HULLS OF DEFORMATIONS IN \( \mathbb{C}^n \)

BY

H. ALEXANDER

ABSTRACT. A problem of E. Bishop on the polynomially convex hulls of deformations of the torus is considered. Let the torus \( T^2 \) be the distinguished boundary of the unit polydisc in \( \mathbb{C}^2 \). If \( t \mapsto T_t^2 \) is a smooth deformation of \( T^2 \) in \( \mathbb{C}^2 \) and \( g_0 \) is an analytic disc in \( \mathbb{C}^2 \) with boundary in \( T^2 \), a smooth family of analytic discs \( t \mapsto g_t \) is constructed with the property that the boundary of \( g_t \) lies in \( T_t^2 \). This construction has implications for the polynomially convex hulls of the tori \( T_t^2 \). An analogous problem for a 2-sphere in \( \mathbb{C}^2 \) is also considered.

Introduction. By a result of A. Browder [3], a compact real orientable manifold \( X \) of dimension \( \geq n \) in \( \mathbb{C}^n \) is never polynomially convex. Beginning with the work of E. Bishop [1], a great deal of effort has gone into giving a constructive explanation of this phenomenon. Bishop showed that there exist analytic discs in \( \mathbb{C}^n \) with boundaries in \( X \) near certain "exceptional" points of \( X \). His method was amplified by a number of others; a report on these developments is given in [6].

We shall consider a related problem which has been attributed to E. Bishop in [2, p. 234, Problem 17]. Let \( \{F_t\} \) be a deformation of the torus \( T^2 \) in \( \mathbb{C}^2 \); i.e., \( F_0 = \text{identity} \) and \( \{F_t\} \) is a family of diffeomorphisms of \( T^2 \) into \( \mathbb{C}^2 \) which vary smoothly with \( t \). Put \( T_t^2 = F_t(T^2) \); \( T_t^2 \) is a torus in \( \mathbb{C}^2 \) which is "close" to \( T^2 \) for small \( t \). By Browder's result \( T_t^2 \) is not polynomially convex. Also, every point of the closed unit polydisc \( \overline{U}^2 \), the polynomially convex hull of \( T_t^2 = T^2 \), lies on some analytic disc \( g_0 \) with boundary in \( T^2 \). The problem is to show that associated to the smooth deformation \( \{F_t\} \) there exists a smooth family of analytic discs \( \{g_t\} \) such that the boundary of \( g_t \) lies in \( T_t^2 \). We shall construct such a family \( \{g_t\} \), at least for \( t \) sufficiently small. This will account for the nonpolynomial convexity of \( T_t^2 \) in the most direct manner. As \( T^2 \) is deformed to \( T_t^2 \), its hull \( \overline{U}^2 \) is deformed to a set \( D_t \) which is close to \( \overline{U}^2 \) \((= D_0)\) and which is related to \( T_t^2 \) in a way that parallels the relationship of \( \overline{U}^2 \) to \( T^2 \); namely, \( D_t \) has a topological boundary composed of analytic discs whose own boundaries lie in \( T_t^2 \) and thus \( D_t \) has \( T_t^2 \) as a distinguished boundary. Eric Bedford [7] has proved that \( D_t \) is in fact the polynomially convex hull of \( T_t^2 \).

We shall also obtain an analogous result for a particular 2-sphere in \( \mathbb{C}^2 \):

\[
S^2 = \{(z_1, z_2) \in \mathbb{C}^2: \text{Im } z_2 = 0, |z_1|^2 + (\text{Re } z_2)^2 = 1 \}.
\]
Specifically, if \( g_0 \) is an analytic disc in \( \mathbb{C}^2 \) with boundary in \( S^2 \) and if \( t \to S_t^2 \) is a smooth deformation in \( \mathbb{C}^2 \), then there exists a smooth family of analytic discs \( t \to g_t \) in \( \mathbb{C}^2 \) with the boundary of \( g_t \) in \( S_t^2 \). This result is in some sense complementary to Bishop’s construction of analytic discs near the exceptional points of \( S_t^2 \). Bedford and Gaveau [8] have considered this problem from a different point of view. They obtain the complete hull by a process of continuation applied to the Bishop discs.

In order to produce analytic discs whose boundaries lie in prescribed sets, we must solve certain functional equations for unknown holomorphic functions on the unit disc. Bishop attacked this problem in [1] by iterating a contradiction mapping. We shall find it more convenient to use the implicit function theorem for Banach spaces [4]; in fact, the same approach can be applied to Bishop’s case in order to avoid an explicit iteration. Of course, iteration still lurks in the proof of the implicit function theorem. Banach spaces enter from the fact that rather than deal with a deformation consisting of a 1-parameter family of diffeomorphisms of \( T^2 \) we shall consider all diffeomorphisms of \( T^2 \) into \( \mathbb{C}^2 \) which are close to the identity. This infinite dimensional space can be associated with a Banach space such that a small deformation of \( T^2 \) is then viewed as a curve through the origin in the Banach space.

Finally, I want to thank Eric Bedford for an observation which simplifies Theorem 3 which was overly complicated in its original version.

1. Preliminaries. We shall denote the open unit disc in \( \mathbb{C} \) by \( U \) and the unit circle by \( T \). Thus \( U^n \) and \( T^n \) are the open unit polydisc and torus in \( \mathbb{C}^n \), respectively. If \( f \) is a continuous complex valued function on a set \( S \), we denote \( \sup\{|f(x)|: x \in S\} \) by \( |f|_S \). For \( K \) a compact set in \( \mathbb{C}^n \), \( \hat{K} \), the polynomially convex hull of \( K \), is the set \( \{z \in \mathbb{C}^n: |f(z)| < |f|_K \text{, for all polynomials } f \text{ in } \mathbb{C}^n\} \). An analytic disc in \( \mathbb{C}^n \) is a map \( g: U \to \mathbb{C}^n \) such that \( g \) is continuous and nonconstant on \( U \) and holomorphic on \( U \); we do not require that \( g \) be injective. We shall also refer to the image of \( g \) in \( \mathbb{C}^n \) as an analytic disc and shall call the set \( g(T) \) the boundary of \( g \). The maximum principle implies that the polynomially convex hull of the boundary of \( g \) contains the full analytic disc \( g \). If \( g = (g_1, g_2, \ldots, g_n) \) is an analytic disc in \( \mathbb{C}^n \) with boundary in \( T^n \), then each \( g_k \) has unit modulus on \( T \) and therefore is a finite Blaschke product, where we admit unimodular constants as Blaschke products of order zero. For \( K \) compact in \( \mathbb{C}^n \), we distinguish a number of subalgebras of the algebra \( C(K) \) of all continuous complex valued functions on \( K \): \( A(K) \) consists of those functions in \( C(K) \) which are holomorphic on the interior of \( K \); \( P(K) \), \( R(K) \) and \( H(K) \) are the closures in \( C(K) \) of the polynomials, the rational functions holomorphic on \( K \), and the functions holomorphic on \( K \), respectively. We shall say that a map is \( C^k \) if it has continuous derivatives of order \( k \); \( X \) is a compact \( C^\infty \) manifold with or without boundary, then \( C^k(X) \) with the topology of uniform convergence of derivatives of order \( k \) is a Banach algebra for an appropriate norm.

For functions \( \varphi \) on the unit circle \( T \) we have some special notation: \( \|\varphi\|_\infty = |\varphi|_T = \sup\{|\varphi(e^{i\theta})|: 0 \leq \theta \leq 2\pi\} \) and \( \|\varphi\|_2 = (\int_0^{2\pi} |\varphi(e^{i\theta})|^2 \, d\theta)^{1/2} \); \( L^2(T) \) is the usual Hilbert space associated with the second norm. Let \( \mathbb{C}_2 \) be the set of all complex
valued functions on $T$ which have distributional derivatives of order $< 2$ in $L^2(T)$. Here derivatives are with respect to $\theta$ where $T = \{e^{i\theta}: 0 < \theta < 2\pi\}$. One can also describe $\mathcal{H}_2$ as the space of complex valued $C^1$ functions on $T$ whose first derivatives are absolutely continuous with square integrable derivatives. From this it is easy to see that $\mathcal{H}_2$ is a ring. Although $\mathcal{H}_2$ can be normed as a Hilbert space, we shall use an equivalent norm which is more convenient for working with the multiplicative structure; namely, we define $\|f\| = \|f\|_\infty + \|f'\|_\infty + \frac{1}{2} \|f''\|_2$ for $f \in \mathcal{H}_2$. It is easy to check that $\|fg\| < \|f\| \cdot \|g\|$ and that $\mathcal{H}_2$ is a Banach algebra with this norm. We shall find it convenient to take the scalar field of $\mathcal{H}_2$ to be the real numbers.

