A TANGENTIAL CONVERGENCE FOR BOUNDED HARMONIC FUNCTIONS ON A RANK ONE SYMMETRIC SPACE

BY

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ABSTRACT. Let \( u \) be a bounded harmonic function on a noncompact rank one symmetric space \( M = G/K \approx N^-A, N^-AK \) being a fixed Iwasawa decomposition of \( G \). We prove that if for an \( a_0 \in A \) there exists a limit \( \lim u(na_0) = c_{a_0} \) as \( n \in N^- \) goes to infinity, then for any \( a \in A \), \( \lim u(na) = c_0 \). For \( M = SU(n, 1)/S(U(n) \times U(1)) = B^n \), the unit ball in \( \mathbb{C}^n \) with the Bergman metric, this is a result of Hulanicki and Ricci, and in this case it reads (via the Cayley transformation) as a theorem on convergence of a bounded harmonic function to a boundary value at a fixed boundary point, along appropriate, tangent to \( \partial B^n \), surfaces.

0. Introduction. Let \( M \) be a noncompact symmetric space of rank one. \( M \) can be expressed as a homogeneous space \( G/K \) where \( G \) is a semisimple group of isometries of \( M \) and \( K \) is a maximal compact subgroup of \( G \). Let \( g, f \) denote the Lie algebras of \( G \) and \( K \), \( B \) the Killing form of \( g \), and \( p \) the orthogonal complement of \( f \) in \( g \) relative to \( B \). If \( \pi: G \to G/K \) denotes the canonical projection, its differential at the identity, \( \pi_* \), identifies the subspace \( p \) of \( g \) with \( T_0(M) \), the tangent space of \( M \) at the origin \( o = \pi(e) \), and the invariant metric \( g \) on \( M \) can be chosen so that \( g_0 \) corresponds to the restriction of \( B \) to \( p \times p \) under the above identification. We denote by \( \Delta \) the corresponding \((G\text{-invariant})\) Laplace-Beltrami operator on \( M \). A function \( u \in C^\infty(M) \) is called harmonic if \( \Delta u = 0 \). Let \( a \) be a maximal (one-dimensional) abelian subspace of \( p \), \( a \) and possibly \( 2a \) in \( a^* \), the corresponding system of positive restricted roots relative to the fixed choice of a "positive part" \( a^+ \) in \( a \). Let \( g_{-\alpha} \) and \( g_{-2\alpha} \) denote the root spaces corresponding to \( -\alpha \) and \( -2\alpha \). Then \( n^- = g_{-\alpha} \oplus g_{-2\alpha} \) is a nilpotent subalgebra of \( g \) and one has the Iwasawa decomposition \( G = N^-AK \), with \( N^- = \exp n^- \), \( A = \exp a \). The above decomposition shows that every \( p \in M \) can be uniquely written as \( p = na \cdot o \) \((n \in N^-, a \in A)\). We regard the nilpotent group \( N^- \) as a boundary for the symmetric space \( M \) in the following sense. The bounded harmonic functions \( u \) on \( M \) have boundary values on \( N^- \), i.e. \( \lim_{\log a \to \infty} u(na \cdot o) \equiv \varphi(n) \) exists a.e. (relative to the Haar measure on \( N^- \)) and \( \varphi \in L^\infty(N^-) \). \( \log a \to \infty \) is understood with respect to the ordering induced on \( a \) by \( a^+ \). Moreover,

\[ u(na \cdot o) = \varphi \cdot P_a(n) = \int_{N^-} \varphi(n_1) P_a(n_1^{-1}n) \, dn_1. \]

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The function $P_a(n)$ on $N^- \times A$ is called the Poisson kernel for the symmetric space $M$ and is given by (Helgason [4])

$$P_a(n) = ce^{d/2} \left[ \left( e + \frac{1}{2} Q(X_{-a}) \right)^2 + 2Q(X_{-2a}) \right]^{-d/2},$$

where

$$n = \exp(X_{-a} + X_{-2a}), \quad X_{-a} \in g_{-a}, \quad X_{-2a} \in g_{-2a};$$

$$e = e^{-\alpha((\log a)), \quad Q(X) = (X, X)_\theta/2(m_a + 4m_{2a})}$$

with $(X, X)_\theta = -B(X, \theta X)$ for $X \in g, \theta$ denoting the Cartan involution associated with the pair $(g, \mathfrak{f}); m_a = \dim g_{-a}, m_{2a} = \dim g_{-2a}, d = m_a + 2m_{2a}.$ The constant $c$ is such that the integral of $P_a$ over $N^-$ is equal to 1.

The following theorem on "tangential" convergence for bounded harmonic functions on the Siegel domain

$$D_{r-1} = \left\{ (z_1, \ldots, z_r) \in \mathbb{C}^r : \text{Im} z_r > \sum_{j=1}^{r-1} |z_j|^2 \right\},$$

$r > 2$, (or, equivalently, on $M = SU(r, 1)/S(U(r) \times U(1))$–the complex hyperbolic space) has been obtained by Hulanicki and Ricci [5]. We formulate it below in terms of a homogeneous space $M$.

**THEOREM.** Let $u$ be a bounded harmonic function on a noncompact rank one symmetric space $M$. In the notation above, assume that for an $a_0 \in A$, $\lim_{N^- \ni n \to \infty} u(na_0 \cdot o) = c_0.$ Then for any $a \in A$, $\lim_{N^- \ni n \to \infty} u(na \cdot o) = c_0.$

Our aim here is to prove the above Theorem and the proof is based on the classification of symmetric spaces. That is, we discuss separately the cases of $M$ being the real, complex (to see how the $M = D_{r-1}$ case fits to our scheme), quaternion and octonion hyperbolic space, which corresponds respectively to $G$ being the classical group $SO_{q}(r, 1), SU(r, 1), Sp(r, 1)$ and the exceptional group $F_{4(-20)}.$ Following the Hulanicki-Ricci method, for each case we construct a suitable commutative subalgebra $\mathfrak{g}$ of (multi) radial functions in $L^1(N^-)$, to which the Poisson kernel $P_a$ belongs. We describe the set $\mathfrak{M}(\mathfrak{g})$ of the maximal ideals in $\mathfrak{g}$ and check that the Gel’fand transform $\hat{P}_a$ of $P_a$ never vanishes on $\mathfrak{M}(\mathfrak{g}).$ The Theorem may then be stated as a theorem on certain ideals in $L^1(N^-)$ and is a consequence of the Wiener property of the algebra $\mathfrak{g}$. To study the algebra $\mathfrak{g}$ we use the holomorphically induced (realizations of the irreducible unitary) representations of $N^-$. 

