cdot 17 \cdot 23 \)), 275,264 (= \( 2^6 \cdot 11 \cdot 17 \cdot 23 \)). Hence by Lemma 16 we see that there exists \( g \in G \) (in fact we may take \( g \in J - C_G(\pi_1) \)) with \( C_G(\pi_1) \cap C_G(\pi_1^g) \) of order 2 \( \cdot |M(22)| \) and index 31,671 in \( C_G(\pi_1) \). From the remarks before Lemma 4, there is an element \( \lambda \sim_H \sigma_1 \sigma_2^{-1} \) which is inverted by \( \pi_1 \) and such that \( C_G(\pi_1) \supset C_G(\lambda) \). Thus \( C_H(\pi_1) \cap C_H(\lambda) = \Lambda \) where \( C_H(\lambda) = \langle \lambda \rangle \times \Lambda \) (because \( \pi_1 \sim_H \pi_1 z \)). As any Sylow 2-subgroup of \( \Lambda \) has centre equal to \( \langle z \rangle \), it follows that \( |C_G(\pi_1) \cap C_G(\pi_1^g)| = 2^{13} \cdot n \), \( n \) odd.
We have shown therefore that $G$ is a rank three extension of $C_G(\pi_1)$, whence $\pi_1^G$ is a class of 3-transpositions. Since $G'' = G_0 = G_0$ by Lemma 14, Fischer's result [1] yields $G \cong M(24)$.

The proof of the theorem is now complete.

**PART III**

The following result is proved.

**Theorem.** Let $G$ be a finite group, $z$ an involution in $G$ and suppose $H = C_G(z)$ satisfies:

(i) $J = O_2(H)$ is extra-special of order $2^{13}$ with $C_H(J) \subseteq J$;
(ii) $H$ contains a normal subgroup $K$ with $K/O_{2,3}(H) \cong U_4(3)$ and $O_{2,3}(H)/J \cong Z_3$;
(iii) $G \neq H \cdot O(G)$ and $|H : K| = 2$.

Then $G \cong M(24)'$.

The following corollary is an immediate consequence of Part II.

**Corollary.** Suppose $G, H, z$ satisfy only (i) and (ii) of the theorem. Then one of the following holds:

(a) $G = H \cdot O(G)$;
(b) $H/K \cong Z_2$ and $G \cong M(24)'$;
(c) $H/K \cong Z_2 \times Z_2$ and $G \cong M(24)$.

Throughout this part we will assume conditions (i), (ii), (iii) hold. In Part II we have shown that under these hypotheses $G$ is a simple group of the same order as $M(24)'$ and that if $j$ is an involution in $G$ with $j \varpropto_G z$, then $C_G(j)$ is a nonsplit extension of a group of order two by Aut $M(22)$. We will begin by listing properties of $G$ which were determined in Part II, and as far as possible we will use the same notation.

Let $T$ be a Sylow 2-subgroup of $H$ with $T \cap C_H(j)$ a Sylow 2-subgroup of $C_G(j)$. Let $E$ be the unique elementary abelian subgroup of order $2^{11}$ in $T$ so that $C_G(E) = E$ and $N_G(E)/E \cong M_{24}$ (the Mathieu group). Put $C = C_G(j)$ so that $C/\langle j \rangle \cong M(22)$. In $N_G(E), j$ has 276 conjugates in $E$ which break up into 4 classes in $N_C(E)$ with representatives $j$ (1 conjugate), $z_1$ (22), $z_1 j$ (22) and $j z$ (231). We have $z_1 \sim z_1 j$ in $C_G(j) \cap N_G(E)$, and $N_C(E)/E \cong M_{22}$.

