

ARC-SMOOTH CONTINUA

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ABSTRACT. Continua admitting arc-structures and arc-smooth continua are introduced as higher dimensional analogues of dendroids and smooth dendroids, respectively. These continua include such spaces as: cones over compacta, convex continua in I_2 , strongly convex metric continua, injectively metrizable continua, as well as various topological semigroups, partially ordered spaces, and hyperspaces. The arc-smooth continua are shown to coincide with the freely contractible continua and with the metric K -spaces of Stadtlander. Known characterizations of smoothness in dendroids involving closed partial orders, the set function T , radially convex metrics, continuous selections, and order preserving mappings are extended to the setting of continua with arc-structures. Various consequences of the special contractibility properties of arc-smooth continua are also obtained.

Introduction. The purpose of this paper is to introduce and study a well-behaved class of arc-wise connected metric continua called *arc-smooth continua*.¹ The class of arc-smooth continua, which may be considered as a higher dimensional analogue of the smooth dendroids [8], includes (a) smooth dendroids; (b) convex (more generally star-like) continua in I_2 ; (c) cones over compacta; (d) strongly convex metric continua; (e) injectively metrizable continua [17]; (f) continua ruled by arcs [22]; (g) continuum semilattices with unit; (h) various hyperspaces of continua; and (i) certain partially ordered spaces. The arc-smooth continua coincide with the (a) freely contractible continua [17]; (b) metric K -spaces [36]; and (c) ruled metric spaces [9].

Our study of these continua involves a new concept called an *arc-structure* which is patterned after the "natural arc-structure" possessed by dendroids. In order to motivate the definition, we begin by considering the special case of dendroids.

Let X be a dendroid (i.e., an arc-wise connected, hereditarily unicoherent metric continuum). Each pair of points $x \neq y$ in X determines a unique arc from x to y denoted by xy . By convention, $xx = \{x\}$. Denote by $C(X)$ the hyperspace of subcontinua of X with the Hausdorff metric (see [19]). The function $A: X \times X \rightarrow C(X)$ defined by $A(x, y) = xy$ satisfies the following metric-like axioms for all x, y and z in X .

$$(a) A(x, x) = \{x\},$$

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¹The concept of arc-smooth continua and some of the results in this paper were announced in [12]. The reader is referred to [12] for a discussion of the relationship of arc-smooth continua to several other generalizations of smooth dendroids which have been studied in recent years.

- (b) $A(x, y) = A(y, x)$, and
 (c) $A(x, z) \subseteq A(x, y) \cup A(y, z)$

with equality prevailing whenever y belongs to $A(x, z)$.

Furthermore, it is easy to see that the dendroid X is smooth [8] at the point p precisely when the induced function $A_p: X \rightarrow C(X)$ defined by $A_p(x) = A(p, x)$ is continuous.

We define an *arc-structure* on an arbitrary continuum X to be a function $A: X \times X \rightarrow C(X)$ such that for $x \neq y$ in X , the set $A(x, y)$ is an arc from x to y and such that conditions (a)–(c) are satisfied. The pair (X, A) is called *arc-smooth at the point p* in X provided the induced function $A_p: X \rightarrow C(X)$ is continuous. The pair (X, A) is *arc-smooth* in case there exists a point in X at which (X, A) is arc-smooth.

An arbitrary continuum X is said to be *arc-smooth at the point p* provided there exists an arc-structure A on X for which (X, A) is arc-smooth at p . The continuum X is *arc-smooth* if it is arc-smooth at some point. We shall observe in §II that this definition is equivalent to the definition originally given in [12].

The paper is divided into two major sections. In §I we study pairs (X, A) where X is a continuum with a fixed arc-structure A , and we consider two general questions: (1) What properties must X and A possess if (X, A) is arc-smooth? (2) What properties of X and A imply that (X, A) is arc-smooth? Many of the results generalize known theorems concerning smoothness in dendroids. In §II we study analogous questions for arbitrary continua: (1) What properties must arc-smooth continua possess? (2) What properties imply that a continuum is arc-smooth?

Preliminaries. By a *continuum* we mean a compact connected metric space. The continuum X is *unicoherent* provided the intersection of any two subcontinua whose union is X is connected; and X is *hereditarily unicoherent* in case each subcontinuum is unicoherent. A *dendroid* is an arcwise connected hereditarily unicoherent continuum. A *dendrite* is a locally connected dendroid.

If X is a continuum, then $C(X)$ denotes the hyperspace of subcontinua of X with the Hausdorff metric. The reader is referred to [19] and [33] for basic definitions and facts concerning $C(X)$.

If Z is a subset of the continuum X , then $\text{cl } Z$ will denote the closure of Z and $\text{int } Z$ will denote the interior of Z . If $\{Z_n\}$ is a sequence of subsets of X , then $\text{Li } Z_n$, $\text{Ls } Z_n$, and $\text{Lim } Z_n$ will denote the limit inferior, the limit superior, and the limit of $\{Z_n\}$, respectively.

I. Arc-smoothness for continua with arc-structures.

Convention. Throughout §I, X denotes a continuum with a given arc-structure $A: X \times X \rightarrow C(X)$ as defined in the Introduction. Given x and y in X , the arc $A(x, y)$ will be denoted by xy .

Recall that the pair (X, A) is said to be *arc-smooth* at p provided the induced function $A_p: X \rightarrow C(X)$ is continuous.

I.1. *The order \leq_p .* For each p in X we define the partial order \leq_p by letting $x \leq_p y$ whenever $px \subseteq py$. When X is a dendroid, \leq_p is simply the familiar weak cutpoint order with respect to p (see [21]).

For a subset H of X the lower set $L_p(H)$ is the set $\{x \in X \mid x \prec_p y \text{ for some } y \in H\}$; and the upper set $M_p(H)$ is the set $\{y \in X \mid x \prec_p y \text{ for some } x \in H\}$.

THEOREM I-1-A. *The following are equivalent.*

- (a) (X, A) is arc-smooth at p .
 - (b) \prec_p is closed in $X \times X$.
 - (c) For each closed subset H , the upper set $M_p(H)$ is closed.
 - (d) For each closed subset H , the lower set $L_p(H)$ is closed; and the upper set $M_p(x)$ is closed for each x in X .
 - (e) Whenever x belongs to an open set W such that $W = M_p(W)$ there exists an open set U such that $U = M_p(U)$ and $x \in U \subseteq \text{cl } U \subseteq W$.
- (Observe that (e) is a kind of regularity condition for open upper sets.)

