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TENSEGRITY FRAMEWORKS

BY

B. ROTH[1] AND W. WHITELEY

ABSTRACT. A tensegrity framework consists of bars which preserve the distance between certain pairs of vertices, cables which provide an upper bound for the distance between some other pairs of vertices and struts which give a lower bound for the distance between still other pairs of vertices. The present paper establishes some basic results concerning the rigidity, flexibility, infinitesimal rigidity and infinitesimal flexibility of tensegrity frameworks. These results are then applied to a number of questions, problems and conjectures regarding tensegrity frameworks in the plane and in space.

1. Introduction. The rigidity and flexibility of frameworks have been extensively studied in recent years and, although many questions remain unanswered, there now exists a substantial body of useful and interesting results. However, our knowledge of tensegrity frameworks (consisting of bars which preserve the distance between certain pairs of vertices, cables which provide an upper bound for the distance between some pair of vertices and struts which give a lower bound for the distance between other pairs of vertices) is in an embryonic state. Tensegrity frameworks are obviously of interest to architects and engineers (for example, see Calladine [**4**] and Fuller [**6**]); perhaps more surprising is their appearance in the work of the sculptor Kenneth Snelson. The symbiosis of frameworks and tensegrity frameworks has only lately become evident. For example, recent work of Connelly [**5**] on the rigidity of frameworks given by triangulated surfaces relies heavily on tensegrity frameworks. On the other hand, one theme of the present paper is that knowledge of frameworks frequently enhances our understanding of tensegrity frameworks. This mutually advantageous interplay between the study of frameworks and that of tensegrity frameworks seems likely to remain a prominent feature of the subject.

The present paper establishes some basic results concerning tensegrity frameworks and then applies these to several open problems and conjectures in the plane and in space. More specifically, in §3 we define rigidity and flexibility for tensegrity frameworks and show that these notions are invariant under various changes in the definitions. In §4, after defining infinitesimal rigidity and infinitesimal flexibility for tensegrity frameworks, we establish the equivalence of infinitesimal rigidity and static rigidity for tensegrity frameworks using standard results from finite-dimensional convexity theory. §5 deals with the relationships between infinitesimal

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rigidity and rigidity for a tensegrity framework $G(p)$ and the framework $\bar{G}(p)$ obtained from $G(p)$ by replacing all its members (bars, cables, and struts) by bars. Among other things, we show that $G(p)$ is infinitesimally rigid if and only if $\bar{G}(p)$ is infinitesimally rigid and there exists a stress of $\bar{G}(p)$ which assigns a negative coefficient to every cable of $G(p)$ and a positive coefficient to every strut of $G(p)$. We also examine various generic properties of tensegrity frameworks in §5. In particular, we identify a dense open set of realizations for which rigidity and infinitesimal rigidity are equivalent. §6 focuses on tensegrity frameworks in the plane, examining in detail some open problems and conjectures regarding tensegrity polygons posed by Grünbaum and Shephard [9]. Finally, in §7 we examine tensegrity frameworks in $\mathbf{R}^3$.

2. Notation and terminology. An *abstract tensegrity framework* $G = (V;\, B,\, C,\, S)$ is a set $V = \{1, \ldots, v\}$ whose elements are called *vertices* together with pairwise disjoint sets B, C, and S of the two-element subsets of V, referred to as *bars*, *cables* and *struts*, respectively. A *member* of an abstract tensegrity framework is an element of the set $E = B \cup C \cup S$. When $C = S = \varnothing$ we often refer to $G = (V;\, B,\, C,\, S)$ as an *abstract framework* and write $G = (V, B)$. A *tensegrity framework* $G(p)$ *in* $\mathbf{R}^n$ is an abstract tensegrity framework $G = (V;\, B,\, C,\, S)$ together with a point

$$p = (p_1, \ldots, p_v) \in \mathbf{R}^n \times \cdots \times \mathbf{R}^n = \mathbf{R}^{nv}.$$

$G(p)$ is the realization of G in $\mathbf{R}^n$ obtained by locating vertex i at point p_i in $\mathbf{R}^n$. We frequently refer to $G(p)$ as a *framework in* $\mathbf{R}^n$ if $C = S = \varnothing$.

For an abstract tensegrity framework $G = (V;\, B,\, C,\, S)$ we let $\bar{G}$ denote the abstract framework $\bar{G} = (\bar{V}, \bar{B})$ where $\bar{V} = V$ and $\bar{B} = B \cup C \cup S$. Thus $\bar{G}(p)$ is the framework in $\mathbf{R}^n$ obtained by replacing all the members of the tensegrity framework $G(p)$ in $\mathbf{R}^n$ by bars.

Throughout the paper $|A|$ denotes the cardinality of the set A.

3. Rigidity and flexibility. A *rigid motion* L of $\mathbf{R}^n$ is a map $L\colon \mathbf{R}^n \to \mathbf{R}^n$ satisfying $\|Lx - Ly\| = \|x - y\|$ for all $x, y \in \mathbf{R}^n$. We say that $p = (p_1, \ldots, p_v)$ and $q = (q_1, \ldots, q_v)$ in $\mathbf{R}^{nv}$ are *congruent* if there exists a rigid motion $L\colon \mathbf{R}^n \to \mathbf{R}^n$ such that $Lp_i = q_i$ for $1 \leqslant i \leqslant v$. Let $M(p)$ denote the smooth manifold in $\mathbf{R}^{nv}$ of points congruent to p. An *algebraic set* is the set of common zeros of a collection of polynomials. Then $M(p)$ is the algebraic set

$$\{q = (q_1, \ldots, q_v) \in \mathbf{R}^{nv}\colon \|p_i - p_j\|^2 = \|q_i - q_j\|^2,\ 1 \leqslant i, j \leqslant v\}.$$

Let $G(p)$ be a tensegrity framework in $\mathbf{R}^n$. Each bar of $G(p)$ preserves the distance between a pair of vertices while each cable (respectively, strut) places an upper bound (respectively, lower bound) on the distance between a pair of vertices. Thus we are led to

$$\begin{aligned} X(p) = \{x = (x_1, \ldots, x_v) \in \mathbf{R}^{nv}\colon \|x_i - x_j\| = \|p_i - p_j\| \text{ for all } \\ \{i,j\} \in B,\ \|x_i - x_j\| \leqslant \|p_i - p_j\| \text{ for all } \{i,j\} \in C \\ \text{and } \|x_i - x_j\| \geqslant \|p_i - p_j\| \text{ for all } \{i,j\} \in S\}, \end{aligned}$$

the set of points in $\mathbf{R}^{nv}$ satisfying the constraints imposed by the members of the tensegrity framework $G(p)$. Implicit in our definition of $X(p)$ is the requirement that the cable (or strut) joining vertices p_i and p_j has length $\|p_i - p_j\|$. Since rigid motions are distance preserving, we have $M(p) \subset X(p)$.

DEFINITION 3.1. Let $G(p)$ be a tensegrity framework in $\mathbf{R}^n$. $G(p)$ is *rigid in* $\mathbf{R}^n$ if there exists a neighborhood U of p in $\mathbf{R}^{nv}$ such that $X(p) \cap U = M(p) \cap U$. $G(p)$ is *flexible in* $\mathbf{R}^n$ if there exists a continuous path x: $[0, 1] \to \mathbf{R}^{nv}$ with $x(0) = p$ and $x(t) \in X(p) - M(p)$ for all $t \in (0, 1]$. Such a path is called a *flexing* of $G(p)$.

Some authors refer to flexible frameworks as "mechanisms" and use the word "stiff" as a synonym for rigid. Some examples of rigid and flexible tensegrity frameworks are given in the next section following Definition 4.1. The following proposition establishes the equivalence of nonrigidity and flexibility and also gives two other equivalent forms of the definition of flexibility for tensegrity frameworks.

PROPOSITION 3.2. *Suppose $G(p)$ is a tensegrity framework in $\mathbf{R}^n$. Then the following are equivalent*:

(a) *$G(p)$ is not rigid in $\mathbf{R}^n$*;

(b) *there exists a real analytic path x: $[0, 1] \to \mathbf{R}^{nv}$ with $x(0) = p$ and $x(t) \in X(p) - M(p)$ for all $t \in (0, 1]$*;

(c) *$G(p)$ is flexible in $\mathbf{R}^n$*;

(d) *there exists a continuous path x: $[0, 1] \to X(p)$ with $x(0) = p$ and $x(t_1) \notin M(p)$ for some $t_1 \in (0, 1]$*.

PROOF. The fact that (a) implies (b) follows from path selection results in algebraic geometry, although a bit of preliminary work is required since $X(p)$ is not an algebraic set. Let A be the algebraic set in $\mathbf{R}^{nv+|C|+|S|}$ consisting of those points $(x_1, \dots, x_v, \dots, y_{\{i,j\}}, \dots, z_{\{k,m\}}, \dots) \in \mathbf{R}^{nv+|C|+|S|}$ such that $\|x_g - x_h\|^2 = \|p_g - p_h\|^2$ for all $\{g, h\} \in B$, $\|x_i - x_j\|^2 + y^2_{\{i,j\}} = \|p_i - p_j\|^2$ for all $\{i,j\} \in C$ and $\|x_k - x_m\|^2 - z^2_{\{k,m\}} = \|p_k - p_m\|^2$ for all $\{k, m\} \in S$. Clearly if $(x_1, \dots, x_v) \in X(p)$ then $(x_1, \dots, x_v, \dots, y_{\{i,j\}}, \dots, z_{\{k,m\}}, \dots) \in A$ where $y^2_{\{i,j\}} = \|p_i - p_j\|^2 - \|x_i - x_j\|^2$ for $\{i,j\} \in C$ and $z^2_{\{k,m\}} = \|x_k - x_m\|^2 - \|p_k - p_m\|^2$ for $\{k, m\} \in S$. Conversely, if $(x_1, \dots, x_v, \dots, y_{\{i,j\}}, \dots, z_{\{k,m\}}, \dots) \in A$ then $y^2_{\{i,j\}}$ and $z^2_{\{k,m\}}$ are as before and $(x_1, \dots, x_v) \in X(p)$. Now suppose $G(p)$ is not rigid in $\mathbf{R}^n$. Then every neighborhood of p in $\mathbf{R}^{nv}$ contains points of $X(p) - M(p)$ and thus every neighborhood of $(p, 0) = (p_1, \dots, p_v, \dots, 0, \dots, 0, \dots)$ in $\mathbf{R}^{nv+|C|+|S|}$ contains points of $A - M(p, 0)$ where $M(p, 0)$ is the algebraic set $M(p) \times \mathbf{R}^{|C|+|S|}$. Therefore, by the curve selection lemma of Milnor [**10**, Lemma 3.1, p. 25], there exists a real analytic path

$$(x_1, \dots, x_v, \dots, y_{\{i,j\}}, \dots, z_{\{k,m\}}, \dots): [0, 1] \to \mathbf{R}^{nv+|C|+|S|}$$

beginning at $(p, 0)$ and belonging to $A - M(p, 0)$ for $t \in (0, 1]$. Then $x(t) = (x_1(t), \dots, x_v(t))$ is a real analytic path with $x(0) = p$ and $x(t) \in X(p) - M(p)$ for $t \in (0, 1]$.

Clearly (b) implies (c) and (c) implies (d). Finally, if (d) holds then there exists $t_0 \in [0, t_1)$ such that $x(t_0)$ is the last point in $M(p)$ as t increases. Since $x(t_0) = (x_1(t_0), \dots, x_v(t_0)) \in M(p)$ there exists a rigid motion L: $\mathbf{R}^n \to \mathbf{R}^n$ with $Lx_i(t_0) = p_i$, $1 \leqslant i \leqslant v$. Therefore $Lx = (Lx_1, \dots, Lx_v)$ maps $(t_0, t_1]$ into $X(p) - M(p)$ and

$Lx(t_0) = p$. Thus every neighborhood of p intersects $X(p) - M(p)$ so $G(p)$ is not rigid in $\mathbf{R}^n$. $\square$

4. Infinitesimal rigidity and statics. The concept of infinitesimal flexibility for tensegrity frameworks arises from the notion of flexibility by focusing on the tangential conditions imposed by the inequalities defining $X(p)$. Suppose $G = (V; B, C, S)$ is an abstract tensegrity framework with v vertices and $p \in \mathbf{R}^{nv}$. Let $x = (x_1, \ldots, x_v)$ be a smooth function on $[0, 1]$ with $x(0) = p$ and $x(t) \in X(p)$ for $t \in [0, 1]$. Examining the derivative of $\|x_i(t) - x_j(t)\|^2$ at $t = 0$, we find that

$$(x_i(0) - x_j(0)) \cdot (x_i'(0) - x_j'(0)) = (p_i - p_j) \cdot (x_i'(0) - x_j'(0))$$

equals zero for $\{i, j\} \in B$ and is less than or equal to zero (respectively, greater than or equal to zero) for $\{i, j\} \in C$ (respectively, S). Thus a smooth flexing of $G(p)$ assigns a velocity vector $\mu_i = x_i'(0) \in \mathbf{R}^n$ to each vertex p_i of $G(p)$ in such a way that

$$(p_i - p_j) \cdot (\mu_i - \mu_j) \begin{cases} = 0 & \text{for } \{i, j\} \in B, \\ \leqslant 0 & \text{for } \{i, j\} \in C, \\ \geqslant 0 & \text{for } \{i, j\} \in S. \end{cases} \tag{4.1}$$

According to Definition 3.1, a flexing $x(t)$ of a tensegrity framework $G(p)$ begins at p, belongs to $X(p)$ for all t but does not belong to $M(p)$ for all $t > 0$. In the same spirit, we require that an infinitesimal flexing μ instantaneously satisfy the constraints imposed by the members, i.e., satisfy (4.1), but not belong to the tangent space $T(p)$ of the manifold $M(p)$ at the point p. Let

$$I(p) = \{\mu \in \mathbf{R}^{nv} : \mu \text{ satisfies } (4.1)\},$$

the space of *infinitesimal motions* of $G(p)$. Note that the tangent space $T(p) \subset I(p)$ since if $\mu \in T(p)$ then $(p_i - p_j) \cdot (\mu_i - \mu_j) = 0$ for all $1 \leqslant i, j \leqslant v$.

