A TANGENTIAL CONVERGENCE
FOR BOUNDED HARMONIC FUNCTIONS
ON A RANK ONE SYMMETRIC SPACE

BY

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ABSTRACT. Let \( u \) be a bounded harmonic function on a noncompact rank one
symmetric space \( M = G/K \approx N^{-}A, N^{-}AK \) being a fixed Iwasawa decomposition
of \( G \). We prove that if for an a\(_0\) \( \in A \) there exists a limit lim \( u(na_{0}) = c_{0} \) as
\( n \in N^{-} \) goes to infinity, then for any \( a \in A \), lim \( u(na) = c_{0} \). For \( M = SU(n, 1)/S(U(n) \times U(1)) = B^{n} \), the unit ball in \( C^{n} \) with the Bergman metric, this
is a result of Hulanicki and Ricci, and in this case it reads (via the Cayley
transformation) as a theorem on convergence of a bounded harmonic function to a
boundary value at a fixed boundary point, along appropriate, tangent to \( \partial B^{n} \),
surfaces.

0. Introduction. Let \( M \) be a noncompact symmetric space of rank one. \( M \) can be
expressed as a homogeneous space \( G/K \) where \( G \) is a semisimple group of
isometries of \( M \) and \( K \) is a maximal compact subgroup of \( G \). Let \( \mathfrak{g}, \mathfrak{f} \) denote the Lie
algebras of \( G \) and \( K \), \( B \) the Killing form of \( \mathfrak{g} \), and \( \mathfrak{p} \) the orthogonal complement of
\( \mathfrak{f} \) in \( \mathfrak{g} \) relative to \( B \). If \( \pi: G \to G/K \) denotes the canonical projection, its differential
at the identity, \( \pi_{*} \), identifies the subspace \( \mathfrak{p} \) of \( \mathfrak{g} \) with \( T_{0}(M) \), the tangent space of
\( M \) at the origin \( o = \pi(e) \), and the invariant metric \( g \) on \( M \) can be chosen so that \( g_{0} \)
corresponds to the restriction of \( B \) to \( \mathfrak{p} \times \mathfrak{p} \) under the above identification. We
denote by \( \Delta \) the corresponding (\( G \)-invariant) Laplace-Beltrami operator on \( M \). A
function \( u \in C^{\infty}(M) \) is called harmonic if \( \Delta u = 0 \). Let \( \alpha \) be a maximal (one-dimen-
sional) abelian subspace of \( \mathfrak{p} \), \( \alpha \) and possibly \( 2\alpha \) in \( a^{+} \), the corresponding system of
positive restricted roots relative to the fixed choice of a “positive part” \( a^{+} \) in \( a \). Let
\( \mathfrak{g}_{-\alpha} \) and \( \mathfrak{g}_{-2\alpha} \) denote the root spaces corresponding to \(-\alpha \) and \(-2\alpha \). Then
\( \mathfrak{n}^{-} = \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha} \) is a nilpotent subalgebra of \( \mathfrak{g} \) and one has the Iwasawa decom-
position \( G = N^{-}AK \), with \( N^{-} = \exp \mathfrak{n}^{-}, A = \exp \alpha \). The above decomposition
shows that every \( p \in M \) can be uniquely written as \( p = na \cdot o \) (\( n \in N^{-}, a \in A \)).
We regard the nilpotent group \( N^{-} \) as a boundary for the symmetric space \( M \) in the
following sense. The bounded harmonic functions \( u \) on \( M \) have boundary values
on \( N^{-} \), i.e. \( \lim_{\log a \to \infty} u(na \cdot o) \equiv \varphi(n) \) exists a.e. (relative to the Haar measure on
\( N^{-} \)) and \( \varphi \in L^{\infty}(N^{-}) \). \( \log a \to \infty \) is understood with respect to the ordering
induced on \( a \) by \( a^{+} \). Moreover,

\[ u(na \cdot o) = \varphi \cdot P_{a}(n) = \int_{N^{-}} \varphi(n_{1})P_{a}(n_{1}^{-1}n) \, dn_{1}. \]
The function $P_a(n)$ on $N^\sim \times A$ is called the Poisson kernel for the symmetric space $M$ and is given by (Helgason [4])

$$P_a(n) = ce^{d/2} \left[ \left( \epsilon + \frac{1}{2} Q(X^{-a}) \right)^2 + 2Q(X^{-2a}) \right]^{-d/2},$$

where

$$n = \exp(X^{-a} + X^{-2a}), \quad X^{-a} \in \mathfrak{g}^{-a}, \quad X^{-2a} \in \mathfrak{g}^{-2a};$$

$$\epsilon = e^{-\alpha(\log a)}; \quad Q(X) = (X, X)(e^{-D/2}(ma + 4m_{2a}))$$

with $(X, X) = -B(X, \theta X)$ for $X \in \mathfrak{g}$, $\theta$ denoting the Cartan involution associated with the pair $(\mathfrak{g}, \mathfrak{f})$; $m_a = \dim \mathfrak{g}^{-a}$, $m_{2a} = \dim \mathfrak{g}^{-2a}$, $d = m_a + 2m_{2a}$. The constant $c$ is such that the integral of $P_a$ over $N^\sim$ is equal to 1.

The following theorem on "tangential" convergence for bounded harmonic functions on the Siegel domain

$$D_{r-1} = \left\{ (z_1, \ldots, z_r) \in \mathbb{C}^r : \Im z_r > \sum_{j=1}^{r-1} |z_j|^2 \right\},$$

$r > 2$, (or, equivalently, on $M = SU(r, 1)/SU(r) \times U(1)$)–the complex hyperbolic space) has been obtained by Hulanicki and Ricci [5]. We formulate it below in terms of a homogeneous space $M$.

**Theorem.** Let $u$ be a bounded harmonic function on a noncompact rank one symmetric space $M$. In the notation above, assume that for an $a_0 \in A$,

$$\lim_{N^\sim \ni n \to \infty} u(na_0 \cdot o) = c_0.$$ 

Then for any $a \in A$, $\lim_{N^\sim \ni n \to \infty} u(na \cdot o) = c_0$.

