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POWER SERIES METHODS OF SUMMABILITY: POSITIVITY AND GAP PERFECTNESS

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ABSTRACT. A class of power series methods of summability is defined. By means of a positivity argument (Bohman-Korovkin) it is shown that each method of the class is gap perfect. This fact facilitates the proof of Tauberian gap theorems. Several examples are given.

1. Introduction. We consider summability methods Q belonging to a class PTR (Power series, Totally monotone, Regular–see the definition in §2). The class, e.g., contains (generalized) Abel summability and logarithmic summability (see §8). The summability domain is considered as an FK space (§3). Each Q has an important positivity property (§4). Using the Bohman-Korovkin pattern we deduce sectional summability Q in the summability domain (§5). This implies gap perfectness (§6), which leads to a rather general Tauberian gap theorem (§7). Our examples (§8) include Hardy-Littlewood's classical high indices theorem for Abel's method and two results obtained rather recently by Krishnan.

2. The class PTR. We denote sequences of complex numbers by $\mathbf{x} = \{x_0, x_1, \dots\}$. Particularly we are dealing with pairs

$$(\mathbf{a}, \mathbf{s}), \quad \text{where } s_k = a_0 + \cdots + a_k \qquad (k = 0, 1, \dots). \tag{2.1}$$

Such a pair is considered as a series with terms a_m $(m = 0, 1, \dots)$ and partial sums s_k; we shortly write $\sum a_m$.

A summability method Q is a way to assign a value to each series of a certain set. This is mostly accomplished by a transformation, here of the following type:

$$q(\mathbf{a}, x) = \sum_{k=0}^{\infty} b(x, k) a_k \qquad (0 < x < 1),$$

combined with a limit process, here:

$$Q\text{-}\sum a_m = \lim_{x \to 1-} q(\mathbf{a}, x).$$

More exactly, a series is summable Q to the latter value if $q(\mathbf{a}, x)$ exists for $0 < x < 1$ and has a limit as x approaches $1-$.

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Now we define the special properties of the class (type)

$$\text{PTR: Power series, Totally monotone, Regular.} \tag{2.2}$$

The transformation kernel $b(x, k)$ is defined by means of a power series:

$$b(x, k) = \frac{1}{q(x)} \sum_{m=k}^{\infty} q_m x^m \quad \text{where } q(x) = \sum_{m=0}^{\infty} q_m x^m.$$

We assume that the sequence $\{q_m\}$ is totally monotone or, equivalently, admits a representation

$$q_m = \int_0^1 t^m \, d\alpha(t) \quad (\alpha \text{ increasing and bounded}) \tag{2.3}$$

[2, Theorems 204, 207]. Finally we demand $\Sigma\, q_m = \infty$, a condition which under the given circumstances is necessary and sufficient for regularity and implies $q_0 \geqslant q_1 \geqslant \cdots > 0$. Using (2.1) and formal summation by parts, we are led to the transformation

$$q^*(\mathbf{s}, x) = \sum_{k=0}^{\infty} a(x, k) s_k \qquad (0 \leqslant x < 1),$$

where

$$a(x, k) = (1/q(x)) q_k x^k \qquad (0 \leqslant x < 1;\ k = 0, 1, \ldots),$$

and hence to a sequence transform and limitability method Q^*. As in many cases, Q and Q^* are equivalent methods:

LEMMA 1. *Given a method Q of type PTR and a series* $(\mathbf{a}, \mathbf{s})$. *If* $q(\mathbf{a}, x)$ *exists for* $0 \leqslant x < 1$ *then* $q^*(\mathbf{s}, x)$ *also exists for* $0 \leqslant x < 1$, *and vice versa. In the case of existence we have*

$$q^*(\mathbf{s}, x) = q(\mathbf{a}, x) \qquad (0 \leqslant x < 1), \tag{2.4}$$

i.e.