Every $f \in \mathcal{H}_2$ has a Fourier series $f = \sum_{n=-\infty}^\infty c_n e^{in\theta}$ with $\sum |n^2 c_n|^2 < \infty$ and, conversely, any integrable function whose Fourier coefficients satisfy this bound is in $\mathcal{H}_2$. We shall utilize two closed subspaces of $\mathcal{H}_2$: $\mathcal{H}_2^\alpha$ is the set of all functions in $\mathcal{H}_2$ which extend to be holomorphic in the unit disc, $f \in \mathcal{H}_2$ is in $\mathcal{H}_2^\alpha$ if and only if $c_n = 0$ for $n < 0$, i.e., $f = \sum_{n=0}^\infty c_n e^{in\theta}$; $\mathcal{H}_2(\mathbb{R})$ is the subspace of real valued functions in $\mathcal{H}_2$. Each $u \in \mathcal{H}_2(\mathbb{R})$ has a Fourier series of the form

$$u = a_0 + \sum_{n=1}^\infty a_ne^{in\theta} + \overline{a}_n e^{-in\theta}$$

with $a_0$ real. The conjugate function $\bar{u}$ of $u$ is defined by

$$\bar{u} = \sum_{n=1}^\infty -ia_n e^{in\theta} + i\overline{a}_n e^{-in\theta}.$$ 

Then $u + i\bar{u} \in \mathcal{H}_2$ and the map $J:\ \mathcal{H}_2(\mathbb{R}) \to \mathcal{H}_2$, $Ju = u + i\bar{u}$ is a bounded linear transformation. This is a theorem of M. Riesz and is easily seen when $\mathcal{H}_2$ is given its equivalent Hilbert space norm.

2. A special case. We begin by treating a rather special class of deformations of $T^2$. The problem of finding analytic discs with boundaries in the associated deformed tori involves the solution of a certain functional equation (2.3). This functional equation plays a key role in the subsequent discussion of the general deformations of $T^n$.

Consider the image $T^2_\varphi$ in $\mathbb{C}^2$ of $T^2$ under the map

$$(z_1, z_2) \mapsto (z_1, z_2 + \varphi(z_1)) = (w_1, w_2)$$

(2.1)

where $\varphi$ is a smooth function on $T$ which is close to zero in $\mathcal{H}_2$-norm. If $g$ is a finite Blaschke product, then $\lambda \mapsto (\lambda, g(\lambda))$ is an analytic disc in $\mathbb{C}^2$ with boundary in $T^2$. We seek an analytic disc in $\mathbb{C}^2$ of the form $\lambda \mapsto (\lambda, f_\varphi(\lambda))$ with boundary in $T^2_\varphi$. The inverse of (2.1) is given explicitly by

$$z_1 = w_1, \quad z_2 = w_2 - \varphi(w_1).$$

(2.2)

Thus $(w_1, w_2) \in T^2_\varphi$ if and only if $|w_1| = 1$ and $|w_2 - \varphi(w_1)| = 1$. What we require of $f_\varphi$ then it is that it satisfy the functional equation

$$|f_\varphi(\lambda) - \varphi(\lambda)| = 1 \quad \text{for} \quad |\lambda| = 1,$$

(2.3)
and that $f_0 = g$ when $\varphi \equiv 0$. We shall now solve (2.3) for all sufficiently small $\varphi$ with functions $f_\varphi$ which are analytic in the unit disc and which vary smoothly with $\varphi$.

**Theorem 1.** Let $g$ be a finite Blaschke product. There exist an open neighborhood $\Omega$ of 0 in $\mathcal{K}_2$ and a $C^1$ nonlinear operator $E : \Omega \to \mathcal{B}_2$ such that $E(0) = g$ and, for $\varphi \in \Omega$, (2.3) holds for $f_\varphi = E(\varphi)$.

**Remark.** To put this into the setting of the Introduction, let $t \mapsto F_t$ with $F_t(z_1, z_2) = (z_1, z_2 + \varphi_t(z_1))$ be a deformation of $T^2$ of the special type under consideration and fix $g_0(\lambda) = (\lambda, g(\lambda))$, an analytic disc in $C^2$ with boundary in $T^2$. Then for $t$ sufficiently small, $\varphi_t \in \Omega$ and so we may define $g_t = (\lambda, E(\varphi_t)) \in \mathcal{B}_2 \oplus \mathcal{B}_2$. Viewing elements of $\mathcal{B}_2$ as holomorphic functions, we see that $t \mapsto g_t$ is a smooth family of analytic discs in $C^2$ with the boundary of $g_t$ in $T^2$.

**Proof.** Let $W = \{ \psi \in \mathcal{K}_2 : \| \psi \|_\infty < 1 \}$, an open set in $\mathcal{K}_2$. Define $S : W \to \mathcal{K}_2(\mathbb{R})$ by $S(\psi) = u$ where $u(\lambda) = \log |g(\lambda) - \psi(\lambda)|$ for $\lambda \in T$. Since $|g - \psi| > 0$, one can compute the derivatives of $u$ to check that $u \in \mathcal{K}_2(\mathbb{R})$. Put $h = Ju = u + i\overline{u} \in \mathcal{B}_2$ and define $A : \mathcal{K}_2 \to \mathcal{B}_2$ by $A(\psi) = e^{-h} = e^{-J \cdot S(\psi)}$.

Now set

$$f = ge^{-h} = g \cdot A(\psi) \quad (2.4)$$

and note that $f \in \mathcal{B}_2$ since $h \in \mathcal{B}_2$. From (2.4), $g = fe^h$ and we get

$$1 = |g| = |f|e^w = |f| |g - \psi| = |f| |1 - \overline{g}\psi|$$

$$= |f - \overline{g}\psi| = |f - e^{-h}\psi|$$

holding on $T$. Now define

$$Q : W(\subset \mathcal{K}_2) \to \mathcal{K}_2$$

by $Q(\psi) = \psi \cdot e^{-h} = \psi \cdot A(\psi)$.

We claim

(a) $A : W \to \mathcal{B}_2$ is a $C^1$ function with $A(0) = 1$.

(b) $Q : W \to \mathcal{K}_2$ is $C^1$.

(c) $Q(0) = 0$ and $Q'(0) =$ identity.

(2.6)

We shall check these conditions below. Assuming them for now, we appeal to the inverse function theorem for Banach spaces [4] to conclude that there exist open neighborhoods $W_1$ and $\Omega$ of 0 in $\mathcal{K}_2$ with $W_1 \subset W$ such that $Q$ is a $C^1$ diffeomorphism of $W_1$ onto $\Omega$. Now define $E : \Omega \to \mathcal{B}_2$ by $E = M_g \circ A \circ Q^{-1}$ where $M_g$ is the bounded linear transformation of $\mathcal{B}_2$ defined by $M_g(f) = g \cdot f$. Then $E$ is clearly a $C^1$ map and $E(0) = g$ because $Q(0) = 0$ and $A(0) = 1$.

Given $\varphi \in \Omega$, put $\psi = Q^{-1}(\varphi)$. Then $\varphi = Q(\psi) = \psi \cdot A(\psi) = \psi e^{-h}$ and $E(\varphi) = g \cdot A(\psi) = ge^{-h} = f$ and so (2.5) can be restated as $1 = |E(\varphi)(\lambda) - \varphi(\lambda)|$ for $|\lambda| = 1$. Thus $E$ has the desired properties.

It remains to show that (2.6)(a), (b) and (c) are valid. This will follow from the following observations. The fact that $\mathcal{K}_2$ has been normed as a Banach algebra will simplify some of the indicated computations which have been left to the reader.
(i) $S: W \to \mathcal{H}_2(\mathbb{R})$ is a $\mathcal{C}^1$ map. To see this fix $\psi \in W$ and write, for small $\sigma \in \mathcal{H}_2$,

$$S(\psi + \sigma) = \log|g - \psi - \sigma| = \log|g - \psi| + \log\left|1 - \frac{\sigma}{g - \psi}\right|$$

$$= \log|g - \psi| + \frac{1}{2} \log\left(1 - \frac{\sigma}{g - \psi}\right) + \frac{1}{2} \log\left(1 - \frac{-\sigma}{g - \psi}\right).$$

Using the Taylor series for $\log(1 - z)$ at $z = 0$ one can expand each of the last two terms in a series. Now easy estimates of these series in the $\mathcal{H}_2$-norm show that $S$ is differentiable at $\psi$ and that $S'(\psi)(\sigma) = -\text{Re}(\sigma/(g - \psi))$. From this expression it follows that the map $\psi \to S'(\psi)$ of $W$ into $\text{Hom}(\mathcal{H}_2, \mathcal{H}_2(\mathbb{R}))$ is continuous when the latter space of bounded linear transformations is given the norm topology.