1. **Nilpotent group $N^-$.** Let $F$ denote the field $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or the Cayley numbers $\mathbb{O}$ (octonions); $F_0 = \{ q \in F : q + \bar{q} = 0 \}$, $\bar{q}$ being the usual conjugation in $F$. According to the notation of the previous section, $g = \mathfrak{f} + \mathfrak{p}$ and for the classical $G$ we have (cf., e.g., [3, pp. 348–351])

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TANGENTIAL CONVERGENCE

\[ \mathfrak{f} = \left\{ \begin{pmatrix} Z & 0 \\ 0 & p \end{pmatrix} : \begin{array}{l} Z \text{ an } r \times r \text{ skew-Hermitian matrix over } F, \\ p \in F_0, \text{ tr } Z = -p \text{ in case of } F = C \end{array} \right\}, \]

\[ \mathfrak{p} = \left\{ \begin{pmatrix} 0 & \ 'q \\ \ 'q & 0 \end{pmatrix} : q \in F^r = F \times \cdots \times F \right\}, \]

\[ \alpha = \{ tE_{1,r+1} + tE_{r+1,1} : t \in \mathbb{R} \}, \]

where \( E_{kl} \) denotes the \( (r+1) \times (r+1) \) matrix \( (\delta_{ak} \delta_{bl})_{1 \leq a,b \leq r+1}, r > 2 \). We choose a basis \( H = E_{1,r+1} + E_{r+1,1} \) in \( \alpha \) and fix an ordering so that \( H \in \alpha^+ \). Then \( \alpha \in \alpha^* \) such that \( \alpha(H) = 1 \) is a positive restricted root, and we have

\[ \mathfrak{g}_{-\alpha} = \left\{ \begin{pmatrix} 0 & -\ 'q \\ \ 'q & 0 \end{pmatrix} : q = (q_2, \ldots, q_r) \in F^{r-1} \right\}, \]

\[ \mathfrak{g}_{-2\alpha} = \left\{ \begin{pmatrix} p & 0 & p \\ 0 & 0 & \bar{p} \\ \bar{p} & 0 & \bar{p} \end{pmatrix} : p \in F_0 \right\} \quad (= \{ 0 \} \text{ for } F = \mathbb{R}). \]

We shall identify \( n^- = \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha} \) with \( F^{r-1} \times F_0 \) by the correspondence

\[ \begin{pmatrix} p & -\ 'q \\ \ 'q & p \end{pmatrix} \leftrightarrow (q, -p). \]

In these coordinates on \( n^- \) the commutator of \( (q, p) = (q_1, \ldots, q_{r-1}, p) \) and \( (q', p') = (q'_1, \ldots, q'_{r-1}, p') \) in \( F^{r-1} \times F_0 \) is given by

\[ [ (q, p), (q', p') ] = (0, 2 \text{ Im}(\bar{q} \cdot q')), \quad (1) \]

where we have put \( \bar{q} \cdot q' \) for \( \Sigma_{i=1}^{r-1} \bar{q}_i q'_i \). We also have the formula (cf., e.g., [11, p. 39])

\[ ((q, p), (q', p'))_e = 4(m_4 + 4m_{2a}) \text{Re}(\bar{q} \cdot q' + \bar{p} p'). \quad (2) \]

For the exceptional \( G \) (cf., e.g., [10, pp. 522–530]), \( \mathfrak{g} = \mathfrak{f}_{4(-20)} \) is isomorphic to the Lie algebra \( \text{Der}(\mathfrak{g}) \) of derivations of the Jordan algebra \( (\mathfrak{g}, \circ) \) of \( 3 \times 3 \) octonion matrices \( A \) of the form

\[ A = \begin{pmatrix} \alpha_1 & a_3 & a_2 \\ \bar{a}_3 & a_2 & a_1 \\ -\bar{a}_2 & -\bar{a}_1 & a_3 \end{pmatrix}, \quad a_i \in \mathbb{O}, \quad \alpha_i \in \mathbb{R}, \quad i = 1, 2, 3, \]

with multiplication given by \( A \circ B = \frac{1}{2}(AB + BA) \), \( A, B \in \mathfrak{g}, \) \( AB \) denoting the usual matrix multiplication. We have

\[ \mathfrak{f} = \{ D \in \text{Der}(\mathfrak{g}) : D(E_{33}) = 0 \}, \]

\[ \mathfrak{p} = \{ D_Q \in \text{Der}(\mathfrak{g}) : Q = \begin{pmatrix} 0 & \ 'q \\ \ 'q & 0 \end{pmatrix}, \quad q \in \mathbb{O}^2 \}, \]

where \( D_Q(B) = QB - BQ, \quad B \in \mathfrak{g} \).
\[ a = \{ D_\varphi \in \text{Der}(\mathfrak{g}) : Q = tE_{13} + tE_{31}, \quad t \in \mathbb{R} \}. \]

We choose \( H = D_\varphi \) with \( Q = E_{13} + E_{31} \in \mathfrak{a}^+ \) and \( \alpha \in \mathfrak{a}^* \) such that \( \alpha(H) = 1 \). Then

\[
\mathfrak{g}_{-\mathfrak{a}} = \left\{ D_{Q(q)} : Q(q) = \begin{bmatrix} 0 & -\bar{q} & 0 \\ q & 0 & q \\ 0 & \bar{q} & 0 \end{bmatrix}, \quad q \in \mathfrak{O} \right\},
\]

\[
\mathfrak{g}_{-2\mathfrak{a}} = \left\{ D_{Q(p)} : Q(p) = \begin{bmatrix} p & 0 & p \\ 0 & 0 & 0 \\ \bar{p} & 0 & \bar{p} \end{bmatrix}, \quad p \in \mathfrak{O}_0 \right\}.
\]

Identifying \( D_{Q(q)} + D_{Q(p)} \) in \( \mathfrak{g}_{-\mathfrak{a}} \oplus \mathfrak{g}_{-2\mathfrak{a}} \) with \( (q, -p) \) in \( \mathfrak{O} \times \mathfrak{O}_0 \), we obtain the same formulas for the commutator and the inner product of \((q, p)\) and \((q', p')\) in \( \mathfrak{O} \times \mathfrak{O}_0 \) as those given by (1) and (2) above.