For the element $\sigma_1 \sigma_2^{-1}$ of order three we have that

$$C_H(\langle \sigma_1 \sigma_2^{-1} \rangle) = C_J(\sigma_1 \sigma_2^{-1} \cdot M \cdot \langle t, uw, \pi_2 \rangle \langle \sigma_3 \rangle$$

where

$$C_J(\sigma_1 \sigma_2^{-1}) = Q_8 \ast Q_8 \ast Q_8 \ast Q_8, \quad M \cong E_3,$$

and

$$C_H(\langle \sigma_1 \sigma_2^{-1} \rangle)/O_{2,3}(C_H(\langle \sigma_1 \sigma_2^{-1} \rangle)) \cong Z_2 \times \Omega_4.$$

In addition,

$$N_H(\langle \sigma_1 \sigma_2^{-1} \rangle) = C_H(\langle \sigma_1 \sigma_2^{-1} \rangle) \cdot \langle uz \rangle, \quad uz \sim_G j,$$

$$N_H(\langle \sigma_1 \sigma_2^{-1} \rangle)/O_{2,3}(C_H(\langle \sigma_1 \sigma_2^{-1} \rangle)) \cong Z_2 \times \Sigma_4.$$
and \(N_H(\langle \sigma_1, \sigma_2^{-1} \rangle) - H(\sigma_1, \sigma_2^{-1})\) contains precisely one class of involutions (with representative \(uz\)) which are conjugate to \(j\) in \(G\). Finally,

\[C_H(uz) \cap H(\sigma_1, \sigma_2^{-1}) = C_j(\langle uz, \sigma_1, \sigma_2^{-1} \rangle) \cdot M_0 \cdot \langle t, \pi_2, uv \rangle\]

where

\[C_j(\langle uz, \sigma_1, \sigma_2^{-1} \rangle) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times Q_8 \times Q_8, \quad M_0 = C_H(uz) \cong E_{2g}\]

and in fact \(C_H(uz) \cap H(\sigma_1, \sigma_2^{-1})\) is isomorphic to the centralizer of an involution in \(\Omega_2(3)\).

**Lemma 1.** We have \(\Omega_1 = C_G(uz) \cap C_G(\sigma_1, \sigma_2^{-1}) = \Omega_2(3)\) and \(C_G(\Omega_1) = \langle uz, \sigma_1, \sigma_2^{-1} \rangle \cong \Sigma_3\).

**Proof.** As \(C_G(\sigma_1, \sigma_2^{-1}) \supset C_G(\sigma_3)\) and \(z \sim \pi_2 u v\) in \(C_G(\sigma_3)\) (see Part II, Lemma 10), Theorem D of Part I yields that \(\Omega_1 \cong \Omega_2(3)\). To complete the proof it is enough to show that \(C_H(\Omega_1) = \langle z, uz, \sigma_1, \sigma_2^{-1} \rangle\). Now \(\Omega_1 \cap H \supseteq \langle c, \sigma_1 \sigma_2^{-1} \rangle\) and we may assume \(j_2 \in \Omega_1 \cap J (j_2 \sim_G z)\). The structures of \(C_H(\langle c, \sigma_1 \sigma_2^{-1} \rangle)\) and \(C_H(j_2)\) now give the required result.

1. **The construction of the subgroup** \(X = M(23)\). Let \(N_C(E) \subset L \subset N_G(E)\) with \(L/E \cong M_{23}\) and \(j \sim_L z_1\), \((L\text{ exists because of the structure of } M_{24} - \text{see [23]})\). As \(j\) has 23 conjugates in \(L, j \sim_L z_1 j\). Put \(C_1 = C_G(z_1)\) and note that \(z_1^C, j^C\) are classes of 3-transpositions in \(C_1, C\), respectively, with \(z_1^C \sim_G \sigma_1, \sigma_2^{-1}\) or \([z_1^C, z_1^C] = 1\) for all \(x \in C\).

Take \(k \in C_1 - C\) with \(k \sim_G j\) (thus \(k^C \sim_G \sigma_1, \sigma_2^{-1}\)). We now define \(\mathfrak{g} = \{j\} \cup z_1^C \cup k^C\) and let \(X = N_G(\mathfrak{g})\). By Lemma 1, \(\Omega = C_G(k) \cap C_G(j) = C_G(k) = \Omega_2(3)\); and we also have \(C_G(z_1) = C_G(z_1) \cap C_G(j)\). It follows therefore that

\[\mathfrak{g} \supseteq \mathfrak{g} \supseteq \{ g \in G | g \sim_G j, g \sim_G \sigma_1, \sigma_2^{-1} \}\]

(1)

Since \(N_H(\langle \sigma_1, \sigma_2^{-1} \rangle)\) contains a Sylow 2-subgroup of \(N_G(\langle \sigma_1, \sigma_2^{-1} \rangle)\) and \(N_H(\langle \sigma_1, \sigma_2^{-1} \rangle) - H(\sigma_1, \sigma_2^{-1})\) contains precisely one class of involutions conjugate to \(j\) in \(G\),

\[\mathfrak{g} \supseteq \mathfrak{g} \supseteq \{ g \in G | g \sim_G j, g \sim_G \sigma_1, \sigma_2^{-1} \}\]

(2)

(Note that \(\mathfrak{g} = \mathfrak{g} \cap \mathfrak{g}\) for \(s, s_1 \in C_G(j) - C\).)