(ii) $J: \mathcal{H}_2(\mathbb{R}) \to \mathcal{H}_2$ is a bounded linear transformation and consequently is $\mathcal{C}^1$. Similarly $M_g : \mathcal{H}_2 \to \mathcal{H}_2$ and $M_g : \mathcal{H}_2 \to \mathcal{H}_2$ are $\mathcal{C}^1$.

(iii) The map $h \to e^{-h}$ of $\mathcal{H}_2$ to $\mathcal{H}_2$ is easily seen to be $\mathcal{C}^1$.

(iv) Now (a) follows because $A(\psi) = e^{-J \cdot S'(\psi)}$ is a composition of $\mathcal{C}^1$ maps. Also $S(0) = 0$ and $J(0) = 0$ imply $A(0) = 1$.

(v) Using $Q(\psi) = A(\psi) \cdot \psi$, a short computation shows that $Q$ is differentiable at each $\psi \in W$ and that

$$Q'(\psi)(\sigma) = \psi \cdot A'(\psi)(\sigma) + A(\psi) \cdot \sigma. \quad (2.7)$$

It follows that $\psi \mapsto Q'(\psi)$ is continuous. This gives (b).

(vi) Finally $Q(0) = A(0) \cdot 0 = 0$ and (2.7) implies $Q'(0)(\sigma) = A(0) \cdot \sigma = \sigma$, as claimed in (c).

3. The general case for $T^n$. Let $F$ be a $\mathcal{C}^3$ diffeomorphism of the torus $T^n$ into $\mathbb{C}^n$ which is close to the identity, $T^n_F$ the image of $T^n$ under $F$. Fix $g = (g_1, g_2, \ldots , g_n)$, an analytic disc in $T^n$ with boundary in $T^n$. Our problem is to find an analytic disc $f_F$ in $\mathbb{C}^n$ with boundary in $T^n_F$ such that $f_F$ varies smoothly with $F$ when $F$ lies in some neighborhood of the identity $I$ and such that $f_I = g$.

We can write $F(z) = z + \beta(z)$ where $\beta = (\beta_1, \beta_2, \ldots , \beta_n)$ is an $n$-tuple of complex valued $\mathcal{C}^3$ functions on $T^n$ which are close to zero in $\mathcal{C}^3(T^n)$-norm. Put $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and define the retraction $\rho: \mathbb{C}^n \to T^n$ by $\rho(z_1, z_2, \ldots , z_n) = (z_1/|z_1|, \ldots , z_n/|z_n|)$. Then $F$ can be extended to a map $\tilde{F}: \mathbb{C}^n \to \mathbb{C}^n$ by setting $\tilde{F}(z) = z + \beta(\rho(z))$. Fix $0 < s < r < 1$ and define $K_t = \{z \in \mathbb{C}^n: t < |z_k| < 1/t\}$ for $1 < k < n$; $T^n \subseteq K_{\tau} \subseteq K_s$.

**Lemma.** If $\beta$ is sufficiently small in $\mathcal{C}^3(T^n)$, then $\tilde{F}$ is a diffeomorphism of the interior of $K_s$ onto a domain $\Sigma$ in $\mathbb{C}^n$ containing $K_r$.

**Proof.** If $\beta$ is sufficiently small then $\tilde{F}$ is easily seen to be injective on $K_s$ with nonvanishing Jacobian determinant. Hence $\tilde{F}$ maps $\text{int}(K_s)$ diffeomorphically onto a domain $\Sigma$ in $\mathbb{C}^n$. If $\beta$ is small, then $\tilde{F}$ is close to the identity and so $\partial \Sigma$ is close to $\partial K_s$ and consequently $\partial \Sigma$ is disjoint from $K_r$. As $K_r \cap \Sigma$ is nonempty, it follows from the connectedness of $K_r$ that $K_r \subseteq \Sigma$. 

We can write the inverse $G$ of $\tilde{F}$ as $G(z) = z - \alpha(z)$ where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is an $n$-tuple of $C^3$ complex valued functions on $\Sigma$ which are small in $C^3(K_r)$ norm when $\beta$ is small in $C^3(T^n)$, since the map $\beta \to \alpha$ of a neighborhood of 0 in $C^3(T^n)$ into $C^3(K_r)$ is norm continuous. When $\beta$ is sufficiently small, $T^n$ is contained in $K_r$ and then, for $z \in K_r$, $z \in T^n$ if and only if $G(z) = z - \alpha(z) \in T^n$; i.e., $|z_k - \alpha_k(z)| = 1$ for $1 \leq k \leq n$. Hence for any analytic disc $f = (f_1, f_2, \ldots, f_n)$ in $\mathbb{C}^n$ with boundary in $K_r$, the boundary of $f$ lies in $T^n$ if and only if

$$|f_k(\lambda) - \alpha_k(f_1(\lambda), \ldots, f_n(\lambda))| = 1, \quad 1 \leq k \leq n, \quad \text{(3.1)}$$

for each $\lambda \in T$. Our problem is thus to solve the system (3.1) for $f$ as a smooth function of $\alpha$.

**Theorem 2.** Let $g = (g_1, g_2, \ldots, g_n)$ be an analytic disc in $\mathbb{C}^n$ with boundary in $T^n$. There exist a neighborhood $N$ of 0 in $(C^3(K_r))^n$ and a $C^1$ map $u: N \to (\mathbb{S}_2)^n$ with the following properties:

(a) $u(0) = g$, and

(b) if $u(\alpha) = f = (f_1, f_2, \ldots, f_n)$ where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, then $|f_k - \alpha_k(f)| = 1$ on $T$ for $1 \leq k \leq n$.

**Remark 1.** To again relate this to our original formulation of the problem, let $\{F_t\}, -\epsilon < t < \epsilon$, be a smooth 1-parameter family of $C^3$ diffeomorphisms of $T^n$ into $\mathbb{C}^n$ with $F_0 = \text{identity}, T^n_0 = F_t(T^n)$ and let $g$ be an analytic disc in $\mathbb{C}^n$ with boundary in $T^n$. Write $F_t(z) = z + \beta_t(z)$. If $\alpha_t$ is associated to $F_t$ as above, then $t \to \alpha_t$ is a smooth curve through 0 in $N \subset (C^3(K_r))^n$, for $t$ sufficiently small. Then, for fixed $t$, $u(\alpha_t)$ is an analytic disc in $\mathbb{C}^n$ with boundary in $T^n_t$ and $u(\alpha_0) = u(0) = g$. Thus the map $t \mapsto u(\alpha_t) \equiv g_t$ yields the desired smooth family of analytic discs.

**Remark 2.** The differentiability assumptions are somewhat arbitrary. With the assumption of more (or less) smoothness on the $\alpha$'s the same method yields a family of analytic discs which is more (or less) smooth. Likewise the choices of $0 < s < r < 1$ are arbitrary.

We shall solve (3.1) by applying the implicit function theorem in Banach space. We begin with some simple facts on compositions.

**Lemma 1.** Let $X$ and $Y$ be Banach spaces with $\Omega$ an open set in $X$. Let $E_k: \Omega \to Y$ be a $C^1$ map for $1 \leq k \leq n$. Define $E: \Omega^n(\subseteq X^n) \to Y^n$ by $E(x_1, x_2, \ldots, x_n) = (E_1(x_1), E_2(x_2), \ldots, E_n(x_n))$. Then $E$ is a $C^1$ map.