$$\sum_{k=0}^{\infty} s_k q_k x^k = \sum_{k=0}^{\infty} a_k \sum_{m=k}^{\infty} q_m x^m \qquad (0 \leqslant x < 1),$$

as well as

$$\sum_{k=0}^{\infty} \tilde{s}_k q_k x^k = \sum_{k=0}^{\infty} |a_k| \sum_{m=k}^{\infty} q_m x^m \qquad (0 \leqslant x < 1) \tag{2.5}$$

$(\tilde{s}_k = |a_0| + \cdots + |a_k|)$, *existence of both sides of* (2.5) *being guaranteed.*

The easy proof is omitted. For completeness we state

LEMMA 2. *Each method Q of type PTR is regular.*

3. *FK* spaces. We consider a fixed method Q of type PTR and its series convergence domain

$$\mathbf{Q}\text{: the set of all } \mathbf{a} \text{ for which } Q\text{-}\Sigma\, a_m \text{ exists.} \tag{3.1}$$

LEMMA 3. *$\mathbf{Q}$ is an FK space with the seminorms*

$$p(\mathbf{a}) = \sup_{0 \leq x < 1} |q(\mathbf{a}, x)|,$$

$$p_j(\mathbf{a}) = \sum_{m=0}^{\infty} b(1 - 1/j, m)|a_m|, \qquad (j = 1, 2, \dots).$$

The proof follows standard lines (cf. [**17**, No. 22], Wlodarski [**16**, p. 173]) using the absolute convergence exhibited in connection with (2.5).

The space $\mathbf{Q}$ contains every

$$\text{finite } \mathbf{a} \quad (\text{with } a_m = 0 \text{ for } m > m_0(\mathbf{a})). \tag{3.2}$$

(We could also work with the $\mathbf{a}$ representing convergent series.) An important question is whether the finite $\mathbf{a}$ are dense in the *FK* space $\mathbf{Q}$ (perfectness). Here we have an even stronger property: Given any set of indices $0 \leq k_0 < k_1 < \cdots$, the finite $\mathbf{a}$ satisfying $a_k = 0$ for $k \neq k_0, k_1, \dots$ are dense in the subspace of $\mathbf{Q}$ defined by the same condition (gap perfectness, cf. [**11**], [**12**]). So we announce

THEOREM 4. *A summability method Q of type PTR is gap perfect.*

The importance of this property will be explained in §6. Here we outline the main steps of the proof. A given $\mathbf{a}$ in $\mathbf{Q}$ will be approximated in two steps. First we define (for $0 \leq y < 1$ and $m = 0, 1, \dots$)

$$\mathbf{a}^{(y)} = \{b(y, 0)a_0, b(y, 1)a_1, \dots\}, \tag{3.3}$$

$$\mathbf{a}^{(y,m)} = \{b(y, 0)a_0, \dots, b(y, m)a_m, 0, 0, \dots\}. \tag{3.4}$$

Every $\mathbf{a}^{(y)}$ represents an (absolutely) convergent series. This by *FK* principles or by an easy computation yields

LEMMA 5. *For given $\mathbf{a} \in \mathbf{Q}$ and $0 \leq y < 1$ we have $\mathbf{a}^{(y,m)} \to \mathbf{a}^{(y)}$ $(m \to \infty)$ in the sense of the FK topology.*

More difficult and depending on the announced positivity property is

LEMMA 6. *For given $\mathbf{a} \in \mathbf{Q}$ we have $\mathbf{a}^{(y)} \to \mathbf{a}$ $(y \to 1-)$ in the sense of the FK topology.*

Both lemmas together show that any $\mathbf{a}$ in $\mathbf{Q}$ can be approximated (arbitrarily closely) by finite elements, even under the side condition described above (gap perfectness, see also §6). Our next steps are to state and verify the positivity property and to use it for the proof of Lemma 6.