Writing \( N^- \) as the manifold \( \mathfrak{n}^- \) with the group multiplication given by the Campbell-Hausdorff formula we obtain

**Proposition 1.** The underlying manifold for the nilpotent group \( N^- \) is \( F^r \times F_0 \) with \( r > 2 \) for \( F = \mathbb{R}, \mathbb{C}, \mathbb{H} \) \((F_0 = \{0\}\) if \( F = \mathbb{R}\)) and with \( r = 2 \) for \( F = \mathbb{O}\). The group law is

\[
(q, p)(q', p') = (q + q', p + p' + \text{Im}(q \cdot q')).
\]

The Haar measure on \( N^- \) is the ordinary Lebesgue measure on \( \mathbb{R}^k \approx F^r \times F_0 \), \( k = r^2 - 1 \). We normalize it so that the volume of the unit cube in \( \mathbb{R}^k \) is 1 and denote by \( dqdp \). The Poisson kernel is given by

\[
P_{\exp(tH)}(q, p) = c_{r,F} e^{d/2} \left( \frac{1}{|q|^2 + \epsilon} + 4|p|^2 \right)^{-d/2},
\]

where \( \epsilon = e^{-t} \), \( d = (r + 1)\sigma - 2 \), \( |q|^2 = \bar{q} \cdot q \), \( c_{r,F} = 2^{-d} r^{-\alpha} \Gamma(rs) \) with \( \sigma = 2s = \text{dim}_R F \).

2. Holomorphically induced representations of \( N^- \). The adjoint and coadjoint action of \( N^- \) on \( \mathfrak{n}^- \) and \( \mathfrak{n}^-* \), respectively, is given by

\[
\text{Ad}_{(q,p)}(q'', p'') = (q'', p'' + 2 \text{Im}(q \cdot q')),
\]

\[
\text{Ad}^*_{(q,p)}(q', p') = (q' + 2q\bar{p}', p'),
\]

\((q, p) \in N^-, (q'', p'') \in \mathfrak{n}^-, (q', p') \in \mathfrak{n}^-* \), and we have identified \( \mathfrak{n}^-* \), the dual space of \( \mathfrak{n}^- \), with \( \mathfrak{n}^- \) by \( \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_0/4(m_n + 4m_2) \).

The single points \((q', 0) \in \mathfrak{n}^-* \) are 0-dimensional orbits of \( \text{Ad}^* \) on \( \mathfrak{n}^-* \) and the corresponding (1-dimensional) representations of \( N^- \) are given by the characters

\[
\chi_{(q, 0)}(q, p) = \exp(\sqrt{-1} \text{ Re}(\bar{q} \cdot q)), \quad (q, p) \in N^-.
\]

The remaining (maximal dimensional) orbits, for \( F = \mathbb{C}, \mathbb{H} \) and \( \mathbb{O} \), are of the form \( F^r \times \{p''\}, p'' \neq 0 \), so they are parameterized, e.g., by the functionals \( f = (0, p'') \in \mathfrak{n}^-* \) with \( p'' \in F_0 \setminus \{0\} \). For such \( f \) and \((q, p), (q', p') \in \mathfrak{n}^- \) we have

\[
\langle f, [(q, p), (q', p')] \rangle = 2 \text{ Re}((pq'')^- \cdot q') = -2 \text{ Re}(\bar{q} \cdot (q'p''')),
\]
i.e. the operator $R_{p} \cdot q \rightarrow qp''$ is skew-symmetric on $F^{r-1}$ with respect to the $R$-bilinear symmetric form $\langle \cdot, \cdot \rangle$ on $F^{r-1}$ given by $\langle q, q' \rangle = 2 \Re(\bar{q} \cdot q')$ the usual inner product on $R$, $k = (r - 1)\sigma$. Let 1, i denote the usual basis of $C$ (over $R$); 1, i, j, k the basis of $H$, and 1, i, j, k, e, ie, je, ke the basis of $O (= H + He$ with the multiplication defined by $(ae)b = (ab)e$, $a(be) = (ba)e$, $(ae)(be) = -\bar{b}a$ for $a, b \in H$). Suppose now that $f = (0, i\lambda) \in n^{-*}$ with $\lambda$ positive real. In the above bases of $F$ the matrix of $R_{i\lambda}$ acting on $F^{r-1}$ with $r = 2$ is equal to $\lambda(E_{21} - E_{12})$ for $F = C$, to $\lambda(E_{21} - E_{12}) - \lambda(E_{43} - E_{34})$ for $F = H$ and to $\lambda(E_{21} - E_{12}) - \lambda(E_{43} - E_{34}) - \lambda(E_{65} - E_{56}) + \lambda(E_{87} - E_{78})$ for $F = O$. Put

$$e_{1} = \frac{1}{2}(1 + \sqrt{-1} i), \quad e_{2} = \frac{1}{2}(\sqrt{-1} j + k),$$

$$e_{3} = \frac{1}{2}(\sqrt{-1} e + ie), \quad e_{4} = \frac{1}{2}(je + \sqrt{-1} ke)$$

for elements in $F^{C} = F + \sqrt{-1} F$—the complexification of $F$. Now define a subspace $W$ of $F^{C}$ by

$$W = \begin{cases} Ce_{1} & \text{if } F = C, \\ Ce_{1} + Ce_{2} & \text{if } F = H, \\ Ce_{1} + Ce_{2} + Ce_{3} + Ce_{4} & \text{if } F = O. \end{cases}$$

Thus

$$\mathfrak{h} = W^{r-1} \times F_{0}^{C}$$

(2.1)

is a positive polarization at $f = (0, i\lambda)$ such that

$$\mathfrak{h} + \mathfrak{h} = n^{-C}, \quad \mathfrak{h} \cap \mathfrak{h} = \{0\} \times F_{0}^{C}, \quad \mathfrak{h}/\{0\} \times F_{0}^{C} = W,$$

(2.2)

where $\tilde{z} = x - \sqrt{-1} y$ for $z = x + \sqrt{-1} y \in n^{-} + \sqrt{-1} n^{-} = n^{-C}$. For arbitrary $f = (0, p'')$ with $p'' \in F_{0} \times \{0\}$, there exists an orthogonal transformation $\Omega$ on $R^{o} \approx F$, such that

$$\langle R_{p} \cdot q, q' \rangle = \langle R_{i\lambda}, \Omega q, \Omega q' \rangle, \quad q, q' \in F^{r-1}.$$ 