PROOF. We shall show that (b) is equivalent to each of the other conditions.

(a) implies (b). Assume that $\{x_n\}$ and $\{y_n\}$ are sequences converging to x and y , respectively, and that $x_n \prec_p y_n$ for each n . By arc-smoothness, $\{py_n\}$ converges to py . Since $x_n \in py_n$, it follows that $x \in py$. Thus $x \prec_p y$ as required.

(b) implies (a). To prove that $A_p: X \rightarrow C(X)$ is continuous, it suffices to show that if $\{y_n\}$ converges to y in X , then $\{py_n\}$ converges to py in $C(X)$. Let $x \in \text{Ls } py_n$. There is a subsequence $\{py_{n_j}\}$ such that each py_{n_j} contains a point x_{n_j} so that $\{x_{n_j}\}$ converges to x . Since \prec_p is closed, $x \prec_p y$. Consequently $\text{Ls } py_n \subseteq py$. But $\text{Ls } py_n$ is a continuum (Theorem 2-101 of [16]) containing p and y ; hence $\text{Ls } py_n = py$. Furthermore, $\text{Li } py_n = py$. For if $x \in py \setminus \text{Li } py_n$, there is an open set U containing x and a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that $U \cap py_{n_j} = \emptyset$ for all j . It follows that $\text{Ls } py_{n_j}$ is a subcontinuum of py which contains p and y but not x . This contradiction shows that $\text{Ls } py_n = \text{Li } py_n = py$. Thus $\{py_n\}$ converges to py .

(c) implies (b). To see that \prec_p is closed, suppose that $x \not\prec_p y$. Choose an open set U such that $x \in U \subseteq \text{cl } U \subseteq X \setminus py$. Thus $M_p(\text{cl } U)$ is closed and $py \cap M_p(\text{cl } U) = \emptyset$. Let V be an open set such that $y \in V \subseteq X \setminus M_p(\text{cl } U)$. Now if $(s, t) \in U \times V$, then $s \not\prec_p t$ (otherwise $t \in M_p(\text{cl } U) \cap V = \emptyset$).

(b) implies (c). Obvious.

(d) implies (b). To prove that \prec_p is closed, suppose that $x \not\prec_p y$. Since $M_p(x)$ is closed there is an open set V such that $y \in V \subseteq \text{cl } V \subseteq X \setminus M_p(x)$. Now $L_p(\text{cl } V)$ is a closed set which misses $M_p(x)$. There is an open set U such that $M_p(x) \subseteq U \subseteq X \setminus L_p(\text{cl } V)$. Now if $(s, t) \in U \times V$, then $s \not\prec_p t$ as required.

(b) implies (d). Obvious.

(e) implies (b). Suppose that $x \not\prec_p y$. Then $W = X \setminus L_p(y)$ is an open set containing x such that $W = M_p(W)$. By hypothesis there is an open set U such that $U = M_p(U)$ and $x \in U \subseteq \text{cl } U \subseteq W$. Let V be an open set such that $y \in V \subseteq X \setminus \text{cl } U$. It follows that $s \not\prec_p t$ whenever $(s, t) \in U \times V$.

(b) implies (e). Let W be an open set containing x such that $W = M_p(W)$. Let V be an open set such that $x \in V \subseteq \text{cl } V \subseteq W$ and let $U = M_p(V)$. To see that U is open let $y \in U$ and let $\{y_n\}$ be a sequence converging to y . Since $y \in M_p(V)$, the arc py intersects V . Thus, by (a), for sufficiently large n , the arc py_n intersects V and hence $y_n \in U$. A similar argument shows that $\text{cl } U \subseteq W$.

REMARK. Some related results for uniquely arcwise connected continua appear in [35].

I.2. *Convex sets.* A subset Z of X is said to be *convex* if for each pair of points x and y of Z the arc xy is a subset of Z . Observe that if Z is a convex subcontinuum of X , then $A|Z \times Z$ is an arc-structure on Z .

THEOREM I-2-A. (a) *The intersection of any collection of convex sets is convex.* (b) *For each $p \in X$ and subset H the lower set $L_p(H)$ is convex.* (c) *For each $p \in X$ and convex subset Z the upper set $M_p(Z)$ is convex.* (d) *If Z is a convex subcontinuum of X , then for each $p \in X$ the set Z has a zero relative to \leq_p .*

PROOF. (a), (b) and (c) follow immediately from the definition. To prove (d) let x be an arbitrary point in Z and let z be the first point on the arc px which belongs to Z . If $z \not\leq_p y$ for some $y \in Z$, then $z \notin py$ and $pz \cap py \subseteq X \setminus Z$. Since $zy \subseteq pz \cup py$ it follows that zy is not contained in Z which contradicts the assumption that Z is convex.

LEMMA I-2-B. *Suppose that (X, A) is arc-smooth at p .*

(a) *If $\{Z_n\}$ is a sequence of convex subsets of X and $\text{Li } Z_n \neq \emptyset$, then $\text{Ls } Z_n$ and $\text{Li } Z_n$ are convex.*

(b) *If $\{x_n\}$ and $\{y_n\}$ are sequences converging to x and y , respectively, and $x_n \leq_p y_n$ for every n , then the sequence of arcs $\{x_n y_n\}$ converges to the arc xy .*

(c) *If Z is a convex subcontinuum of X and z is the zero of Z relative to \leq_p , then $(Z, A|Z \times Z)$ is arc-smooth at z .*

(d) *If Z is a convex subset of X , then $\text{cl } Z$ is convex.*

PROOF. (a) To show that $\text{Ls } Z_n$ is convex it suffices to prove that $xy \subseteq \text{Ls } Z_n$ whenever $x \in \text{Li } Z_n$ and $y \in \text{Ls } Z_n$. Passing to subsequences if necessary, we may assume that there are sequences $\{x_n\}$ and $\{y_n\}$ converging to x and y , respectively, with x_n and y_n belonging to Z_n for each n . Notice that for each n the arc $x_n y_n$ is contained in $Z_n \cap (px_n \cup py_n)$. By arc-smoothness, $\text{Ls}(px_n \cup py_n) = px \cup py$. Thus $\text{Ls } x_n y_n$ is a subcontinuum of $(\text{Ls } Z_n) \cap (px \cup py)$ which contains x and y . It follows that $xy \subseteq \text{Ls } x_n y_n \subseteq \text{Ls } Z_n$ as required.