4. Positivity. Again we have under consideration a fixed method $\mathbf{Q}$ of type PTR. Let T_y be the linear operator which carries over each $\mathbf{a} \in \mathbf{Q}$ into $\mathbf{a}^{(y)} \in \mathbf{Q}$ (cf. (3.3)):

$$T_y\mathbf{a} = \mathbf{a}^{(y)} \qquad (\mathbf{a} \in \mathbf{Q}, 0 \leq y < 1). \tag{4.1}$$

The main key to our considerations is

THEOREM 7. *Each operator T_y $(0 \leq y < 1)$ is positive with respect to the order relation given by $q(\mathbf{a}, \cdot)$.*

In more detail this means: Let $\mathbf{a} \in \mathbf{Q}$ and a value y $(0 < y < 1)$ be given; then

$$q(\mathbf{a}, x) > 0 \text{ (for } 0 < x < 1) \text{ implies } q(\mathbf{a}^{(y)}, x) > 0 \text{ (for } 0 < x < 1).$$

Theorem 7 is an immediate consequence of

LEMMA 8. *Let* $\mathbf{a} \in \mathbf{Q}$. *Then, for* $0 < x < 1$ *and* $0 < y < 1$,

$$q(\mathbf{a}^{(y)}, x) = \int_0^1 K(x, y, t) q(\mathbf{a}, xyt)\, d\alpha(t)$$

with

$$K(x, y, t) = \frac{q(xyt)}{q(x)q(y)}\left(\frac{1}{1 - xt} + \frac{1}{1 - yt} - 1\right) \qquad (0 < t < 1).$$

PROOF OF LEMMA 8. We have to verify the equation

$$\sum_{k=0}^{\infty} a_k\left(\sum_{m=k}^{\infty} q_m x^m\right)\left(\sum_{n=k}^{\infty} q_n y^n\right)$$
$$= \int_0^1\left(\sum_{k=0}^{\infty} s_k q_k x^k y^k t^k\right)\left(\frac{1}{1 - xt} + \frac{1}{1 - yt} - 1\right) d\alpha(t)$$

(cf. (2.4)). Here the left-hand side, which is convergent even if a_k is replaced by $|a_k|$ (cf. (2.5)), can be written in the form

$$\sum_{k=0}^{\infty} a_k S_k^{(1)}(x, y) + \sum_{k=0}^{\infty} a_k S_k^{(2)}(x, y) - \sum_{k=0}^{\infty} a_k S_k^{(3)}(x, y)$$
$$\equiv U_1(x, y) + U_2(x, y) - U_3(x, y),$$

where

$$S_k^{(1)}(x, y) = \sum_{m=k}^{\infty}\sum_{n=k}^{m} q_m q_n x^m y^n = \sum_{n=k}^{\infty}\sum_{m=n}^{\infty} q_m q_n x^m y^n,$$
$$S_k^{(2)}(x, y) = \sum_{m=k}^{\infty}\sum_{n=m}^{\infty} q_m q_n x^m y^n = S_k^{(1)}(y, x),$$
$$S_k^{(3)}(x, y) = \sum_{m=k}^{\infty} q_m q_m x^m y^m.$$

Furthermore (cf. (2.3))

$$U_1(x, y) = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} a_k\left(\sum_{m=n}^{\infty} q_m q_n x^m y^n\right)$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k\left(\sum_{m=n}^{\infty} q_m q_n x^m y^n\right)$$
$$= \sum_{n=0}^{\infty} s_n q_n y^n \sum_{m=n}^{\infty} x^m \int_0^1 t^m\, d\alpha(t)$$
$$= \sum_{n=0}^{\infty} s_n q_n x^n y^n \int_0^1 t^n \frac{d\alpha(t)}{1 - xt}.$$

The latter series is convergent even if we replace s_n by $|a_0| + \cdots + |a_n|$, whence it follows that it is convergent if we replace s_n by $|s_n|$. Since it is therefore allowed to

interchange the order of summation and integration we get

$$U_1(x, y) = \int_0^1 \left(\sum_{n=0}^{\infty} s_n q_n x^n y^n t^n \right) \frac{d\alpha(t)}{1 - xt}.$$

Together with $U_2(x, y) = U_1(y, x)$ and

$$U_3(x, y) = \sum_{k=0}^{\infty} a_k \sum_{m=k}^{\infty} q_m q_m x^m y^m = \sum_{m=0}^{\infty} s_m q_m x^m y^m \int_0^1 t^m \, d\alpha(t)$$
$$= \int_0^1 \left(\sum_{n=0}^{\infty} s_n q_n x^n y^n t^n \right) d\alpha(t),$$

the proof of Lemma 8 is complete.