Our aim here is to prove the above Theorem and the proof is based on the classification of symmetric spaces. That is, we discuss separately the cases of $M$ being the real, complex (to see how the $M = \mathbb{P}_{r-1}$ case fits to our scheme), quaternion and octonion hyperbolic space, which corresponds respectively to $G$ being the classical group $SO_q(r, 1)$, $SU(r, 1)$, $Sp(r, 1)$ and the exceptional group $F_{4(-20)}$. Following the Hulanicki-Ricci method, for each case we construct a suitable commutative subalgebra $\mathcal{G}$ of (multi) radial functions in $L^1(N^-)$, to which the Poisson kernel $P_a$ belongs. We describe the set $\mathcal{M}(\mathcal{G})$ of the maximal ideals in $\mathcal{G}$ and check that the Gel'fand transform $\hat{P}_a$ of $P_a$ never vanishes on $\mathcal{M}(\mathcal{G})$. The Theorem may then be stated as a theorem on certain ideals in $L^1(N^-)$ and is a consequence of the Wiener property of the algebra $\mathcal{G}$. To study the algebra $\mathcal{G}$ we use the holomorphically induced (realizations of the irreducible unitary) representations of $N^\sim$.

1. **Nilpotent group $N^-$.** Let $\mathbf{F}$ denote the field $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or the Cayley numbers $\mathbb{O}$ (octonions); $\mathbb{F}_0 = \{ q \in \mathbb{F} : q + \bar{q} = 0 \}$, $\bar{q}$ being the usual conjugation in $\mathbb{F} = \mathbb{C}, \mathbb{H}, \mathbb{O}$ and $\bar{q} = q$ for $q \in \mathbb{R}$; $\Im q = \frac{1}{2}(q - \bar{q})$, $\sigma = 2s = \dim_{\mathbb{R}} \mathbb{F}$. According to the notation of the previous section, $\mathfrak{g} = \mathfrak{f} + \mathfrak{p}$ and for the classical $G$ we have (cf., e.g., [3, pp. 348–351])

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\[ \mathfrak{f} = \left\{ \begin{pmatrix} Z & 0 \\ 0 & p \end{pmatrix} : Z \text{ an } r \times r \text{ skew-Hermitian matrix over } F, \quad p \in F_0, \tr Z = -p \text{ in case of } F = \mathbb{C} \right\}, \]

\[ \mathfrak{p} = \left\{ \begin{pmatrix} 0 & t'q \\ \bar{q} & 0 \end{pmatrix} : q \in F^r = F \times \cdots \times F \right\}, \]

\[ \alpha = \{ tE_{1,r+1} + tE_{r+1,1} : t \in \mathbb{R} \}, \]

where \( E_{kl} \) denotes the \((r + 1) \times (r + 1)\) matrix \((\delta_{ak} \delta_{bl})_{1 \leq a,b \leq r+1}, r > 2\). We choose a basis \( H = E_{1,r+1} + E_{r+1,1} \) in \( \alpha \) and fix an ordering so that \( H \in \alpha^+ \). Then \( \alpha \in \alpha^* \) such that \( \alpha(H) = 1 \) is a positive restricted root, and we have

\[ \mathfrak{g}_{-\alpha} = \left\{ \begin{pmatrix} 0 & -\bar{q} & 0 \\ t'q & 0 & t'q \\ \bar{q} & 0 \end{pmatrix} : q = (q_2, \ldots, q_r) \in F^{r-1} \right\}, \]

\[ \mathfrak{g}_{-2\alpha} = \left\{ \begin{pmatrix} p & 0 & p \\ 0 & 0 & 0 \\ \bar{p} & \bar{q} & \bar{p} \end{pmatrix} : p \in F_0 \right\} \quad (= \{0\} \text{ for } F = \mathbb{R}). \]

We shall identify \( \mathfrak{n}^- = \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha} \) with \( F^{r-1} \times F_0 \) by the correspondence

\[ \begin{pmatrix} p & -\bar{q} \\ t'q & 0 \\ \bar{q} & \bar{p} \end{pmatrix} \leftrightarrow (q, -p). \]

In these coordinates on \( \mathfrak{n}^- \) the commutator of \((q, p) = (q_1, \ldots, q_{r-1}, p)\) and \((q', p') = (q'_1, \ldots, q'_{r-1}, p')\) in \( F^{r-1} \times F_0 \) is given by

\[ [ (q, p), (q', p') ] = (0, 2 \Im(\bar{q} \cdot q')), \]

where we have put \( \bar{q} \cdot q' \) for \( \Sigma_{i=1}^{r-1} \bar{q}_i q_i' \). We also have the formula (cf., e.g., [11, p. 39])

\[ ((q, p), (q', p'))_{\theta} = 4(m_4 + 4m_2)\Re(\bar{q} \cdot q' + \bar{p}p'). \]

For the exceptional \( G \) (cf., e.g., [10, pp. 522–530]), \( \mathfrak{g} = \mathfrak{f}_{4(-20)} \) is isomorphic to the Lie algebra \( \text{Der}(\mathfrak{g}) \) of derivations of the Jordan algebra \( (\mathfrak{g}, \circ) \) of \( 3 \times 3 \) octonion matrices \( A \) of the form

\[ A = \begin{pmatrix} \alpha_1 & a_3 & a_2 \\ \bar{a}_3 & \alpha_2 & a_1 \\ -\bar{a}_2 & -\bar{a}_1 & a_3 \end{pmatrix}, \quad a_i \in \mathbb{O}, \quad \alpha_i \in \mathbb{R}, \quad i = 1, 2, 3, \]

with multiplication given by \( A \circ B = \frac{1}{2}(AB + BA) \), \( A, B \in \mathfrak{g} \), \( AB \) denoting the usual matrix multiplication. We have

\[ \mathfrak{f} = \{ D \in \text{Der}(\mathfrak{g}) : D(E_{33}) = 0 \}, \]

\[ \mathfrak{p} = \left\{ D_Q \in \text{Der}(\mathfrak{g}) : Q = \begin{pmatrix} 0 & t'q \\ \bar{q} & 0 \end{pmatrix}, \quad q \in \mathbb{O}^2 \right\}, \]

where \( D_Q(B) = QB - BQ, B \in \mathfrak{g} \).
We choose $H = D_Q$ with $Q = E_{13} + E_{31}$ and $a \in \mathfrak{a}^+$ such that $\alpha(H) = 1$. Then

$$\mathfrak{u}_{-\alpha} = \left\{ D_Q(q): Q(q) = \begin{bmatrix} 0 & -\bar{q} & 0 \\ q & 0 & q \\ 0 & \bar{q} & 0 \end{bmatrix}, \ q \in O \right\},$$

$$\mathfrak{u}_{-2\alpha} = \left\{ D_Q(p): Q(p) = \begin{bmatrix} p & 0 & p \\ 0 & 0 & 0 \\ \bar{p} & 0 & \bar{p} \end{bmatrix}, \ p \in O_0 \right\}.$$

Identifying $D_Q(q) + D_Q(p)$ in $\mathfrak{u}_{-\alpha} \oplus \mathfrak{u}_{-2\alpha}$ with $(q, -p)$ in $O \times O_0$, we obtain the same formulas for the commutator and the inner product of $(q, p)$ and $(q', p')$ in $O \times O_0$ as those given by (1) and (2) above.