Suppose that $\text{Li } Z_n$ is not convex and let x and y be points of $\text{Li } Z_n$ for which $xy \not\subseteq \text{Li } Z_n$. Let $z \in xy \setminus \text{Li } Z_n$ and let U be an open set such that $z \in U \subseteq X \setminus \text{Li } Z_n$. There exists a subsequence $\{Z_{n_i}\}$ of $\{Z_n\}$ and points $x_{n_i}, y_{n_i} \in Z_{n_i}$ such that $Z_{n_i} \cap U = \emptyset$ for each i , $\{x_{n_i}\}$ converges to x , and $\{y_{n_i}\}$ converges to y . Thus $\text{Ls}(px_{n_i} \cup py_{n_i}) \cap U = \emptyset$. By arc-smoothness, $xy \subseteq px \cup py = \text{Ls}(px_{n_i} \cup py_{n_i})$. Thus $xy \cap U = \emptyset$ contradicting the choice of z .

(b) By (a) $xy \subseteq \text{Li } x_n y_n$; hence it suffices to show that $\text{Ls } x_n y_n \subseteq xy$. Let $z \in \text{Ls } x_n y_n$. Without loss of generality there is a sequence $\{z_n\}$ converging to z with $z_n \in x_n y_n$. Thus $x_n \leq_p z_n \leq_p y_n$, and, by Theorem I-1-A, $x \leq_p z \leq_p y$. Consequently $z \in xy$.

(c) Given $\{y_n\}$ in Z converging to y , apply (b) to see that $\text{Lim } zy_n = zy$.

(d) Let $Z_n = Z$ for each n . Then, by (a), $\text{cl } Z = \text{Ls } Z_n$ is convex.

We define X to be *locally convex at a subcontinuum* M if for each open set U containing M , there is a convex set Z such that $M \subseteq \text{int } Z \subseteq \text{cl } Z \subseteq U$. In case M consists of a single point p we say that X is *locally convex at* p .

If Y is a subset of X , then by a *convex component* of Y we mean a maximal convex subset of Y .

THEOREM I-2-C. *The following are equivalent.*

- (a) (X, A) is arc-smooth at p .
- (b) For each open set U containing p , the convex component of U which contains p is open.
- (c) X is locally convex at each convex subcontinuum which contains p .

PROOF. (a) implies (b). If Z denotes the convex component of U which contains p , then $Z = \{x \in U \mid px \subseteq U\}$. If Z is not open, then for some $x \in Z$ there exists a sequence $\{x_n\}$ in $X \setminus Z$ which converges to x . Then for each n , the set $px_n \cap (X \setminus U)$ is nonempty. Consequently $(\text{Ls } px_n) \cap (X \setminus U) \neq \emptyset$. By arc-smoothness $px \cap (X \setminus U) \neq \emptyset$ which contradicts the fact that $px \subseteq U$.

(b) implies (c). Let $p \in M \subseteq U$ where M is a convex subcontinuum and U is an open set. Let V be an open set such that $M \subseteq V \subseteq \text{cl } V \subseteq U$. If Z denotes the convex component of V which contains p , then Z is open by (b), and, furthermore, $M \subseteq \text{int } Z = Z \subseteq \text{cl } Z \subseteq U$ as needed.

(c) implies (a). By Theorem I-1-A, it suffices to show that \prec_p is closed. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences with $x_n \prec_p y_n$ for each n which converge to x and y , respectively. If $x \not\prec_p y$, then there exists an open set U which contains py and misses x . By (c) there is a convex set Z such that $py \subseteq \text{int } Z \subseteq \text{cl } Z \subseteq U$. This contradicts the fact that for sufficiently large n , the arc py_n lies in Z .

REMARK. An analogous result is valid for continua which are smooth in the sense of Mackowiak [31, Theorem (3.1)].

COROLLARY I-2-D. *If (X, A) is arc-smooth at p , then X is locally convex at p .*

THEOREM I-2-E. *The following are equivalent.*

- (a) (X, A) is arc-smooth at each point.
- (b) X is locally convex at each point.
- (c) X is a dendrite.

PROOF. (a) implies (b). Corollary I-2-D.

(b) implies (c). It suffices to show that each pair of distinct points x and y can be separated by a third point (see [42, p. 88]). Let $z \in xy \setminus \{x, y\}$ and observe that x and y lie in distinct convex components of $X \setminus \{z\}$. Moreover, every convex component of $X \setminus \{z\}$ is open by (b). Thus z separates x from y in X .

(c) implies (a). Dendrites are smooth (hence arc-smooth) at each point by Corollary 4 of [8].

Notation. Let $I(X) = \{p \in X \mid (X, A) \text{ is arc-smooth at } p\}$. $I(X)$ is called the *initial set* of X . Let $N(X) = \{x \in X \mid X \text{ is not locally convex at } x\}$.

The sets $I(X)$ and $N(X)$ were introduced in [8] for dendroids. Our next result generalizes Theorem 2 of [8]. Since the proof carries over with only minor modifications, it will be omitted.

THEOREM I-2-F. *If (X, A) is arc-smooth at p , then the initial set $I(X)$ is the convex component of $X \setminus N(X)$ which contains p .*

I.3. A generalization of the set function T . Jones [18] defined the set function T (which he called L) on any continuum M by the equation $T(x) = \{y \in M \mid \text{each subcontinuum of } M \text{ with } y \text{ in its interior contains } x\}$. This set function has been used to characterize smoothness in dendroids [8] and hereditarily unicoherent continua [13].

Here we modify T so that it is applicable to continua with arc-structures and use it to characterize arc-smoothness.

For each point x of the continuum X with arc-structure A we define $T_A(x)$ to be the set $\{y \in X \mid \text{each convex subcontinuum of } X \text{ with } y \text{ in its interior contains } x\}$.

It is clear from the definitions that (a) $T(x) \subseteq T_A(x)$ for all $x \in X$, (b) $T(x)$ and $T_A(x)$ are always closed, and (c) T and T_A agree whenever X is a dendroid. The simple example which follows shows that, unlike $T(x)$, the set $T_A(x)$ need not be connected. Observe that $T_A(x)$ is connected whenever (X, A) is arc-smooth by Theorem I-3-B.