5. Sectional summability. We have seen that the operators T_y are positive, and hence we can suspect convergence (Bohman-Korovkin). Indeed we shall show

$$T_y \mathbf{a} \to \mathbf{a} \qquad (y \to 1 -) \tag{5.1}$$

for each $\mathbf{a} \in \mathbf{Q}$ in the sense of the *FK* topology. Going back to the definition of the operators T_y and using the notation $\mathbf{e}_0 = \{1, 0, 0, \ldots\}$, $\mathbf{e}_1 = \{0, 1, 0, 0, \ldots\}, \ldots$ we can express this convergence by

$$\mathbf{a}^{(y)} = \sum_{k=0}^{\infty} b(y, k) a_k \mathbf{e}_k \to \mathbf{a} \qquad (y \to 1 -),$$

the series being convergent for $0 \leq y < 1$. Using Q as a summability method for series in the *FK* space $\mathbf{Q}$ this means

$$Q\text{-}\sum a_m \mathbf{e}_m = \mathbf{a} \tag{5.2}$$

(for each $\mathbf{a} \in \mathbf{Q}$, in the sense of the *FK* topology; cf. [**17**, pp. 45, 111]). We express this fact in the following way:

THEOREM 9. *The FK space* $\mathbf{Q}$, *where* Q *is of type PTR, possesses sectional summability* Q.

The word "sectional" refers to the fact that the partial sums of the series in question are the sections of $\mathbf{a}$:

$$a_0 \mathbf{e}_0 + \cdots + a_k \mathbf{e}_k = \{a_0, \ldots, a_k, 0, 0, \ldots\}.$$

The statement of Theorem 9 means the same as the conjuction of Lemmas 5 and 6.

It remains to prove Lemma 6. First we verify convergence with respect to each seminorm p_j:

$$p_j(\mathbf{a}^{(y)} - \mathbf{a}) \to 0 \qquad (y \to 1 - ; j = 1, 2, \ldots ; \mathbf{a} \in \mathbf{Q}).$$

The left-hand side is equal to

$$\sum_{m=0}^{\infty} b(1 - 1/j, m) |a_m| \, (1 - b(y, m)).$$

This means that the original absolutely convergent series is modified by the factors $(1 - b(y, m)) \geq 0$ (observe $q_m > 0$). These factors tend to zero as $y \to 1 -$ and are

uniformly bounded (upper bound 1). Hence they transform the absolutely convergent series into a null sequence: Use direct estimation or the corresponding theorem of Toeplitz type (see, e.g., [**13**, No. 28]).

Secondly we verify convergence with respect to the seminorm p. We have to show

$$q(\mathbf{a}^{(y)}, x) \to q(\mathbf{a}, x) \qquad (y \to 1-\,;\, 0 \leqslant x < 1)$$

uniformly in x. Uniform convergence in any interval $0 \leqslant x \leqslant 1 - \delta$ is already established by the preceding argument, and "uniform convergence near 1" will now be established by a Bohman-Korovkin argument (cf. Beekmann and Zeller [**1**], Kershaw [**6**]). To begin with we note ($0 \leqslant x < 1$, $0 \leqslant y < 1$)

$$q(\mathbf{e}_0, x) = 1, \quad q(\mathbf{e}_1, x) = 1 - q_0/q(x),$$
$$q(\mathbf{e}_0^{(y)}, x) = 1, \quad q(\mathbf{e}_1^{(y)}, x) = (1 - q_0/q(x))(1 - q_0/q(y)).$$