**Proof.** It is straightforward to check that $E$ is differentiable at $(x_1, x_2, \ldots, x_n) \in \Omega^n$ and that

$$E'(x_1, \ldots, x_n)(u_1, \ldots, u_n) = (E'_1(x_1)(u_1), \ldots, E'_n(x_n)(u_n)).$$

**Lemma 2.** Let $W = \{f = (f_1, f_2, \ldots, f_n): f_k \in \mathbb{S}_2 \text{ and } f(T) \subseteq \text{the interior of } K_r\}$, an open set in $\mathbb{S}_2^n$. The map $B: C^1(K_r) \times W \to \mathbb{S}_2$ given by $B(\alpha; f_1, f_2, \ldots, f_n) = \alpha(f_1, f_2, \ldots, f_n)$ is $C^1$. 

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Proof. That $B$ maps into $\mathcal{H}_2$ is easily seen from a computation of the first two derivatives of $\alpha(f_1, \ldots, f_n)$. Fix $\alpha \in C^2(K)$ and $f \in W$. Write, for small $\sigma \in C^2(K)$ and $h \in \mathbb{C}^n$,

$$B(\alpha + \sigma, f + h) = \alpha(f + h) + \sigma(f + h) = \alpha(f) + \nabla \alpha(f) \cdot h + \gamma + \sigma(f) + \delta$$

where $\gamma = \alpha(f + h) - \alpha(f) - \nabla \alpha(f) \cdot h$ and $\delta = \sigma(f + h) - \sigma(f)$; here $\nabla \alpha$ is the real gradient of $\alpha$ and so $\nabla \alpha(f) \cdot h$ is a linear combination of $h_k$ and $\tilde{h}_k$. By tedious, but direct computations of $\gamma$, $\gamma'$ and $\gamma''$ one sees that $||y||/||n|| \to 0$ as $||n|| \to 0$ in $\mathcal{H}_2$-norms; likewise, with $C^3(K)$ norm on $\sigma$, $||\delta||/(||\sigma|| + ||h||) \to 0$ as $||h|| + ||\sigma|| \to 0$. It follows that $B$ is differentiable at $(\alpha, f)$ and that $B'(\alpha, f)(\sigma, h) = \nabla \alpha(f) \cdot h + \sigma(f)$. Another straightforward computation shows that $(\alpha, f) \to B'(\alpha, f)$ is a continuous map into $\text{Hom}(C^3(K) \oplus C^2, \mathcal{H}_2)$ with operator norm.

Proof of Theorem 2. Apply Theorem 1 $n$ times to each of the Blaschke functions $g_1, g_2, \ldots, g_n$ to get $n$ $C^1$ functions $E_1, E_2, \ldots, E_n$, all defined on a single open neighborhood $\Omega$ of $0$ in $\mathcal{H}_2$. The map $B$ of Lemma 2 is continuous at $(0, g)$ and $B(0, g) = 0$. Hence there is a neighborhood $N_1$ of $0$ in $C^1(K)$ and a neighborhood $D$ of $g$ in $\mathbb{C}^n$ such that $B(N_1 \times D) \subseteq \Omega$. Therefore we can define a map

$$R: N_1 \times D \to \mathbb{C}^n$$

by

$$R(\alpha, f) = f - (E_1(\alpha_1(f)), E_2(\alpha_2(f)), \ldots, E_n(\alpha_n(f))).$$

It follows from Lemmas 1 and 2 that $R$ is a $C^1$ map. We have $R(0, f) = f - (E_1(0), \ldots, E_n(0)) = f - g$ and therefore (a) $R(0, g) = 0$ and (b) $(D_2R)(0, g)$ = identity map: $\mathbb{C}^n \to \mathbb{C}^n$, where $D_2R$ is the “partial derivative” with respect to the second variable $f$. We can now invoke the implicit function theorem for Banach spaces [4] to conclude from (a) and (b) that there exist a neighborhood $N \subseteq \mathbb{C}^n$ of $0$ in $C^1(K)$ and a $C^1$ map $u: N \to \mathbb{C}^n$ such that $u(N) \subseteq D$, $u(0) = g$ and $R(\alpha, u(\alpha)) \equiv 0$ for all $\alpha \in N$. The latter implies that if $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in N$ and $u(\alpha) = (f_1, f_2, \ldots, f_n) \in \mathbb{C}^n$, then $f_k = E_k(\alpha_k(f_1, f_2, \ldots, f_n))$ for $1 < k < n$. Hence $|f_k - \alpha_k(f_1, f_2, \ldots, f_n)| \equiv 1$ on $T$ for $1 < k < n$. Thus $u$ has the desired properties.

4. Varying the initial analytic disc. In the last section we dealt with a fixed analytic disc $g$ with boundary in $T^n$. Now we allow the initial analytic disc to vary simultaneously with the deformations. For simplicity, we shall limit the discussion to the two dimensional case.

We first need to extend Theorem 1: we let $g$ vary and show that the solution $f_\varphi$ of (2.3) can be chosen to depend smoothly on both $g$ and $\varphi$. To this end, take $Q$, $A$, $E$ to be the nonlinear operators and $\Omega$ and $W_1$ the neighborhoods of $0$ in $\mathcal{H}_2$ given in the proof of Theorem 1 for the case $g(\lambda) \equiv \lambda$; then $Q^{-1}: \Omega \to W_1$ is $C^1$ and $E(\varphi)(\lambda) \equiv \lambda \circ A \circ Q^{-1}(\varphi)$ for $\varphi \in \Omega$ satisfies $|E(\varphi) - \varphi| \equiv 1$ on $T$ and $E(0)(\lambda) \equiv \lambda$. We shall let $\lambda$ denote the variable in $\mathbb{C}$ as well as the identity function on $\mathbb{C}$.
Fix a Blaschke product \( g \) and choose a neighborhood \( \Omega_1 \) of 0 in \( \mathbb{H}_2 \) such that \( \lambda \cdot \tilde{g} \varphi \in \Omega \) whenever \( \varphi \in \Omega_1 \). Then \( \psi = Q^{-1}(\lambda \cdot \tilde{g} \cdot \varphi) \in \mathbb{W}_1 \). Put \( f = g \cdot A(\psi) \in \mathbb{E}_2 \); then \( A(\psi) = \tilde{g} f \). We have \( |E(\lambda \tilde{g} \varphi) - \lambda \tilde{g} \varphi| = 1 \) on \( T \), where \( E(\lambda \tilde{g} \varphi) = \lambda A \cdot Q^{-1}(\lambda \tilde{g} \varphi) = \lambda A(\psi) \). Thus \( |\lambda \tilde{g} f - \lambda \tilde{g} \varphi| = 1 \) and so \( |f - \varphi| \equiv 1 \) on \( T \) where \( f = g \cdot A \cdot Q^{-1}(\lambda \tilde{g} \varphi) \). Also, if \( \phi = 0 \), then clearly \( f = g \). Since \( A \) and \( Q \) are independent of \( g \), this shows that \( f \) is a jointly \( C^1 \) function of \( g \) and \( \varphi \) in the following sense.

**Proposition 1.** Let \( t \rightarrow g_t \) be a \( C^1 \) map into \( \mathbb{E}_2 \) such that each \( g_t \) is a finite Blaschke product where \( t \) varies over some open set \( S \in \mathbb{R}^d \). Suppose that \( \Omega_1 \) is a neighborhood of 0 in \( \mathbb{H}_2 \) such that \( \lambda \tilde{g}_t \varphi \in \Omega \) for each \( t \) and each \( \varphi \in \Omega_1 \). Then the correspondence \((t, \varphi) \rightarrow f \equiv g_t \cdot A \cdot Q^{-1}(\lambda \tilde{g}_t \varphi)\) is a \( C^1 \) map of \( S \times \Omega_1 \) into \( \mathbb{E}_2 \) such that \( |f - \varphi| = 1 \) on \( T \) and \( f = g_t \) when \( \varphi = 0 \).

One would like to use this map together with the method of §3 to show that for a small but fixed \( \alpha \), each member of a family of analytic discs with boundaries in \( T^2 \) which completely fills up \( U^2 \) can be deformed to a disc with boundary in \( T^2 \) (where we write \( T^2_a = F(T^2) \) when \( z \mapsto z - \alpha(z) \) is the inverse of \( z \mapsto F(z) = z + \beta(z) \)). Proposition 1 requires, in order that \( f \) be defined, that \( \lambda \tilde{g}_t \varphi \in \Omega = \) the domain of \( Q^{-1} \). This means that for a fixed \( \varphi \) (corresponding to a fixed \( \alpha \)), only \( g_t \)'s whose norms are not too large are admissible. The following result shows however that in order to exhaust \( U^2 \) by analytic discs with boundaries in \( T^2 \), one must use Blaschke products of arbitrarily large \( \mathbb{H}_2 \)-norm and thus for no fixed \( \alpha \neq 0 \) can all of these analytic discs be deformed by this method to discs with boundaries in \( T^2_a \).