EXAMPLE. Let X denote the unit circle in the plane, let $p \in X$, and let \leq order X in a clockwise manner away from p . Define $A: X \times X \rightarrow C(X)$ by the equation $A(x, y) = \{z \in X \mid x \leq z \leq y\}$. Now A is an arc-structure on X , although (X, A) is not arc-smooth at any point. For any $q \neq p$, we have $T_A(q) = \{p, q\}$.

THEOREM I-3-A. *If (X, A) is arc-smooth at p , then $T_A(x) \subseteq M_p(x)$ for all $x \in X$.*

PROOF. Suppose $y \notin M_p(x)$. Then $X \setminus M_p(x)$ is an open set which contains py by Theorem I-1-A. By Theorem I-2-C, X is locally convex at py . It follows that $y \notin T_A(x)$.

THEOREM I-3-B. *(X, A) is arc-smooth at p if and only if for each x in X the following hold.*

- (a) $T_A(x)$ is convex, and
- (b) $px \cap T_A(x) = \{x\}$.

PROOF. Suppose that (X, A) is arc-smooth at p . Then (b) follows immediately from Theorem I-3-A. To prove (a), let $y \in T_A(x)$ and note that $x \leq_p y$. If $xy \not\subseteq T_A(x)$ then there is a point $z \in xy$ and a convex continuum Z with $z \in \text{int } Z \subseteq Z \subseteq X \setminus \{x\}$. Observe that $x \leq_p w$ for each $w \in Z$ (otherwise $x \notin pw$ and hence $x \notin pz \subseteq pw \cup wz$). Now $M_p(Z)$ is a convex subcontinuum of X with y in its interior which misses x . This contradicts $y \in T_A(x)$. Thus $xy \subseteq T_A(x)$ and $T_A(x)$ is convex.

Suppose that (a) and (b) hold. It suffices by Theorem I-1-A to show that \leq_p is closed. Suppose $x \not\leq_p y$ and observe that (a) and (b) imply that $y \notin T_A(x)$. Thus there is a convex subcontinuum Z such that $y \in \text{int } Z \subseteq Z \subseteq X \setminus \{x\}$. Thus

$py \cup Z$ is a convex continuum missing x . Now if $(s, t) \in [X \setminus (py \cup Z)] \times \text{int } Z$, then $s \notin pt \subseteq py \cup Z$. Hence $s \not\ll_p t$ and \ll_p is closed.

THEOREM I-3-C. X is a dendrite if and only if $T_A(x) = \{x\}$ for all x in X .

PROOF. If X is a dendrite, then $T_A(x) = T(x) = \{x\}$ for each x .

If $T_A(x) = \{x\}$ for each x , then (a) and (b) of Theorem I-3-B are satisfied for each p in X . Consequently, X is arc-smooth at each point, and, by Theorem I-2-E, X is a dendrite. (Alternately, one can easily establish local convexity at each point.)

THEOREM I-3-D. (X, A) is arc-smooth if and only if for each x and y in X the following hold.

- (a) $T_A(x)$ is convex, and
- (b) $xy \cap T_A(x) = \{x\}$ or $xy \cap T_A(y) = \{y\}$.

PROOF. The proof parallels the one for dendroids (Theorem 6 of [8]) provided it is shown that conditions (a) and (b) imply that if $\{x_n\}$ is a sequence of points converging to x such that $x_n \ll_p x_{n+1}$ for every n , then $\text{cl}(\cup px_n) = px$. Observe that it suffices to show that $x_n \in px$ for each n . Suppose this is not the case, and without loss of generality assume that $x_n \notin px$ for all n . Let $y = \sup(px_1 \cap px)$ and observe that $y \in xx_n$ for all n . Suppose that Z is a convex subcontinuum containing x in its interior. Then, for sufficiently large n , it follows that $xx_n \subseteq Z$, hence $y \in Z$ and $x_n \in Z$ for all n . Consequently, $x \in T_A(y)$ and $x \in T_A(x_n)$ for all n . By (a), $T_A(x_n)$ is convex, and hence $y \in T_A(x_n)$ for all n . It follows that $y \in T_A(x)$. Thus $x \in T_A(y)$ and $y \in T_A(x)$, contradicting (b).

I.4. Radially convex metrics. A metric d on X is called *radially convex* at the point p provided that $d(p, z) = d(p, y) + d(y, z)$ whenever $y \in pz$. Observe that this implies that each arc pz is congruent to a line segment. Also, for each $\epsilon > 0$, the set $N_\epsilon(p) = \{x \in X \mid d(p, x) \leq \epsilon\}$ is a convex subcontinuum of X .

THEOREM I-4-A. (X, A) is arc-smooth at p if and only if the following hold.

- (a) $T_A(x)$ is convex for each x in X , and
- (b) X admits a metric d which is radially convex at p .

PROOF. Assume that (X, A) is arc-smooth at p . Theorem I-3-B establishes (a). By Theorem I-1-A, \ll_p is a closed partial order. Let H denote the Hilbert cube with the coordinatewise partial order, and let the distance between points $\{x_n\}$ and $\{y_n\}$ in H be given by $\sum |x_n - y_n|/2^n$. Carruth has shown (Theorem 1 of [6] or Corollary 1.5 of [5]) that there is an order preserving embedding of X into H . The induced metric d on X is easily seen to be radially convex at p .

Now suppose that (a) and (b) hold. By Theorem I-3-B, it suffices to show that $px \cap T_A(x) = \{x\}$ for each $x \in X$. Suppose that $y \in px \setminus \{x\}$, and let ϵ be a real number such that $d(p, y) < \epsilon < d(p, x)$. Then $N_\epsilon(p)$ is a convex subcontinuum containing y in its interior and missing x . Hence $y \notin T_A(x)$ as required.

THEOREM I-4-B. X is a dendrite if and only if for each p in X there exists a metric on X which is radially convex at p .

PROOF. By Theorem I-2-E, it suffices to note that the existence of a metric on X which is radially convex at p implies that X is locally convex at p (consider $N_\varepsilon(p)$).

I.5. *Continuous selections.* Let $\text{Convex}(X)$ denote the subspace of $C(X)$ consisting of all convex subcontinua of X .