Writing $N^-$ as the manifold $n^-$ with the group multiplication given by the Campbell-Hausdorff formula we obtain

**Proposition 1.** The underlying manifold for the nilpotent group $N^-$ is $F^{r-1} \times F_0$ with $r > 2$ for $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$ ($F_0 = \{0\}$ if $F = \mathbb{R}$) and with $r = 2$ for $F = \mathbb{O}$. The group law is

$$(q, p)(q', p') = (q + q', p + p' + 2m(q - q')).$$

The Haar measure on $N^-$ is the ordinary Lebesgue measure on $\mathbb{R}^k \approx F^{r-1} \times F_0$, $k = r - 1$. We normalize it so that the volume of the unit cube in $\mathbb{R}^k$ is 1 and denote by $dq dp$. The Poisson kernel is given by

$$P_{\exp(iH)}(q, p) = c_{r,F} e^{d/2} \left[ (|q|^2 + \varepsilon)^2 + 4|p|^2 \right]^{-d/2},$$

where $\varepsilon = e^{-\varepsilon}$, $d = (r + 1)\sigma - 2$, $|q|^2 = \bar{q} \cdot q$, $c_{r,F} = 2^{d-1} \pi^{-\alpha} \Gamma(rs)$ with $\sigma = 2s = \dim_F F$.

2. **Holomorphically induced representations of $N^-$**. The adjoint and coadjoint action of $N^-$ on $n^-$ and $n^-\ast$, respectively, is given by

$$\text{Ad}_{(q,p)}(q'', p'') = (q'' + 2Im(\bar{q} \cdot q''),$$

$$\text{Ad}_{(q,p)}^\ast(q', p') = (q' + 2q\bar{p}', p').$$

$(q, p) \in N^-$, $(q'', p'') \in n^-$, $(q', p') \in n^-\ast$, $q\bar{p'} = (q_1\bar{p'}, \ldots, q_{r-1}\bar{p'})$, and we have identified $n^-\ast$, the dual space of $n^-$, with $n^-$ by $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_0/4(m_\alpha + 4m_{2\alpha})$. The single points $(q', 0) \in n^-\ast$ are 0-dimensional orbits of $\text{Ad}^\ast$ on $n^-\ast$ and the corresponding (1-dimensional) representations of $N^-$ are given by the characters

$$\chi_{(q',0)}(q, p) = \exp(\sqrt{-1} \text{ Re}(\bar{q} \cdot q)), \quad (q, p) \in N^-.$$

The remaining (maximal dimensional) orbits, for $F = \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$, are of the form $F^{r-1} \times \{p''\}, p'' \neq 0$, so they are parameterized, e.g., by the functionals $f = (0, p'') \in n^-\ast$ with $p'' \in F_0 \setminus \{0\}$. For such $f$ and $(q, p)$, $(q', p') \in n^-$ we have

$$\langle f, \left[ (q, p), (q', p') \right] \rangle = 2 \text{ Re}((q p'') \cdot q) = -2 \text{ Re}(\bar{q} \cdot (q' p'')),$
i.e. the operator $R_{\rho} \cdot q \rightarrow q \rho''$ is skew-symmetric on $F^{r-1}$ with respect to the $R$-bilinear symmetric form $\langle \cdot, \cdot \rangle$ on $F^{r-1}$ given by $\langle q, q' \rangle = 2 \Re(\overline{q} \cdot q')$ the usual inner product on $R^k$, $k = (r - 1)/2$. Let $1, i, j, k$ denote the usual basis of $C$ (over $R$); $1, i, j, k$ the basis of $H$, and $1, i, j, k, e, \overline{e}, je, \overline{je}, ke$ the basis of $O(= H + He$ with the multiplication defined by $(ae)b = (ab)e, a(be) = (ba)e, (ae)(be) = -\overline{ba}$ for $a, b \in H$). Suppose now that $f = (0, i\lambda) \in \eta^{-*}$ with $\lambda$ positive real. In the above bases of $F$ the matrix of $R_{\rho}$ acting on $F^{r-1}$ with $r = 2$ is equal to $\lambda(E_{21} - E_{12})$ for $F = C$, $E(\pm 2, E_{34} - E_{43})$ for $F = H$ and to $\lambda(E_{21} - E_{12}) - \lambda(E_{34} - E_{43}) - \lambda(E_{65} - E_{56}) + \lambda(E_{87} - E_{78})$ for $F = O$. Put

$$e_1 = \frac{1}{2}(1 + \sqrt{-1} i), \quad e_2 = \frac{1}{2}(\sqrt{-1} j + k),$$

$$e_3 = \frac{1}{2}(\sqrt{-1} e + ie), \quad e_4 = \frac{1}{2}(je + \sqrt{-1} ke)$$

for elements in $F^C = F + \sqrt{-1} F$ - the complexification of $F$. Now define a subspace $W$ of $F^C$ by

$$W = \begin{cases} Ce_1 & \text{if } F = C, \\ Ce_1 + Ce_2 & \text{if } F = H, \\ Ce_1 + Ce_2 + Ce_3 + Ce_4 & \text{if } F = O. \end{cases}$$

Thus

$$\mathfrak{b} = W^{r-1} \times F^C_0$$

is a positive polarization at $f = (0, i\lambda)$ such that

$$\mathfrak{b} + \mathfrak{b} = n^{-c}, \quad \mathfrak{b} \cap \mathfrak{b} = \{0\} \times F^C_0, \quad \mathfrak{b}/ \{0\} \times F^C_0 = W,$$

where $\hat{z} = x - \sqrt{-1} y$ for $z = x + \sqrt{-1} y \in n^{-} + \sqrt{-1} n^{-} = n^{-C}$. For arbitrary $f = (0, p'')$ with $p'' \in F_0 \times \{0\}$, there exists an orthogonal transformation $\Omega$ on $R^n \approx F$, such that

$$\langle R_{\rho} \cdot q, q' \rangle = \langle R_{i\rho''} \cdot \Omega q, \Omega q' \rangle, \quad q, q' \in F^{r-1}.$$  