Hence,

$$\langle f, [(q, p), (q', p')] \rangle = \langle (0, i|p''|), [(\Omega q, p), (\Omega q', p')] \rangle,$$

and $\mathfrak{h}' = \Omega \mathfrak{h}$, with $\mathfrak{h}$ as in (2.1), is a positive polarization at $f$ and $\mathfrak{h}'$ satisfies (2.2) with $W' = \Omega W$ instead of $W$. Here $\Omega(q + \sqrt{-1} q')$ is understood as $\Omega q + \sqrt{-1} \Omega q'$, $q, q' \in F^{r-1}$, and $\Omega q = (\Omega q_{1}, \ldots, \Omega q_{r-1}) \in F^{r-1}$. As in [1, pp. 158-162] one obtains that the space $\mathcal{K}(f, \mathfrak{h})$ of the representation $\rho(f, \mathfrak{h})$ corresponding to the chosen $f$ and $\mathfrak{h}$ may be realized as a space of complex $C^{\infty}$ functions $\psi$ on the complex space $\tilde{W}'$, square integrable with respect to the measure $\exp(-\sqrt{-1} \langle f, [Y, \tilde{Y}] \rangle) dYd\tilde{Y}$ ($dYd\tilde{Y}$ denoting the Lebesgue measure) on $\tilde{W}'$ and satisfying the following functional equation:

$$[\tau(\sqrt{-1} \bar{X})\psi](\bar{Y}) = \sqrt{-1} [\tau(X)\psi](\bar{Y}), \quad X, Y \in W'.$$
where $[\tau(\bar{X})\psi](\bar{Y}) = (d/dt)\psi(\bar{Y} + t\bar{X})|_{t=0}$. The representation $\rho$ is given by

$$
\rho(f, b)(\exp(\bar{X}_0 + X_0 + Z_0))\psi(\bar{X})
= \exp\left(\sqrt{-1} \langle f, [X_0, \bar{X}] \rangle - (\sqrt{-1}/2)\langle f, [X_0, \bar{X}_0] \rangle\right)\chi_f(Z_0) \cdot \psi(\bar{X} - \bar{X}_0),
$$

(2.3)

where $\bar{X}_0 + X_0 + Z_0$ is in $\mathfrak{n}^-$ for $X_0 \in W'$, $Z_0 \in F_0$; $\chi_f(Z_0) = \exp(\sqrt{-1} \langle f, Z_0 \rangle)$.

Passing to the complex coordinates $(z_1, \ldots, z_{r-1})$ on $W'$, according to the identifications

- $X = (z, \tilde{z}, \ldots, z_{r-1}) \mapsto (z_1, \ldots, z_{r-1})$ for $F = \mathbb{C}$,
- $X = (z_1, \tilde{z}_2, \ldots, z_{2(r-1)} \tilde{z}_2) \mapsto (z_1, \ldots, z_{2(r-1)})$ for $F = \mathbb{H}$,
- $X = (z_1, \tilde{z}_2 + z_2 \tilde{z}_1, \ldots, z_{2(r-1)} \tilde{z}_2) \mapsto (z_1, \ldots, z_{2(r-1)})$ for $F = \mathbb{O}$,

we have

$$
\sqrt{-1} \langle f, [X, \bar{X}] \rangle = |p''| \bar{z} \cdot \bar{z}', \quad X = (\bar{X})^-, \quad \bar{X}, \bar{X}' \in W',
$$

where $f = (0, p'')$, $z = (z_1, \ldots, z_{2(r-1)}) \in \mathbb{C}^{r(r-1)}$.

Rewriting (2.3) in these coordinates we obtain

**Proposition 2.1.** All the inequivalent irreducible unitary representations of $N^-$ fall into two classes:

(a) a family of 1-dimensional characters $\chi_{q'}$ parameterized by $q' \in F^{-1}$ and given by

$$
\chi_{q'}(q, p) = \exp(\sqrt{-1} \Re(\bar{q} \cdot q)), \quad (q, p) \in N^-;
$$

(b) a family of infinite-dimensional representations $\rho_{p'}$ parameterized by $p' \in F_0 \setminus \{0\}$. The Hilbert space $\mathcal{H}_{p'}$ of the representation $\rho_{p'}$ consists of holomorphic functions $\psi$ on $\mathbb{C}^{r(r-1)}$, such that

$$
\|\psi\|_{p'}^2 = \int_{\mathbb{C}^{r(r-1)}} |\psi(z)|^2 \exp(-|p'| |z|^2) \, dz \, d\bar{z} < \infty,
$$

with

$$
dz \, d\bar{z} = \prod_{j=1}^{z(r-1)} 2d \Re z_j d \Im z_j.
$$

The action of $\rho_{p'}$ on $\psi \in \mathcal{H}_{p'}$ is given by

$$
(\rho_{p'}(q, p)\psi)(z) = \exp(\sqrt{-1} \Re(\bar{p} \cdot p) + |p'|(z_0 \cdot z - \frac{1}{2}|z_0|^2))\psi(z - z_0),
$$

$z \in \mathbb{C}^{r(r-1)}$, $(q, p) \in N^-$ with $q = q(z_0)$, $z_0 \in \mathbb{C}^{r(r-1)}$, where

- $q(z) = \Omega(P(z_1, \ldots, z_s), \ldots, P(z_{2(r-2)+1}, \ldots, z_{2(r-1)}))$,

and $P(z_{2(l-1)+1}, \ldots, z_{2s})$, $l = 1, \ldots, r-1$, is defined as

- $\Re z_l + i \Im z_l$ for $F = \mathbb{C}$,
- $\Re z_{2l-1} + i \Im z_{2l-1} + j \Im z_{2l} + k \Re z_{2l}$ for $F = \mathbb{H}$,
- $\Re z_1 + i \Im z_1 + j \Im z_2 + k \Re z_2 + e \Im z_3 + i e \Re z_3 + je \Re z_4 + ke \Im z_4$ for $F = \mathbb{O}$;
Im(a + √−1 \ b) = b, a, b ∈ R. The functions

\[ \psi^p_n(z) = (2\pi)^{-m/2}|p'|(n+m)/(2(n!))^{-1/2}z^n, \quad z \in \mathbb{C}^m, \]

\[ n = (n_1, \ldots, n_m) \in \mathbb{N}^m, \quad \text{with} \quad n! = n_1! \cdots n_m!, \quad z^n = z_1^{n_1} \cdots z_m^{n_m}, \quad |n| = n_1 + \cdots + n_m, \quad m = s(r - 1), \]

form an orthonormal basis of \( \mathcal{H}_p \) as \( n \) runs over \( \mathbb{N}^m \).