DEFINITION 4.1. Suppose $G(p)$ is a tensegrity framework in $\mathbf{R}^n$. $G(p)$ is *infinitesimally rigid in* $\mathbf{R}^n$ if $T(p) = I(p)$ and *infinitesimally flexible in* $\mathbf{R}^n$ otherwise. Elements of $I(p) - T(p)$ are called *infinitesimal flexings* of $G(p)$.

One immediate consequence of Definition 4.1 is that interchanging the cables and struts of a tensegrity framework preserves its infinitesimal classification. More formally, if $G = (V; B, C, S)$ and $G' = (V; B, C', S')$ where $C' = S$ and $S' = C$ then $G(p)$ is infinitesimally rigid in $\mathbf{R}^n$ if and only if $G'(p)$ is infinitesimally rigid in $\mathbf{R}^n$.

We next present a few simple examples of tensegrity frameworks. Throughout the paper bars will be denoted by solid lines, cables by dashes and struts by double lines. The tensegrity framework $G(p)$ in $\mathbf{R}^2$ shown in Figure 4.1 is both rigid and infinitesimally rigid in $\mathbf{R}^2$ as is the tensegrity framework $G'(p)$ obtained by interchanging the cables and struts of $G(p)$. Now consider $G(p)$ and $G'(p)$ as tensegrity frameworks in $\mathbf{R}^3$ (with the four vertices coplanar). Then $G(p)$ is rigid but infinitesimally flexible in $\mathbf{R}^3$. (The assignment to any one vertex of a nonzero vector perpendicular to the plane of the four vertices and zero vectors to the remaining vertices gives an infinitesimal flexing of $G(p)$.) However, $G'(p)$ is both flexible and infinitesimally flexible in $\mathbf{R}^3$. Thus the "interchangeability" of cables and struts fails for rigid tensegrity frameworks. §§6 and 7 contain many more examples.

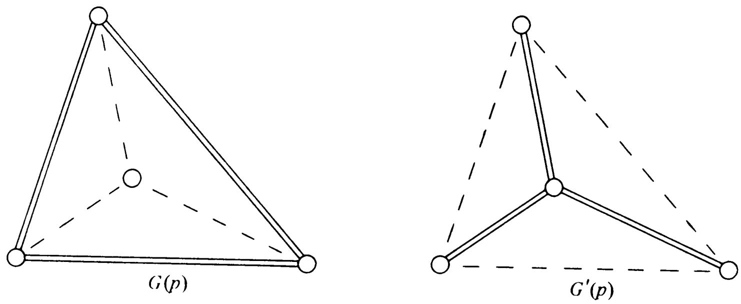


FIGURE 4.1

One basic result in the study of frameworks is the equivalence of infinitesimal rigidity and "static rigidity" (which is defined to mean all equilibrium forces are resolvable). The analogous result for tensegrity frameworks seems to play an even more important role in the theory and its proof is more interesting too. Before establishing that equivalence, we define the basic concepts of statics for tensegrity frameworks.

We begin by creating in each member of a tensegrity framework $G(p)$ in $\mathbf{R}^n$ a force of tension or compression directed along the member with the stipulation that only tension is allowed in a cable and only compression in a strut. More precisely, suppose there is associated with each $\{i,j\} \in B \cup C \cup S$ a scalar $\omega_{\{i,j\}}$ such that $\omega_{\{i,j\}}(p_i - p_j)$ is the force exerted by the member on vertex p_i (and $\omega_{\{i,j\}}(p_j - p_i)$ is the force exerted on p_j). If $\omega_{\{i,j\}} \leqslant 0$ the force is called a *tension* in the member while if $\omega_{\{i,j\}} \geqslant 0$ it is referred to as a *compression* in the member. A *stress* of a tensegrity framework $G(p)$ in $\mathbf{R}^n$ is a collection $\omega = (\ldots, \omega_{\{i,j\}}, \ldots)$ of scalars, one for each $\{i,j\} \in E = B \cup C \cup S$, such that $\omega_{\{i,j\}} \leqslant 0$ (respectively, $\geqslant 0$) for all $\{i,j\} \in C$ (respectively, S) and

$$\sum_{\{j:\ \{i,j\} \in E\}} \omega_{\{i,j\}}(p_i - p_j) = 0, \qquad 1 \leqslant i \leqslant v. \tag{4.2}$$

Condition (4.2) says the forces at each vertex are in equilibrium, i.e., their sum is zero. It proves convenient to replace the v equations in $\mathbf{R}^n$ in (4.2) by a single equation in $\mathbf{R}^{nv}$. To accomplish this, let

$$F_{\{i,j\}} = (x_1, \ldots, x_v) \in \mathbf{R}^n \times \cdots \times \mathbf{R}^n = \mathbf{R}^{nv}$$

where $x_k = 0$ for $k \neq i, j$, $x_i = p_i - p_j$ and $x_j = p_j - p_i$. Then $\omega = (\ldots, \omega_{\{i,j\}}, \ldots)$ is a stress of $G(p)$ if and only if $\omega_{\{i,j\}} \leqslant 0$ for cables, $\omega_{\{i,j\}} \geqslant 0$ for struts and

$$\sum_{\{i,j\} \in E} \omega_{\{i,j\}} F_{\{i,j\}} = 0.$$

We next allow external forces to act on the vertices of the tensegrity framework $G(p)$ in $\mathbf{R}^n$. Since our interest is in statics, we restrict our attention to systems of external forces which are in "equilibrium". A vector $F = (F_1, \ldots, F_v) \in \mathbf{R}^{nv}$ is an *equilibrium force* for $p = (p_1, \ldots, p_v) \in \mathbf{R}^{nv}$ if $F \in T(p)^{\perp}$, the orthogonal complement of the tangent space $T(p)$ at p to the manifold of points congruent to p. (The reader who is accustomed to thinking of equilibrium forces in dimensions two and three in terms of moments may think this a strange definition. To become comfortable with it one should verify that for $p = (p_1, \ldots, p_v) \in \mathbf{R}^{3v}$, $F = (F_1, \ldots, F_v) \in T(p)^{\perp}$ if and only if the application of the forces F_i to the vertices p_i produces zero torque about every axis, which is equivalent to $\sum_{i=1}^{v} F_i = 0$ and $\sum_{i=1}^{v} p_i \times F_i = 0$ where "$\times$" denotes the cross product in $\mathbf{R}^3$. Similarly, for $p = (p_1, \ldots, p_v) \in \mathbf{R}^{2v}$, $F = (F_1, \ldots, F_v) \in T(p)^{\perp}$ if and only if $\sum_{i=1}^{v} F_i = 0$ and $\sum_{i=1}^{v} p_i \cdot F_i^* = 0$ where $(a, b)^* = (b, -a)$ for $(a, b) \in \mathbf{R}^2$.)

A vector $F = (F_1, \ldots, F_v) \in \mathbf{R}^{nv}$ is a *resolvable force* for the tensegrity framework $G(p)$ in $\mathbf{R}^n$ if there exist scalars $\omega_{\{i,j\}}$, $\{i,j\} \in E = B \cup C \cup S$, such that $\omega_{\{i,j\}} \leqslant 0$ (respectively, $\geqslant 0$) for all $\{i,j\} \in C$ (respectively, S) and

$$\sum_{\{j:\ \{i,j\} \in E\}} \omega_{\{i,j\}}(p_i - p_j) = F_i, \qquad 1 \leqslant i \leqslant v,$$

or, equivalently,

$$\sum_{\{i,j\} \in E} \omega_{\{i,j\}} F_{\{i,j\}} = F.$$

Note that a stress of a tensegrity framework $G(p)$ is simply a resolution of the trivial force $F = (0, \ldots, 0) \in \mathbf{R}^{nv}$.

The space $T(p)^{\perp}$ of equilibrium forces for p is a vector space. On the other hand, the set $\mathfrak{R}$ of resolvable forces for $G(p)$ is a convex cone, which means it is a convex set which is closed under multiplication by nonnegative scalars. It is not hard to show that every resolvable force for $G(p)$ is an equilibrium force for p. Each $F_{\{i,j\}} \in T(p)^{\perp}$ since for $\mu \in T(p)$ we have

$$F_{\{i,j\}} \cdot \mu = (p_i - p_j) \cdot (\mu_i - \mu_j) = 0$$

by the comment preceding Definition 4.1. Since every resolvable force F for $G(p)$ is a linear combination of the vectors $F_{\{i,j\}}$, $\{i,j\} \in B \cup C \cup S$, we have $\mathfrak{R} \subset T(p)^{\perp}$. Static rigidity is defined by the opposite inclusion.

DEFINITION 4.2. A tensegrity framework $G(p)$ is *statically rigid in* $\mathbf{R}^n$ if every equilibrium force for p is a resolvable force for $G(p)$, i.e., if $T(p)^{\perp} \subset \mathfrak{R}$.

We are now in a position to establish the equivalence of infinitesimal rigidity and static rigidity for tensegrity frameworks. The proof relies on standard results from finite-dimensional convexity theory. For a subset $X \subset \mathbf{R}^n$, let

$$X^{+} = \{\mu \in \mathbf{R}^n : \mu \cdot x \geqslant 0 \text{ for all } x \in X\}.$$

If X is a closed convex cone in $\mathbf{R}^n$ and $x_0 \in \mathbf{R}^n - X$, then standard separation results (for example, see [**8**, Theorem 1, p. 11] or [**11**, Theorem 11.3, p. 97]) imply that there exists $\mu \in \mathbf{R}^n$ with $\mu \in X^{+}$ but $\mu \cdot x_0 < 0$. Armed with this, it is quite easy to show that

(i) for $Y \subset \mathbf{R}^n$ and X a closed convex cone in $\mathbf{R}^n$, $Y \subset X$ if and only if $X^{+} \subset Y^{+}$,

(ii) for $Y \subset \mathbf{R}^n$, Y^{++} is the smallest closed convex cone in $\mathbf{R}^n$ containing Y.

Finally, it is obvious geometrically but not entirely trivial to prove (see [**11**, Theorem 19.1, p. 171]) that

(iii) if $Y = \{y_1, \ldots, y_k\}$ is a finite subset of $\mathbf{R}^n$ then the convex cone $\{\sum_{i=1}^k \lambda_i y_i: \lambda_i \geq 0 \text{ for } 1 \leq i \leq k\}$ generated by Y is closed in $\mathbf{R}^n$ (and therefore equals Y^{++}).

THEOREM 4.3. *Suppose $G(p)$ is a tensegrity framework in $\mathbf{R}^n$ where $G = (V; B, C, S)$. Then $G(p)$ is infinitesimally rigid in $\mathbf{R}^n$ if and only if $G(p)$ is statically rigid in $\mathbf{R}^n$.*

PROOF. Let $Y = \{-F_{\{i,j\}}: \{i,j\} \in B \cup C\} \cup \{F_{\{i,j\}}: \{i,j\} \in B \cup S\}$. Then the space $I(p)$ of infinitesimal motions of $G(p)$ is Y^+ and the set $\mathfrak{R}$ of resolvable forces for $G(p)$ is the convex cone generated by Y. By (i) we have $Y^+ = I(p) \subset T(p)$ if and only if $T(p)^+ \subset I(p)^+ = Y^{++}$. But $T(p)^+ = T(p)^\perp$ since $T(p)$ is a subspace and $Y^{++} = \mathfrak{R}$ by (iii). Therefore every infinitesimal motion belongs to $T(p)$ if and only if every equilibrium force is resolvable. $\square$

A similar argument shows that the equilibrium force $F_{\{i,j\}}$ fails to be a resolvable force for $G(p)$ if and only if there exists an infinitesimal motion μ of $G(p)$ which instantaneously decreases the distance between p_i and p_j, i.e., satisfies $F_{\{i,j\}} \cdot \mu = (p_i - p_j) \cdot (\mu_i - \mu_j) < 0$.