Hence,

$$\langle f, [(q, p), (q', p')] \rangle = \langle (0, i|p''|), [(\Omega q, p), (\Omega q', p')] \rangle,$$

and $\mathfrak{b}' = \Omega \mathfrak{b}$, with $\mathfrak{b}$ as in (2.1), is a positive polarization at $f$ and $\mathfrak{b}'$ satisfies (2.2) with $W' = \Omega W$ instead of $W$. Here $\Omega(q + \sqrt{-1} q')$ is understood as $\Omega q + \sqrt{-1} \Omega q'$, $q, q' \in F^{r-1}$, and $\Omega q = (\Omega q_1, \ldots, \Omega q_{r-1}) \in F^{r-1}$. As in [1, pp. 158-162] one obtains that the space $\mathcal{X}(f, \mathfrak{b})$ of the representation $\rho(f, \mathfrak{b})$ corresponding to the chosen $f$ and $\mathfrak{b}'$ may be realized as a space of complex $C^\infty$ functions $\psi$ on the complex space $\overline{W}'$, square integrable with respect to the measure $\exp(-\sqrt{-1} \langle f, [Y, \overline{Y}] \rangle) dYd\overline{Y}$ ($dYd\overline{Y}$ denoting the Lebesgue measure) on $\overline{W}'$ and satisfying the following functional equation:

$$\tau(\sqrt{-1} X) \psi)(\overline{Y}) = \sqrt{-1} \left[ \tau(\overline{X}) \psi)(\overline{Y}) \right], \quad X, Y \in W'.$$
where \([\tau(\vec{X})\psi](\vec{Y}) = (d/dt)\psi(\vec{Y} + t\vec{X})|_{t=0}\). The representation \(\rho\) is given by

\[
\rho(f, b) \left( \exp(\vec{X}_0 + X_0 + Z_0) \right) \psi(\vec{X})
\]

\[
= \exp\left( \sqrt{-1} \left( f, \left[ X_0, \vec{X} \right] \right) - \frac{1}{2} \sqrt{-1} \left( f, \left[ X_0, \vec{X}_0 \right] \right) \right) \chi_f(Z_0) \cdot \psi(\vec{X} - \vec{X}_0),
\]

(2.3)

where \((\vec{X}_0 + X_0 + Z_0)\) is in \(n^-\) for \(X_0 \in W', Z_0 \in F_0\); \(\chi_f(Z_0) = \exp(\sqrt{-1} \left( f, Z_0 \right))\).

Passing to the complex coordinates \((z_1, \ldots, z_{d(r-1)})\) on \(W'\), according to the identifications

\[
\begin{align*}
X = (z, z_1, \ldots, z_r) &\leftrightarrow (z, z_1, \ldots, z_{r-1}) \quad \text{for } F = C, \\
X = (z, z_1^2, \ldots, z_{r-1}^2) &\leftrightarrow (z, z_1, \ldots, z_{r-1}) \quad \text{for } F = H, \\
X = (z, z_1^3, \ldots, z_r^3) &\leftrightarrow (z, z_1, \ldots, z_{r-1}) \quad \text{for } F = O,
\end{align*}
\]

we have

\[
\sqrt{-1} \left( f, \left[ X, \vec{X}' \right] \right) = |p|^2 \bar{z} \cdot z', \quad X = (\vec{X})^-, \quad \vec{X}, \vec{X}' \in W',
\]

\[
f = (0, p^0), \quad z = (z_1, \ldots, z_{d(r-1)}) \in C^{r(r-1)}.
\]

Rewriting (2.3) in these coordinates we obtain

PROPOSITION 2.1. All the inequivalent irreducible unitary representations of \(N^-\) fall into two classes:

(a) a family of 1-dimensional characters \(\chi_q\) parameterized by \(q \in F^{r-1}\) and given by

\[
\chi_q(q, p) = \exp(\sqrt{-1} \Re(q^j \cdot q)), \quad (q, p) \in N^-;
\]

(b) a family of infinite-dimensional representations \(\rho_{p'}\) parameterized by \(p' \in F_0 \setminus \{0\}\). The Hilbert space \(\mathcal{H}_{p'}\) of the representation \(\rho_{p'}\) consists of holomorphic functions \(\psi\) on \(C^{r(r-1)}\), such that

\[
\|
\psi
\|_{p'}^2 = \int_{C^{r(r-1)}} |\psi(z)|^2 \exp(-|p'| |z|^2) \, dz \, d\bar{z} < \infty,
\]

with

\[
dz \, d\bar{z} = \prod_{j=1}^{z(r-1)} 2 \Re z_j d \Im z_j.
\]

The action of \(\rho_{p'}\) on \(\psi \in \mathcal{H}_{p'}\) is given by

\[
(\rho_{p'}(q, p)\psi)(z_j) = \exp(\sqrt{-1} \Re(p^j p) + |p'| |z_0 \cdot z - \frac{1}{2} |z_0|^2) \psi(z - z_0),
\]

\(z \in C^{r(r-1)}, (q, p) \in N^-\) with \(q = q(z_0), z_0 \in C^{r(r-1)}\), where

\[
q(z) = \Omega(P(z_1, \ldots, z_s), \ldots, P(z_{d(r-2)+1}, \ldots, z_{d(r-1)}))
\]

and \(P(z_{d(l-1)+1}, \ldots, z_d), l = 1, \ldots, r - 1\), is defined as

\[
\begin{align*}
\Re z_i + i \Im z_i &\quad \text{for } F = C, \\
\Re z_{2l-1} + i \Im z_{2l-1} + j \Im z_{2l} + k \Re z_{2l} &\quad \text{for } F = H, \\
\Re z_1 + i \Im z_1 + j \Im z_2 + k \Re z_2 + e \Im z_3 &\quad \text{for } F = O;
\end{align*}
\]
\[ \text{Im}(a + \sqrt{-1} b) = b, a, b \in \mathbb{R}. \] The functions

\[ \psi_n^{(r)}(z) = (2\pi)^{-m/2} |p'|^{(n_1 + m)/2}(n!)^{-1/2} z^n, \quad z \in \mathbb{C}^m, \]

\( n = (n_1, \ldots, n_m) \in \mathbb{N}^m, \) with \( n! = n_1! \cdots n_m!, \) \( z^n = z_1^{n_1} \cdots z_m^{n_m}, \) \( |n| = n_1 + \cdots + n_m, \) \( m = s(r - 1), \) form an orthonormal basis of \( \mathcal{K}_r \) as \( n \) runs over \( \mathbb{N}^m. \)

We also note the following symmetry properties of \( \chi_f \) and \( \rho_f \) relative to the orthogonal and the unitary transformations.