We also note the following symmetry properties of \( \chi_q^p \) and \( \rho_p^q \) relative to the orthogonal and the unitary transformations.

**Proposition 2.2.** (a) Let \( o_1, \ldots, o_{r-1} \in O(\sigma, \mathbb{R}) \); then for \( q' \in F^{r-1} \),

\[ \chi_q^p(o_1 q_1, \ldots, o_{r-1} q_{r-1}, p) = \chi_{o_1 q_1, \ldots, o_{r-1} q_{r-1}}(q, p), \quad (q, p) \in N^{-}. \]

(b) Let \( u_1, \ldots, u_{r-1} \in U(s) \); then for \( p' \in F_0 \setminus \{0\} \),

\[ A_u^{-1} \rho_p(q, p) A_u = \rho_p(q, p). \]

\( (q, p) \in N^{-} \) with \( q = q(z_0), \quad z_0 \in \mathbb{C}^{s(r-1)} \) and \( q^u = q(uz_0) \) with \( uz = (u_1(z_1, \ldots, z_s), \ldots, u_{r-1}(z_{s(r-2)+1}, \ldots, z_{s(r-1)})); \quad ((A_u)\psi)(z) = \psi(uz) \) for \( \psi \in \mathcal{H}_p; \quad u = (u_1, \ldots, u_{r-1}), \quad \sigma = 2s = \dim_{\mathbb{R}} F. \)

3. Algebra of multiradial functions.

**Definition (cf. Geller [2]).** We say that a function \( F \) on \( N^{-} = F^{r-1} \times F_0, \quad F = C, H, O, \) is multiradial if there is a function \( f \) on \( \mathbb{R}^{r-1} \times F_0 \) such that

\[ F(q, p) = f(|q_1|, \ldots, |q_{r-1}|, p), \quad (q, p) \in N^{-}. \]  

**Proposition 3.1.** Let \( \mathcal{E} \) denote the space of multiradial functions in \( L^1(N^{-}) \). Then \( \mathcal{E} \) is a commutative closed *-subalgebra of \( L^1(N^{-}) \) and \( \mathcal{E} \) is symmetric.

**Proof.** 1°. If \( F, G \in \mathcal{E} \) then \( F \ast G \in \mathcal{E} \). For we have

\[ F \ast G(q', p') = \int f(|q_1|, \ldots, |q_{r-1}|, p) \times g(|q_1' - q_1|, \ldots, |q_{r-1}' - q_{r-1}|, p' - p - \text{Im}(\bar{q} \cdot q')) dq dp. \]

Substituting \( q = ((q'_1/|q'_1|)\bar{q}_1, \ldots, (q'_{r-1}/|q'_{r-1}|)\bar{q}_{r-1}) \) we get (since \((ab)b = a(bb)\) for \( a, b \in F\))

\[ \int f(|\bar{q}_1|, \ldots, |\bar{q}_{r-1}|, p) g \left( |q_1'| |1 - \bar{q}_1/|q_1'||, \ldots, |q_{r-1}'| |1 - \bar{q}_{r-1}/|q_{r-1}'||, p' - p - \text{Im} \left( \sum_{i=1}^{r-1} \bar{q}_i |q'_i| \right) \right) d\bar{q} dp, \]

i.e. \( F \ast G \) is multiradial. Obviously \( \mathcal{E} \) is closed.

2°. \( \mathcal{E} \) is commutative (cf. Kaplan and Putz [6, p. 377]). Under the orthogonal change of variables

\[ q_l \mapsto q''_l = q'_l \cdot 2 \text{Re}(\bar{q}_l q'_l)/|q'_l|^2 - q_l, \quad l = 1, \ldots, r - 1, \]
one has $|q'_i - q''_i| = |q'_i - q_i|$ and $\text{Im}(\bar{q}_i q'_i) = -\text{Im}(\bar{q}_i q'_i)$. Thus (3.1) is equal to

$$\int f(|q''_1|, \ldots, |q''_{r-1}|, p) \times g \left(|q'_1 - q''_1|, \ldots, |q'_{r-1} - q''_{r-1}|, p' - p - \sum_{i=1}^{r-1} - \text{Im}(\bar{q}'_i q'_i)\right) dq'' dp$$

$$= \int_{N} F(q'', p) G((q', p')(q'', p)^{-1}) dq'' dp$$

$$= G \ast F(q', p').$$

3°. Since $L^1(N^-)$ is symmetric (Leptin [8, p. 205]), its *-subalgebra $\mathfrak{A}$ is also symmetric.

**Proposition 3.2.** For $F \in \mathfrak{A}$ and $u = (u_1, \ldots, u_{r-1}) \in U(s) \times \cdots \times U(s)$, the operators $\rho_p(F) = \int_{N} F(q, p)\rho_p(q, p) \, dq \, dp$ and $A_u$ commute on $\mathcal{K}_u$.

**Proof.** By Proposition 2.2(b),

$$A_u^{-1} \int \rho_p(q(z_0), p) F(q(z_0), p) \, dq(z_0) \, dp \, A_u$$

$$= \int \rho_p(q(uz_0), p) F(q(z_0), p) \, dq(z_0) \, dp.$$

Since

$$q_i(uz_0) = \Omega(P(u_i z_i)) = (\Omega Pu_i P^{-1}\Omega^{-1})(\Omega Pz_i)$$

with $z_l = (z_{l(i-1)+1}, \ldots, z_{l})$, $l = 1, \ldots, r - 1$, and since $\Omega Pu_i P^{-1}\Omega^{-1}$ is an orthogonal transformation on $\mathbb{R}^r \cong F$, and $\Omega Pz_i = q_i(z_0)$, the last integral is equal to $\rho_p(F)$.

**Remark.** For $F = R$, the corresponding group $N^-$ is $\mathbb{R}^{r-1}$, so the algebra $L^1(N^-)$ is already commutative, and, as in the case $M = \mathbb{R}^r \times \mathbb{R}^+$ with the Euclidean metric [5], we consider $\mathfrak{A} = L^1(N^-)$.