Next we consider an element $\mathbf{a} \in \mathbf{Q}$; we may assume $\mathbf{a}$ real and Q- $\Sigma\, a_m = 0$. We majorize $\mathbf{a}$ and $-\mathbf{a}$ by a suitable element

$$\mathbf{a}^*(\varepsilon) = \varepsilon \mathbf{e}_0 + \gamma(\varepsilon)(\mathbf{e}_0 - \mathbf{e}_1)$$

in the sense of the order relation used in Theorem 7. Since the operators T_y are positive, the elements $T_y\mathbf{a}$ and $-T_y\mathbf{a}$ are majorized by $T_y\mathbf{a}^*(\varepsilon)$. The formulas above show that there exist δ and y^* such that

$$q(T_y\mathbf{a}^*(\varepsilon), x) < 2\varepsilon \quad \text{for } \delta \leqslant x < 1 \text{ and } y > y^*.$$

Hence the same inequality is true if we replace $\mathbf{a}^*(\varepsilon)$ by $\mathbf{a}$ or $-\mathbf{a}$. This completes the proof. We remark that the Bohman-Korovkin principle can also be applied to cover the full interval [0, 1) and that we could employ other tools for the convergence proof.

6. Gap perfectness. As in §3 we consider a fixed

$$\text{set } \mathbf{k} \text{ of indices } 0 \leqslant k_0 < k_1 < \cdots \tag{6.1}$$

and the corresponding gap condition

$$G(\mathbf{k})\colon a_m = 0 \text{ for } m \neq k_0, k_1, \ldots. \tag{6.2}$$

We have defined and proved (see Theorem 4) gap perfectness of a method Q of type PTR and describe it now in the following way: An $\mathbf{a} \in \mathbf{Q}$ satisfying a certain gap condition $G(\mathbf{k})$ can be approximated by finite elements $\tilde{\mathbf{a}}$ satisfying the same condition $G(\mathbf{k})$. This property is important for the proof of Tauberian gap theorems (cf. [**11**], [**12**]). The essential point is contained in

LEMMA 10. *If a method Q of type PTR sums a divergent series $\Sigma\, a_m$ satisfying a certain gap condition $G(\mathbf{k})$, then it also sums such a series which, in addition, has bounded partial sums.*

The construction of the latter series uses the approximations mentioned above and a gliding hump technique (cf. also Mazur and Sternbach [**10**], Wilansky and Zeller [**15**], Kolodziej [**7**]).

Because of Lemma 10, in order to prove a Tauberian gap theorem for a method Q of type PTR it is sufficient to verify it for series with bounded partial sums. In many cases this fact enables us to pass from Q to a simpler summability method, for which it is easier to prove gap theorems.

7. A Tauberian gap theorem. Frequently weighted means will play the part of the simpler summability method which was mentioned just before. Particularly, together with a Q of type PTR we consider the summability method $W(Q)$ based on the sequence to sequence transform

$$t_m = \frac{1}{\bar{q}_m} \sum_{k=0}^{m} q_k s_k \quad \text{with } \bar{q}_m = \bar{q}(m) = q_0 + \cdots + q_m.$$

The following Tauberian gap theorem (which is proved easily; cf. [**11**], [**9**]) holds:

LEMMA 11. *$G(\mathbf{k})$ is a Tauberian condition for the method $W(Q)$ with Q of type PTR if there exists a constant $\lambda > 1$ such that (for $l = 0\ 1, \ldots$)*

$$\bar{q}(k_{l+1}) \geqslant \lambda \bar{q}(k_l). \tag{7.1}$$

As a consequence of Lemmas 10 and 11 we state

THEOREM 12. *Let the method Q of type PTR have the property*

$$Q \to W(Q) \text{ for bounded sequences } \mathbf{s}; \tag{7.2}$$

then $G(\mathbf{k})$ for a given $\mathbf{k}$ is a Tauberian condition for Q if there exists a constant $\lambda > 1$ such that (7.1) *is fulfilled.*

Here (7.2) means that $Q\text{-}\sum a_m = s$ implies $W(Q)\text{-}\sum a_m = s$ for series with bounded partial sums. There are well-known cases in which the transition (7.2) is possible (cf. §8). It is always possible if Q belongs to a certain subclass of the class PTR (see Jakimovski and Tietz [**5**]):