5. Generic properties of tensegrity frameworks. This section deals with the general behavior of tensegrity frameworks, examining questions such as the following. Does each abstract tensegrity framework have a "generic" classification? Is there a "rigidity predictor" for tensegrity frameworks? Is there a dense open set of realizations for which rigidity and infinitesimal rigidity agree? What is the topological nature of the set of infinitesimally rigid realizations? How do projective maps affect tensegrity frameworks?

First, we formulate some definitions. Let $G = (V; B, C, S)$ be an abstract tensegrity framework with v vertices. For each nonempty subset A of $E = B \cup C \cup S$, we order the members in A in some way and define the *edge function* f_A: $\mathbf{R}^{nv} \to \mathbf{R}^{|A|}$ *of the set* A by

$$f_A(p_1, \ldots, p_v) = (\ldots, \|p_i - p_j\|^2, \ldots)$$

where $\{i,j\} \in A$. A point $p \in \mathbf{R}^{nv}$ is a *regular point* of $G = (V; B, C, S)$ if

$$\operatorname{rank} df_E(p) = \max\{\operatorname{rank} df_E(q): q \in \mathbf{R}^{nv}\}$$

and p is said to be in *general position* for $G = (V; B, C, S)$ if

$$\operatorname{rank} df_A(p) = \max\{\operatorname{rank} df_A(q): q \in \mathbf{R}^{nv}\}$$

for every nonempty $A \subset E$. Note that the rows of the matrix $df_E(p)$ are the vectors $2F_{\{i,j\}}$ for $\{i,j\} \in E$. Thus a stress of the framework $\overline{G}(p)$ is just a linear dependency among the rows of $df_E(p)$.

Next, we review some properties of frameworks which might be referred to as "generic" ones. The regular points of a framework G form a dense open subset of $\mathbf{R}^{nv}$ [**1**, p. 283]. For regular points we either always have rigidity or always have flexibility [**1**, Corollary 2]. Since rigidity and infinitesimal rigidity are equivalent at regular points ([**1**, Theorem] and [**2**, §3]), we have a dense open set of realizations for which the classification is constant and the two notions of rigidity agree. An abstract framework G is *generically rigid in* $\mathbf{R}^n$ if $G(p)$ is rigid (or, equivalently, infinitesimally rigid) in $\mathbf{R}^n$ for all regular points p of G. Otherwise G is said to be *generically flexible in* $\mathbf{R}^n$. The generic classification is given by a "rigidity predictor" which involves the rank of the derivative of the edge function [**1**, Theorem]. Since $G(p)$ infinitesimally rigid implies p is a regular point of G [**2**, Theorem, §3], the set of infinitesimally rigid realizations of G is open (it is either empty or coincides with the set of regular points). Finally, the infinitesimal properties of frameworks are projectively invariant.

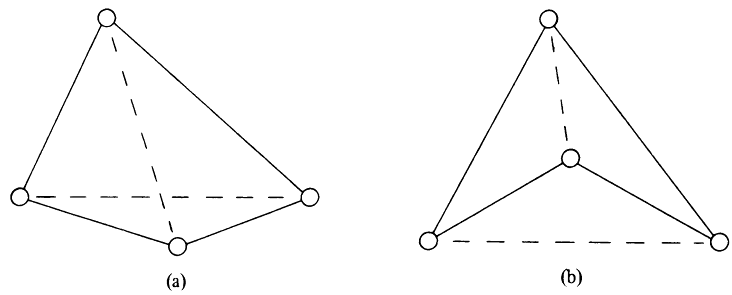


FIGURE 5.1

A first indication of the more complicated nature of the situation for tensegrity frameworks is provided by the fact that an abstract tensegrity framework may not have a generic classification. For example, the tensegrity frameworks in $\mathbf{R}^2$ shown in Figure 5.1 are realizations of the same abstract tensegrity framework $G = (V; B, C, \varnothing)$ where the four bars in B are indicated by solid lines and the two cables in C by dashes. The realizations close to that shown in (a) are all rigid in $\mathbf{R}^2$ while those close to that shown in (b) are all flexible in $\mathbf{R}^2$. Thus there exist nonempty open sets of both rigid and flexible realizations. For all realizations p near those shown in (a) and (b), every nontrivial stress of $\overline{G}(p)$ assigns coefficients of one sign to the interior members (there are two interior members, both cables, in (a) and three interior members, two bars and a cable, in (b)) and coefficients of the opposite sign to the remaining members of $G(p)$. We will soon see that the distribution of signs in these stresses explains the rigidity of realizations near (a) and the flexibility of those near (b).

The following technical lemma is the key to several parts of the first theorem of this section.

LEMMA 5.1. *Suppose* $Y = \{y_1, \ldots, y_k\} \subset \mathbf{R}^n$. *Then* $Y^+ = Y^\perp$ *if and only if there exist positive scalars* $\lambda_1, \ldots, \lambda_k$ *with* $\sum_{i=1}^k \lambda_i y_i = 0$.

PROOF. If $Y^+ = Y^\perp$ then $Y^{++} = \{\sum_{i=1}^k \lambda_i y_i \colon \lambda_i \geqslant 0 \text{ for } 1 \leqslant i \leqslant k\}$ is a subspace. Therefore for each j we have $-y_j = \sum_{i=1}^k \lambda_i y_i$, $\lambda_i \geqslant 0$ for $1 \leqslant i \leqslant k$, which gives $\lambda_1 y_1 + \cdots + (1 + \lambda_j) y_j + \cdots + \lambda_k y_k = 0$. The sum of k such expressions gives a linear dependency of $y_1, \ldots, y_k$ with all positive coefficients.

Conversely, if $\sum_{i=1}^k \lambda_i y_i = 0$ with $\lambda_i > 0$ and $\mu \in Y^+$ then

$$0 = \mu \cdot \sum_{i=1}^k \lambda_i y_i = \sum_{i=1}^k \lambda_i (\mu \cdot y_i).$$

Since $\lambda_i(\mu \cdot y_i) \geqslant 0$ and $\lambda_i > 0$ for all i, we have $\mu \cdot y_i = 0$ for all i. Therefore $\mu \in Y^{\perp}$. $\square$

Closely related to the positive dependency in Lemma 5.1 is a special kind of stress for tensegrity frameworks. A *proper stress* $\omega = (\ldots, \omega_{\{i,j\}}, \ldots)$ of a tensegrity framework $G(p)$ is a stress of $G(p)$ satisfying $\omega_{\{i,j\}} < 0$ for all $\{i, j\} \in C$ and $\omega_{\{i,j\}} > 0$ for all $\{i, j\} \in S$. Note that the zero stress $\omega = (\ldots, 0, \ldots)$ is a proper stress of $G(p)$ if $C = S = \varnothing$. Lemma 5.1 implies that there exists a proper stress of $G(p)$ if and only if the space $I(p)$ of infinitesimal motions of $G(p)$ equals the space of infinitesimal motions of $\bar{G}(p)$. The following theorem establishes some connections between infinitesimal rigidity and rigidity for a tensegrity framework $G(p)$ and the corresponding framework $\bar{G}(p)$. It can be viewed as extending the rigidity predictor from frameworks to tensegrity frameworks.

THEOREM 5.2. *Suppose $G(p)$ is a tensegrity framework in $\mathbf{R}^n$. Then the following are equivalent*:

(a) *$G(p)$ is infinitesimally rigid in $\mathbf{R}^n$*;

(b) *$G(p)$ is infinitesimally rigid in $\mathbf{R}^n$ and there exists a proper stress of $G(p)$*;

(c) *$\bar{G}(p)$ is infinitesimally rigid in $\mathbf{R}^n$ and there exists a proper stress of $G(p)$*;

(d) *$G(p)$ is rigid in $\mathbf{R}^n$, p is a regular point of G and there exists a proper stress of $G(p)$*;

(e) *$\bar{G}(p)$ is rigid in $\mathbf{R}^n$, p is a regular point of G and there exists a proper stress of $G(p)$.*

PROOF. Let $Y = \{-F_{\{i,j\}}\colon \{i,j\} \in B \cup C\} \cup \{F_{\{i,j\}}\colon \{i,j\} \in B \cup S\}$. Then $I(p) = Y^+$ and $T(p) \subset Y^{\perp}$ by the comment preceding Definition 4.1. If $I(p) = T(p)$ then $Y^+ \subset Y^{\perp}$ and thus $Y^+ = Y^{\perp}$. By Lemma 5.1 there exists a positive dependency among the elements of Y which gives a proper stress of $G(p)$. Thus (a) implies (b) and (b) clearly implies (c). We now show that (c) implies (a). Since $G(p)$ has a proper stress there exists a linear dependency among the elements of Y with positive coefficients for all elements of Y arising from cables and struts. Each bar $\{i, j\}$ gives two elements $\pm F_{\{i,j\}}$ of Y so it is easy to find a linear dependency among the elements of Y with all positive coefficients. Thus $I(p) = Y^+ = Y^{\perp}$ by Lemma 5.1. Also the space $Y^{\perp}$ of infinitesimal motions of $\bar{G}(p)$ equals $T(p)$ since $\bar{G}(p)$ is infinitesimally rigid in $\mathbf{R}^n$. Therefore $I(p) = T(p)$, i.e., $G(p)$ is infinitesimally rigid in $\mathbf{R}^n$. This establishes the equivalence of (a)–(c).

Since a framework $\bar{G}(p)$ is infinitesimally rigid in $\mathbf{R}^n$ if and only if $\bar{G}(p)$ is rigid in $\mathbf{R}^n$ and p is a regular point of $\bar{G}$ (see [**2**, Theorem, §3]), (c) and (e) are equivalent. Finally, it is obvious that (d) implies (e) and all that remains is to show that if $G(p)$ is infinitesimally rigid in $\mathbf{R}^n$ then $G(p)$ is rigid in $\mathbf{R}^n$. We delay the proof of this fact a bit; it is Theorem 5.7. $\square$

Theorem 5.2 explains the absence of a generic classification for tensegrity frameworks. Note that the tensegrity framework in Figure 5.1(a) has a proper stress while that in Figure 5.1(b) does not. Therefore the existence of a proper stress for a single realization (or even a nonempty open set of realizations) does not imply the existence of a dense open set of realizations with a proper stress. Thus the infinitesimal rigidity of a single realization does not imply infinitesimal rigidity for a dense open set of realizations. On the other hand, one can think of Theorem 5.2 as a kind of rigidity predictor for tensegrity frameworks since it replaces the question of the infinitesimal rigidity of a tensegrity framework $G(p)$ by that of the infinitesimal rigidity of the framework $\bar{G}(p)$ and the existence of a stress of $\bar{G}(p)$ with opposite signs on the cables and struts of $G(p)$.

Our next corollary says that if $G(p)$ is an infinitesimally rigid tensegrity framework then $\bar{G}(p)$ is an infinitesimally rigid framework which is over-braced.

COROLLARY 5.3. *If $G(p)$ is an infinitesimally rigid tensegrity framework in $\mathbf{R}^n$ then the framework $G'(p)$ obtained by deleting any cable or strut of G and replacing the remaining members of G by bars is infinitesimally rigid in $\mathbf{R}^n$.*

PROOF. Suppose $G = (V; B, C, S)$ and let $G' = (V, E)$ where $E = B \cup C \cup S - \{\{k, m\}\}$ for some $\{k, m\} \in C \cup S$. If $G(p)$ is infinitesimally rigid in $\mathbf{R}^n$ then $I(p) = T(p)$ and there exists a proper stress of $G(p)$. If $\mu \in I'(p)$, i.e., $\mu \cdot F_{\{i,j\}} = 0$ for all $\{i, j\} \in E$, then $\mu \cdot F_{\{k,m\}} = 0$ also since $G(p)$ has a proper stress. Therefore $\mu \in I(p) = T(p)$ so $G'(p)$ is infinitesimally rigid in $\mathbf{R}^n$. $\square$

So far in this paper we have thought of the tensegrity framework $G(p)$ as primary and occasionally considered the associated framework $\bar{G}(p)$. Dually, we could begin with a framework and ask if it has bars which can be replaced by cables or struts. Theorem 5.2 says that if $G(p)$ is an infinitesimally rigid framework with a nontrivial stress then there exist bars of $G(p)$ whose replacement by suitably chosen cables and struts leads to an infinitesimally rigid tensegrity framework. In fact, every infinitesimally rigid tensegrity framework arises in this way according to Theorem 5.2.

We next examine the topological nature of the set of infinitesimally rigid realizations of an abstract tensegrity framework. The techniques used are closely related to those involved in Cramer's rule for solving systems of linear equations.