4. **Multiplicative linear functionals on $\mathfrak{A}$.** Let $\Phi$ be a nonzero multiplicative linear functional on $\mathfrak{A}$. Since $\mathfrak{A}$ is a symmetric *-subalgebra of $L^1(N^-)$, there exist an irreducible *-representation $\pi$ of $L^1(N^-)$ and a unit vector $\xi$ in the Hilbert space $\mathcal{K}_\pi$ such that

$$\pi(F)\xi = \Phi(F)\xi \quad \text{for all } F \in \mathfrak{A}.$$ (4.1)

If $\mathcal{K}_\pi$ is one dimensional, then

$$\pi(F)\xi = \int_{N} F(q, p)\chi_q(q, p) \, dq \, dp \xi$$ (4.2)

for some $q' \in \mathbb{R}^{r-1}$, and by Proposition 2.2(a), if $q'$ and $q''$ in $\mathbb{R}^{r-1}$ are such that $|q'_l| = |q''_l|$, $l = 1, \ldots, r - 1$, the $\Phi$'s corresponding by (4.1) and (4.2) to $\chi_{q'}$ and $\chi_{q''}$ are identical. If $\pi \simeq \rho_p$, then by Proposition 3.2, $\rho_p(F)$ and $A_u$ commute. Now for $\psi(z) = \psi_1(Z_1) \ldots \psi_{r-1}(Z_{r-1})$ with $z = (Z_1, \ldots, Z_{r-1}), Z_l = (z_{l(i-1)+1}, \ldots, z_l)$, $l = 1, \ldots, r - 1$, we have

$$(A_u \psi)(z) = \psi_1(u_1 Z_1) \ldots \psi_{r-1}(u_{r-1} Z_{r-1}).$$
Thus putting $\psi_i(Z_i) = Z_i^{n_i}$ with $n_i = (n_{1i}, \ldots, n_{si}) \in \mathbb{N}^s$, we note that $A_u$ preserves the finite-dimensional subspaces of $\mathcal{H}_{p}$, namely the spaces $\mathcal{H}^n = \mathcal{S}^{\mathbb{Z}_{-1}} \mathcal{H}^{n_i}$, where $n = (|n_1|, \ldots, |n_{s-1}|) \in \mathbb{N}^{s-1}$. $\mathcal{H}^{n_i}$ is the space of homogeneous polynomials in $z_{s(t-1)}^{n_1}, \ldots, z_{s_i}^{n_i}$ of degree $|n_i|$. Moreover, $\mathcal{H}_{p} = \bigoplus_{n} \mathcal{H}^n$—an orthogonal direct sum over $n \in \mathbb{N}^{s-1}$. We also note that $A_u$ restricted to $\mathcal{H}^n$ is equal to $\bigotimes_{i=1}^{s+1} \mathcal{T}^{|n_i|}(u_{1i}, \ldots, u_{si})$ with $T^k$, $k = |n_i|$, being the representation of $U(s)$ on $\mathcal{H}^k$ given by $(T^k u)(Z) = u(Z)$ since $T^k$ is irreducible (cf., e.g., [13, pp. 204–209]). The representations $T^n = \bigotimes \mathcal{T}^{|n|}$ of $U(s) \times \cdots \times U(s)$, $r - 1$ copies of $U(s)$, act irreducibly on $\mathcal{H}^n$, and $T^n \approx T^m$ iff $n = m$. Hence, by Schur’s Lemma, every intertwining operator $S$ for $\bigoplus_n T^n$ on $\mathcal{H}_{p}$ is of the form $S = \bigoplus_n c_n(S) \text{id}_{\mathcal{H}_{p}}$.

In particular, each $\rho_p(F)$ with $F \in \mathcal{O}$ is such. It follows from (4.1) that $\Phi(F)$ is equal to one of the constants $c_n(\rho_p(F))$, $n \in \mathbb{N}^{s-1}$. Conversely, for every fixed $n$, the mapping $F \mapsto c_n(\rho_p(F))$ defines a multiplicative linear functional on $\mathcal{O}$. Now we shall derive explicit formulas for the constants $c_n$ above. Since, e.g.,

$$c_n(\rho_p(F)) = \left(\rho_p(F)\psi_n(\psi_n)\right)(\mathcal{H}_{p}),$$

with $n' = (n_1, 0, \ldots, 0; n_2, 0, \ldots, 0; \ldots; n_{s-1}, 0, \ldots, 0) \in (\mathbb{N}^s)^{s-1}$, we calculate the integral, see Proposition 2.1(b),

$$\int_{\mathbb{C}^{(r-1)}} \left[\rho_p(F)\psi_n(\psi_n)\right](z) \bar{\psi}_n(\bar{z}) \exp(-|p'| |z|^2) \, dz \, d\bar{z},$$

which in expanded form is equal to (with $k = s(r - 1)$)

$$(2\pi)^{\frac{k}{2}}(n!)^{-1} |p'| |n_i| + k \times \int_{\mathbb{C}^s} \int_{F^{r-1}} F(q(z_0), p) \exp\left(\sqrt{-1} \Re(\bar{p}' p) + |p'|(|z_0 - z - \frac{1}{2}|z_0^2)\right) \times (z - z_0)^n \bar{z} \exp(-|p'| |z|^2) \, dq(z_0) \, dp \, dz \, d\bar{z}.$$  

The integral

$$\int_{\mathbb{C}^{(r-1)}} (z - z_0)^n \bar{z} \exp(-|p'| |z|^2) \exp(|p'| |z_0| \cdot z) \, dz \, d\bar{z}$$  

is equal to

$$(2\pi)^{s(r-1)} \left(\frac{1}{2}\pi n_1 |p'| \right)^{-n_{s-1}} \prod_{i=1}^{r-1} \left(\frac{1}{2}\pi n_i |p'| \right)^{-n_i-1} \left(-|p'| |z_{s(t-1)+1}|^2\right)^{\frac{n_i}{2}} (j!)^{-1}$$

$$= (2\pi)^{s(r-1)} |p'|^{-n_s} \left(\frac{1}{2}\pi n_i |p'| \right)^{-n_{s-1}} \prod_{i=1}^{r-1} L_n \left(|p'| |z_{s(t-1)+1}|^2\right)$$

with $L_n$ being the Laguerre polynomial. (4.5a) is obtained (see [5]) by substituting the binomial formula for $(z - z_0)^n$, developing $\exp(-|p'| |z_0| \cdot z)$ in a power series and integrating this series term by term using the orthogonality relations for the
functions $z^n$ in $H_c$. Substituting (4.5a) in (4.4) we obtain that (4.3) is equal to

$$
\int_{F^{-1} \times F_0} F(q(z_0), p) \exp(\sqrt{-1} \Re(\bar{p}' p) - \frac{1}{2}|p'| |z_0|^2) \times \prod_{l=1}^{r-1} L_m(|p'| |z_{s(l-1)+1}|^2) \, dq(z_0) \, dp
$$