We prove next that

\[xz_1 \sim_G \sigma_1, \sigma_2^{-1} \quad \text{for all } x \in \mathfrak{g} - C_1\]

(3)

If \(x \in z_1^C\) this is clear as \(z_1^C\) is a class of 3-transpositions in \(C\). Suppose that \(x \in k^C\). From the character table of \(M(22)\) (see [25]) we have

\[C = \Omega(C_1 \cap C) \cup \Omega w(C_1 \cap C)\]

where \(w \in N_C(E) - C_1, w^2 \in C_1\) and \(z_1^C = z^2 \not\in C_1\). (Since \(k \not\in C_G(E) = E\) and \(N_C(E)/E \cong M_{22}\) is 2-transitive on the 22 conjugates of \(z_1\) in \(N_C(E)\), we can find such a \(w\).) Hence \(k^C = k(C_1 \cap C) \cup k w(C_1 \cap C)\) and it is enough to show that \(k^w z_1 \sim_G \sigma_1, \sigma_2^{-1}\).

As \(k \in C_1, k^w \in C_2 = C_G(z_2)\) and \([k^w, z_1] \not= 1\). Now \(k^w j \sim_G \sigma_1, \sigma_2^{-1}\) also as \(j^C_2\) is a class of 3-transpositions in \(C_2\) and \(j \sim z_1\) in \(L \cap C_2\). Thus \(3\) is proved.
Choose $g \in C_1 \cap N_C(E) - C$ with $g^2 \in C$ and let
\[ z_3 = j^g(\{C_1 \cap N_C(E); C_1 \cap N_C(E)\} = 2 \cdot 11). \]

We will show that $g \in X = N_G(\mathcal{Y})$. From the structure of $L$ we can find $y \in N_C(E)$ with $z_3^y = z_3$, whence $[x, z_3] = 1$ or $xz_3 \sim_G \sigma_1 \sigma_2'^{-1}$ for all $x \in \mathcal{Y}$ by (3).

As $z_3^C \cap C_i \subseteq z_3^C = z_3^C$ and $j_i \cap C = \{j\} \cup z_3^C \cap C_i \cup k \cap C_i$, it follows that $j_i \cap C = z_3^C \cap C_i \cap C - \{z_3\}$, whence $(z_3 \cap C_1)^y = z_3 \cap C_1$.

Let $x \in z_3 \cap C_1$. Thus $xz_3 \sim_G \sigma_1 \sigma_2'^{-1}$ by (3), so $x^g z_3 \sim_G \sigma_1 \sigma_2'^{-1}$ also. If $[x, z_3] = 1$, $x^g \in C$ whence $x^g \in \mathcal{Y}$ as required. Therefore we will assume $xz_3 \sim_G \sigma_1 \sigma_2'^{-1}$ so $x^g \sim_G \sigma_1 \sigma_2'^{-1} \sim_G x^g z_3$. It follows that $\langle x^g, j, z_3 \rangle \cong \Sigma_4$ with $x^g z_3$ of order four.

Suppose $x^g \notin \mathcal{Y}$. By (2) we have $x^g \in \mathcal{Y}$, $s \in C_G(j) - C$. Choose $s$ so that $z_3^s = z_3^j$ (see the remarks above). Let $x^g = x^s_i$ for some $x_i \in \mathcal{Y}$. Then $[z_3^s, x_i] = 1$ or $x_i z_3 \sim_G \sigma_1 \sigma_2'^{-1}$ by (3) so $(x_i z_3)^s$ has order two or three. However $(x_i z_3)^s = x^g z_3 j$ has order four, a contradiction. Thus $x^g \notin \mathcal{Y}$ and therefore $g \in N_G(\mathcal{Y}) = X$.