THEOREM 5.4. *Let $G = (V; B, C, S)$ be an abstract tensegrity framework with v vertices. Then the set $\{p \in \mathbf{R}^{nv}$: $G(p)$ is infinitesimally rigid in $\mathbf{R}^n\}$ of infinitesimally rigid realizations of G is open in $\mathbf{R}^{nv}$.*

PROOF. Suppose $G(p)$ is infinitesimally rigid in $\mathbf{R}^n$. Then p is a regular point of G, say rank $df_E(p) = k$ where $E = B \cup C \cup S$. Then some $k \times k$ submatrix of the $|E| \times nv$ matrix $df_E(p)$ is nonsingular, say for simplicity the submatrix formed by the first k rows and columns. Then $\omega \in \mathbf{R}^{|E|}$ is a stress of $\bar{G}(p)$ if and only if $A(p)\omega = 0$ where $A(p)$ is the $k \times |E|$ matrix which is the transpose of the matrix consisting of the first k columns of $df_E(p)$. Let $a(p)$ be the nonsingular $k \times k$ matrix consisting of the first k columns of $A(p)$. Multiplying by $a(p)^{-1}$ we find that the set of solutions ω of the linear system $A(p)\omega = 0$ is

$$\Big\{\omega = (\omega_1, \ldots, \omega_{|E|}) \in \mathbf{R}^{|E|}\colon \omega_i \text{ arbitrary for } k < i \leqslant |E| \\ \text{and } \omega_i = -\sum_{j=k+1}^{|E|} a_{ij}(p)\omega_j \text{ for } 1 \leqslant i \leqslant k\Big\}$$

where $a_{ij}(p)$ is the determinant of the matrix obtained by replacing the ith column of $a(p)$ by the jth column of $A(p)$ divided by the determinant of $a(p)$.

Since $G(p)$ is infinitesimally rigid in $\mathbf{R}^n$ there exists a proper stress of $G(p)$ by Theorem 5.2. This means that some choice of $\omega_{k+1}, \ldots, \omega_{|E|}$ gives a solution $\omega = (\omega_1, \ldots, \omega_{|E|})$ of $A(p)\omega = 0$ with $\omega_i < 0$ (respectively, > 0) for i's corresponding to cables (respectively, struts) of $G(p)$. The same choice of $\omega_{k+1}, \ldots, \omega_{|E|}$ gives a solution $\omega = (\omega_1, \ldots, \omega_{|E|})$ of $A(q)\omega = 0$ with ω_i negative for cables and positive for struts provided q is sufficiently close to p since

$$\omega_i = -\sum_{j=k+1}^{|E|} a_{ij}(q)\omega_j, \qquad 1 \leqslant i \leqslant k,$$

depends continuously on q for q near p. This gives a proper stress of $G(q)$ for q near p. All that remains is to recall that the set of infinitesimally rigid realizations of the framework $\overline{G}$ is just the open set of regular points of $\overline{G}$ ([**1**, Corollary 2] and [**2**, Theorem, §3]). By Theorem 5.2 we conclude that $G(q)$ is infinitesimally rigid in $\mathbf{R}^n$ for all q sufficiently close to p. $\square$

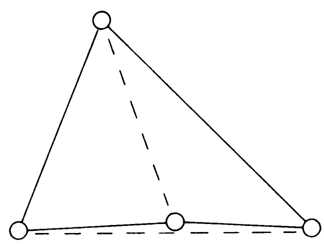


FIGURE 5.2

The infinitesimally flexible realizations of a tensegrity framework may fail to be open. For example, the tensegrity framework $G(p)$ in $\mathbf{R}^2$ with three collinear vertices shown in Figure 5.2 is infinitesimally flexible in $\mathbf{R}^2$. But there exist realizations q of G arbitrarily close to p for which $G(q)$ is infinitesimally rigid in $\mathbf{R}^2$ (as shown in Figure 5.1(a)). However, it is true that the infinitesimally flexible realizations which are in general position form an open set. The proof relies on the following lemma concerning stresses with a minimal number of nonzero coordinates. The *support* of a stress $\omega = (\ldots, \omega_{\{i,j\}}, \ldots)$ of a tensegrity framework, denoted supp ω, is the set $\{\{i, j\}: \omega_{\{i,j\}} \neq 0\}$ of members with nonzero coefficients.

LEMMA 5.5. *Let $G = (V;\ B,\ C,\ S)$ be an abstract tensegrity framework and $\{k, m\} \in C \cup S$. If there exists a stress of $G(p)$ with $\{k, m\}$ in its support then there exists a stress ω of $G(p)$ with $\{k, m\} \in$ supp ω and* rank $df_A(p) = |A|$ *for every proper subset A of* supp ω.

PROOF. Choose a stress $\omega = (\ldots, \omega_{\{i,j\}}, \ldots)$ of $G(p)$ with $\{k, m\} \in$ supp ω for which the cardinality of supp ω is minimal. We show that ω has the desired property. Suppose there exists a proper subset A of supp ω such that $df_A(p)$ has linearly dependent rows. Consider a nontrivial linear dependency among the rows of $df_A(p)$. We regard it as a linear dependency $\lambda = (\ldots, \lambda_{\{i,j\}}, \ldots)$ of the rows of $df_E(p)$ where $E = B \cup C \cup S$ by introducing zero coefficients for members of $E - A$.

For convenience, suppose $\{k, m\} \in C$. If $\lambda_{\{k,m\}} \neq 0$ then either λ or $-\lambda$ has the coefficient of $\{k, m\}$ negative but is not a stress of $G(p)$ by our choice of ω. If $\lambda_{\{k,m\}} = 0$ then for either λ or $-\lambda$ there exists a member $\{g, h\}$ with coefficient opposite in sign to that of $\omega_{\{g,h\}}$. Thus there exists a nontrivial linear dependency ν (either λ or $-\lambda$) among the rows of $df_E(p)$ with $\operatorname{supp} \nu \subset \operatorname{supp} \omega$, $\nu_{\{k,m\}} \leqslant 0$ and $\nu_{\{g,h\}}\omega_{\{g,h\}} < 0$ for some $\{g, h\}$.

Consider the convex combination

$$t\nu + (1 - t)\omega = (\ldots, t\nu_{\{i,j\}} + (1 - t)\omega_{\{i,j\}}, \ldots)$$

for $t \in [0, 1]$. Since $t\nu_{\{g,h\}} + (1 - t)\omega_{\{g,h\}}$ has opposite signs at $t = 0$ and $t = 1$, there exists $t_1 \in (0, 1)$ with $t_1\nu_{\{g,h\}} + (1 - t_1)\omega_{\{g,h\}} = 0$. Let t_0 be the smallest value of t such that some coordinate in $\operatorname{supp} \omega$ of $t\nu + (1 - t)\omega$ equals zero. Then $t_0\nu_{\{k,m\}} + (1 - t_0)\omega_{\{k,m\}} < 0$ and the signs of the other coefficients of $t_0\nu + (1 - t_0)\omega$ are all appropriate (i.e., $\leqslant 0$ for cables and $\geqslant 0$ for struts) by our choice of t_0. Therefore $t_0\nu + (1 - t_0)\omega$ is a stress of $G(p)$ with $\{k, m\}$ in its support. But its support has fewer elements than $\operatorname{supp} \omega$, contradicting our choice of ω. $\square$

THEOREM 5.6. *Let $G = (V; B, C, S)$ be an abstract tensegrity framework with v vertices. Then the intersection of the set of infinitesimally flexible realizations of G and the set of points in general position for G is open in $\mathbf{R}^{nv}$.*

PROOF. Suppose that p is in general position for G and $\{p_l\}$ is a sequence of realizations with $p_l \to p$ where each $G(p_l)$ is infinitesimally rigid in $\mathbf{R}^n$. We show that $G(p)$ is also infinitesimally rigid in $\mathbf{R}^n$ by using Theorem 5.2. First, note that $\bar{G}(p)$ is infinitesimally rigid in $\mathbf{R}^n$ since the infinitesimal flexibility of $\bar{G}(p)$ would imply that $\bar{G}(q)$ is infinitesimally flexible for all regular points q ([**1**, Corollary 2] and [**2**, §3]). Thus $\bar{G}(q)$ (and hence also $G(q)$) is infinitesimally flexible for all q sufficiently close to the regular point p, a contradiction.

Next we establish the existence of a proper stress of $G(p)$ by showing that for each $\{k, m\} \in C \cup S$ there exists a stress of $G(p)$ with $\{k, m\}$ in its support. Since each $G(p_l)$ is infinitesimally rigid there exists a stress of $G(p_l)$ with $\{k, m\}$ in its support. Therefore, by Lemma 5.5, for each l there exists a stress ω_l of $G(p_l)$ with $\{k, m\} \in \operatorname{supp} \omega_l$ and $\operatorname{rank} df_A(p_l) = |A|$ for every proper subset A of $\operatorname{supp} \omega_l$. We now choose a subsequence of $\{p_l\}$ (which we again denote by $\{p_l\}$) for which the stresses given by Lemma 5.5 all have the same support set D.

Let $d = |D|$ and $A = D - \{\{k, m\}\}$. Since p is in general position for G, p_l is also for all sufficiently large l. Therefore $\operatorname{rank} df_A(p) = |A| = d - 1$ but the d rows of $df_D(p)$ are linearly dependent since every p_l has these properties. Since the rows of the $(d - 1) \times nv$ matrix $df_A(p)$ are linearly independent, some $(d - 1) \times (d - 1)$ submatrix of $df_A(p)$ is nonsingular. The corresponding columns of $df_D(p)$ can be used to solve for the linear dependencies among the d rows of $df_D(p)$ (as in the proof of Theorem 5.4). We find the linear dependencies $\omega = (\ldots, \omega_{\{i,j\}}, \ldots)$ among the rows of $df_D(p)$ are given by $\omega_{\{k,m\}}$ arbitrary and

$$\omega_{\{i,j\}} = -a_{\{i,j\}}(p)\omega_{\{k,m\}} \quad \text{for } \{i, j\} \in A$$

where each $a_{\{i,j\}}(p)$ is a quotient of determinants of $(d - 1) \times (d - 1)$ matrices. Now suppose $\{k, m\} \in C$. Letting $\omega_{\{k,m\}} = -1$ and $\omega_{\{i,j\}} = a_{\{i,j\}}(p)$ for $\{i, j\} \in A$

gives a linear dependency among the rows of $df_D(p)$. We need to verify that it is actually a stress of $G(p)$, i.e., the signs of the $\omega_{\{i,j\}}$ are appropriate. In a neighborhood of p the $(d-1) \times (d-1)$ submatrix of df_A is nonsingular while the rows of df_D remain linearly dependent. Thus for q near p the dependencies among the rows of $df_D(q)$ are also given by $\omega_{\{k,m\}}$ arbitrary and $\omega_{\{i,j\}} = -a_{\{i,j\}}(q)\omega_{\{k,m\}}$ for $\{i,j\} \in A$. Clearly each $G(p_l)$ has a stress with support D and $\omega_{\{k,m\}} = -1$. Therefore for l sufficiently large we have $a_{\{i,j\}}(p_l)$ nonpositive for cable and nonnegative for struts. Letting $l \to \infty$ we find that $a_{\{i,j\}}(p)$ is nonpositive for cables and nonnegative for struts. $\square$

One consequence of Theorems 5.4 and 5.6 is that the classification of an abstract tensegrity framework as infinitesimally rigid or infinitesimally flexible is constant on the components of the set of points in general position.

Using Theorem 5.4 and the existence of real analytic flexings it is quite easy to prove that infinitesimal rigidity implies rigidity for tensegrity frameworks. This was first observed by Connelly [**5**, Remark 4.1].

THEOREM 5.7. *If a tensegrity framework $G(p)$ is flexible in $\mathbf{R}^n$ then $G(p)$ is infinitesimally flexible in $\mathbf{R}^n$.*

PROOF. If $G(p)$ is flexible in $\mathbf{R}^n$ then there exists a real analytic path x: $[0, 1] \to \mathbf{R}^{nv}$ with $x(0) = p$ and $x(t) \in X(p) - M(p)$ for all $t \in (0, 1]$ by Proposition 3.2. Since $x(t) \notin M(p)$ for all $t > 0$ there exists a pair of vertices, say k and m, such that $\|x_k(t) - x_m(t)\|$ is not a constant function. Thus $\|x_k(t) - x_m(t)\|^2$ is a nonconstant real analytic function on [0, 1] so its derivative is nonzero for all sufficiently small $t > 0$. Similarly, real analyticity implies that for every $\{i,j\} \in C$ (respectively, S) we either have $\|x_i(t) - x_j(t)\|^2$ constant on [0, 1] or its derivative is negative (respectively, positive) for all small $t > 0$. Therefore for all sufficiently small positive t we have

$$(x_i(t) - x_j(t)) \cdot (x_i'(t) - x_j'(t)) \begin{cases} = 0 & \text{for } \{i,j\} \in B, \\ \leqslant 0 & \text{for } \{i,j\} \in C, \\ \geqslant 0 & \text{for } \{i,j\} \in S, \end{cases}$$

but

$$(x_k(t) - x_m(t)) \cdot (x_k'(t) - x_m'(t)) \neq 0.$$

Therefore $x'(t) \in I(x(t)) - T(x(t))$ which means $G(x(t))$ is infinitesimally flexible in $\mathbf{R}^n$ for all positive t near 0. Therefore $G(p)$ is infinitesimally flexible in $\mathbf{R}^n$ by Theorem 5.4. $\square$

We now show that the rigidity and the infinitesimal rigidity of a tensegrity framework $G(p)$ are equivalent for points p in general position for G. Thus again we find that general position points play a role for tensegrity frameworks analogous to that played by regular points for frameworks.