For each p in X we define a selection $\sigma_p: \text{Convex}(X) \rightarrow X$ by letting $\sigma_p(Z)$ be the unique zero of Z relative to \leq_p (see Theorem I-2-A).

Observe that X is a dendroid precisely when $\text{Convex}(X) = C(X)$. In this case, σ_p is the so-called *least element function*. Our next theorem generalizes a result of Ward [40]: the dendroid X is smooth at p if and only if the least element function σ_p is continuous.

THEOREM I-5-A. (X, A) is arc-smooth at p if and only if the following hold.

(a) $\text{Convex}(X)$ is closed in $C(X)$, and

(b) $\sigma_p: \text{Convex}(X) \rightarrow X$ is continuous.

PROOF. Assume that (X, A) is arc-smooth at p . Lemma I-2-B implies (a). Condition (b) follows easily from the definition of σ_p and the fact that \leq_p is closed.

Suppose (a) and (b) hold. To show that \leq_p is closed, let $\{x_n\}$ and $\{y_n\}$ be sequences in X converging to x and y , respectively, such that $x_n \leq_p y_n$ for each n . Then $\{x_n y_n\}$ is a sequence in $\text{Convex}(X)$ which, without loss of generality, converges to a convex subcontinuum Z . By (b), $\sigma_p(Z) = x$. Hence $x \leq_p y$ and \leq_p is closed.

REMARK. The example in (I.3) shows that Theorem I-5-A fails without the assumption that $\text{Convex}(X)$ be closed in $C(X)$.

I.6. \leq_p -contractibility. Given p in X , we define a \leq_p -contraction to be a homotopy $H: X \times [0, 1] \rightarrow X$ satisfying the following conditions for all x in X :

(a) $H(x, 1) = x$,

(b) $H(x, 0) = p$, and

(c) $H(x, t) \leq_p x$ for all $t \in [0, 1]$.

THEOREM I-6-A. (X, A) is arc-smooth at p if and only if X admits a \leq_p -contraction.

PROOF. Suppose that (X, A) is arc-smooth at p and let d be a metric on X which is radially convex at p and bounded by 1. Following Mohler's construction of a retracting homotopy for smooth dendroids (Theorem 1.16 of [32]) we define $H: X \times I \rightarrow X$ as follows:

(i) If $d(p, x) \leq t$, then $H(x, t) = x$.

(ii) If $d(p, x) > t$, then $H(x, t) = y$

where $y \in px$ and $d(p, y) = t$.

Clearly H satisfies (a), (b) and (c). The continuity of H follows just as in [32].

Now suppose that $H: X \times [0, 1] \rightarrow X$ is a \leq_p -contraction. We shall show that \leq_p is closed. Assume that $\{x_n\}$ and $\{y_n\}$ are sequences in X converging to x and y , respectively, and that $x_n \leq_p y_n$ for every n . The properties of a \leq_p -contraction imply that for each n there is a $t_n \in [0, 1]$ such that $H(y_n, t_n) = x_n$. Without loss of generality we may assume that $\{(y_n, t_n)\}$ converges, say to (y, t) . Then $x = H(y, t)$ by continuity, and $H(y, t) \leq_p y$ in (c). Consequently $x \leq_p y$ and \leq_p is closed.

I.7. Order preserving mappings. Order preserving mappings of various kinds have been applied to obtain characterizations of smoothness (e.g., [8], [10], [14], [27], [29], and [30]). Here we introduce several types of order preserving mappings for continua with arc-structures and use them to characterize arc-smoothness.

Convention. Throughout (I.7) X and Y denote continua with fixed arc-structures A and B , respectively.

We say that a continuous surjective function $f: X \rightarrow Y$ is a \leq_p -mapping in case $x \leq_p y$ in X implies that $f(x) \leq_{f(p)} f(y)$ in Y . If, in addition, $Y \subseteq X$, $B = A|Y \times Y$ and f is a retraction, then f is called a \leq_p -retraction. \leq_p -mappings are defined in a similar manner.

Observe that if $f: X \rightarrow Y$ is a \leq_p -mapping and $x \leq_p y$, then $f(xy) = f(x)f(y)$. From this it follows easily that f preserves convex sets and that f^{-1} preserves convex sets containing $f(p)$.

THEOREM I-7-A. *If $f: X \rightarrow Y$ is a \leq_p -mapping and (X, A) is arc-smooth at p , then (Y, B) is arc-smooth at $f(p)$.*

PROOF. It suffices to show that $\leq_{f(p)}$ is closed. Let $\{y_n\}$ and $\{z_n\}$ be sequences in Y converging to y and z , respectively, and such that $y_n \leq_{f(p)} z_n$ for each n . Let $\{x_n\}$ be a sequence in X such that $f(x_n) = z_n$ for each n . Since $f(px_n) = f(p)f(x_n) = f(p)z_n$, there is a point $w_n \in px_n$ such that $f(w_n) = y_n$ for each n . Without loss of generality we may assume that $\{(w_n, x_n)\}$ converges to (w, x) in $X \times X$. Then $w \leq_p x$ since (X, A) is arc-smooth at p . Consequently $y = f(w) \leq_{f(p)} f(x) = z$ and $\leq_{f(p)}$ is closed.

The cone over the Cantor set is universal in the mapping sense for the class of smooth dendroids (Theorem 11 of [8]). Our next theorem extends this result to arbitrary arc-smooth continua.

THEOREM I-7-B. *Let X be the cone over the Cantor set, A the unique arc-structure on X , and p the vertex of X . Then (Y, B) is arc-smooth at the point q if and only if there exists a \leq_p -mapping $f: X \rightarrow Y$ such that $f(p) = q$.*

PROOF. If f exists, then (Y, B) is arc-smooth at q by Theorem I-7-A. If (Y, B) is arc-smooth at q , then Y admits a metric which is radially convex at q (Theorem I-4-A) and hence the construction of f given in Theorem 11 of [8] for dendroids is valid for arc-smooth continua.

THEOREM I-7-C. *The following are equivalent.*

- (a) (X, A) is arc-smooth at p .
- (b) There exists a \leq_p -mapping $f: X \rightarrow [0, 1]$ with $f(p) = 0$, and for each x in X the set $T_A(x)$ is convex.
- (c) For each x in X , there exists a \leq_p -retraction $r: X \rightarrow px$ and $T_A(x)$ is convex.