We have proved that $C \subset X \subset G$. From above we have $C = C_x(j)$. Since $j \sim_G z_3 \sim_G j z_3$, $O(X) = 1$. Let $N$ be a minimal normal subgroup of $X$. From $C/\langle j \rangle \cong M(22)$ and $j \in C'$ it follows that $j \in N$. Hence $C \subset N$ as $j \sim_X z_3$. As $N$ is minimal normal, the structure of $C$ yields that $N$ is simple whence $N \cong M(23)$ by a result of D. Hunt [12]. Now $T \cap C$ is a Sylow 2-subgroup of $N$ with $Z(T \cap C) = \langle z, j \rangle$. Finally, as $z, j, j z$ have nonisomorphic centralizers in $N$ (see [12] or Part I) the Frattini argument yields $N = X$. We have proved

**Lemma 2.** The group $G$ contains a subgroup $X$ isomorphic to $M(23)$.

2. Some properties of $X$.

**Lemma 3.** We have $G = X \cup XsX \cup XrX$ where $X \cap X^s = C$ and $|X : C| = 31,671$ while $D = X \cap X^r$ satisfies $|X : D| = 275,264$.

**Proof.** From the character table for $M(23)$ in [26], we see that $X$ has only four permutation characters of degree $< 306,936 = |G : X|$: \[ \varphi_0(1) = 1; \quad \varphi_1(1) = 31,671; \quad \varphi_2(1) = 137,632; \quad \varphi_3(1) = 275,264. \]

If $\varphi$ is the permutation character for the representation of $G$ on the cosets of $X$, then by considering values of $\varphi, \varphi_i (i = 1, 2, 3)$ on involutions we get $\varphi = \varphi_0 + \varphi_1 + \varphi_3$. The lemma follows.

**Lemma 4.** Let $D$ be a subgroup of $X$ of index $275,264$. Then $D$ contains precisely two classes of involutions with representatives $z$ ($|C_D(z)| = 2^{12} \cdot 3^5$) and $j z$ ($|C_D(jz)| = 2^{10} \cdot 3^5 \cdot 5 \cdot 7$).

**Proof.** We first note that $|D| = 2^{12} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 13$ and $\varphi_3$ is the permutation character associated with $D$. From the character table, $\varphi_3(z) = 320, \varphi_3(jz) = 2816$ which yields $z_D = z_X \cap D$ and $|C_D(z)| = 2^{12} \cdot 3^5 \cdot j_X \cap D = \emptyset$ and $|C_D(jz)| = 2^{10} \cdot 3^5 \cdot 5 \cdot 7 \cdot n, 1 \leq n \leq 4$. It remains to prove that $n = 1$.

Let $v$ be an element of order 7 in $C_D(jz)$. Then $C_X(v) = Z_7 \times \Sigma_5$ and $|N_X(\langle v \rangle); C_X(v)| = 6$.

From $\varphi_3(v) = 10$ it follows that $|C_D(v)| = 7 \times 12$ whence, as $\varphi_3$ vanishes on elements of order 42, $C_D(v) \cong Z_7 \times \Theta_4$. Hence all involutions which centralize an element of order 7 in $D$ are conjugate in $D$. This yields $n = 1$. 

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Lemma 5. There exists an involution \( d \sim x \) with \( d \in N_X(D) \) and \( C_D(d) = \Omega_7(3) \).

Proof. Without loss we may assume \( z \in D \). If \( M^* \) is a Sylow 3-subgroup of \( C_D(z) \) then \( M^* \) is a Sylow 3-subgroup of \( C_X(z) \). Since \( j^X \cap D = \emptyset \),

\[
[M^*, J] = J \cap D = Q_8 \cdot Q_8 \cdot Q_8 \cdot Q_8.
\]

Hence \( J \cap D - \{ z \} \) contains an involution \( j_2 \sim x \), whence a Sylow 2-subgroup of \( C_D(z) / C_D(z) \cap J \) must be elementary abelian of order 8 by Proposition 7 of [16]. The structure of Aut \( PSp_4(3) \) yields \( C_D(z) / C_D(z) \cap J \) is isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) by Proposition 7 of Part I. It follows from the structure of \( C_X(z) \) that there exists \( d \in N_X(C_D(z)) \) with \( t_z \sim d \) in \( H \cap X \) (\( t_z \sim_g J \) and \( N_X(M^*) \cap C_X(z) / \langle z, M^* \rangle = \mathbb{Z}_2 \times \mathbb{Z}_2 \) see Part I). A computation yields that \( C_X(d) \cap C_D(z) \) is isomorphic to the centralizer of an involution in \( \Omega_7(3) \). Clearly \( O(C_D(d)) = 1 \), and as \( |C_D(d)| > |C_X(d)| \cdot |D| / |X| \), Part I, Theorem D yields \( C_D(d) = \Omega_7(3) \).