THEOREM 5.8. *Suppose $G = (V; B, C, S)$ is an abstract tensegrity framework and $p \in \mathbf{R}^{nv}$ is in general position for G. Then $G(p)$ is rigid in $\mathbf{R}^n$ if and only if $G(p)$ is infinitesimally rigid in $\mathbf{R}^n$.*

PROOF. Theorem 5.7 gives one direction. Conversely, suppose that $G(p)$ is infinitesimally flexible in $\mathbf{R}^n$. If $\overline{G}(p)$ is flexible in $\mathbf{R}^n$ then $G(p)$ is also, so we suppose that $\overline{G}(p)$ is rigid in $\mathbf{R}^n$. Since p is a regular point of G, $\overline{G}(p)$ is even infinitesimally rigid in $\mathbf{R}^n$. Let $I(p)$ be the space of infinitesimal motions of $G(p)$ and let

$$A = \{\{i,j\} \in E = B \cup C \cup S\colon F_{\{i,j\}} \in I(p)^{\perp}\}.$$

Obviously $B \subset A$ and, moreover, $A \neq E$. For the infinitesimal flexibility of $G(p)$ says that there exists $\mu \in I(p) - T(p)$ and for thi μ there exists $\{k, m\} \in E$ with $\mu \cdot F_{\{k,m\}} \neq 0$ since $\overline{G}(p)$ is infinitesimally rigid in $\mathbf{R}^n$. Next choose $\nu \in I(p)$ with $\nu \cdot F_{\{i,j\}} \neq 0$ for all $\{i,j\} \in E - A$. Then we have

$$(p_i - p_j)\cdot(\nu_i - \nu_j)\begin{cases} = 0 & \text{for all } \{i,j\} \in A, \\ < 0 & \text{for all } \{i,j\} \in C - A, \\ > 0 & \text{for all } \{i,j\} \in S - A. \end{cases}$$

We now "integrate" the infinitesimal flexing ν to create a flexing of $G(p)$.

If $A = \varnothing$ then

$$x(t) = p + t\nu = (p_1 + t\nu_1, \ldots, p_v + t\nu_v)$$

satisfies $x(0) = p$ and, furthermore, $x(t) \in X(p) - M(p)$ for small positive t since the derivative of $\|x_i(t) - x_j(t)\|^2$ at $t = 0$ is $2(p_i - p_j)\cdot(\nu_i - \nu_j)$. Therefore $G(p)$ is flexible in $\mathbf{R}^n$. Finally, suppose A is nonempty. Since p is in general position for G, p is a point of maximum rank of df_A and therefore $f_A^{-1}(f_A(p))$ is a manifold near p whose tangent space at p is $\ker df_A(p)$. Since $\nu \in \ker df_A(p)$ there exists a smooth path $x\colon \mathbf{R} \to f_A^{-1}(f_A(p))$ with $x(0) = p$ and $x'(0) = \nu$. Then $\|x_i(t) - x_j(t)\|^2 = \|p_i - p_j\|^2$ for all t and all $\{i,j\} \in A$. And $\|x_i(t) - x_j(t)\|^2 < \|p_i - p_j\|^2$ (respectively, $> \|p_i - p_j\|^2$) for all small positive t and all $\{i,j\} \in C - A$ (respectively, $S - A$). Therefore $G(p)$ is flexible in $\mathbf{R}^n$. $\square$

The following result says that for a framework with "independent" edges, replacing any edge with a cable or strut gives a flexible tensegrity framework.

COROLLARY 5.9. *If $G(p)$, $G = (V, E)$, is a framework in $\mathbf{R}^n$ with* rank $df_E(p) = |E|$ *then $G'(p)$ is a flexible tensegrity framework in $\mathbf{R}^n$ where G' is obtained by replacing any element of E by a cable or strut.*

PROOF. Since the framework $G(p)$ admits only the trivial stress, $G'(p)$ does not have a proper stress and thus $G'(p)$ is infinitesimally flexible in $\mathbf{R}^n$ by Theorem 5.2. But p is in general position for G since rank $df_A(p) = |A|$ for every nonempty $A \subset E$. Therefore $G'(p)$ is flexible in $\mathbf{R}^n$ by Theorem 5.8. $\square$

We conclude this section with a look at the effect of projective maps on tensegrity frameworks. We say that L is a *projective map* of $\mathbf{R}^n$ if

$$Lx = \frac{Ax + b}{c \cdot x + d} \quad \text{for } x \in \mathbf{R}^n \text{ with } c \cdot x + d \neq 0$$

where $A\colon \mathbf{R}^n \to \mathbf{R}^n$ is a linear map, $b, c \in \mathbf{R}^n$, $d \in \mathbf{R}$ and the map from $\mathbf{R}^{n+1}$ to $\mathbf{R}^{n+1}$ given by

$$\left(\begin{array}{c|c} A & b \\ \hline c & d \end{array}\right)$$

is nonsingular. For our purposes, it is useful to think of projective maps in a slightly different way. Consider a nonsingular linear map $\tilde{L}$: $\mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$. For $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ let $\tilde{x} = (x_1, \ldots, x_n, 1) \in \mathbf{R}^{n+1}$. Then the composition

$$x \to \tilde{x} \to \tilde{L}\tilde{x} = (y_1, \ldots, y_{n+1}) \to \left(\frac{y_1}{y_{n+1}}, \ldots, \frac{y_n}{y_{n+1}}\right) \tag{5.1}$$

defines the projective map L where $Lx = (y_1/y_{n+1}, \ldots, y_n/y_{n+1})$ for $x \in \mathbf{R}^n$ with the last coordinate y_{n+1} of $\tilde{L}\tilde{x}$ nonzero.

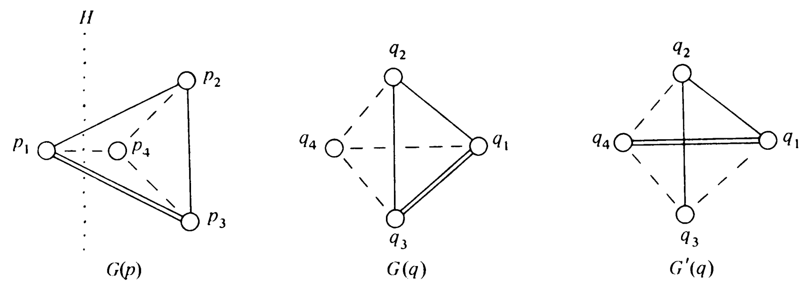


FIGURE 5.3

Consider $p = (p_1, \ldots, p_v)$ and $q = (q_1, \ldots, q_v)$ in $\mathbf{R}^{nv}$. Suppose there exists a projective map L of $\mathbf{R}^n$ with $Lp_i = q_i$ for $1 \leqslant i \leqslant v$. For a framework G with v vertices we have that $G(p)$ is infinitesimally rigid in $\mathbf{R}^n$ if and only if $G(q)$ is infinitesimally rigid in $\mathbf{R}^n$. This result, which is sometimes referred to as the projective invariance of infinitesimal rigidity, fails for tensegrity frameworks. For example, consider the tensegrity frameworks $G(p)$ and $G(q)$ in $\mathbf{R}^2$ shown in Figure 5.3. There exists a projective map L of $\mathbf{R}^2$ with $Lp_i = q_i$ for $1 \leqslant i \leqslant 4$ where the dotted line H shown is mapped by L into the "line at infinity". Clearly $G(p)$ is infinitesimally rigid in $\mathbf{R}^2$ while $G(q)$ is infinitesimally flexible in $\mathbf{R}^2$. However, if we replace the cable of G which crosses H in realization p by a strut and the strut of G which crosses H in realization p by a cable then the resulting tensegrity framework $G'(q)$ shown in Figure 5.3 is infinitesimally rigid in $\mathbf{R}^2$. The final theorem of the section shows such behavior is typical for tensegrity frameworks.

THEOREM 5.10. *Suppose $p = (p_1, \ldots, p_v)$, $q = (q_1, \ldots, q_v) \in \mathbf{R}^{nv}$ and there exists a projective map L of $\mathbf{R}^n$ with $Lp_i = q_i$ for $1 \leqslant i \leqslant v$. Let $G = (V;\, B,\, C,\, S)$ be a tensegrity framework with v vertices. Let $G' = (V;\, B,\, C',\, S')$ be the tensegrity framework obtained by replacing every cable $\{i, j\}$ of G (respectively, strut $\{i, j\}$ of G) for which the line segment $[p_i, p_j]$ intersects the hyperplane H mapped by L to infinity by a strut (respectively, cable) and leaving the remaining members of G unchanged. Then $G(p)$ is infinitesimally rigid in $\mathbf{R}^n$ if and only if $G'(q)$ is infinitesimally rigid in $\mathbf{R}^n$.*

PROOF. Let $\tilde{L}$: $\mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ be the nonsingular linear map for which the composition (5.1) gives L. Let λ_i be the $(n + 1)$st coordinate of $\tilde{L}\tilde{p}_i \in \mathbf{R}^{n+1}$ for $1 \leqslant i \leqslant v$. We first show that if $\omega = (\ldots, \omega_{\{i,j\}}, \ldots)$ is a linear dependency among the rows of $df_E(p)$ where $E = B \cup C \cup S$ then $\omega' = (\ldots, \lambda_i\lambda_j\omega_{\{i,j\}}, \ldots)$ is a linear dependency among the rows of $df_{E'}(q)$ where $E' = B \cup C' \cup S' = E$. This is accomplished by letting the various maps in the composition (5.1) defining L act on the matrix $df_E(p)$.

First, since

$$\sum_{\{i,j\}\in E} \omega_{\{i,j\}} F_{\{i,j\}} = \sum_{\{i,j\}\in E} \omega_{\{i,j\}}(\ldots, p_i - p_j, \ldots, p_j - p_i, \ldots) = 0$$

we have

$$\sum_{\{i,j\}\in E} \omega_{\{i,j\}}(\ldots, \tilde{p}_i - \tilde{p}_j, \ldots, \tilde{p}_j - \tilde{p}_i, \ldots) = 0$$

since the last coordinate of each $\tilde{p}_i$ is one. By the linearity of $\tilde{L}$ we have

$$\sum_{\{i,j\}\in E} \omega_{\{i,j\}}(\ldots, \tilde{L}\tilde{p}_i - \tilde{L}\tilde{p}_j, \ldots, \tilde{L}\tilde{p}_j - \tilde{L}\tilde{p}_i, \ldots) = 0. \tag{5.2}$$

Summing over the ith $(n+1)$-tuple of columns of (5.2) gives

$$\sum_{\{j:\ \{i,j\}\in E\}} \omega_{\{i,j\}}(\tilde{L}\tilde{p}_i - \tilde{L}\tilde{p}_j) = 0 \quad \text{for } 1 \leqslant i \leqslant v. \tag{5.3}$$

Summing over the last coordinate of the vector equation (5.3) gives the scalar equation

$$\sum_{\{j:\ \{i,j\}\in E\}} \omega_{\{i,j\}}(\lambda_i - \lambda_j) = 0 \quad \text{for } 1 \leqslant i \leqslant v. \tag{5.4}$$

By hypothesis the last coordinate λ_i of each $\tilde{L}\tilde{p}_i$ is nonzero. We have

$$\sum_{\{j:\ \{i,j\}\in E\}} \lambda_i\lambda_j\omega_{\{i,j\}}\left(\frac{\tilde{L}\tilde{p}_i}{\lambda_i} - \frac{\tilde{L}\tilde{p}_j}{\lambda_j}\right) = 0 \quad \text{for } 1 \leqslant i \leqslant v \tag{5.5}$$

since the left-hand side of equation (5.5) equals the product of λ_i and the vector in (5.3) minus the product of the scalar in (5.4) and the vector $\tilde{L}\tilde{p}_i$. Thus

$$\sum_{\{i,j\}\in E} \lambda_i\lambda_j\omega_{\{i,j\}}\left(\ldots, \frac{\tilde{L}\tilde{p}_i}{\lambda_i} - \frac{\tilde{L}\tilde{p}_j}{\lambda_j}, \ldots, \frac{\tilde{L}\tilde{p}_j}{\lambda_j} - \frac{\tilde{L}\tilde{p}_i}{\lambda_i}, \ldots\right) = 0.$$

Since the last coordinate of each $\tilde{L}\tilde{p}_i/\lambda_i$ is one, $\omega' = (\ldots, \lambda_i\lambda_j\omega_{\{i,j\}}, \ldots)$ is a linear dependency among the rows of $df_{E'}(q)$. Therefore rank $df_E(p) \geqslant$ rank $df_{E'}(q)$.