**Proposition 2.** Let \( \{p_n = (0, \lambda_n)\} \), where \( |\lambda_n| \rightarrow 1 \), be a sequence of points in \( U^2 \). Let \((h_n, k_n)\) be an analytic disc in \( \mathbb{C}^2 \) passing through \( p_n \) and with boundary in \( T^2 \). Then, in \( \mathbb{H}_2 \)-norm, as \( n \rightarrow \infty \),

\[
\|(h_n, k_n)\| \equiv \|h_n\| + \|k_n\| \rightarrow \infty.
\]

**Proof.** Both \( h_n \) and \( k_n \) are nonconstant Blaschke products. Let \( z_n \in U \) be such that \( h_n(z_n) = 0 \) and \( k_n(z_n) = \lambda_n \). Arguing by contradiction we may suppose that \( \|h_n\| + \|k_n\| \leq M < \infty \) and that \( z_n \rightarrow z_0 \in \overline{U} \). We consider two cases.

**Case 1.** \( |z_0| = 1 \). Then

\[
\frac{h_n(z_0) - h_n(z_n)}{z_0 - z_n} = \frac{1}{z_0 - z_n} \int_{z_n}^{z_0} h'_n(\xi) d\xi
\]

and so we get

\[
\frac{1}{|z_0 - z_n|} \leq \|h'_n\|_\infty \leq \|h_n\|.
\]

Then as \( n \rightarrow \infty \), \( \|h_n\| \rightarrow \infty \), a contradiction.

**Case 2.** \( |z_0| < 1 \). Because \( \{k_n\} \) is a normal family we may assume, without loss of generality, that \( k_n \) converges to a holomorphic function \( k \) uniformly on compact subsets of \( U \) where \( |k(z)| < 1 \). Since \( |k(z_0)| = \lim \|k_n(z_n)\| = 1 \), \( k \) is a unimodular constant; i.e., \( |k_n| \rightarrow 1 \) uniformly on compact subsets of \( U \). Therefore if \( z'_n \) is a zero
of \( k_n \) in \( U \) (it has at least one), we conclude that \( |z'_n| \to 1 \). We can now apply Case 1 to conclude that \( \|k_n\| \to \infty \).

Although we cannot deform all discs in \( U^2 \) with boundary in \( T^2 \) for a fixed \( \alpha \), we shall be able to do this for all discs in \( \partial U^2 \). The topological boundary \( \partial U^2 \) of \( U^2 \) is the union of "faces" \( M_1 \) and \( M_2 \) where \( M_1 = \{(z_1, z_2) \in C^2 : |z_1| < 1 \text{ and } |z_2| = 1 \} \) and \( M_2 \) is obtained by reversing the roles of \( z_1 \) and \( z_2 \) in \( M_1 \). The \( M \) are real 3-dimensional compact manifolds with \( \partial M_2 = T^2 \) and \( M_2 \cap M_2 = T^2 \). \( M_1 \) is a disjoint union of analytic discs \( g(e^{it}) \) where \( g(e^{it})(\lambda) = (\lambda, e^{it}) \) for \( e^{it} \in T \), where the second component is a constant function. Since \( \varphi \to e^{-it}\varphi \in \Omega \) is an isometry of \( \Omega \), it follows that there exists a neighborhood \( \Omega_1 \subseteq \Omega \) of 0 in \( \mathcal{K}_2 \) such that \( \lambda e^{-it}\varphi \in \Omega \) for all \( \varphi \in \Omega_1 \) and all \( t \). This will allow us to deform all of the discs in \( M_1 \) simultaneously.

For \( f \in \partial \Omega_2 \) such that \( f(T^2) \subseteq K \) we have seen that for \( \alpha \in (C^3(K))^2 \), \((\alpha, f) \mapsto \alpha \circ f\) is a \( C^1 \) map \( B \) into \( \mathcal{K}_2 \). Therefore for each \( t \), there exist neighborhoods of 0 in \((C^3(K))^2 \) and of \( g(e^{it}) \) in \( \partial \Omega_2 \) such that \( B \) takes their product into \( \Omega_1 \times \Omega_1 \). By compactness we get a single neighborhood \( N \) of 0 in \((C^3(K))^2 \) and an open set \( W \) in \( \partial \Omega_2 \) such that \( B(N \times W) \subseteq \Omega_1 \times \Omega_1 \) and \( W \) contains each \( g(e^{it}) \). Define a map \( R: T \times N \times W \to \partial \Omega_2 \) by

\[
R(e^{it}; \alpha_1, \alpha_2; f_1, f_2) = (f_1, f_2) - (E(\alpha_1(t)), e^{it} \circ A \circ Q^{-1}(\lambda e^{-it}\alpha_2(t)))
\]

where \( f = (f_1, f_2) \). It follows from our choice of \( N \) and \( W \) that \( R \) is well defined and from previous arguments \( R \) is a \( C^1 \) map. We have \( R(e^{it}; 0; f) = \alpha \circ f - g(e^{it}) \). Hence \( R(e^{it}; 0; g(e^{it})) = 0 \) and \( D_\alpha R(e^{it}; 0; g(e^{it})) = 0 \). Therefore, by the implicit function theorem, for each fixed \( t \), there exist a neighborhood \( V_t \) of \( e^{it} \) in \( T \), a neighborhood \( N_t \subseteq N \) of 0 in \((C^3(K))^2 \) and a \( C^1 \) map \( u_t: V_t \times N_t \to W \) such that \( u_t(e^{it}, 0) = g(e^{it}) \) and \( R(e^{it}; \alpha; u_t(e^{it}, \alpha)) = 0 \) for \((e^{it}, \alpha) \in V_t \times N_t \). Moreover \( u_t(e^{it}, 0) = g(e^{it}) \) for \( e^{it} \in V_t \) by uniqueness. By the compactness of \( T \) we can choose a finite subcover \( \{V_j \} \) of \( \{V_t \} \); let \( \{N_j \} \) and \( \{u_t \} \) be the corresponding finite subfamilies of \( \{N_t \} \) and \( \{u_t \} \) respectively. By the uniqueness in the implicit function theorem, it follows that \( u_t = u_j \) on \( (V_j \cap V_t) \times (N_j \cap N_t) \), provided that \( V_t \cap V_j \) is nonempty. Thus by "patching" we get a \( C^1 \) map \( u: T \times N_0 \to W \) where \( N_0 = \cap N_t \) is a neighborhood of 0 in \((C^3(K))^2 \), such that \( u(e^{it}, 0) = g(e^{it}) \) and \( R(e^{it}; \alpha; u(e^{it}, \alpha)) = 0 \). Now write \( u(e^{it}, \alpha) = (f_1, f_2) \). It follows from the definition of \( R \) that \( f_1 = E(\alpha_1(t)) \) and \( f_2 = e^{it}A \circ Q^{-1}(\lambda e^{-it}\alpha_2(t)) \). Taking \( g_1 \) in Proposition 1 to be \( e^{it} \) we conclude that

\[
|f_1 - \alpha_1(t)| \equiv 1, \quad |f_2 - \alpha_2(t)| \equiv 1
\]
on \( T \); i.e., \( u_e^{it}, \alpha \) is an analytic disc in \( C^2 \) with boundary in \( T_{e^{it}} \) for each \( e^{it} \).

For fixed \( \alpha \in N_0 \) define a map \( P_\alpha: \tilde{U} \times T(= M_1) \to C^2 \) by \( P_\alpha(\lambda, e^{it}) = u(e^{it}, \alpha)(\lambda) \) where \( u(e^{it}, \alpha) \in \partial \Omega_2 \) and we view elements of \( \partial \Omega_2 \) as functions on \( \tilde{U}, \) holomorphic on \( U \). Let \( M_1(\alpha) = P_\alpha(M_1) \subseteq C^2 \). Note that \( P_\alpha = \text{identity} \) and so \( M_1(0) = M_1 \subseteq \partial U^2 \).