**Proposition 2.2.** (a) Let \( \alpha_1, \ldots, \alpha_{r-1} \in O(\sigma, \mathbb{R}); \) then for \( q' \in \mathbb{F}^{r-1}, \)

\[ \chi_f^{(r)}(\alpha_1q_1, \ldots, \alpha_{r-1}q_{r-1}, p) = \chi_f(\alpha_1\sigma_f, \ldots, \alpha_{r-1}\sigma_f)(q, p), \quad (q, p) \in N^- . \]

(b) Let \( u_1, \ldots, u_{r-1} \in U(s); \) then for \( p' \in F_0 \setminus \{0\}, \)

\[ A^{-1}_u \rho_p(q, p) A_u = \rho_p(q, p), \quad (q, p) \in N^- \quad \text{with} \quad q = q(z_0), \quad z_0 \in \mathbb{C}^{(r-1)} \quad \text{and} \quad q^u = q(uz_0) \quad \text{with} \quad uz = (u_1(z_1, \ldots, z_s), \ldots, u_{r-1}(z_{(r-2)+1}, \ldots, z_{(r-1)})); \]

\[ ((A_u \phi)(z) = \phi(uz) \quad \text{for} \quad \phi \in \mathcal{K}_r; \]

\[ u = (u_1, \ldots, u_{r-1}), \quad \sigma = 2s = \dim \mathbb{R} F. \]

**3. Algebra of multiradial functions.**

**Definition (cf. Geller [2]).** We say that a function \( F \) on \( N^- = \mathbb{F}^{r-1} \times F_0, \)

\( F = C, H, O, \) is multiradial if there is a function \( f \) on \( \mathbb{R}^{r-1} \times F_0 \) such that

\[ F(q, p) = A(q, p, F(q, p)), \quad (q, p) \in N^- . \]

\[ (3.0) \]

**Proposition 3.1.** Let \( \mathcal{E} \) denote the space of multiradial functions in \( L^1(N^-). \) Then \( \mathcal{E} \) is a commutative closed \( * \)-subalgebra of \( L^1(N^-) \) and \( \mathcal{E} \) is symmetric.

**Proof.** 1°. If \( F, G \in \mathcal{E} \) then \( F \ast G \in \mathcal{E}. \) For we have

\[ F \ast G(q', p') = \int f(|q_1|, \ldots, |q_{r-1}|, p) \prod g(|q'_1 - q_1|, \ldots, |q'_{r-1} - q_{r-1}|, p' - p - \text{Im}(\bar{q} \cdot q')) \, dq \, dp. \]

\[ (3.1) \]

Substituting \( q = ((q'_1/|q'_1|)\bar{q}_1, \ldots, (q'_{r-1}/|q'_{r-1}|)\bar{q}_{r-1}) \) we get (since \( (ab)b = a(bb) \) for \( a, b \in \mathbb{F} \))

\[ \int f(|\bar{q}_1|, \ldots, |\bar{q}_{r-1}|, p) g(|q'_1| |1 - \bar{q}_1/|q'_1||, \ldots, |q'_{r-1}| |1 - \bar{q}_{r-1}/|q'_{r-1}||, p' - p - \text{Im} \left( \sum_{i=1}^{r-1} \bar{q}_i |q'_i| \right) \, d\bar{q} \, dp, \]

i.e. \( F \ast G \) is multiradial. Obviously \( \mathcal{E} \) is closed.

2°. \( \mathcal{E} \) is commutative (cf. Kaplan and Putz [6, p. 377]). Under the orthogonal change of variables

\[ q_l \mapsto q''_l = q'_l \cdot 2 \Re(\bar{q}_l q''_l)/|q''_l|^2 - q_l, \quad l = 1, \ldots, r - 1, \]

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one has \(|q'_i - q''_i| = |q'_i - q_i|\) and \(\text{Im}(q'_i q_i) = -\text{Im}(q''_i q_i)\). Thus (3.1) is equal to
\[
\int f(|q''_i|, \ldots, |q''_{r-1}|, p) \\
\times g\left(|q'_1 - q''_1|, \ldots, |q'_{r-1} - q''_{r-1}|, p' - p - \sum_{i=1}^{r-1} \text{Im}(\bar{q}'_i q''_i)\right) dq'' dp \\
= \int_{\mathcal{N}} F(q'', p) G((q', p')(q'', p)^{-1}) dq'' dp \\
= G \ast F(q', p') .
\]

3°. Since \(L^1(N^-)\) is symmetric (Leptin [8, p. 205]), its *-subalgebra \(\mathcal{A}\) is also symmetric.

**Proposition 3.2.** For \(F \in \mathcal{A}\) and \(u = (u_1, \ldots, u_{r-1}) \in U(s) \times \cdots \times U(s)\), the operators \(\rho_p(F) = \int_{\mathcal{N}} F(q(p), p) \rho(q(p), p) dq dp\) and \(A_u\) commute on \(\mathcal{K}_{\rho}^r\).

**Proof.** By Proposition 2.2(b),
\[
A_u^{-1} \int \rho_p(q(z_0), p) F(q(z_0), p) dq(z_0) dp = \int \rho_p(q(u z_0), p) F(q(z_0), p) dq(z_0) dp.
\]
Since
\[
q_i(u z_0) = \Omega(P(u_i Z_i)) = (\Omega P u_i P^{-1} \Omega^{-1})(\Omega P Z_i)
\]
with \(Z_i = (z_{l(-1)+1}, \ldots, z_0, z)_l, l = 1, \ldots, r - 1,\) and since \(\Omega P u_i P^{-1} \Omega^{-1}\) is an orthogonal transformation on \(\mathbb{R}^n \approx F\), and \(\Omega P Z_i = q_i(z_0)\), the last integral is equal to \(\rho_p(F)\).

**Remark.** For \(F = \mathbb{R}\), the corresponding group \(N^- = \mathbb{R}^{r-1}\), so the algebra \(L^1(N^-)\) is already commutative, and, as in the case \(M = \mathbb{R}^n \times \mathbb{R}^+\) with the Euclidean metric [5], we consider \(\mathcal{A} = L^1(N^-)\).