Let \( \langle \rho \rangle \) be a Sylow 7-, 13-subgroup, respectively, of \( C_D(d) \). Then \( C_X(\rho) = \mathbb{Z}_{13} \times \Sigma_3 \), \( C_D(\rho) = \mathbb{Z}_{13} \times \mathbb{Z}_3 \), \( C_D(\rho) \cap C_D(d) = \langle \rho \rangle \), and in each case \( N(\langle \rho \rangle) / C(\rho) = \mathbb{Z}_6 \) (see [25] and [26]). Thus \( d \) normalizes \( N_D(\langle \rho \rangle) \). Similarly it follows from \( C_D(d) \cap C_D(v) = \mathbb{Z}_7 \times \mathbb{Z}_2 \), \( N_D(\langle \nu \rangle) \cap C_D(d) / C_D(v) \cap C_D(d) = \mathbb{Z}_6 \) (see [25]) and the structure of \( N_X(\langle \nu \rangle) \), \( N_D(\langle \nu \rangle) \) that \( d \) also normalizes \( N_D(\langle \nu \rangle) \). Hence \( d \in N_X(D) \) by Sylow's theorem.

Lemma 6. We have \( D' = D_4(3) \), \( D / D' \equiv \mathbb{Z}_3 \), and if \( D_0 = \langle D, d' \rangle \), then \( D_0 / D' \equiv \mathbb{Z}_3 \) and \( D_0 \subseteq \text{Aut } D' \). Further \( D_0 - D \) contains precisely one class of involutions conjugate to \( j \) in \( X \) i.e. \( j^X \cap D = d^{D_0} = d^D \).

Proof. The last statement follows from the fact that the representation of \( X \) on the cosets of \( D_0 \) must have permutation character \( \epsilon_2 \). As \( j^X \) is a class of 3-transpositions, \( d^{D_0} \) is a class of 3-transpositions in \( D_0 \). Clearly \( D_0 = \langle d^{D_0} \rangle \), \( Z(D_0) = O_2(D_0) = O_3(D_0) = 1 \), so Fischer's result [1] yields \( D_0' \neq D_0'' \).

If \( T_0 \) is a Sylow 2-subgroup of \( C(z) \cap D_0 \) then \( Z(T_0) = \langle z \rangle \) and \( \langle z^D \cap C_D(z) \rangle = O_{2,3,2} (C_D(z)) \). Thus the Frattini argument yields \( D' = D'' \) and \( |D : D'| < 3 \). Hence \( |D : D'| = 3 \) and \( \langle D', d' \rangle \cong D_4(3) \cdot \mathbb{Z}_3 = O_8^+(3) \) by Fischer [1]. The other assertions in the lemma follow from the structure of \( C_X(z) \cap D_0 \).

Lemma 7. The subgroup \( X \) contains exactly one conjugacy class of subgroups isomorphic to \( D_0 \) (at \( D_4(3) \cdot \Sigma_3 \)).

Proof. Suppose \( D_0^* \subseteq X \), \( D_0^* \cong D_0 \) but \( D_0^* \) is not conjugate to \( D_0 \). We may assume \( d \in D_0 \cap D_0^* \). Now \( C_D(d) = \Omega_7(3) \) and this subgroup contains no conjugates of \( d \). By [30] \((17 \cdot 2 \cdot 4), M(22)\) contains precisely two classes of subgroups isomorphic to \( \Omega_7(3) \). If \( [d^x, d] \neq 1 \) for \( x \in X \) then \( C_X(\langle d^x, d \rangle) \cong \Omega_7(3) \) and this subgroup contains conjugates of \( d \). Thus we may assume that \( D_0 \cap D_0^* \supseteq C_D(d) \times \langle d \rangle \).