**Lemma.** (a) The map \( e: \partial \Omega_2 \times \tilde{U}(\subseteq \mathbb{R}^2) \to C, e(f, \lambda) = f(\lambda) \) is \( C^1 \).
(b) \( P_\alpha: M_1 \to C^2 \) is \( C^1 \).
(c) \( \alpha \mapsto P_\alpha \) is a continuous map of \( N_0 \subseteq (C^3(K))^2 \) into \( C^1(M_1, C^2) \).
Proof. (a) A straightforward computation shows that $e$ is differentiable and that
$$e'(f, \lambda)(g, \xi) = f'(|\lambda| \cdot \xi + g(\lambda)).$$
Using the easily verified fact that the derivatives of functions $f \in \mathcal{O}_2$ satisfy a Hölder condition of the form
$$|f'(|\lambda_1|) - f'(|\lambda_2|)| \leq C \|f\| |\lambda_1 - \lambda_2|^{1/2}$$
for $\lambda_1$ and $\lambda_2$ in $\bar{U}$ where $C$ is a universal constant, one sees that $(f, \lambda) \to e'(f, \lambda)$ is continuous.

(b) Consider the map $\mu: \bar{U} \times T \times N_0 \to \mathbb{C}^2$ given by $\mu(\lambda, e^{\nu}, \alpha) = u(e^{\nu}, \alpha)(\lambda)$. We know that $(\lambda, e^{\nu}, \alpha) \to (u(e^{\nu}, \alpha), \lambda)$ is a $C^1$ map into $\mathbb{C}^2 \times \bar{U}$. It follows from (a) that $\mu$ is $C^1$. Hence for fixed $\alpha$, $P_\alpha(\lambda, e^{\nu}) = \mu(\lambda, e^{\nu}, \alpha)$ is $C^1$ on $M_1$.

(c) We can regard $\mu$ as a function of two variables $p$ and $\alpha$ where $p = (\lambda, e^{\nu}) \in M_1$ and $\alpha \in N_0$. Then $\mu$ is $C^1$ on $M_1 \times N_0$ and $DP_\alpha(\lambda, e^{\nu}) = D_1 \mu(p, \alpha)$ is continuous on $M_1 \times N_0$, as is $P_\alpha(p) = \mu(p, \alpha)$. For fixed $\alpha_0 \in N_0$ a simple compactness argument shows that
$$\|D_1 \mu(p, \alpha) - D_1 \mu(p, \alpha_0)\|_\infty + \|\mu(p, \alpha) - \mu(p, \alpha_0)\|_\infty \to 0$$
as $\alpha \to \alpha_0$ where the sup norms are taken over $M_1$. This gives (c).

Since the set of imbeddings of $M_1$ into $\mathbb{C}^2$ is an open set in $C^1(M_1, \mathbb{C}^2)$ and since $P_0 =$ the identity, it follows from (c) of the lemma that $P_\alpha$ is an imbedding (injective with differential of maximal rank 3 at each point of $M_1$) for $\alpha$ sufficiently small. Thus $M_1(\alpha)$ is an imbedded compact 3 dimensional manifold in $\mathbb{C}^2$ with $\partial M_1(\alpha) = T^{3}_2$; moreover $M_1(\alpha)$ is a disjoint union of analytic discs $\lambda \mapsto P_\alpha(\lambda, e^{\nu})$ for fixed $\nu$. Of course, the same is true for $M_2(\alpha)$ with the obvious definitions of $Q_\alpha: M_2 \to \mathbb{C}^2$ with $M_2(\alpha) = Q_\alpha(M_2)$ and $Q_0 =$ the identity on $M_2$. Clearly $M_1(\alpha) \cap M_2(\alpha)$ contains $T^{3}_2$; moreover $M_1(\alpha)$ and $M_2(\alpha)$ meet transversally at $T^{3}_2$. It follows that $M_1(\alpha) \cap M_2(\alpha) = T_i^2$, for $\alpha$ sufficiently small. Hence $\mathbb{C}^2 \setminus M_1(\alpha) \cup M_2(\alpha)$ is the union of two open sets, the bounded component being homeomorphic to $U^2$. Let $D_\alpha$ be the closure of this bounded component. Then $D_\alpha$ is a compact subset of $\mathbb{C}^2$. If $f$ is holomorphic on $D_\alpha$, then $|f|_{D_\alpha} < |f|_{\partial D_\alpha}$ by the maximum principle and $|f|_{\partial D_\alpha} < |f|_{T^2_i}$ because $\partial D_\alpha = M_1(\alpha) \cup M_2(\alpha)$ and the $M_j(\alpha)$ are fibered by analytic discs with boundaries in $T^{2}_a$. Thus $T^{2}_i$ is a "distinguished" boundary for $D_\alpha$ and, in particular, the polynomially convex hull of $T^{2}_i$ contains $D_\alpha$. The following theorem summarizes the above results. It says that as $T^2$ is smoothly deformed to $T^2_\alpha$, the hull $\hat{\mathbb{T}}^2 = \bar{U}^2$ is, in some sense, smoothly deformed to the set $D_\alpha \subset \hat{T}^2_\alpha$.

Theorem 3. There exists a ball $B$ about 0 in $(C^3(K))^2$ such that if $\alpha \in B$ and if $T^2_\alpha$ is the associated deformation of $T^2$, then the following holds. There exist $C^1$ imbeddings $P_\alpha: M_1 \to \mathbb{C}^2$, $P_\alpha(M_1) = M_1(\alpha)$ and $Q_\alpha: M_2 \to \mathbb{C}^2$, $Q_\alpha(M_2) = M_2(\alpha)$ with $T^2_\alpha = \partial M_j(\alpha)$, $j = 1, 2$. The manifolds $M_j(\alpha)$ depend smoothly on $\alpha$ and each is a disjoint union of analytic discs with boundaries in $T^{2}_j$. $M_1(\alpha) \cap M_2(\alpha) = T^2_i$. There exists a compact set $D_\alpha$ in $\mathbb{C}^2$ with $\partial D_\alpha = M_1(\alpha) \cup M_2(\alpha)$ such that $D_\alpha \subset \hat{T}^2_\alpha$. Moreover the set $D_\alpha$ is close to $\bar{U}^2$ in the following sense: if $\epsilon > 0$ is given, then $U^{1-\epsilon}_i \subset D_\alpha \subset U^{1+\epsilon}_i$ (where $U_i = (\lambda \in \mathbb{C}: |\lambda| < i)$ provided that $\alpha$ is sufficiently small.
Remark. Bedford [7] has shown that $D_\alpha$ is in fact the polynomially convex hull of $T^2_\alpha$.

Proof. We have verified all but the last assertion. This follows from the fact that $M_1(\alpha) \cup M_2(\alpha) \subseteq U^1 + \varepsilon \setminus U^{1-\varepsilon}$ for $\alpha$ sufficiently small, because then $P_\alpha$ and $Q_\alpha$ are uniformly close to identity maps.

The construction of $D_\alpha$ raises a number of questions about the hull of $T^2_\alpha$:

(a) Does every point of $D_\alpha$ (or $T^2_\alpha$) lie in an analytic disc with boundary in $T^2$?

(b) Is $T^2_\alpha$ the Shilov boundary of $A(D_\alpha)$ (or $H(D_\alpha)$ or $P(D_\alpha)$)? Is every point of $T^2_\alpha$ a peak point for these algebras?

(c) When is $T^2_\alpha$ the image of $T^2$ under a biholomorphism of $\overline{U}^2$ into $\mathbb{C}^2$? When is $D_\alpha$ biholomorphic to $\overline{U}^2$?

(d) Let $f$ be a continuous (or smooth) function on $T^2_\alpha$. Suppose that $f$ extends to be holomorphic on each of the analytic discs in $M_1(\alpha)$ and $M_2(\alpha)$. Does $f$ extend to be holomorphic on the interior of $D_\alpha$? For $f$ on $T^2$, the hypothesis amounts to the vanishing of the appropriate Fourier coefficients.

We shall conclude our discussion of perturbations of the torus by considering the algebra $R(T^2_\alpha)$. It follows from the Stone-Weierstrass Theorem that $z, z^2, 1/z$, and $1/z^2$ generate $C(T^2)$, because $\bar{z}_k = 1/z_k$ on $T^2$. The same is true for $C(T^2_\alpha)$.

**Proposition 3.** For $\alpha$ sufficiently small in $\mathcal{O}(K_\alpha)^2$, the functions $z_1, z_2, 1/z_1$ and $1/z_2$ generate $C(T^2_\alpha)$. In particular $R(T^2_\alpha) = C(T^2_\alpha)$.