4. **Multiplicative linear functionals on \(\mathcal{A}\).** Let \(\Phi\) be a nonzero multiplicative linear functional on \(\mathcal{A}\). Since \(\mathcal{A}\) is a symmetric *-subalgebra of \(L^1(N^-)\), there exist an irreducible *-representation \(\pi\) of \(L^1(N^-)\) and a unit vector \(\xi\) in the Hilbert space \(\mathcal{K}_{\pi}^r\) such that
\[
\pi(F)\xi = \Phi(F)\xi \quad \text{for all } F \in \mathcal{A} .
\]
If \(\mathcal{K}_{\pi}^r\) is one dimensional, then
\[
\pi(F)\xi = \int_{\mathcal{N}} F(q, p) \chi_{q'}(q, p) dq dp\xi
\]
for some \(q' \in \mathbb{F}^{r-1}\), and by Proposition 2.2(a), if \(q'\) and \(q''\) in \(\mathbb{F}^{r-1}\) are such that \(|q'_l| = |q''_l|, l = 1, \ldots, r - 1,\) the \(\Phi\)'s corresponding by (4.1) and (4.2) to \(\chi_{q'}\) and \(\chi_{q''}\) are identical. If \(\pi \approx \rho_p\), then by Proposition 3.2, \(\rho_p(F)\) and \(A_u\) commute. Now for \(\psi(z) = \psi_1(Z_1) \cdots \psi_{r-1}(Z_{r-1})\) with \(z = (Z_1, \ldots, Z_{r-1}), Z_l = (z_{l(-1)+1}, \ldots, z_l), l = 1, \ldots, r - 1,\) we have
\[
(A_u \psi)(z) = \psi_1(u_1 Z_1) \cdots \psi_{r-1}(u_{r-1} Z_{r-1}) .
\]

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Thus putting $\psi_j(Z_j) = Z_j^{n_j}$ with $n_j = (n_{1j}, \ldots, n_{sj}) \in \mathbb{N}^s$, we note that $A_u$ preserves the finite-dimensional subspaces of $\mathfrak{H}_p$, namely the spaces $\mathfrak{H}^n = \bigotimes_{j=1}^{s-1} \mathfrak{H}^{n_j}$, where $n = (n_1, \ldots, n_{s-1}) \in \mathbb{N}^{s-1}$. $\mathfrak{H}^{n_j}$ is the space of homogeneous polynomials in $z_{s(1-j)}, \ldots, z_d$ of degree $|n_j|$. Moreover, $\mathfrak{H}_p = \bigoplus_n \mathfrak{H}^n$ — an orthogonal direct sum over $n \in \mathbb{N}^{s-1}$. We also note that $A_u$ restricted to $\mathfrak{H}^n$ is equal to $\bigotimes_{j=1}^{s-1} T^{n_j}(u_1^{-1}, \ldots, u_{s-1}^{-1})$ with $T^k = |n_k|$, being the representation of $U(s)$ on $\mathfrak{H}^k$ given by $(T^k_u \psi)(Z) = \psi(u^{-1}Z)$. Since $T^k$ is irreducible (cf., e.g., [13, pp. 204–209]), the representations $T^n = \bigotimes_j T^{n_j}$ of $U(s) \times \cdots \times U(s)$, $r - 1$ copies of $U(s)$, act irreducibly on $\mathfrak{H}^n$, and $T^n \cong T^m$ iff $n = m$. Hence, by Schur’s Lemma, every intertwining operator $S$ for $\bigoplus_n T^n$ on $\mathfrak{H}_p$ is of the form $S = \bigoplus_n c_n(S) 1_{\mathfrak{H}^n}$. In particular, each $\rho_p(F)$ with $F \in \mathfrak{G}$ is such. It follows from (4.1) that $\Phi(F)$ is equal to one of the constants $c_n(\rho_p(F)), n \in \mathbb{N}^{s-1}$. Conversely, for every fixed $n$, the mapping $F \mapsto c_n(\rho_p(F))$ defines a multiplicative linear functional on $\mathfrak{G}$. Now we shall derive explicit formulas for the constants $c_n$ above. Since, e.g.,

$$c_n(\rho_p(F)) = \left( \rho_p(F) \psi_{p'}^{n'}, \psi_{p'}^{n'} \right)_{\mathfrak{H}_p},$$

with $n' = (n_1, 0, \ldots, 0; n_2, 0, \ldots, 0; \ldots; n_{s-1}, 0, \ldots, 0) \in (\mathbb{N}^s)^{s-1}$, we calculate the integral, see Proposition 2.1(b),

$$\int_{C^{r-1}} \left[ \rho_p(F) \psi_{p'}^{n'} \right](z) \tilde{\psi}_{p'}^{n'}(z) \exp(-|p'| |z|^2) \, dz \, d\tilde{z}, \quad (4.3)$$

which in expanded form is equal to (with $k = s(r - 1)$)

$$(2\pi)^{-k(n!)}^{-1} |p'|^{n_k+k}$$

$$\times \int_{C^{s-1}} \int_{F^{-1} \times F_0} F(q(z_0), p) \exp \left( \sqrt{-1} \left( \text{Re}(\bar{p'}p) + |p'| (\bar{z}_0 \cdot z - \frac{1}{2} |z_0|^2) \right) \right)$$

$$\times (z - z_0)^{n} \tilde{z}^{n'} \exp(-|p'| |z|^2) \, dq(z_0) \, dp \, dz \, d\tilde{z}. \quad (4.4)$$

The integral

$$\int_{C^{r-1}} (z - z_0)^{n} \tilde{z}^{n'} \exp(-|p'| |z|^2) \exp(|p'| \bar{z}_0 \cdot z) \, dz \, d\tilde{z} \quad (4.5)$$

is equal to

$$(2\pi/|p'|)^{s(r-1)} \prod_{i=1}^{r-1} 2\pi n_i |p'|^{-n_i-1} \sum_{j=0}^{n_i} \binom{n_i}{j} \binom{\eta_j}{j!}^{-1}$$

$$= (2\pi)^{s(r-1)} |p'|^{-n} \prod_{i=1}^{r-1} L_n(|p'| |z_i^0|^{2(r-1)+1}) \quad (4.5a)$$

with $L_n$ being the Laguerre polynomial. (4.5a) is obtained (see [5]) by substituting the binomial formula for $(z - z_0)^{n}$, developing $\exp(-|p'| \bar{z}_0 \cdot z)$ in a power series and integrating this series term by term using the orthogonality relations for the
functions $z^n$ in $\mathcal{H}_p$. Substituting (4.5a) in (4.4) we obtain that (4.3) is equal to

$$\int_{F^{r-1} \times F_0} F(q(z_0), p) \exp(\sqrt{-1} \ \text{Re}(\bar{p}^r p) - \frac{1}{2} |p'|^2 |z_0|^2)$$