$$
= \int_0^\infty dt_1 \ldots \int_0^\infty dt_{r-1} \left( \int_{F_0} f(t_1, \ldots, t_{r-1}, p) \exp(\sqrt{-1} \Re(\bar{p}' p)) \, dp \right) \times \exp\left(-\frac{1}{2}|p'| (t_1^2 + \ldots + t_{r-1}^2) \right) \prod_{j=1}^{r-1} t_j^{s-1} \mathcal{g}_j,
$$

with $f$ as in (3.0) and $\mathcal{g}_j$ given by

$$
\mathcal{g}_j = \int_{S(\sigma - 1)} L_m(|p'| |z_{-1}|^2) \, dS(q(Z)),
$$

$Z = (z_1, \ldots, z_r) \in C^r$, $|Z| = t_r$,

$S(\sigma - 1)$ being the unit sphere in $F$. Since here $q(Z) = \Omega(PZ)/t_r$, with $P$ as in Proposition 2.1(b) and $\Omega \in O(\sigma, \mathbb{R})$, in order to compute $\mathcal{g}_j$ one has to calculate the integrals

$$
\int_{S(\sigma - 1)} |z_j(q)|^{2j} \, dS(q), \quad j = 0, \ldots, n_r, \quad (4.6)
$$

where $|z_j(q)|^2 = t_j^2((q_1)^2 + (q')^2)$ with $q_1, q', \ldots$ denoting the coordinates of $q$ in the (standard) basis $\{1, i, \ldots\}$ of $F$ over $\mathbb{R}$. Now (4.6) is equal to

$$
\int_0^{\pi/2} \cos^{2j} \theta \cos \theta \sin^{2\sigma-2\theta} d\theta \cdot 2\pi \cdot 2\pi^{s-1}[s - 2)!]^{-1} t_j^{2j}
$$

$$
= 2\pi t_j^2 j! / (j + s - 1)!
$$

We summarize the results of this section in the following:

**Proposition 4.** The multiplicative linear functionals on $\mathcal{G}$ fall into two classes:

(a) the functionals corresponding to $(r - 1)$-tuples $(t_1, \ldots, t_{r-1})$ of nonnegative real numbers and given by

$$
F \mapsto \hat{F}(t_1, \ldots, t_{r-1}) = \int_{F^{-1} \times F_0} F(q, p) \exp(\sqrt{-1} \Re(\bar{q} \cdot q)) \, dq \, dp
$$

with $q' \in F^{-1}$ arbitrary provided $(|q_1|, \ldots, |q_{r-1}|) = (t_1, \ldots, t_{r-1})$.

(b) the functionals corresponding to pairs $(p', n) \in F_0 \setminus \{0\} \times \mathbb{N}^{r-1}$ and given by

$$
F \mapsto \hat{F}(p', n) = (2\pi)^{r-1} \int_{\mathbb{R}^{r-1}_+} \exp\left(-\frac{|p'|^2}{2} (t_1^2 + \ldots + t_{r-1}^2) \right) \prod_{j=1}^{r-1} L_m(|p'| t_j^2) \times \left( \int_{F_0} f(t_1, \ldots, t_{r-1}, p) \exp(\sqrt{-1} \Re(\bar{p}' p)) \, dp \right) dt_1 \ldots dt_{r-1},
$$
where \( f(|q_1|, \ldots, |q_{r-1}|, p) = F(q_1, \ldots, q_{r-1}, p) \) and
\[
L^{(m)}_k(x) = \sum_{j=0}^{k} \frac{(-1)^j}{(j+m)!} \binom{k}{j} = \left[ (k+m)! \right]^{-1} x^{-m}e^x (d^k/dx^k)(x^{k+m}e^{-x}).
\] (4.7)

5. Nonvanishing of the Gel’fand transform of \( P_a \).

**Lemma 1.** For \( 1 < m < 2k + \frac{1}{2}, k > \frac{3}{2} \) and \( Q > 0 \), the following formula holds:
\[
\int_{\mathbb{R}^m} \frac{\exp(-\sqrt{-1} x_0 \cdot x)}{(Q^2 + 4|x|^2)^k} \, dx = 2^{-m\mu/2} \frac{\Gamma(k - m/2)}{\Gamma(k)} Q^{m-2k}, \quad \text{for } x_0 = 0,
\]
\[
= 2^{m+1} 4^k \Gamma(m+1/2) 2^{k-m-1} e^{-(r/2)Q} \frac{\Gamma(k) \Gamma(k - (m-1)/2)}{\Gamma(k) \Gamma(k - (m-1)/2)}
\]
\[
\times \int_0^\infty e^{-(r/2)Qt}((t+1)^2 - 1)^{k-(m+1)/2} \, dt,
\]
\[
\text{for } r = |x_0| \neq 0. \quad (5.1)
\]

**Proof.** For \( x_0 = 0 \), the integral is equal to the “area” of the unit sphere in \( \mathbb{R}^m \)
\((=2 \text{ when } m=1) \) times \( \int_{\mathbb{R}} r^{m-1}(Q^2 + 4r^2)^{-k} dr \) and we substitute \( r = r'Q/2 \).

For \( x_0 \neq 0 \), the function \( [4((1/2)Q^2 + x^2)]^{-k} \) is radial on \( \mathbb{R}^m \), hence its Fourier transform \((5.1)\) is equal to, see, e.g., [9, p. 155],
\[
4^{-k} (2\pi)^{m/2} r^{-(m-2)/2} \int_0^\infty \left( \left( \frac{1}{2} Q \right)^2 + t^2 \right)^{-k} J_{(m-2)/2}(rt) t^{m/2} \, dt,
\]
\[
m > 1, \quad k > 3/2.
\]

Combining now the Sonine formula \([12, \text{ p. 434, (2)}]\),
\[
\int_0^\infty x^{r+1} J_r(ax) \, dx = \frac{a^\mu k^{-\mu}}{2^\mu \Gamma(\mu+1)} K_{\mu-r}(ak),
\]
valid when \(-1 < \text{Re}(\mu) < 2 \text{Re}(\mu) + \frac{3}{2}\), with the following expression for the function \( K \) \([12, \text{ p. 172, (4)}]\),
\[
K_\nu(z) = \frac{\Gamma(1/2)(1/2 z)^\nu}{\Gamma(\nu + 1/2)} \int_1^\infty e^{-zt}(t^2 - 1)^{-1/2} \, dt,
\]
valid for \( \text{Re}(\nu + 1/2) > 0, |\text{arg } z| < \pi/2 \), we obtain \((5.1)\).