PROOF. (a) implies (b). $T_A(x)$ is convex by Theorem I-3-B. Let d be a metric on X which is radially convex with respect to p and such that $\sup\{d(p, x) | x \in X\} = 1$. Then the mapping $f: X \rightarrow [0, 1]$ defined by $f(x) = d(p, x)$ is a \leq_p -mapping.

(b) implies (a). By Theorem I-3-B it suffices to show that $px \cap T_A(x) = \{x\}$ for each $x \in X$. Suppose that $y \in px \setminus \{x\}$, and choose $t \in [0, 1]$ such that $f(y) < t < f(x)$. Then $f^{-1}([0, t])$ is a convex subcontinuum of X which contains y in its interior and misses x . Consequently, $y \notin T_A(x)$ and $px \cap T_A(x) = \{x\}$.

(a) implies (c). $T_A(x)$ is convex by Theorem I-3-B. Let d be a metric which is radially convex with respect to p and which is bounded by 1. Fix $x \in X$, and let $t = d(p, x)$. Let H be the \leq_p -contraction defined in Theorem I-6-A, and let $f = H|X \times \{t\}$. Then f retracts X onto $N_t(p)$. Now define $g: N_t(p) \rightarrow [0, t]$ by $g(y) = d(p, y)$, and define $h: [0, t] \rightarrow px$ to be the isometry with $h(0) = p$. Then $r = h \circ g \circ f$ is the required retraction.

(c) implies (a). The proof is similar to the proof that (b) implies (a).

REMARK. If X is a dendroid, then the assumption that $T_A(x)$ be convex in Theorem I-7-C is superfluous. Thus for the special case of dendroids, the equivalence of (a) and (b) is Corollary 4 of [14], and the equivalence of (a) and (c) is the main result of [29]. The following example shows that the assumption that $T_A(x)$ be convex is not superfluous in general.

EXAMPLE. Let X denote the unit circle in the plane and let $p = (0, -1)$. For each x in X , if $p \neq x \neq (0, 1)$ define px to be the shorter arc joining p to x . For $x = (0, 1)$ let px be the set of points with nonnegative first coordinates. Let A be the unique arc-structure on X compatible with the collection of arcs just defined. Then (X, A) is not arc-smooth at p even though the function $f: X \rightarrow [0, 1]$ defined by the equation $f(x) = d(p, x)/2$ is a \leq_p -mapping.

COROLLARY I-7-D. *The following are equivalent.*

- (a) X is a dendrite.
- (b) For each p in X there exists a \leq_p -mapping $f: X \rightarrow [0, 1]$ with $f(p) = 0$.
- (c) For each arc xy in X there exists a \leq_x -retraction $r: X \rightarrow xy$.

PROOF. That (a) implies (b) and (c) follows immediately from Theorem I-7-C. It is easy to verify that (b) and (c) each implies that $T_A(x) = \{x\}$ for each x in X . Thus the desired conclusion follows from Theorem I-7-C.

I.8. *Convex-monotone mappings.* Monotone mappings are known to preserve the class of smooth dendroids [8]. In contrast, simple examples show that monotone mappings need not preserve arc-smooth continua. In this section we define a special class of monotone mappings for continua with arc-structures and show that they preserve arc-smoothness.

Convention. In this section X denotes, as usual, a continuum with a fixed arc-structure A , and Y denotes an arbitrary continuum.

We say that a mapping $f: X \rightarrow Y$ is *convex-monotone* provided that $f^{-1}(y)$ is a convex subcontinuum of X for each y in Y .

REMARK. Observe that when X is a dendroid, the convex-monotone mappings coincide with the monotone mappings.

We now observe that if $f: X \rightarrow Y$ is a convex-monotone mapping, then f induces a natural arc-structure, say B , on Y . Given y_1 and y_2 in Y , let x_1x_2 be an arc in X which is irreducible from $f^{-1}(y_1)$ to $f^{-1}(y_2)$. Using the convexity of $f^{-1}(y_1)$ and

$f^{-1}(y_2)$ it is easy to see that x_1x_2 is unique. For each point z in $f(x_1x_2)$ the set $f^{-1}(z) \cap x_1x_2$ is convex, hence connected. Consequently, $f|_{x_1x_2}$ is monotone, and $f(x_1x_2)$ is an arc from y_1 to y_2 . The function $B: Y \times Y \rightarrow C(Y)$ given by $B(y_1, y_2) = f(x_1x_2)$ is easily seen to be an arc-structure on Y .

THEOREM I-8-A. *Let $f: X \rightarrow Y$ be a convex-monotone mapping, and let B be the arc-structure on Y induced by f . Then f is a \leq_p -mapping for each p in X and $f(I(X)) \subseteq I(Y)$.*

PROOF. By the definition of the induced arc-structure B , the mapping $f|_{px}$ is monotone for each p and x in X . Thus f is a \leq_p -mapping for each p in X . Now $f(I(X)) \subseteq I(Y)$ by Theorem I-7-A.

I.9. The end set E . Following Lelek's definition for end point in the classical sense [25], we define the end set E of X to be $\{e \in X \mid \text{if } e \in xy, \text{ then } e = x \text{ or } e = y\}$.

Even in the special case when X is a dendroid the end set E can be very complicated (see [1], [24], and [25]). Our purpose here is to show that strong conclusions can be drawn when (X, A) is arc-smooth and the end set E is sufficiently well behaved.

The example in (I.3) shows that the end set is sometimes empty. However, whenever (X, A) is arc-smooth, the end set is not empty as a consequence of the next lemma.

LEMMA I-9-A. *If (X, A) is arc-smooth at p , then each arc px is contained in an arc pe with e in the end set E .*

PROOF. Suppose there is no such point e . Then there is a sequence of points $\{y_n\}$ greater than x such that $y_n \leq_p y_{n+1}$ for each n and such that no point z satisfies $y_n \leq_p z$ for all n . Passing to subsequences if necessary, we can assume that $\{y_n\}$ converges to some point y . Since \leq_p is closed by Theorem I-1-A, it follows that $y_n \leq_p y$ for each n , which is a contradiction.