Let \( \langle \rho \rangle \) be a Sylow 13-subgroup of \( C_D(d) \). Then

\[
\langle \rho \rangle \times \langle d^x \rangle \subseteq C_D(d) \times \langle d \rangle \subseteq D_0 \cap D_0^*,
\]

\[
R_0(\rho) = C_X(\rho) \cong \mathbb{Z}_{13} \times \Sigma_3,
\]

whence \( D_0 \cap D_0^* \supseteq C_D(d) \). It follows immediately that \( D_0 = D_0^* \) and the lemma is proved.
3. **The graph for $G$.** In this section we will consider various graphs, all of which will be undirected without loops or multiple edges. If $\Gamma$ is a graph and $x$ a vertex in $\Gamma$ then $\Gamma_x$ will denote the subgraph of $\Gamma$ whose vertices are those connected (by an edge) to $x$.

**The graph $\Gamma(\mathcal{O})$.** Let $\mathcal{O} = j^X$, the class of 3-transpositions of $X$. The graph $\Gamma(\mathcal{O})$ has vertex set $\mathcal{O}$ and $\{d, e\}$ is an edge for $d, e \in \mathcal{O}$ if and only if $1 \neq de = ed$.

**The graph $\Gamma(\mathcal{O}, D_0, \Lambda_d)$.** Let $\pi : D_0^X \to D_0^X\pi$ be a bijection with $D_0^X \cap D_0^X\pi = \emptyset$. Put $A = D_0^X \cup D_0^X\pi$. Fischer [30] defines the graph $\Gamma = \Gamma(\mathcal{O}, D_0, \Lambda_d)$ in the following way: $\Gamma$ has vertex set $\{\pi\} \cup \mathcal{O} \cup A$ with edges as given below:

- $\{\pi, \gamma\}$ is an edge if and only if $\gamma \in \mathcal{O}$;
- $\{d, e\}$ is an edge, $d, e \in \mathcal{O}$, if and only if $1 \neq de = ed$;
- $\{d, D_0^X\pi\}$ is an edge, $d \in \mathcal{O}$, $x \in X$, if and only if $d \in D_0^X$;
- $\{d, D_0^X\pi\}$ is an edge, $d \in \mathcal{O}$, $x \in X$, if and only if $\{d, D_0^X\}$ is an edge.

Fix $d \in \mathcal{O} \cap D_0$; let $\Gamma_d$ be the graph with vertex set $\Gamma_d$ and edges as defined above plus a set of edges $\Lambda_d$ with vertices in $\Delta$ so that $\Gamma_d \cong \Gamma(\mathcal{O})$.

$\Lambda_d$ is a set of edges in $\Gamma$.

For each $g \in D_0$, define the map $\mathcal{g}$ on $\mu$ by

$$
\pi(\mathcal{g}) = \pi; \quad d(\mathcal{g}) = d^g \quad (d \in \mathcal{O});
$$

$$
\delta(\mathcal{g}) = \begin{cases} D_0^g & \text{if } \delta = D_0^g \in \Delta, \\ D_0^{\sigma g} & \text{if } \delta = D_0^{\sigma g} \in \Delta.
\end{cases}
$$

$\{\delta_1, \delta_2\}$ is an edge, $\delta_1, \delta_2 \in \Delta$ if and only if there exists $g \in D_0$ with $\{\delta_1(\mathcal{g}), \delta_2(\mathcal{g})\} \in \Lambda_d$.

The graph $\Gamma = \Gamma(\mathcal{O}, D_0, \Lambda_d)$ is a central extension of $\Gamma(\mathcal{O})$ with respect to $D_0, \Lambda_d$. In [30] Fischer proves that the definition of $\Gamma$ is independent of the choice of $d \in \mathcal{O}$ [30, 19.1.4], $\Gamma_d \cong \Gamma(\mathcal{O})$ [30, 19.1.2], $\Gamma$ has transitive group of automorphisms isomorphic to $M(24)$ [30, 19.1.5; 19.2.7] and the central extension is unique [30, §20]. Further $D_0\mu \subseteq \text{Aut} \Gamma$ [30, 10.6.3].