Proof. Write $1/z_k = \bar{z}_k + R_k(z_1, z_2)$ for $z = (z_1, z_2) \in T^2_\alpha$. By a theorem of Hörmander and Wermer [5], it suffices to check that $R = (R_1, R_2)$ extends to a neighborhood of $T^2_\alpha$ where it satisfies a Lipschitz condition $\|R(z) - R(z')\| < k\|z - z'\|$ for some $k < 1$ where $\| \cdot \|$ is the Euclidean norm in $\mathbb{C}^2$.

Recall that $T^2_\alpha$ is the image of $T^2$ under the map $F(w) = w + \beta(w)$ where $z \mapsto z - \alpha(z)$ is the inverse of $F$. It follows that if $z = (z_1, z_2) \in T^2_\alpha$ corresponds to $w = (w_1, w_2) \in T^2$, then $\alpha(z) = \beta(w)$ and $|w_k| = 1$ for $k = 1, 2$. Hence, for $z \in T^2_\alpha$,

$$R_k(z) = \frac{1}{z_k} - \bar{z}_k = \frac{1}{w_k + \beta_k(w)} - \bar{w}_k - \frac{1}{w_k + \alpha_k(z)}$$

$$= \frac{1}{w_k + \beta_k(w)} - \frac{1}{w_k - \alpha_k(z)}$$

$$= \frac{\beta_k(w)}{(w_k + \beta_k(w))w_k} - \frac{\alpha_k(z)}{w_k}.$$

Thus

$$R_k = -\frac{\alpha_k}{z_k(z_k - \alpha_k)} - \overline{\alpha_k}.$$

Clearly the Lipschitz condition on $R$ holds on $K_\alpha$ if $\alpha$ is small.

**5. Deformations of a two-sphere.** We consider small deformations of the 2-sphere $S^2 = \{(z_1, z_2) \in \mathbb{C}^2: \text{Im } z_2 = 0 \text{ and } |z_1|^2 + (\text{Re } z_2)^2 = 1\}$, again restricting to $\mathbb{C}^2$ for simplicity. The polynomially convex hull of $S^2$ is easily described: $\tilde{S}^2 \setminus S^2$ is the
disjoint union of analytic discs $g_i(U)$ where $g_i(\lambda) = (\sqrt{1 - t^2} \lambda, t)$ for $-1 < t < 1$. In fact, the $g_i$’s are, except for a reparameterization, the only analytic discs in $C^2$ with boundary in $S^2$. We shall show that given a $g_i$, every sufficiently small deformation of $S^2$ in $C^2$ induces a deformation of $g_i$ to analytic discs with boundaries in the deformed 2-spheres.

For $s > 0$, let $K_s = \{z \in C^2: \text{distance } (z, S^2) < s\}$, a compact neighborhood of $S^2$ in $C^2$, where $s$ is fixed small enough so that there exists a smooth retraction $\rho$ of $K_s$ to $S^2$. Let $F$ be a $C^2$ diffeomorphism of $S^2$ into $C^2$ which is close to the identity. Write $F(z) = z + \beta(z)$ where $\beta \in (C^0(S^2))^2$ is close to 0. Extend $F$ to $K_s$ by $\tilde{F}(z) = z + \beta \circ \rho(z)$. Then, for $0 < r < s$, if $\beta$ is sufficiently small, just as in §2, $\tilde{F}$ is a diffeomorphism of the interior of $K_s$ onto a neighborhood of $K_s$ and the inverse $G$ of $\tilde{F}$ can be written $G(z) = z - \alpha(z)$ where $\alpha \in (C^0(K_s))^2$. Let $S^2_a = F(S^2)$ be the deformed 2-sphere; $S^2_a$ is contained in $K_s$ when $\alpha$ is sufficiently small. Then, when $z \in K_s$, $z \in S^2_a$ if and only if $G(z) = z - \alpha(z) \in S^2$; i.e., $|z_1 - \alpha_1(z)|^2 + (\Re(z_2 - \alpha_2(z)))^2 = 1$ and $\Im(z_2 - \alpha_2(z)) = 0$ where $\alpha = (\alpha_1, \alpha_2)$. Hence, if $\lambda \mapsto (f_1(\lambda), f_2(\lambda))$ is an analytic disc $f$ in $C^2$, then the boundary of $f$ lies in $S^2_a$ if and only if the boundary of $f$ lies in $K_s$ and

$$|f_1 - \alpha_1(f)|^2 + (\Re(f_2 - \alpha_2(f)))^2 = 1,$$

$$\Im(f_2 - \alpha_2(f)) = 0 \quad (4.1)$$

on $T$. Thus our problem is to solve the system (4.1) for $f = (f_1, f_2) \in \mathcal{B}_2^2$ given small $\alpha \in C^3(K_s)^2$ such that $f$ varies smoothly with $\alpha$ and with the initial condition $f = g_i$ when $\alpha = 0$. In fact, we shall solve (4.1) with the $t$ of the initial condition as an additional parameter. We shall need a refinement of Theorem 1.

**Theorem 4.** There exists an open set $\Omega$ in $\mathcal{H}_2 \oplus \mathcal{H}_2(\mathbb{R})$ such that $\Omega$ contains each pair $(0, \sigma)$ if $\sigma > 0$ and conversely if $(\varphi, \sigma) \in \Omega$, then $\sigma > 0$ and there exists a $C^1$ map $E: \Omega \to \mathcal{O}_2$ such that

(i) if $(\varphi, \sigma) \in \Omega$ and $E(\varphi, \sigma) = f$, then $|f - \varphi| = \sigma$ on $T$,

(ii) $E(0, c)(\lambda) \equiv c\lambda$ when $c$ is a positive constant function.

**Proof** Let $E_0: \Omega_0 \to \mathcal{O}_2$ be the $C^1$ map given in Theorem 1 for $g(\lambda) \equiv \lambda$; we have used a zero subscript to avoid confusion with the current notation. Then $\Omega_0$ is a neighborhood of 0 in $\mathcal{H}_2$, $E(0)(\lambda) \equiv \lambda$ and, if $\varphi \in \Omega_0$ and $f = E_0(\varphi)$, then $|f - \varphi| \equiv 1$ on $T$.

Let $W = \{\sigma \in \mathcal{H}_2(\mathbb{R}): \sigma > 0\}$ and define for $\sigma \in W$, $E_1(\sigma) = f \in \mathcal{O}_2$ by $f = e^{J(\log \sigma)}$ where $J(u) = u + i\bar{u}$ (with $\bar{u}$ the conjugate of $u$) is the map given in §1 and $|f| = \sigma$. Previous arguments show that $E_1: W \to \mathcal{O}_2$ is $C^1$; the same is true for $E_2: W \to \mathcal{O}_2$ given by $E_2(\sigma) = e^{-J(\log \sigma)}$.

Define $\Omega = \{(\varphi, \sigma) \in \mathcal{H}_2 \oplus \mathcal{H}_2(\mathbb{R}): \sigma > 0 \text{ and } \varphi \cdot E_2(\sigma) \in \Omega_0\}$; $\Omega$ is an open set in $\mathcal{H}_2 \oplus \mathcal{H}_2(\mathbb{R})$. Now define $E: \Omega \to \mathcal{O}_2$ by $E(\varphi, \sigma) = E_1(\sigma) \cdot E_0(E_2(\sigma) \cdot \varphi)$. Then $E$ is clearly a $C^1$ map with $E(0, c) = E_1(c)E_0(0) = c\lambda$. To check (i), let $f_1 = E_1(\sigma)$ and let $f_2 = E_0(\varphi/f_1)$, then $f = E(\varphi, \sigma) = f_1 \cdot f_2$ and $|f_2 - \varphi/f_1| = 1$ on $T$ implies $|f - \varphi| = |f_1| = \sigma$. 

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Remark. To have \((\varphi, \sigma) \in \Omega\) we required that \(\varphi \cdot E_2(\sigma) = \varphi \cdot h \in \Omega_0\) where \(h = E_2(\sigma)\) and \(|h(\lambda)| = 1/\sigma(\lambda)| \) for \(\lambda \in T\). Since \(\|h\| \to \infty\) as \(\inf \sigma \to 0\), we see that for fixed \(\varphi\), in order that \((\varphi, \sigma) \in \Omega\), the value of \(\inf \sigma\) cannot be too small.