Next, observe that $\tilde{L}^{-1}$ defines a projective map L^{-1} by composition (5.1) with $L^{-1}q_i = p_i$ for $1 \leqslant i \leqslant v$. Moreover, λ_i^{-1} is the $(n+1)$st coordinate of $\tilde{L}^{-1}\tilde{q}_i \in \mathbf{R}^{n+1}$ for $1 \leqslant i \leqslant v$. Thus if $\omega' = (\ldots, \omega'_{\{i,j\}}, \ldots)$ is a linear dependency among the rows of $df_{E'}(q)$ then $\omega = (\ldots, \lambda_i^{-1}\lambda_j^{-1}\omega'_{\{i,j\}}, \ldots)$ is a linear dependency among the rows of $df_E(p)$. Hence rank $df_E(p) =$ rank $df_{E'}(q)$. Since the dimension of the affine span of $\{p_1, \ldots, p_v\}$ obviously equals that of $\{q_1, \ldots, q_v\}$, the "rigidity predictor" ([**1**, Theorem] and [**2**, §3]) gives that the framework $\overline{G}(p)$ is infinitesimally rigid in $\mathbf{R}^n$ if and only if the framework $\overline{G}'(q)$ is infinitesimally rigid in $\mathbf{R}^n$.

Finally, suppose the tensegrity framework $G(p)$ is infinitesimally rigid in $\mathbf{R}^n$. Then $\overline{G}(p)$ is infinitesimally rigid in $\mathbf{R}^n$ and there exists a proper stress $\omega = (\ldots, \omega_{\{i,j\}}, \ldots)$ of $G(p)$. Then $\omega' = (\ldots, \lambda_i\lambda_j\omega_{\{i,j\}}, \ldots)$ is a proper stress of $G'(q)$ since $\lambda_i\lambda_j < 0$ if and only if p_i and p_j lie on opposite sides of H, the hyperplane mapped to infinity by L. Therefore $G'(q)$ is also infinitesimally rigid in $\mathbf{R}^n$. Conversely, if $G'(q)$ is infinitesimally rigid in $\mathbf{R}^n$ then a proper stress $\omega' = (\ldots, \omega'_{\{i,j\}}, \ldots)$ of $G'(q)$ gives the proper stress $\omega = (\ldots, \lambda_i^{-1}\lambda_j^{-1}\omega'_{\{i,j\}}, \ldots)$ of $G(p)$. Hence $G(p)$ is infinitesimally rigid in $\mathbf{R}^n$. $\square$

6. Tensegrity frameworks in the plane. We now apply the general results of the previous sections to a number of very specific problems and questions concerning tensegrity frameworks in the plane. Some of the methods used in this section arise from joint work with Rachad Antonius and Janos Baracs of the Groupe de Recherche Topologie Structurale. Much of the section deals with bracing convex polygons whose edges are struts (or bars) by various cables across the interior of the polygon. We begin with a method for combining two infinitesimally rigid tensegrity frameworks in the plane. The result, although somewhat unwieldy to formulate and prove, has interesting applications to convex polygons in the plane. It was observed in examples by Janos Baracs and involves a form of circuit exchange in the underlying combinatorial geometry.

Consider two infinitesimally rigid tensegrity frameworks $G'(p')$ and $G''(p'')$ in $\mathbf{R}^2$ which share some vertices (where we suppose the common vertices have the same indices in G' and G''). Then any one member $\{k, m\}$ which is a cable of G' and a strut of G'' can be deleted and a judicious choice of bars, cables and struts for the remaining members gives an infinitesimally rigid tensegrity framework in the plane. To minimize the possibility of confusion we assume the vertices of $G'(p')$ are distinct, i.e., $p'_i \neq p'_j$ for $i \neq j$, as are those of $G''(p'')$.

THEOREM 6.1. *Suppose $G'(p')$ and $G''(p'')$ are infinitesimally rigid tensegrity frameworks in $\mathbf{R}^2$ (where the vertices of $G'(p')$ and also $G''(p'')$ are distinct and indexed so that $p'_i = p''_j$ if and only if $i = j$). Let E be the set of pairs of vertices which are joined by a different member of G' than G''. That is, let*

$$E = (B' \cap (C'' \cup S'')) \cup (C' \cap (B'' \cup S'')) \cup (S' \cap (B'' \cup C'')).$$

Suppose $\{k, m\} \in (C' \cap S'') \cup (S' \cap C'')$. Let $G = (V;\, B,\, C,\, S)$ where $V = V' \cup V''$, $B = B' \cup B'' \cup E - \{\{k, m\}\}$, $C = C' \cup C'' - E$ and $S = S' \cup S'' - E$. Let $p = (\ldots, p_i, \ldots)$ where $p_i = p'_i$ if $i \in V'$ and $p_i = p''_i$ if $i \in V''$. Then $G(p)$ is an infinitesimally rigid tensegrity framework in $\mathbf{R}^2$.

PROOF. Recall that in $\mathbf{R}^2$ we have $\mu \in T(p)$ if and only if $\mu = (\mu_1, \ldots, \mu_v) = (t + rp_1^*, \ldots, t + rp_v^*)$ where $t \in \mathbf{R}^2$, $r \in \mathbf{R}$ and $(a, b)^* = (b, -a)$ for $(a, b) \in \mathbf{R}^2$. Now we apply Theorem 5.2. By Corollary 5.3, $\overline{G}'(p')$ with bar $\{k, m\}$ deleted is infinitesimally rigid as is $\overline{G}''(p'')$ with bar $\{k, m\}$ deleted. This implies $\overline{G}(p)$ is infinitesimally rigid. To see this, let μ be an infinitesimal motion of $\overline{G}(p)$. Then there exist $t' \in \mathbf{R}^2$ and $r' \in \mathbf{R}$ such that

$$\mu_i = t' + r'p_i'^* \quad \text{for all } i \in V'$$

since $\overline{G}'(p')$ with bar $\{k, m\}$ deleted is infinitesimally rigid. Similarly, there exists $t'' \in \mathbf{R}^2$ and $r'' \in \mathbf{R}$ such that

$$\mu_i = t'' + r''p_i''^* \quad \text{for all } i \in V''.$$

For $i = k, m$, we have $t' + r'p_i'^* = t'' + r''p_i''^*$ since $p_i' = p_i'' = p_i$. Thus $(t' - t'') + (r' - r'')p_i^* = 0$ for $i = k, m$ so $(r' - r'')(p_k^* - p_m^*) = 0$. Since $p_k \neq p_m$ we have $r' = r''$ and therefore $t' = t''$. Thus $\mu_i = t' + r'p_i^*$ for all vertices p_i of $\overline{G}(p)$ which says $\mu \in T(p)$. Hence $\overline{G}(p)$ is infinitesimally rigid in $\mathbf{R}^2$.

To complete the proof we show that $G(p)$ has a proper stress. Suppose $\{k, m\} \in C' \cap S''$. Since $G'(p')$ is infinitesimally rigid there exists a proper stress ω' of $G'(p')$ (so $\omega'_{\{k,m\}} < 0$). Similarly, there exists a proper stress ω'' of $G''(p'')$ (so $\omega''_{\{k,m\}} > 0$). By introducing zero coefficients for certain edges one can regard both ω' and ω'' as stresses of the tensegrity framework $G(p)$ augmented by the bar $\{k, m\}$. It is then easy to check that

$$\omega''_{\{k,m\}}\omega' - \omega'_{\{k,m\}}\omega''$$

is a proper stress of $G(p)$. $\square$

The analog of Theorem 6.1 in $\mathbf{R}^n$ is valid provided the two frameworks share n vertices whose affine span has dimension $n - 1$. Theorem 6.1 and its higher dimensional generalizations hold for rigidity also. That is, two rigid tensegrity frameworks combine to give another rigid tensegrity framework.

The proof of Theorem 6.4 furnishes several examples of the use of Theorem 6.1. Next we prove a technical lemma regarding the distribution of signs in the stress coefficients at a vertex. This simple result is the two-dimensional analog of the index lemma connected with Cauchy's rigidity theorem (for example, see [**7**, Lemma 5.3] or [**12**, Theorem 6.1]).

LEMMA 6.2. *Suppose $p_j \in \mathbf{R}^2 - \{0\}$ and $\omega_j \in \mathbf{R} - \{0\}$ for $1 \leqslant j \leqslant n$. If $\sum_{j=1}^n \omega_j p_j = 0$ then there does not exist a line through the origin such that $\{p_j\colon \omega_j > 0\}$ is contained in one open half space determined by the line and $\{p_j\colon \omega_j < 0\}$ is contained in the other open half space.*

PROOF. If such a line exists then every $\omega_j p_j$ belongs to one open half space and thus $\sum_{j=1}^n \omega_j p_j \neq 0$. $\square$

We now focus on tensegrity frameworks in the plane of a rather special kind. A *tensegrity polygon* $G(p)$ is a tensegrity framework in $\mathbf{R}^2$ with the following properties:

(i) the vertices of $G(p)$ are distinct and form the set of vertices of a convex polygon P in the plane;

(ii) the struts of $G(p)$ are the edges of P;

(iii) the set of bars of G is empty.

One consequence of this definition is that every cable of a tensegrity polygon runs through the interior of P. A parallel theory develops if one considers convex polygons with bars for edges and no struts. We make contact with this theory at the end of the section.

Next we consider various schemes for producing infinitesimally rigid tensegrity polygons. Our proofs rely on a vertex by vertex analysis of the signs of the coefficients in a stress based on Lemma 6.2. But there are other possibilities. For instance, one can employ Maxwell's theorem on projections of polyhedra to study stresses (in this theory proper stresses on tensegrity polygons with planar graphs correspond to projections of convex polyhedra (see **[16]**)) or one can examine in detail the infinitesimal motions allowed by the members.

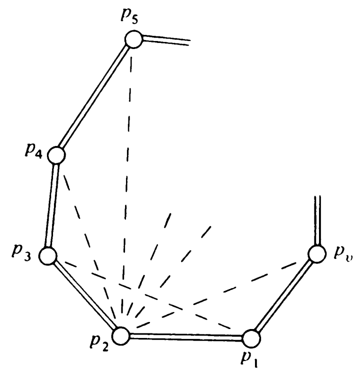


FIGURE 6.1

Perhaps the simplest tensegrity polygons are *Grünbaum polygons* introduced in **[9]** which have cables joining one vertex to all the nonadjacent vertices of the polygon and one additional cable joining the two vertices adjacent to that vertex. See Figure 6.1 where as usual struts are indicated by double lines and cables by dashes. Actually, it is no harder to prove the infinitesimal rigidity of *generalized Grünbaum polygons* in which two adjacent vertices are chosen and every other vertex is joined by a cable to exactly one of these two vertices (as shown in Figure 6.2).

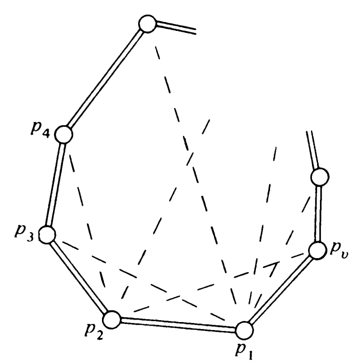


FIGURE 6.2

THEOREM 6.3. *Generalized Grünbaum polygons are infinitesimally rigid in* $\mathbf{R}^2$.

PROOF. Let $G(p)$ be the generalized Grünbaum polygon shown in Figure 6.2. Clearly the framework $\overline{G}(p)$ is infinitesimally rigid in $\mathbf{R}^2$ since it can be built up fr. m the triangle $p_1\, p_2\, p_3$ by the addition of two-valent vertices (with one bar left over at the end). This extra bar is essential since its presence guarantees that $\overline{G}(p)$ has a nontrivial stress. We have rank $df_E(p) = 2v - 3$ where E is the set of bars of $\overline{G}$ since $\overline{G}(p)$ is infinitesimally rigid in $\mathbf{R}^2$ (see [**2**, Equation 2, §3] or [**13**, §6.1] and [**14**, Corollary 5.2]). But $df_E(p)$ has $2v - 2$ rows, one for each bar of $\overline{G}$, and thus there exists a nontrivial linear dependency among the rows of $df_E(p)$, i.e., a nontrivial stress $\omega = (\ldots, \omega_{\{i,j\}}, \ldots)$ of $\bar{G}(p)$. We need only verify that ω (or its negative) is a proper stress of $G(p)$. By applying Lemma 6.2 to the three-valent vertices $p_3, \ldots, p_v$, we find that the two edges of the polygon (the struts) incident with the vertex are of one sign while the remaining member (the cable) incident with the vertex has the opposite sign. Another application of Lemma 6.2 shows that $[p_1, p_2]$ has the same sign as all the other struts of $G(p)$. Therefore $\pm\omega$ is a proper stress of $G(p)$ and thus $G(p)$ is infinitesimally rigid by Theorem 5.2. □

The flexible tensegrity polygon shown in Figure 6.3 demonstrates the necessity of the requirement that the two vertices to which all others are joined by cables in generalized Grünbaum polygons be consecutive vertices.