$$\times \prod_{l=1}^{r-1} L_n(|p'| |z_{x_{(l-1)+1}}^2) \ dq(z_0) \ dp$$

$$= \int_0^\infty dt_1 \ldots \int_0^\infty dt_{r-1} \left( \int_{F_0} f(t_1, \ldots, t_{r-1}, p) \exp(\sqrt{-1} \ \text{Re}(\bar{p}^r p)) \ dp \right)$$

$$\times \exp\left(-\frac{1}{2} |p'| (t_1^2 + \ldots + t_{r-1}^2) \right) \prod_{l=1}^{r-1} t_l^{\sigma-1} g_l,$$

with $f$ as in (3.0) and $g_l$ given by

$$g_l = \int_{S(\sigma-1)} L_n(|p'| |z_l^2) \ dS(q(Z)),$$

$$Z = (z_1, \ldots, z_r) \in \mathbb{C}, \quad |Z| = t_l,$$

$S(\sigma-1)$ being the unit sphere in $F$. Since here $q(Z) = \Omega(PZ)/t_l$, with $P$ as in Proposition 2.1(b) and $\Omega \in O(\sigma, \mathbb{R})$, in order to compute $g_l$ one has to calculate the integrals

$$\int_{S(\sigma-1)} |z_j(q)|^2 \ dS(q), \quad j = 0, \ldots, n,$$

(4.6)

where $|z_j(q)|^2 = t_j^2((q_1)^2 + (q')^2)$ with $q_1, q', \ldots \ldots$ denoting the coordinates of $q$ in the (standard) basis $\{1, i, \ldots\}$ of $F$ over $\mathbb{R}$. Now (4.6) is equal to

$$\int_0^{\pi/2} \cos^2 \theta \cos \theta \sin^{s-3} \theta \ d\theta \cdot 2\pi \cdot 2\pi^{s-1} [(s-2)!]^{-1} t_l^{j}$$

$$= 2\pi t_l^{j} j! / (j + s - 1)!.$$

We summarize the results of this section in the following:

**Proposition 4.** The multiplicative linear functionals on $\mathcal{H}_p$ fall into two classes:

(a) The functionals corresponding to $(r-1)$-tuples $(t_1, \ldots, t_{r-1})$ of nonnegative real numbers and given by

$$F \mapsto \hat{F}(t_1, \ldots, t_{r-1}) = \int_{F^{r-1} \times F_0} F(q, p) \exp(\sqrt{-1} \ \text{Re}(\bar{q'} \cdot q)) \ dq \ dp$$

with $q' \in F^{r-1}$ arbitrary provided $(|q_1|, \ldots, |q_{r-1}|) = (t_1, \ldots, t_{r-1})$.

(b) The functionals corresponding to pairs $(p', n) \in F_0 \setminus \{0\} \times \mathbb{N}^{r-1}$ and given by

$$F \mapsto \hat{F}(p', n) = (2\pi^r)^{-1} \int_{R^{r-1}_+} \exp\left(-\frac{|p'|}{2} (t_1^2 + \ldots + t_{r-1}^2) \right) \prod_{l=1}^{r-1} L_n(|p'| t_l^2) t_l^{\sigma-1}$$

$$\times \left( \int_{F_0} f(t_1, \ldots, t_{r-1}, p) \exp(\sqrt{-1} \ \text{Re}(\bar{p} p)) \ dp \right) dt_1 \ldots dt_{r-1},$$

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where \( f(q_1, \ldots, q_{r-1}, p) = F(q_1, \ldots, q_{r-1}, p) \) and
\[
L_k^{(m)}(x) = \sum_{j=0}^{k} \frac{(-x_j)^j}{(j+m)!} \binom{k}{j} (k + m)!^{-1} x^{-m} e^{x}(d^k/dx^k)(x^{k+m} e^{-x}).
\] (4.7)

5. Nonvanishing of the Gel’fand transform of \( P_a \).

**Lemma 1.** For \( 1 < m < 2k + \frac{1}{2} \), \( k > \frac{3}{2} \) and \( Q > 0 \), the following formula holds:
\[
\int_{\mathbb{R}^m} \frac{\exp(\sqrt{-1} x_0 \cdot x)}{(Q^2 + 4|x|^2)^k} dx = 2^{-m} \pi^{m/2} \frac{\Gamma(k - m/2)}{\Gamma(k)} Q^{m-2k},
\]
for \( x_0 = 0 \),
\[
= 2^{m+1} 4 k^{(m+1)/2} \frac{\Gamma(2k-m-(r/2)Q)}{\Gamma(k) \Gamma(k - (m-1)/2)} \times \int_0^\infty e^{-(r/2)Q(t^2 - 1)^{k-(m+1)/2}} dt,
\]
for \( r = |x_0| \neq 0 \). (5.1)

**Proof.** For \( x_0 = 0 \), the integral is equal to the “area” of the unit sphere in \( \mathbb{R}^m \)
(= 2 when \( m = 1 \)) times \( \int_{\mathbb{R}} r^{m-1} (Q^2 + 4r^2)^{-k} dr \) and we substitute \( r = r'Q/2 \).
For \( x_0 \neq 0 \), the function \( 4((r_0^2 + x^2))^{-m} \) is radial on \( \mathbb{R}^m \), hence its Fourier transform (5.1) is equal to, see, e.g., [9, p. 155],
\[
4^{-k}(2\pi)^{m/2} r^{-(m-2)/2} \int_0^\infty \left( \left[ \frac{1}{2} Q^2 + t^2 \right]^{-k} J_{(m-2)/2} (rt) \right)^{m/2} dt,
\]
where \( m > 1 \), \( k > 3/2 \).

Combining now the Sonine formula [12, p. 434, (2)],
\[
\int_0^\infty x^{r+1} J_r(ax) dx \quad \left( \frac{x^2 + k^2}{2} \right)^{r+1} = \frac{a^n k^{-\mu}}{2^n \Gamma(\mu + 1)} K_{\mu-r}(ak),
\]
valid when \( -1 < \text{Re}(\nu) < 2 \text{Re}(\mu) + \frac{3}{2} \), with the following expression for the function \( K \) [12, p. 172, (4)],
\[
K_\nu(z) = \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu + \frac{1}{2})} \int_1^\infty e^{-zt} (t^2 - 1)^{r-1/2} dt,
\]
valid for \( \text{Re}(\nu + \frac{1}{2}) > 0 \), \( |\arg z| < \pi/2 \), we obtain (5.1).