Define \(\tilde{S}^2(t) = \{z \in \mathbb{C}^2: -t < \text{Re} z_2 < t\} \) for \(0 < t < 1\). Then \(\tilde{S}^2(t) \subseteq \mathbb{C}^2\) is a noncompact 3-dimensional manifold with boundary; \(\partial \tilde{S}^2(t)\) is an open subset of \(S^2\). We shall now show that if \(S^2\) is smoothly deformed to \(S_a^2\), then, if \(\alpha\) is small, \(\tilde{S}^2(t)\) is smoothly deformed to a 3-manifold whose boundary is contained in \(S_a^2\) and which consists of a disjoint union of analytic discs also with boundaries in \(S_a^2\).

**Theorem 5.** There exists an open neighborhood \(V\) of \((-1, 1) \times \{0\}\) in \((-1, 1) \times \mathbb{C}^2(K_r)^{2}\) and there exists a \(C^1\) map \(u: V \to \mathbb{C}^2\) such that for \((t, \alpha) \in V\), if \(u(t, \alpha) = f = (f_1, f_2)\), then \(f\) is an analytic disc in \(\mathbb{C}^2\), \(f = g_1\) when \(\alpha = 0\), and the boundary of \(f\) lies in \(S_a^2\); i.e., (4.1) holds on \(T\).

Moreover for fixed \(t, 0 < t < 1\), there is a neighborhood \(N = N(t)\) of \(0\) in \(\mathbb{C}^2(K_r)^{2}\) such that \([-t, t] \times N \subseteq V\) and, for \(\alpha \in N\), the map \(P_a: \tilde{S}^2(t) \to \mathbb{C}^2\) defined by

\[
P_a(z_1, s) = u(s, \alpha)
\]

is a \(C^1\) imbedding into \(\mathbb{C}^2\). The image set \(M(\alpha) = P_a(\tilde{S}^2(t))\) is a disjoint union of analytic discs with boundaries in \(S_a^2\) and \(M(\alpha) \subseteq \tilde{S}^2_a\). \(M(\alpha)\) is a \(C^1\) 3-dimensional manifold whose boundary \(\partial M(\alpha)\) is an open subset of \(S_a^2\). The \(M(\alpha)\) depend smoothly on \(\alpha\) and \(M(0) = \tilde{S}^2(t)\).

**Proof.** Define a bounded linear transformation \(H: \mathbb{K}_2(K_r) \to \mathbb{K}_2(K_r)\) by \(H(u) = -\bar{u}\), where \(\bar{u}\) is the conjugate function of \(u\) discussed in §1. For \(u \in \mathbb{K}_2(K_r)\),

\[
H(u) + iu = \bar{u}(u + i\bar{u}) = \bar{u}_2.
\]

For \((t, \alpha, f)\) in \((-1, 1) \times \mathbb{C}^2(K_r)^{2} \times \mathbb{C}^2\) such that \(f(T) \subset K_r\) and \(\alpha\) small we can define

\[
\Phi(t, \alpha, f) = \left(\alpha_1(f), \sqrt{1 - [t + H(\alpha_2(f)) - \text{Re}(\alpha_2(f))]}\right).
\]

an element of \(\mathbb{K}_2 \oplus \mathbb{K}_2(K_r)\). In order that \(\Phi\) be defined at \((t, \alpha, f)\), the expression in square brackets must be less than one in absolute value on \(T\). Then \(\Phi\) is a \(C^1\) map on its (open) domain which includes all points \((t, 0, g_1)\) for \(-1 < t < 1\) and \(\Phi(t, 0, g_1) = (0, \sqrt{1 - t^2}) \in \Omega \subseteq \mathbb{K}_2 \oplus \mathbb{K}_2(K_r)\). Hence there exists an open set \(W\) in \(\mathbb{R} \oplus \mathbb{C}^2(K_r)^{2} \oplus \mathbb{C}^2\) such that for \((t, \alpha, f) \in W\), we have \(-1 < t < 1\), \(W \subseteq\) domain of \(\Phi, \Phi(W) \subseteq \Omega\) and \((t, 0, g_1) \in W\) for all \(t, \alpha\) small.

Now we can define a map \(R: W \to \mathbb{C}^2\) by

\[
R(t, \alpha, f) = f - (E \circ \Phi(t, \alpha, f), t + H(\text{Im} \alpha_2(f)) + i \text{Im} \alpha_2(f)).
\]

\(R\) is a \(C^1\) map. We have \(R(t, 0, f) = f = (E(0, \sqrt{1 - t^2}), t) = f - g_1\). Hence \(R(t, 0, g_1) = 0\) and \(D_3 R(t, 0, g_1) = \) the identity map on \(\mathbb{C}^2\) for \(-1 < t < 1\).

By the implicit function theorem [4], for each \(t, -1 < t < 1\), there exist a neighborhood \(J_t\) of \(t\) in \((-1, +1)\) and a neighborhood \(N_t\) of \(0\) in \(\mathbb{C}^2(K_r)^{2}\) and a \(C^1\) function \(u_t: J_t \times N_t \to \mathbb{C}^2\) such that \(u_t(t, 0) = g_1\) and \(R(s, \alpha, u(s, \alpha)) \equiv 0\) for \(s \in J_t, \alpha \in N_t\). By uniqueness \(u_t = u_q\) whenever the domains overlap. It follows that there
exists an open set $V$ in $(-1, 1) \times C^3(K)$ such that $V$ contains each point $(t, 0)$ and there exists a $C^1$ map $u: V \to \partial_2^3$ such that $u(t, 0) = g_t$ and $R(s, \alpha, u(s, \alpha)) = 0$ for all $(s, \alpha) \in V$.

Let $f = (f_1, f_2) = u(t, \alpha)$. We shall show that $f$ is an analytic disc with boundary in $S_a^2$. From $R(t, \alpha, f) = 0$ we get

$$f_1 = E\left(\alpha_1(f), \sqrt{1 - \left[t + H(\text{Im} \ \alpha_2(f)) - \text{Re} \ \alpha_2(f)\right]^2}\right)$$

and

$$f_2 = t + H(\text{Im} \ \alpha_2(f)) + i \text{Im} \ \alpha_2(f).$$

Now (4.3) implies $\text{Re} \ f_2 = t + H(\text{Im} \ \alpha_2(f))$. Using this in (4.2) and then taking the imaginary parts in (4.3) yields

$$|f_1 - \alpha_1(f)| = \sqrt{1 - \left[\text{Re}(f_2 - \alpha_2(f))^2\right]},$$

$$\text{Im} \ f_2 = \text{Im} \ \alpha_2(f).$$

The system (4.4) is clearly equivalent to the system (4.1). Hence the boundary of $f$ lies in $S_a^2$.

For a fixed $t$, $0 < t < 1$, a compactness argument shows that there is a neighborhood $N = N(t)$ of 0 in $C^3(K)$ such that $[-t, t] \times N \subseteq V$. Now define $P_a$ for $\alpha \in N$ as above. Using the fact that $u$ is $C^1$ we can argue as in §5 to see that $P_a$ is a $C^1$ imbedding for $\alpha$ sufficiently small and that $M(\alpha)$ depends smoothly on $\alpha$. Since $\partial M(\alpha) = P_a(S_2 \cap \hat{S}(t))$ and since $P_a | S^2$ is an open map into $S_a^2$ it follows that $\partial M(\alpha)$ is open in $S_a^2$. $P_a$ is clearly holomorphic in $z_1$ for fixed $s$ and so $M(\alpha)$ is a union of discs with boundaries in $S_a^2$. This implies that $M(\alpha) \subseteq S_a^2$.

**Remark.** In general the open set $V$ of this theorem is not a product set in $(-1, 1) \times C^3(K)$; in fact, we would expect that as $t \to \pm 1$, the set of $\alpha$ for which $(t, \alpha) \in V$ will shrink to 0. This is because, by the remark after Theorem 4, for a fixed $\varphi$ corresponding to a fixed $\alpha$, the quantity $\sqrt{1 - t^2}$ cannot be too small if the operator $E$ is to be applied; i.e., for fixed $\alpha$, $|t|$ must be bounded away from 1 if we want to deform $g_t$ to $S_a^2$. Thus our method is in some sense complementary to that of Bishop [1] who produces analytic discs near the "exceptional" points of $S_a^2$; our construction yields discs away from the exceptional points. Bedford and Gaveau [8] have shown by other methods that the Bishop discs can be continued from the exceptional points to fill up the full hull. A uniqueness result for discs plays an important role in their approach.

**REFERENCES**


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT CHICAGO CIRCLE, CHICAGO, ILLINOIS 60680