**Lemma 2.** For \( \epsilon > 0, m > 0, x \in \mathbb{R}^n \),
\[
(\epsilon + |x|^2)^{-m} = (4\pi)^{-n/2} \Gamma(m)^{-1} \int_{\mathbb{R}^n} \exp(-\sqrt{-1} x \cdot y) \times \left( \int_0^\infty t^{m-1} e^{-t} e^{-|y|^2/4t} \, dt \right) dy,
\]
i.e. \((\epsilon + |x|^2)^{-m}\) is a Fourier transform of a positive function in \( L^1(\mathbb{R}^n) \).
PROOF. Combine
\[
(e + |x|^2)^{-m} = \Gamma(m)^{-1} \int_0^\infty t^{m-1} e^{-(e + |x|^2)t} \, dt, \quad m > 0,
\]
with
\[
\exp(-|x|^2 t) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} \exp(-|y|^2/4t) \exp(-\sqrt{-1} x \cdot y) \, dy,
\]
and note that the obtained double integral is absolutely convergent.

PROPOSITION 5. For every \( a \in A \), \( \hat{P}_a \) does not vanish on \( \mathcal{M}(\mathcal{O}) \) — the maximal ideal space of \( \mathcal{O} \).

PROOF. (a) For the points \((t_1, \ldots, t_{r-1}) \in \mathcal{M}(\mathcal{O})\), integrating over \( F_0 \) in the formula of Proposition 4(a), according to Lemma 1 (with \( x_0 = 0 \)) we get
\[
\hat{P}_a(t_1, \ldots, t_{r-1}) = c \int_{\mathbb{R}^{(r-1)}} \frac{\exp(\sqrt{-1} \text{Re}(\bar{q} \cdot q))}{(e + |q|^2)^{d+1-\sigma}} \, dq,
\]
with
\[
c = 2^{1-\sigma-1/2} e^{d/2} c_{e, F} \Gamma(d/2 - s + 1/2) \Gamma(d/2)^{-1},
\]
and this is positive by Lemma 2 and the Fourier inversion formula.

(b) For the points \((p', n) \in \mathcal{M}(\mathcal{O})\) with \( p' \in F_0 \setminus \{0\}, n = (n_1, \ldots, n_{r-1}) \in \mathbb{N}^{r-1}\), we use the formula from Proposition 4(b) for \( \hat{P}_a(p', n) \). Applying Lemma 1 to the integral over \( F_0 \) there (with \( Q = e + t_1^2 + \cdots + t_{r-1}^2, m = \sigma - 1, x_0 = p', k = d/2 \)), then interchanging the order of integration from \( dt_1 \ldots dt_{r-1} dt \) to \( dt_1 \ldots dt_{r-1} dt \), making the change of variables \((|p'|t_1^2, \ldots, |p'|t_{r-1}^2) = (x_1, \ldots, x_{r-1})\), and finally applying (4.7), we obtain
\[
\hat{P}_a(p', n) = c \int_0^\infty e^{-|p'|t_1^2/2}((t + 1)^2 - 1)^{d/2 - s} \prod_{i=1}^{r-1} \delta_j(t) \, dt,
\]
with
\[
c = 2^{d(r-1)}(e|p'|)^{d/2} \Gamma(d/2)^{-1} \exp(-|p'|e/2)
\]
and
\[
\delta_j(t) = ((n_i + s - 1)!)^{-1} \int_0^\infty e^{-tx/2}(d^n/dx^n)(x^{n+s-1}e^{-x}) \, dx.
\]
Integrating by parts get
\[
\delta_j(t) = ((n_i + s - 1)!)^{-1} \int_0^\infty e^{-tx/2}x^{n+s-1}e^{-x} \, dx \cdot (t/2)^n
\]
\[
= (t/2)^n(t/2 + 1)^{-(n+s)}.
\]
Thus \( \hat{P}_a(p', n) \) is positive.

REMARK. For \( M \) being the real hyperbolic space, i.e. for \( F = \mathbb{R} \), we have \( N^- = \mathbb{R}^d, \mathcal{M}(L^1(N^-)) = \hat{N}^-, P_a(X_{-a}) = c_{d, \mathbb{R}} e^{d/2}(e + |q|^2)^{-d} \) and, by Lemma 2, \( \hat{P}_a > 0 \) on \( \hat{N}^- \).
6. **Theorem on ideals in** $L^1(N^-)$. Since the algebras $\mathfrak{a}$ we consider here have the same qualitative properties as the one considered in [5], similar facts can be proved about them. In particular, the following statement about ideals in $L^1(N^-)$ is a consequence of the Wiener property of $\mathfrak{a}$ and existence of the approximate identity for $L^1(N^-)$ in $\mathfrak{a}$ (the dilations $\delta_t$, $t > 0$, on $N^-$ used in the construction of the approximate identity are given by $\delta_t(q, p) = (t^{-1/2}q, t^{-1}p)$).

**Proposition 6.** If $\mathcal{J}$ is a proper closed right ideal in $L^1(N^-)$, then there is a $\Phi$ in $\mathfrak{M}(\mathfrak{a})$ such that $\hat{F}(\Phi) = 0$ for every $F \in \mathcal{J} \cap \mathfrak{a}$.

7. **Proof of the Theorem** [5]. The Theorem follows now from Proposition 6, for if we put

$$\mathcal{J} = \left\{ f \in L^1(N^-) : \lim_{N^- \ni n \to \infty} q \cdot f(n) = c_0 \int_{N^-} f(n) \, dn \right\},$$

with $q \in L^\infty(N^-)$ being the boundary value of the bounded harmonic function $u$ on $M$, then $P_a \in \mathcal{J} \cap \mathfrak{a}$ and $\hat{P}_a \neq 0$ on $\mathfrak{M}(\mathfrak{a})$, so $\mathcal{J} = L^1(N^-)$. Hence $P_a \in \mathcal{J}$ for every $a$ in $A$.

**Added in proof.** Meanwhile Korányi [14] described the Gel’fand space, as well as the related Plancherel formula, for the commutative algebra $\mathfrak{a}$ of *biradial* functions in $L^1(N^-)$, i.e. the functions $F$ such that

$$F(q, p) = f(|q|, |p|), \quad (q, p) \in N^-,$$

for some $f$ on $R_+ \times R_+$, cf. §3. His approach uses neither the classification of symmetric spaces nor the representations of nilpotent groups.

**References**


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