**Lemma 2.** For \( \epsilon > 0 \), \( m > 0 \), \( x \in \mathbb{R}^n \),
\[
(\epsilon + |x|^2)^{-m} = (4\pi)^{-n/2} \Gamma(m)^{-1} \int_{\mathbb{R}^n} \exp(-\sqrt{-1} x \cdot y) \times \left( \int_0^\infty t^{m-1-m/2} e^{-\epsilon t} e^{-|y|^2/4t} dt \right) dy,
\]
i.e. \((\epsilon + |x|^2)^{-m}\) is a Fourier transform of a positive function in \( L^1(\mathbb{R}^n) \).
Proof. Combine
\[ (\epsilon + |x|^2)^{-m} = \Gamma(m)^{-1} \int_0^\infty t^{m-1} e^{-(\epsilon + |x|^2)t} \, dt, \quad m > 0, \]
with
\[ \exp(-|x|^2 t) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} \exp(-|y|^2/4t) \exp(-\sqrt{-1} x \cdot y) \, dy, \]
and note that the obtained double integral is absolutely convergent.

Proposition 5. For every \( a \in A \), \( \hat{P}_a \) does not vanish on \( \mathcal{M}(\hat{\mathcal{E}}) \) — the maximal ideal space of \( \hat{\mathcal{E}} \).

Proof. (a) For the points \((t_1, \ldots, t_{r-1}) \in \mathcal{M}(\hat{\mathcal{E}})\), integrating over \( F_0 \) in the formula of Proposition 4(a), according to Lemma 1 (with \( x_0 = 0 \)) we get
\[
\hat{P}_a(t_1, \ldots, t_{r-1}) = c \int_{\mathbb{R}^{r-1}} \frac{\exp(\sqrt{-1} \operatorname{Re}(\bar{q} \cdot q))}{(\epsilon + |q|^2)^{d+1-\sigma}} \, dq,
\]
with
\[ c = 2^{1-s} \pi^{-(d-1)/2} e^{d/2} c_{r, \epsilon} \Gamma(d/2 - s + 1/2) \Gamma(d/2)^{-1}, \]
and this is positive by Lemma 2 and the Fourier inversion formula.

(b) For the points \((p', n) \in \mathcal{M}(\hat{\mathcal{E}})\) with \( p' \in F_0 \setminus \{0\}, \quad n = (n_1, \ldots, n_{r-1}) \in \mathbb{N}^{r-1} \), we use the formula from Proposition 4(b) for \( \hat{P}_a(p', n) \). Applying Lemma 1 to the integral over \( F_0 \) there (with \( Q = \epsilon + t_1^2 + \cdots + t_{r-1}^2, \quad m = \sigma - 1, \quad x_0 = p', \quad k = d/2 \)), then interchanging the order of integration from \( dt_1 \cdots dt_{r-1} \) to \( dt_1 \cdots dt_{r-1} dt \), making the change of variables \((|p'|t_1^2, \ldots, |p'|t_{r-1}^2) = (x_1, \ldots, x_{r-1})\), and finally applying (4.7), we obtain
\[
\hat{P}_a(p', n) = c \int_0^\infty e^{-|p'|\tau^{d/2}}((\tau + 1)^2 - 1)^{d/2-s} \prod_{i=1}^{r-1} \int_0^\infty e^{-x/2}(d^n/dx^n)(x^{n+s-1}e^{-x}) \, dx.
\]
with
\[ c = 2^{(r-1)(d/2)} \Gamma(d/2)^{-1} \exp(-|p'|d/2)
\]
and
\[ \hat{g}_j(t) = ((n_j + s - 1))^{-1} \int_0^\infty e^{-tx/2}(d^n/dx^n)(x^{n+s-1}e^{-x}) \, dx.
\]
Integrating by parts get
\[ \hat{g}_j(t) = ((n_j + s - 1))^{-1} \int_0^\infty e^{-tx/2}x^{n+s-1}e^{-x} \, dx \cdot (t/2)^n.
\]
Thus \( \hat{P}_a(p', n) \) is positive.

Remark. For \( M \) being the real hyperbolic space, i.e. for \( F = \mathbb{R} \), we have \( N^- = \mathbb{R}^d, \quad \mathcal{M}(L^1(N^-)) = \hat{N}^-, \quad P_a(X_{-a}) = c_{d, \epsilon} \epsilon^{d/2}(\epsilon + |q|^2)^{-d} \) and, by Lemma 2, \( \hat{P}_a > 0 \) on \( \hat{N}^- \).
6. Theorem on ideals in \(L^1(N^-)\). Since the algebras \(\mathfrak{A}\) we consider here have the same qualitative properties as the one considered in [5], similar facts can be proved about them. In particular, the following statement about ideals in \(L^1(N^-)\) is a consequence of the Wiener property of \(\mathfrak{A}\) and existence of the approximate identity for \(L^1(N^-)\) in \(\mathfrak{A}\) (the dilations \(\delta_t, t > 0\), on \(N^-\) used in the construction of the approximate identity are given by \(\delta_t(q, p) = (t^{-1/2}q, t^{-1}p)\)).

**Proposition 6.** If \(\mathfrak{g}\) is a proper closed right ideal in \(L^1(N^-)\), then there is a \(\Phi\) in \(\mathcal{M}(\mathfrak{A})\) such that \(\hat{\Phi}(\Phi) = 0\) for every \(F \in \mathfrak{g} \cap \mathfrak{A}\).

7. Proof of the Theorem [5]. The Theorem follows now from Proposition 6, for if we put

\[
\mathfrak{g} = \left\{ f \in L^1(N^-) : \lim_{N^- \ni n \to \infty} \varphi \ast f(n) = c_0 \int_{N^-} f(n) \, dn \right\},
\]

with \(\varphi \in L^\infty(N^-)\) being the boundary value of the bounded harmonic function \(u\) on \(M\), then \(P_{a_0} \in \mathfrak{g} \cap \mathfrak{A}\) and \(\hat{P}_{a_0} \neq 0\) on \(\mathcal{M}(\mathfrak{A})\), so \(\mathfrak{g} = L^1(N^-)\). Hence \(P_a \in \mathfrak{g}\) for every \(a\) in \(A\).

**Added in proof.** Meanwhile Korányi [14] described the Gel'fand space, as well as the related Plancherel formula, for the commutative algebra \(\mathfrak{A}\) of biradial functions in \(L^1(N^-)\), i.e. the functions \(F\) such that

\[
F(q, p) = f(|q|, |p|), \quad (q, p) \in N^-,
\]

for some \(f\) on \(\mathbb{R}_+ \times \mathbb{R}_+\), cf. §3. His approach uses neither the classification of symmetric spaces nor the representations of nilpotent groups.

**References**


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