**The graph $\Gamma(G)$**. The vertex set for $\Gamma(G)$ is $\{Xg \mid g \in G\}$ while $\{Xg, Xy\}$ is an edge if and only if $X^g \cap X^y = C = Z_2 \cdot M(22)$. We will use below the fact that $N_G(X) = X$. (Without loss $c \in C_X(z)$ and as $C_H(c) \cap \langle c, z \rangle = U_4(3)$ while $C_H(c) \cap X/\langle c, z \rangle \cong PSp_4(3)$, we have $N_G(C_X(z)) = C_X(z)$. The Frattini argument yields $N_G(X) = X$.)

Our aim, of course, is to prove that $\Gamma(G) \cong \Gamma$. Let $X \cap X^* = C$ (with $Z(C) = \langle j \rangle$) and $X^* \cap X = D$, with $N_X(D) = \langle d, D \rangle = D_0$. We note that for $x \in X, C \cap C_X(j^x) = C, (Z_2 \times Z_2) \cdot U_6(2)$ (if $1 \neq j \cdot j^x = j^x \cdot j$) or $\Omega_7(3)$ and $D_0 \cap D_0^x \cong D_0$.

**Let $\alpha : \{\pi\} \cup \mathcal{O} \to \{X\} \cup \{Xsx \mid x \in X\}$ be defined by $\pi \alpha = X; \quad j^x \alpha = Xsx$ for $x \in X$.**

Note that $\langle j^x \rangle = Z(C^x) = Z(X^x \cap X)^x = Z(X^{sx} \cap X)$.

By definition $\{\pi, j^x\}$ is an edge and $\{X, Xsx\} = \{\pi \alpha, j^x \alpha\}$ is an edge for all $x \in X$. Suppose $\{Xsx, Xsy\}$ is an edge $(x, y \in X)$ so that $X^{sx} \cap X^{sy} = C$. If

---

2In this section, results of the form [30, 19.1.4] refer to Lemma 19.1.4 of [30], etc.
$[j^x, j^y] \neq 1$ then $\Omega^* = X^{x^*} \cap X^{y^*} \cap X = \Omega_2(3)$, and by Lemma 1, $C_G(\Omega^*) = \langle j^x, j^y \rangle$. However if $\langle e \rangle = Z(X^{x^*} \cap X^{y^*}) \subseteq C_G(\Omega^*)$, which is a contradiction. Thus $[j^x, j^y] = 1$. Conversely, suppose $[j^x, j^y]$ is an edge so that $[j^x, j^y] = 1$. Then $(X^{x^*} \cap X) \cap (X^{y^*} \cap X) \cong (Z_2 \times Z_2) : U(2)$, whence $X^{x^*} \cap X^{y^*} = C$ as required. Therefore $(X^{x^*}, X^{y^*})$ is an edge in $\Gamma(G)_X$ if and only if $(j^x, j^y)$ is an edge in $\Gamma$. In particular, we have proved that $\Gamma(G)_X \cong \Gamma(0)$ and, as $G$ is transitive on $\Gamma(G)$, $\Gamma(G)_{X^g} \cong \Gamma(0)$ for all $g \in G$.

For convenience take $j = d \in D_0$ and extend the definition of $\alpha$ to $\Gamma_d$ by taking $\alpha|_{\Gamma_d}$ to be an isomorphism from $\Gamma_d$ to $\Gamma(G)_{X^x}$. (This is possible because of the way $\Lambda_d$ is chosen in the definition of $\Gamma$.) As in Lemma 3 take $X' \cap X = D$ (for some $r \in G$). If $D_0^r \in \Gamma_d$ for $x \in G$ we claim that $\{D_0^x, D_0^{y^*}\} = \{X^{x^*}, X^{y^*}\}$. Note that $\{D_0, D_0\}$ is uniquely determined by the set $\{X, X^x\}$ of involutions joined to $D_0, D_0$. Let $e = d^x$ so that $ea = X^{x^*}$. Since $\alpha$ is an isomorphism we must show therefore that $\{e, D_0\}$ is an edge if and only if both $\{X^{x^*}, X^{y^*}\}$ is an edge in $\Gamma(G)_X$. In particular, we have proved that $\Gamma(G)_X \cong \Gamma(0)$ and, as $G$ is transitive on $\Gamma(G)$, $\Gamma(G)_{X^g} \cong \Gamma(0)$ for all $g \in G$.