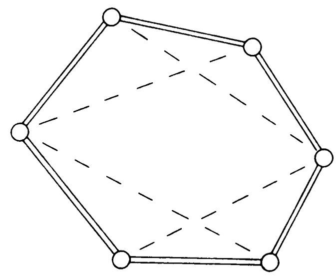


FIGURE 6.3

We next consider *Cauchy polygons* which are tensegrity polygons $G(p)$ of the kind shown in Figure 6.4 with cables joining vertices p_1 and p_3, p_2 and $p_4, \ldots, p_{v-2}$ and p_v. As Grünbaum [**9**, p. 2.13] observes, the rigidity of Cauchy polygons follows from standard arguments related to Cauchy's rigidity theorem. Connelly [**5**, Theorem 4.1] establishes the infinitesimal rigidity of Cauchy polygons by examining their infinitesimal motions. Our approach to the infinitesimal rigidity of Cauchy polygons is based on the observation that Grünbaum polygons and Cauchy polygons are related to each other in a simple way. Indeed, one can get from the Grünbaum polygon in Figure 6.1 to the Cauchy polygon in Figure 6.4 by simply replacing cable $\{2, 5\}$ by $\{3, 5\}$, cable $\{2, 6\}$ by $\{4, 6\}, \ldots,$ cable $\{2, v\}$ by $\{v - 2, v\}$. Thus of the $v - 2$ cables, $v - 4$ have been replaced while cables $\{1, 3\}$ and $\{2, 4\}$ remain unchanged. We now show that *mixed polygons*, obtained by terminating the replacement procedure at any stage, are infinitesimally rigid in $\mathbf{R}^2$.

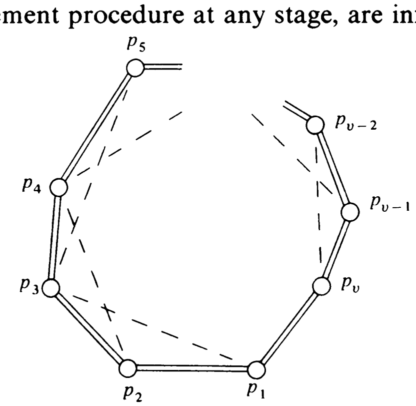


FIGURE 6.4

THEOREM 6.4. *Suppose the i cables $\{2, 5\}$, $\{2, 6\}$, ..., $\{2, i + 4\}$ of a Grünbaum polygon have been replaced by the cables $\{3, 5\}$, $\{4, 6\}$, ..., $\{i + 2, i + 4\}$. Then the resulting mixed polygon is infinitesimally rigid in $\mathbf{R}^2$.*

PROOF. We use induction on the number i of replacements and employ Theorem 6.1. For one replacement, the resulting mixed polygon is infinitesimally rigid since it is the combination (in the sense of Theorem 6.1) of the Grünbaum polygon in Figure 6.1 and the Grünbaum polygon shown in Figure 6.5. Note that member $\{2, 5\}$ disappears since it is a cable in the former and a strut in the latter.

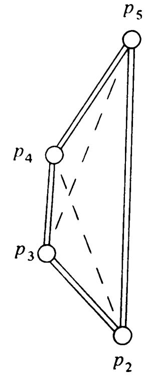


FIGURE 6.5

Suppose all mixed polygons obtained by no more than i replacements are infinitesimally rigid and consider the mixed polygon shown in Figure 6.6 obtained by the $i + 1$ replacements $\{2, 5\}$ by $\{3, 5\}$, $\{2, 6\}$ by $\{4, 6\}$, ..., $\{2, i + 5\}$ by $\{i + 3, i + 5\}$. The mixed polygon in Figure 6.6 is the combination (in the sense of Theorem 6.1) of the mixed polygons shown in Figure 6.7. The mixed polygon in (a) involves only i replacements and thus is infinitesimally rigid. And the Cauchy polygon in (b) with vertices $p_2, p_3, \ldots, p_{i+5}$ is obtained from the corresponding Grünbaum polygon by i replacements (when the Grünbaum polygon is cabled from vertex p_3). Therefore the mixed polygon in Figure 6.6 is infinitesimally rigid by Theorem 6.1. □

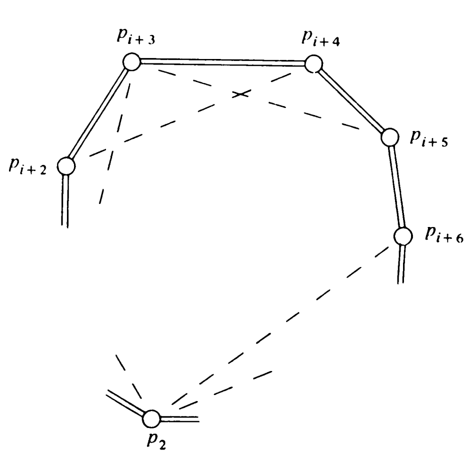


FIGURE 6.6

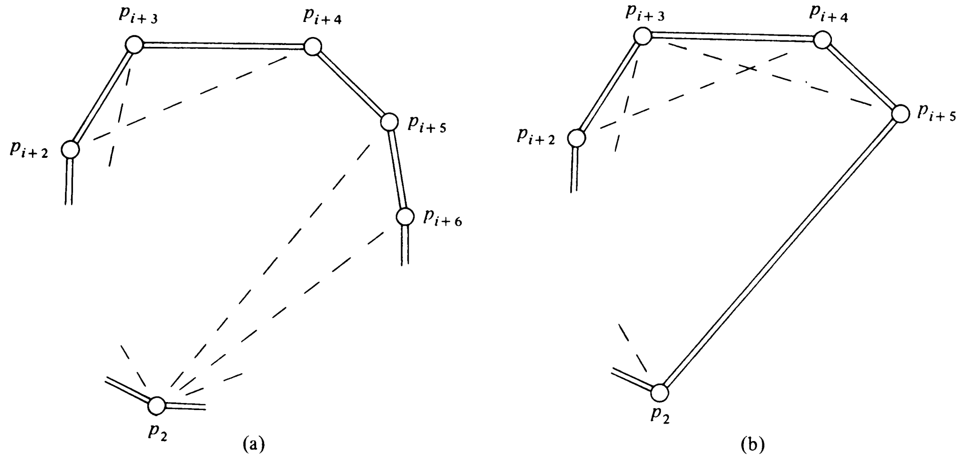


FIGURE 6.7

One might hope that the replacements involved in passing from Grünbaum to Cauchy polygons need not be done sequentially. However, the tensegrity hexagon $G(p)$ shown in Figure 6.8 which is obtained from a Grünbaum polygon with six vertices by replacing cable $\{2, 6\}$ by $\{4, 6\}$ is infinitesimally flexible in $\mathbf{R}^2$. In fact, $\mu_1 = \mu_2 = \mu_3 = 0$ and $\mu_4 = \mu_5 = \mu_6$ as shown is an infinitesimal flexing of $G(p)$.

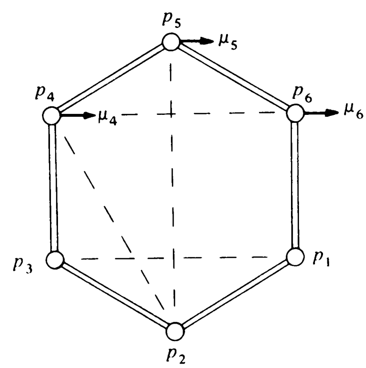


FIGURE 6.8

Note that the various schemes for generating infinitesimally rigid tensegrity polygons previously considered work for all convex realizations of the polygon. Their infinitesimal rigidity is independent of vertex location, edge lengths and other metric properties of the realization of the convex polygon. Grünbaum [**9**, p. 2.13] asks if this is typical. That is, if $G(p)$ and $G(q)$ are tensegrity polygons, are they either simultaneously rigid or simultaneously flexible? Actually Grünbaum's question was posed for tensegrity polygons with bars as edges of the polygons and no struts, but it makes no difference–the answer in both cases is negative.

EXAMPLE 6.5. Consider the tensegrity hexagon $G(p)$ shown in Figure 6.9. Since $|E| = 10 = 2v - 2$, $\overline{G}(p)$ admits a nontrivial stress for all values of x. The idea is that when $x = 2$ classical results about realizations of the complete bipartite graph $K_{3,3}$ in the plane (see [**3**, Theorem 14] or [**13**, Example 2.4]) give a nontrivial stress of $\overline{G}(p)$ with zero coefficient for the bar $\{1, 5\}$. It would not be surprising if the coefficient of this bar changes sign at $x = 2$. Solving for the set of dependencies $\omega = (\ldots, \omega_{\{i,j\}}, \ldots)$ among the rows of the matrix $df_E(p)$ where E is the set of bars of $\bar{G}$, we find that for $x \neq 1, -2$ the stresses of $\bar{G}(p)$ are of the form

$$\omega_{\{1,2\}} = \omega_{\{2,3\}} = \omega_{\{3,4\}} = \omega_{\{4,5\}} = \lambda,$$
$$\omega_{\{1,4\}} = \omega_{\{2,5\}} = -\lambda/2, \qquad \omega_{\{1,6\}} = \omega_{\{5,6\}} = \lambda/(x-1),$$
$$\omega_{\{1,5\}} = \lambda(x-2)/2(x-1), \qquad \omega_{\{3,6\}} = -2\lambda/(x+2)$$

where λ is an arbitrary real number. For $x = 3/2$ the tensegrity polygon $G(p)$ is infinitesimally rigid in $\mathbf{R}^2$ since $\bar{G}(p)$ is infinitesimally rigid and any positive λ gives a proper stress of $G(p)$. On the other hand, $G(p)$ is infinitesimally flexible in $\mathbf{R}^2$ when $x = 3$ since the sign of $\omega_{\{1,5\}}$ is then opposite to the signs of the coefficients of the other cables of $G(p)$ for all nontrivial stresses. Note also that for both values of x the form of the stresses implies that rank $df_A(p) = |A|$ for every proper subset A of E. Thus $x = 3/2$ leads to a point p in general position as does $x = 3$. Therefore $G(p)$ is rigid for $x = 3/2$ and flexible for $x = 3$ by Theorem 5.8. ($G(p)$ is also flexible for $x = 2$ but verifying this is messy since p is not in general position when $x = 2$.) Furthermore, all our conclusions hold whether the six edges of the hexagon are bars or struts.

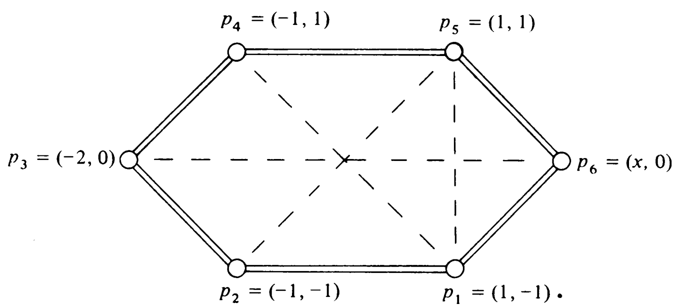


FIGURE 6.9

Little seems to be known about what distinguishes cabling schemes which give infinitesimally rigid tensegrity polygons for all convex realizations from those that do not.

Finally, the results of this paper shed some light on what is perhaps the least understood of the conjectures of Grünbaum [**9**, Conjecture 6, p. 2.14]. This conjecture deals with tensegrity polygons $G(p)$ with bars as edges of the polygon (and no struts) and says that if $G(p)$ is rigid in $\mathbf{R}^2$ then so is the tensegrity framework $G'(p)$ obtained by replacing the bars of $G(p)$ by cables and the cables of $G(p)$ by bars. The fact that the converse of the conjecture is known to be false (see Example 6.7 or [**9**, Figure 13]) adds to the mystery surrounding this conjecture. We here prove the infinitesimal version of the conjecture and offer a conjecture of our own.

THEOREM 6.6. *Let $G(p)$ be a tensegrity polygon with bars rather than struts as edges of the polygon and at least four vertices. If $G(p)$ is infinitesimally rigid in $\mathbf{R}^2$ then the tensegrity framework $G'(p)$ with $B' = C$, $C' = B$ and $S' = \varnothing$ is also infinitesimally rigid in $\mathbf{R}^2$.*

PROOF. Since $G(p)$ is infinitesimally rigid, $\bar{G}(p)$ is infinitesimally rigid and $G(p)$ has a proper stress $\omega = (\ldots, \omega_{\{i,j\}}, \ldots)$ by Theorem 5.2. The infinitesimal rigidity of $\bar{G}(p)$ together with the fact that $v \geqslant 4$ implies $C \neq \varnothing$. Thus there exists a vertex having at least one cable incident with it. Since the stress is proper all the cables incident with this vertex have negative coefficients. Therefore, by Lemma 6.2, the two edges of the polygon (the bars) incident with this vertex have positive coefficients. Applying Lemma 6.2 to the consecutive vertices of the polygon, we find that all the bars $\{i, j\}$ of the tensegrity polygon have $\omega_{\{i,j\}} > 0$. Of course, all the cables have negative coefficients since the stress is proper. Since $\bar{G}'(p) = \bar{G}(p)$ is infinitesimally rigid and $-\omega$ is a proper stress of $G'(p)$, $G'(p)$ is infinitesimally rigid in $\mathbf{R}^2$. $\square$

Clearly what makes this argument work is the fact that the existence of a stress with negative coefficients for all interior members implies that all the edges of the polygon have positive coefficients. However, the existence of a stress with negative coefficients for all the edges of a polygon does not force all the interior members to have positive coefficients. This accounts for the failure of the converse of Theorem 6.6, an example of which we now give.