THEOREM I-9-B. *Let X be a three-manifold with boundary a two-sphere S^2 . If the end set E coincides with S^2 and (X, A) is arc-smooth at some point $p \in X \setminus E$, then X is a three-cell.*

PROOF. Clearly \leq_p is a closed partial order whose set of maximal elements coincides with S^2 . Consequently X is a three-cell by Theorem 3 of [37].

Our next result is a generalization of the theorem that if P is the pseudo-arc, then the hyperspace $C(P)$ is a two-dimensional Cantor manifold [34]. Theorem II-1-A together with the observation that $C(P)$ has unique segments and that subcontinuum inclusion is a closed partial order show that $C(P)$ is arc-smooth at the point P . The same is true for any hereditarily indecomposable continuum. For an arbitrary continuum Z , the hyperspace $C(Z)$ need not be arc-smooth since $C(Z)$ need not be contractible. On the other hand, $C(Z)$ is sometimes homeomorphic to a cone in which case Theorem II-1-A shows that $C(Z)$ is arc-smooth. The reader is referred to [33] for these and related facts about hyperspaces.

THEOREM I-9-C. *Suppose that the continuum X contains no separating points and that the end set E is a continuum. If (X, A) is arc-smooth at some point $p \in X \setminus E$, then no closed zero-dimensional set separates X .*

PROOF. First observe that X is contractible with respect to the circle by Theorem I-6-A. Now according to Property 3 of [23] it suffices to show the following: (a) for each $e \in E$, $pe \cap E = \{e\}$, and (b) for each $z \neq p$ in $X \setminus E$ there are points x and y in E such that $z \in xy$. Condition (a) is immediate from the definition of E . To prove (b), let x be a point in E such that $pz \subseteq px$ (use Lemma I-9-A). By hypothesis, no point of pz separates X . It follows easily that there is a point $w \in X \setminus px$ such that $z \notin pz \cap pw$. Now let y be a point of E such that $pw \subseteq py$. Thus xy is a subarc of $px \cup py$ which contains z .

II. Arc-smoothness for arbitrary continua.

Convention. Throughout this section of the paper X denotes an arbitrary continuum.

In the setting of arbitrary continua the following alternate definition of arc-smoothness (introduced in [12]) is often convenient.

ALTERNATE DEFINITION. The continuum X is said to be *arc-smooth at the point p* provided there exists a function $F: X \rightarrow C(X)$ such that for $x \neq p$ the set $F(x)$ is an arc from p to x and the following conditions are satisfied:

- (a) $F(p) = \{p\}$,
- (b) if $x \in F(y)$, then $F(x) \subseteq F(y)$, and
- (c) F is continuous.

To see that the two definitions are equivalent, we first suppose that A is an arc-structure on X such that (X, A) is arc-smooth at p . Letting $F = A_p$ it is clear that F satisfies the conditions of the alternate definition. Now suppose that $F: X \rightarrow C(X)$ satisfies the alternate definition. For each (x, y) in $X \times X$, let $A(x, y)$ denote the unique arc joining x to y in the subcontinuum $F(x) \cup F(y)$. Then $A: X \times X \rightarrow C(X)$ is an arc-structure on X and $A_p = F$. Consequently A_p is continuous and (X, A) is arc-smooth at p .

The alternate definition for arc-smoothness makes it apparent that cones over arbitrary compacta and star-like continua in I_2 are arc-smooth.

If X is a convex continuum in I_2 , then X is clearly *arc-smooth at each point* in the sense that for each point p in X there is a function F satisfying the conditions of the alternate definition. Observe that this is very different from requiring that X admit a fixed arc-structure A for which (X, A) is arc-smooth at each point. The latter requirement forces X to be a dendrite by Theorem I-2-E.

II.1. A partial order characterization. Purely order theoretic characterizations for smooth dendroids (i.e., metrizable generalized trees), smooth continua, and weakly smooth continua were obtained in [39], [26], and [28], respectively. In this section we present such a characterization for arc-smooth continua which is closely related to Theorems 2.6 and 2.8 of [9].

THEOREM II-1-A. *The continuum X is arc-smooth at the point p if and only if X admits a partial order \leq such that*

- (a) \leq is closed,
- (b) p is the zero of \leq , and
- (c) for each $y \in X$ the lower set $L(y) = \{x \in X \mid x \leq y\}$ is an order arc.

PROOF. Let X be arc-smooth at p and let A be an arc-structure on X such that (X, A) is arc-smooth at p . Then the partial order \leq_p defined in (I.1) is closed by Theorem I-1-A, and \leq_p clearly satisfies (b) and (c).

Now suppose that \leq is a partial order on X satisfying (a), (b) and (c). Define $F: X \rightarrow C(X)$ by the equation $F(x) = L(x)$. Clearly $F(p) = \{p\}$ and $F(x) \subseteq F(y)$ whenever $x \in F(y)$. The proof that F is continuous is analogous to the proof that (b) implies (a) in Theorem I-1-A. Thus X is arc-smooth at p by the alternate definition.

II.2. Semigroups and semigroup actions. In their study of topological semigroups with zero and unit, Koch and McAuley [22] introduced a class of spaces called *continua ruled by arcs*. It is easy to see that the first four of the eight conditions in their definition are equivalent to our definition of arc-smooth continua. Continua satisfying their first four conditions were studied under the name *ruled spaces* by Eberhart [9] and under the name *K-spaces* by Stadtlander [36]. The reader is referred to [9] for various results concerning semigroup and semilattice structures on arc-smooth continua. In particular, it is worth noting that Corollary 2.3 of [9] shows that every continuum admitting a semilattice structure with identity is arc-smooth.

We next reinterpret the main theorem in [36] to obtain a useful characterization of arc-smooth continua in terms of thread actions. First we need some definitions.

By a *thread* T we mean any topological semigroup (written multiplicatively) on the interval $[0, 1]$ with 0 acting as a zero and 1 acting as a unit. We say that the thread T *acts naturally* on the pointed continuum (X, p) if there is a mapping $m: T \times X \rightarrow X$ satisfying the following conditions for all x in X :

- (a) $m(0, x) = p$,
- (b) $m(1, x) = x$, and
- (c) $m(s, m(t, x)) = m(st, x)$ for all $s, t \in T$.