If $(X^{x^*}, X^{y^*})$ is an edge, $X^{x^*} \cap X^{y^*} = C$ from which we get $X \cap X^{x^*} \cap X^{y^*} = \Omega_2(3)$. Since $X \cap X^{y^*} = X \cap X^{x^*} \cap X^{y^*}$ the possible intersections of conjugates of $D_0$ in $X$ yield $X^{x^*} \cap X^{y^*} = C$ also. The same argument yields the converse so we have $(X^{x^*}, X^{y^*})$ is an edge if and only if $(X^{x^*}, X^{y^*})$ is an edge. Since $C_X(e) \cap X^{x^*} \subseteq X \cap X^{x^*} \cap X^{y^*}$, the argument of Lemma 5 yields that if $X^{x^*} \cap X^{y^*} \subseteq C$ then $e \in D_0^x \cap D_0^y$ ($D^x = X^{x^*} \cap X$). Suppose now that $e \in D_0^x$. Then $C_X(e) \cap D^x = \Omega_2(3)$ and so $X \cap X^{x^*} \cap X^{y^*}$ contains a subgroup isomorphic to $\Omega_2(3)$. As above this yields that $X^{x^*} \cap X^{y^*} = C$. This completes the proof that $\{D_0^x, D_0^y\} = \{X^{x^*}, X^{y^*}\}$. Without loss we may take $D_0 \alpha = Xr, D_0 \alpha^n = X^{rdx}$.

Let $\beta$ denote the representation of $G$ on the cosets of $X$, i.e. $X\gamma(g\beta) = X\gamma g$ for $\gamma, g \in G$. For each $e = d^x \in D_0 \cap \Omega$, $g \in D$, define

$$\alpha_e : \Gamma_e \rightarrow \Gamma(G)_{X^g} \text{ by } \alpha_e = (g^{-1} \mu)\alpha(g\beta).$$

From the definition we have $\pi \alpha = \pi \alpha = X$, and if $d^x \in \Gamma_e$, then $d \alpha = d \alpha = X^{x^*}$. We show that the definition of $\alpha_e$ is independent of $g$. Since $\mu, \beta$ are homomorphisms it is enough to show that for $g \in D_0(d), (g^{-1} \mu)\alpha(g\beta) = \alpha$ on $\Gamma_d$. However this follows by Fischer [30, 10.5] as $(g^{-1} \mu)\alpha(g\beta)\alpha^{-1}$ fixes $\Gamma(d \cap \Gamma) \cup \{D_0, D_0\}$ (we need here that $g \in D$).

We next prove that if $e \in \Gamma_e, f \in \Gamma_f$ (for some $f \in \Omega, D_0$) then $\gamma \alpha_e = \gamma \alpha_f$. Again, as $\mu, \beta$ are homomorphisms, it is enough to show that if $\gamma \in \Gamma_d \cap \Gamma_e, \gamma \alpha = \gamma \alpha$. Suppose $\gamma \in \{D_0^x, D_0^y\}$, so that $\langle d, e \rangle \subseteq D_0^x$. As the definition of $\alpha_e$ is independent of the choice of $g (e = d^x)$ we take $g = ed \in D^x$. Thus $\gamma(g^{-1})\alpha(g\beta) = \gamma$ and $(\gamma \alpha)(g\beta) = \gamma$, whence $\gamma \alpha = \gamma \alpha_e$ as required.

Since $D_0^x \cap D_0^y \neq \emptyset$ for all $x \in X$ [30, 19.1.5], we can define $\alpha : \Gamma \rightarrow \Gamma(G)$ by $\alpha|_{\Gamma_e} = \alpha_e$ for each $e \in D_0 \cap \Omega$. Each $\alpha_e$ is an isomorphism, whence $\alpha$ is an isomorphism and $\Gamma \cong \Gamma(G)$.

Lemma 8. We have $G \cong M(24)'$.

Proof. As we have proved $\Gamma \cong \Gamma(G)$, $G$ is isomorphic to a subgroup of $\text{Aut} \Gamma = M(24)$. The lemma follows as both $G, M(24)'$ are simple groups of order $\frac{1}{2} | M(24)|$. 

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