EXAMPLE 6.7. Let $G(p)$ be the tensegrity hexagon of Example 6.5 with $x = 3$ and bars instead of struts for the six edges of the hexagon. Then $G'(p)$ is infinitesimally rigid in $\mathbf{R}^2$ since $\bar{G}'(p)$ is infinitesimally rigid and any $\lambda < 0$ gives a proper stress of $G'(p)$. But, as we saw in Example 6.5, $G(p)$ is infinitesimally flexible in $\mathbf{R}^2$ when $x = 3$.

Of course, Theorem 6.6 and its converse are both valid for tensegrity polygons with struts as edges of the polygon. This is just a special case of the "interchangeability" of struts and cables in an infinitesimally rigid tensegrity framework which was mentioned following Definition 4.1.

We conclude this section with a conjecture which, if correct, would establish Grünbaum's Conjecture 6.

Conjecture. Rigidity and infinitesimal rigidity are equivalent for tensegrity polygons with bars as edges of the polygon.

7. Tensegrity frameworks in space. Interest in tensegrity frameworks in the plane and, in particular, tensegrity polygons stems partly from their connection with the infinitesimal rigidity of certain tensegrity frameworks in $\mathbf{R}^3$ arising from convex polyhedra. As was suggested by Grünbaum [**9**, p. 2.12] and confirmed by Whiteley [**15**, Corollary 3.5 and Corollary 3.6] and Connelly [**5**, Theorem 5.1], a convex polyhedron with faces that are infinitesimally rigid tensegrity polygons forms an infinitesimally rigid tensegrity framework in space. In this setting the infinitesimal rigidity of the spacial tensegrity framework follows from the planar infinitesimal rigidity of its faces.

The same methods establish a variety of results of the same type (see [**15**, Corollary 3.5]). For example, suppose $G(p)$ is a tensegrity framework in $\mathbf{R}^3$ satisfying the following properties:

(i) the vertices of $G(p)$ are distinct and their affine span is $\mathbf{R}^3$;

(ii) every vertex of $G(p)$ belongs to an edge of the convex hull P of $p_1, \ldots, p_v$ (so no vertex lies in the interior of a face of P or in the interior of P);

(iii) $[p_i, p_j]$ is contained in ∂P for every member $\{i, j\}$ of G (so no member of G passes through the interior of P);

(iv) for each face of P, the set of vertices and members of $G(p)$ which belong to the face forms an infinitesimally rigid tensegrity framework in the plane of the face. Then $G(p)$ is infinitesimally rigid in $\mathbf{R}^3$. Either the infinitesimal motions of Connelly or the statics of Whiteley can be used to prove this result. In both cases, the proof is based on Alexandrov's rigidity theorem which says that a triangulated convex polyhedron with all its vertices on the natural edges of the polyhedron is infinitesimally rigid in $\mathbf{R}^3$ (see [**2**, §6, Corollary 1] or [**15**, Theorem 3.1]). In the proof it is not necessary to show the existence of a proper stress (although by Theorem 5.2 the existence of a proper stress is a consequence of the result).

On the other hand, proper stresses can be used in various ways to establish the infinitesimal rigidity of certain tensegrity frameworks which have some faces that fail to be infinitesimally rigid in the plane. One very simple way to construct such examples relies on the fact that the addition of stresses of the individual faces can lead to a stress of the entire tensegrity framework with a rather bizarre distribution of signs.

EXAMPLE 7.1. Consider the tensegrity cube $G(p)$ in $\mathbf{R}^3$ shown in Figure 7.1. Each face of the cube is braced by both diagonals. The top and bottom faces consist entirely of cables while the four remaining faces have struts for diagonal braces. Since a nontrivial stress of the doubly braced square in the plane has coefficients of one sign for the two diagonals and opposite sign for the four edges of the square, the sum of four such stresses gives a stress of $G(p)$ with appropriate signs on all but the top and bottom faces of the cube. Adding suitably small stresses of the top and bottom faces leads to a proper stress of $G(p)$. The infinitesimal rigidity of $\overline{G}(p)$ is a consequence of Alexandrov's rigidity theorem. By Theorem 5.2, $G(p)$ is infinitesimally rigid in $\mathbf{R}^3$ even though the top and bottom faces of $G(p)$ are obviously flexible in the plane.

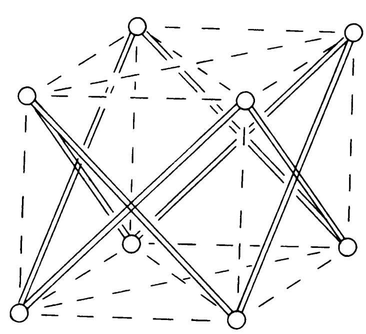


FIGURE 7.1

More exotic examples can be created with the help of the following general theorem for modifying an infinitesimally rigid tensegrity framework. We say that a subset F of the set B of bars of a tensegrity framework $G(p)$ in $\mathbf{R}^n$ is *free* if there exists a proper stress ω of $G(p)$ with $F \cap \operatorname{supp} \omega = \varnothing$. The idea of our next result is that some of the free bars of an infinitesimally rigid tensegrity framework $G(p)$

can be replaced by suitably chosen cables and struts and a cable or strut of $G(p)$ can be deleted in such a way that the resulting tensegrity framework is also infinitesimally rigid.

THEOREM 7.2. *Suppose $G'(p')$ is an infinitesimally rigid tensegrity framework in $\mathbf{R}^n$ where the set $F' \subset B'$ of bars of $G'(p')$ is free. Suppose the tensegrity framework $G''(p'')$ has no bars ($B'' = \emptyset$), its vertices are a subset of the set of vertices of $G'(p')$ (where the vertices of $G'(p')$ and also $G''(p'')$ are distinct and indexed so that $p'_i = p''_j$ if and only if $i = j$) and there exists a proper stress ω'' of $G''(p'')$. Suppose $\{k, m\} \in (C' \cap S'') \cup (S' \cap C'')$. Consider $G = (V; B, C, S)$ with $V = V'$, $B = (C' \cap S'') \cup (S' \cap C'') \cup (B' - (F' \cap E'')) - \{\{k, m\}\}$, $C = (C' \cap C'') \cup (F' \cap C'') \cup (C'' - E') \cup (C' - E'')$ and $S = (S' \cap S'') \cup (F' \cap S'') \cup (S'' - E') \cup (S' - E'')$ where E' and E'' are the members of G' and G'', respectively. Then $G(p')$ is infinitesimally rigid in $\mathbf{R}^n$.*

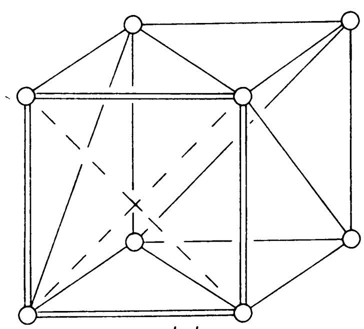
PROOF. Let ω' be a proper stress of $G'(p')$ with $F' \cap \operatorname{supp} \omega' = \emptyset$. We regard both ω' and ω'' as stresses of the framework given by $\bar{G}(p')$ together with the bar $\{k, m\}$. Suppose that $\{k, m\} \in C' \cap S''$ so $\omega'_{\{k,m\}} < 0$ while $\omega''_{\{k,m\}} > 0$. Then one can verify without difficulty that

$$\omega = \omega''_{\{k,m\}}\omega' - \omega'_{\{k,m\}}\omega''$$

is a proper stress of $G(p')$. Furthermore, the framework $\bar{G}'(p')$ with bar $\{k, m\}$ deleted is infinitesimally rigid in $\mathbf{R}^n$ by Corollary 5.3 and therefore $\bar{G}(p')$ is also infinitesimally rigid in $\mathbf{R}^n$ since it has even more bars. Thus $G(p')$ is infinitesimally rigid in $\mathbf{R}^n$ by Theorem 5.2. $\square$

Theorem 7.2 can be used in place of Theorem 6.1 to prove the infinitesimal rigidity of mixed polygons in Theorem 6.4. Several applications of Theorem 7.2 are involved in the construction of our next example which is an infinitesimally rigid tensegrity cube for which every face is flexible in the plane.

EXAMPLE 7.3. Consider the infinitesimally rigid tensegrity cube $G'(p')$ shown in Figure 7.2 where every face except the front one is braced by a single bar. Note that the set B' of all bars of $G'(p')$ is free. Next consider the planar tensegrity framework $G''(p'')$ shown in Figure 7.2. Letting p'' coincide with the top face of



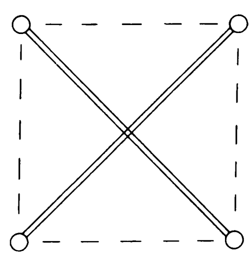
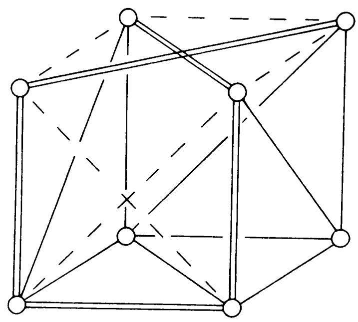
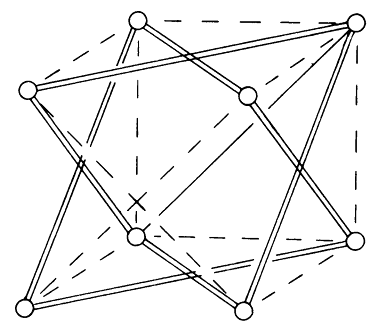


FIGURE 7.2

$G'(p')$ and combining $G'(p')$ and $G''(p'')$ by Theorem 7.2, we obtain the infinitesimally rigid tensegrity framework $G'''(p')$ shown in Figure 7.3. Now let p'' coincide with a side face of $G'''(p')$ and again apply Theorem 7.2. Continuing in this fashion for both the other side and the bottom face, we arrive at the infinitesimally rigid tensegrity cube $G(p')$ shown in Figure 7.3. Note that $G(p')$ has $3v - 5$ members, only one of which is a bar. It is easy to continue the procedure for another step and find an infinitesimally rigid tensegrity cube with $3v - 4$ members, none of which are bars. We do not know if there exists an infinitesimally rigid tensegrity cube with $3v - 5$ members and no bars.



$G'''(p')$



$G(p')$

FIGURE 7.3

We conclude with a short discussion of tensegrity frameworks in $\mathbf{R}^3$ arising from the complete bipartite graph $K_{5,5}$. Consider a (framework) realization $G(p)$ of $K_{5,5}$ in $\mathbf{R}^3$ such that both five-sets of vertices have affine span $\mathbf{R}^3$ and all ten vertices do not lie in a quadric surface in $\mathbf{R}^3$. Then the stress space of $G(p)$ is one-dimensional and the framework $G(p)$ is infinitesimally rigid in $\mathbf{R}^3$ (see [**3**, Example 17]). Moreover, the stresses of $G(p)$ are quite easy to describe (and even to compute). Suppose $a_1, \ldots, a_5$ and $b_1, \ldots, b_5$ are the two five-sets of vertices. Let $(\alpha_1, \ldots, \alpha_5)$ be an affine dependency of $(a_1, \ldots, a_5)$, which means $\Sigma_{i=1}^5 \alpha_i a_i = 0$ and $\Sigma_{i=1}^5 \alpha_i = 0$. Let $(\beta_1, \ldots, \beta_5)$ be an affine dependency of $(b_1, \ldots, b_5)$. Then it is a simple matter to verify that assigning coefficient $\alpha_i \beta_j$ to edge $[a_i, b_j]$ for $1 \leqslant i, j \leqslant 5$ gives a stress of $G(p)$. Introducing cables and struts in accordance with the signs of the products $\alpha_i \beta_j$ leads to an infinitesimally rigid tensegrity framework in $\mathbf{R}^3$ by Theorem 5.2.

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Department of Mathematics, University of Wyoming, Laramie, Wyoming 82071

Department of Mathematics, Champlain Regional College, St. Lambert, Québec, J4P 3P2, Canada

Groupe de Recherche Topologie Structurale, Université de Montréal, Montréal, Québec H3C 3J7, Canada