THEOREM II-2-A. *Let (X, p) be any pointed continuum, and let T be any thread. Then X is arc-smooth at p if and only if T acts naturally on (X, p) .*

PROOF. Stadtlander [36, p. 487] has observed that a *K-space* over a single point (called a *K-space*) satisfies the first four axioms for a continuum ruled by arcs. The result now follows immediately from our previous remarks and Theorem 1 of [36].

II.3. Contractibility of arc-smooth continua. Arc-smooth continua are contractible by Theorem I-6-A. Of course contractible continua need not be arc-smooth even in the class of dendroids. Our next result shows that a special kind of contractibility utilized by Isbell [17] in the study of injective metric spaces actually characterizes the arc-smooth continua.

A *free contraction* [17] of a space Z to a point p is a homotopy $H: Z \times [0, 1] \rightarrow Z$ satisfying the following conditions for all z in Z :

- (a) $H(z, 0) = p$,
- (b) $H(z, 1) = z$, and
- (c) $H(H(z, s), t) = H(z, \min\{s, t\})$ for all $s, t \in [0, 1]$.

THEOREM II-3-A. *The continuum X is arc-smooth at p if and only if X is freely contractible to p .*

PROOF. Suppose that H is a free contraction of X to p . Let T be the thread with $st = \min\{s, t\}$ for $s, t \in [0, 1]$. Define a mapping $m: T \times X \rightarrow X$ by $m(t, x) = H(x, t)$. It follows that T acts naturally on (X, p) . Hence X is arc-smooth at p by Theorem II-2-A.

The converse follows in a similar manner.

REMARK. It follows from Theorem II-3-A and Theorem 1.1 of [17] that every injectively metrizable continuum is arc-smooth at each of its points.

REMARK. A metric d on the continuum X is called *strongly convex* if for each pair of distinct points x and y in X there is a unique arc from x to y which is isometric to the real line segment $[0, d(x, y)]$.

If X admits such a metric, then it is easy to see that X is arc-smooth at each point. The converse is an apparently difficult unsolved problem. By using ordinary multiplication in place of min multiplication in the proof of Theorem II-3-A, one sees that the converse is equivalent to a problem posed by Bing in [2]. Does X admit a strongly convex metric provided that for each point p in X there exists a homotopy $H: X \times [0, 1] \rightarrow X$ satisfying the following conditions for all x in X :

- (a) $H(x, 0) = p$,
- (b) $H(x, 1) = x$, and
- (c) $H(H(x, s), t) = H(x, st)$ for all s, t in $[0, 1]$?

THEOREM II-3-B. *If X is arc-smooth at p , then X has a basis of closed contractible neighborhoods at p .*

PROOF. Let A be an arc-structure on X such that (X, A) is arc-smooth at p , and let d be a metric on X which is radially convex at p (see Theorem I-4-A). Then for each $\epsilon > 0$, the ϵ -neighborhood $N_\epsilon(p)$ is clearly arc-smooth, hence contractible.

II.4. Some consequences of contractibility. In this section we collect a variety of results about arc-smooth continua which depend on their special contractibility properties.

THEOREM II-4-A. *A finite dimensional continuum which is arc-smooth at each point is an absolute retract.*

PROOF. Such a continuum is contractible and locally contractible, hence an absolute retract by Corollary 10.5 of [4].

REMARK. Theorems I-2-E and II-4-A can be viewed as different ways of generalizing the fact that a dendroid which is smooth at each point is a dendrite (i.e., a one-dimensional absolute retract).

THEOREM II-4-B. *A one-dimensional continuum is arc-smooth if and only if it is a smooth dendroid.*

PROOF. By Theorem 1 of [7], every contractible one-dimensional continuum is a dendroid.

COROLLARY II-4-C. *A continuum is hereditarily arc-smooth (i.e., every subcontinuum is arc-smooth) if and only if it is a smooth dendroid.*

PROOF. Smooth dendroids are hereditarily smooth by Corollary 6 of [8]. Conversely, hereditarily arc-smooth continua are hereditarily arcwise connected, hence one-dimensional. Now apply Theorem II-4-B.

Our next theorem is a restatement of the main result in [38].

THEOREM II-4-D. *A two-dimensional polyhedron is arc-smooth if and only if it is collapsible.*

PROOF. This follows directly from Theorem II-3-A and Theorem 1.8 of [17].

The continuum X is called *homogeneous* if for each pair of points x and y in X there exists a homeomorphism from X onto X taking x to y . The Hilbert cube is a homogeneous arc-smooth continuum (see [11]).

THEOREM II-4-E. *Every nondegenerate homogeneous arc-smooth continuum is infinite dimensional.*

PROOF. The proof in [17] of the analogous statement for injectively metrizable compacta depends on their free contractibility, and hence applies to arc-smooth continua as well.

The continuum X is said to have the *fixed point property* if for each self-mapping f there is a point x in X such that $f(x) = x$. Dendroids are known to have the fixed point property [3]; however, continua with arc-structures need not have the fixed point property even when arc-smooth since some cones over continua do not [20].

THEOREM II-4-F. *If X is an arc-smooth continuum which is uniquely arcwise connected or embeddable in the plane, then X has the fixed point property.*

PROOF. Uniquely arcwise connected contractible continua have the fixed point property by Theorem 5 of [43]. Arcwise connected nonseparating plane continua have the fixed point property by Theorem 3 of [15].

THEOREM II-4-G. *If the continuum X is arc-smooth at the point p , then each closed set containing p is the fixed point set of some self-mapping on X .*

PROOF. As in the remark on strongly convex metrics in (II.3), let $H: X \times [0, 1] \rightarrow X$ be a homotopy satisfying the following conditions for all x in X :

- (a) $H(x, 0) = p$,
- (b) $H(x, 1) = x$, and
- (c) $H(H(x, s), t) = H(x, st)$ for all s, t in $[0, 1]$.

By Theorem 1 of [41] it suffices to prove that $H(x, t) \neq x$ for $x \neq p$ and $t \neq 1$. To this end, suppose that $t \neq 1$ and $H(x, t) = x$. Then $H(x, t) = H(H(x, t), t) = H(x, t^2)$. Similarly, $H(x, t) = H(x, t^n)$ for all n . By continuity $H(x, t) = H(x, 0)$ and hence $x = p$ as desired.

COROLLARY II-4-H. *If the continuum X is arc-smooth at each point, then each of its closed sets is the fixed point set of some self-mapping on X .*

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