LEMMA 13. *Let Q be of type PTR. Then* (7.2) *holds if $q_m = R(m)$ for $m = 0, 1, \ldots,$ where $R(x) > 0$ is a function which is continuous for $x \geqslant 0$, and, furthermore, is regular in the sense that there exists a constant $\rho \geqslant -1$ such that $R(\lambda x)/R(x) \to \lambda^\rho$ for each $\lambda > 0$ as $x \to \infty$.*

8. Examples.

EXAMPLE 1. Let $\alpha(t) = 0$ for $0 \leqslant t < 1$, $\alpha(1) = 1$, $q_m = 1$ ($m = 0, 1, \ldots$). Then Q is Abel's method A_0, $W(Q)$ is Cesàro's method C_1. Since in this case, as is well known, (7.2) is fulfilled, Theorem 12 informs us that $G(\mathbf{k})$ is a Tauberian condition for A_0 if the condition

$$\text{there exists a constant } \lambda > 1 \text{ such that } k_{l+1} \geqslant \lambda k_l \tag{8.1}$$

($l = 0, 1, \ldots$) is fulfilled. In this way our results imply Hardy-Littlewood's high indices theorem for Abel's method (cf. [**17**, p. 78]; for a corresponding proof of the more general high indices theorem which works with Dirichlet series instead of power series see [**11**, p. 218]).

EXAMPLE 2. For any fixed real β $(-1 < \beta < 0)$ let

$$\alpha(t) = \frac{1}{\Gamma(-\beta)\Gamma(1+\beta)} \int_0^t u^\beta (1-u)^{-\beta-1}\, du \qquad (0 \leqslant t \leqslant 1),$$

$$q_m = \binom{m+\beta}{m}, \quad \bar{q}_m = \binom{m+\beta+1}{m} \qquad (m = 0, 1, \ldots).$$

Then Q is the generalized Abel method A_β (cf. [**4**, p. 18], [**17**, p. 186]). Together with the fact that (7.2) holds in the present case (Rajagopal [**14**]), Theorem 12 yields a result of Krishnan [**8**] who proved by means of Wiener-Pitt's theory: $G(\mathbf{k})$ is a Tauberian condition for A_β if (as in the case of Abel's method) condition (8.1) is fulfilled.

EXAMPLE 3. For any fixed real γ $(-1 \leqslant \gamma < 0)$ let

$$\alpha(t) = \frac{1}{\Gamma(-\gamma)} \int_0^t \left(\log \frac{1}{u}\right)^{-\gamma-1} du \qquad (0 \leqslant t \leqslant 1),$$

$$q_m = (m+1)^\gamma \qquad (m = 0, 1, \ldots), \qquad \bar{q}_m \cong \frac{m^{1+\gamma}}{1+\gamma} \quad \text{for } -1 < \gamma < 0,$$

$$\bar{q}_m \cong \log m \quad \text{for } \gamma = -1 \qquad (m \to \infty).$$

(a) *Case* $-1 < \gamma < 0$. Since (7.2) holds (see Lemma 13), $G(\mathbf{k})$ is a Tauberian condition for our Q under consideration if (as in the previous examples) condition (8.1) is fulfilled.

(b) *Case* $\gamma = -1$. Then Q is the logarithmic method L given by

$$L\text{-}\sum a_m = \lim_{x \to 1-} \frac{-x}{\log(1-x)} \sum_0^\infty \frac{1}{k+1} s_k x^k$$

and $W(Q)$ is the logarithmic method l given by

$$l\text{-}\sum_0^\infty a_m = \lim_{m \to \infty} \frac{1}{\log m} \sum_{k=0}^m \frac{s_k}{k+1}.$$

Summability L implies summability l for series with bounded partial sums (Ishiguro [**3**]; cf. [**4**, p. 21]). Therefore $G(\mathbf{k})$ is a Tauberian condition for the method L if there exists a constant $\lambda > 1$ such that (for $l = 0, 1, \ldots$)

$$\log k_{l+1} \geqslant \lambda \log k_l. \tag{8.2}$$

This result was obtained recently (with a proof ad hoc) by Krishnan [**9**].

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