ISOMORPHISM THEOREMS FOR OCTONION PLANES
OVER LOCAL RINGS

BY
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Abstract. It is proved that there is a collineation between two octonion planes over local rings if and only if the underlying octonion algebras are isomorphic as rings. It is shown that every isomorphism between the little or middle projective groups of two octonion planes over local rings is induced by conjugation with a collineation or a correlation of the planes when the local rings contain $\frac{1}{2}$.

Octonion planes over local rings were defined and studied in [3]. In this paper we prove two main theorems about such planes. In §1 we show that there is a collineation between two such planes if and only if there is a semilinear algebra isomorphism of the underlying octonion algebras. This was proved for octonion algebras over fields by Faulkner [6, p. 20].

Second, we prove that every isomorphism between the little or middle projective groups of two octonion planes over local rings is induced by conjugation with a collineation or a correlation of the planes when 2 is a unit in the local rings. This was proved for octonion division algebras over fields of characteristic $\neq 2$ by Suh [10, p. 338] and Veldkamp [11, p. 287]. The corresponding result for octonion division algebras over fields of characteristic 2 was proved by Faulkner [6, p. 57]. Thus our theorem extends known results to split octonion algebras over fields of characteristic $\neq 2$ and to arbitrary octonion algebras over local rings containing $\frac{1}{2}$. In the case of fields of characteristic $\neq 2$, we further generalize known results by extending our theorem to include isomorphisms between subgroups of the full collineation groups of two planes when each subgroup contains the little projective group. This extension follows directly from the classification of the subgroups of the full collineation group normalized by the little projective group [3, Corollary 7.2].

The results of the last paragraph are proved in §§2–6. In §2 we study two kinds of involutions in the little projective group $PS$: those that fix all the points on a line and a point not connected to the line, and those that fix a four-point. We prove in §3 that every involution in the middle projective group $PG$ is of one of these two kinds. We show in §4 that involutions of the first kind are distinguished in $PS$ by group-theoretic properties, so they are preserved by isomorphisms of little projective groups. In §5 we determine the geometric conditions for two involutions of the...
first kind to commute and for their product to be an involution of the first kind. We apply this criterion in §6 to prove the results of the preceding paragraph.

All notation is as in [3]. $\mathcal{O}$ is an octonion algebra over a local ring $(R, m)$ with norm $n(x)$, trace $t(x)$, and involution $x \to x^\sharp$. $J = H(\mathcal{O}, \gamma)$ is the quaternion Jordan algebra of Hermitian 3-by-3 matrices over $\mathcal{O}$. If $x, y \in J$, $N(x)$ is the generic norm of $x$, $x^\sharp$ is the adjoint of $x$, and $x \times y = (x + y)^\sharp - x^\sharp - y^\sharp$. $\Gamma = \Gamma(J)$ is the group of norm semisimilarities of $J$, $G = G(J)$ is the group of norm similarities, and $S = S(J)$ is the group of norm preserving transformations [3, Definition 1.2]. The octonion plane $PJ$ consists of points $x_\cdot = Rx$ and lines $x^\star = Rx$ for $x \in J - mJ$, $x^\sharp = 0$, with relations:

- $x_\cdot \parallel y^\star$, $x_\cdot$ “on” $y^\star$, if $T(x, y) = 0$,
- $x_\cdot \sim y^\star$, $x_\cdot$ “connected” to $y^\star$, if $T(x, y) \in m$,
- $x_\cdot \sim y^\star$, $x_\cdot$ “connected” to $y^\star$, if $x \times y \in mJ$,
- $x^\star \sim y^\star$, $x^\star$ “connected” to $y^\star$, if $x \times y \in mJ$.

If $x_\cdot \sim y^\star$, $(x \times y)^\star$ is the unique line on $x_\cdot$ and $y^\star$; if $x^\star \sim y^\star$, $(x \times y)_\cdot$ is the unique point on $x^\star$ and $y^\star$ [3, Lemma 2.2]. A three-point is an ordered triple of points $(a_1, a_2, a_3)$ such that $a_1 \sim (a_2 \times a_3)^\star$, a condition symmetrical in the $a_i$. A four-point is an ordered quadruple of points such that any three form a three-point. A collineation of two octonion planes consists of a bijection of their points and a bijection of their lines preserving the relations “on” and “connected to”. $\phi \in \Gamma(J)$ induces a collineation $P\phi$ of $PJ$ by $P\phi(x_\cdot) = (\phi x)_\cdot$ and $P\phi(y^\star) = (\phi^\star y)^\star$. If $H \subset \Gamma$, let $PH = \{ P\phi | \phi \in H \}$. $P\Gamma$ is the full collineation group of $PJ$, and $R - m$ is the kernel of the homomorphism $\phi \to P\phi$ taking $\Gamma$ onto $P\Gamma$ [3, Lemma 3.3 and Theorem 8.4]. Let $\mathcal{O}'$ and $J'$ be defined analogously over a local ring $(R', m')$.

1. Isomorphism of octonion planes. In this section we prove that there is a collineation between two octonion planes if and only if their underlying octonion algebras are isomorphic as rings. Since every collineation between two octonion planes is induced by a norm semisimilarity [3, Theorem 8.4], it suffices to prove that there is a norm semisimilarity of $J$ onto $J'$ if and only if there is a ring isomorphism of $\mathcal{O}$ onto $\mathcal{O}'$.

Define a norm semisimilarity $(\phi, \sigma)$ of $\mathcal{O}$ onto $\mathcal{O}'$ to be an additive group isomorphism $\phi: \mathcal{O} \to \mathcal{O}'$, a ring isomorphism $\sigma: R \to R'$, and a unit $\rho' \in R'$ such that $\phi(\alpha x) = \alpha^\sigma \phi x$ and $n'(\phi x) = \rho' n(x)^\rho$ for $\alpha \in R$ and $x \in \mathcal{O}$. If $\phi$ takes $1 \in \mathcal{O}$ to $1' \in \mathcal{O}'$, setting $x = 1$ in the last sentence implies that $n'(\phi x) = n(x)^\rho$ for $x \in \mathcal{O}$.

**Lemma 1.1.** There is a ring isomorphism of $\mathcal{O}$ onto $\mathcal{O}'$ if and only if there is a norm semisimilarity of $\mathcal{O}$ onto $\mathcal{O}'$ taking $1 \in \mathcal{O}$ to $1' \in \mathcal{O}'$. In fact, any ring isomorphism $\phi$ of $\mathcal{O}$ onto $\mathcal{O}'$ induces a norm semisimilarity $(\phi, \sigma)$ taking $1$ to $1'$ such that $(\phi x)^d = \phi(x^d)$ and $t'(\phi x) = t(x)^\rho$ for $x \in \mathcal{O}$.

**Proof.** Let $\phi$ be a ring isomorphism of $\mathcal{O}$ onto $\mathcal{O}'$. $\phi(R1) = R'1'$ [3, Lemma 1.11]. Since $R' \cong R'1'$, we can define a ring isomorphism $\sigma$ of $R$ onto $R'$ by $r^\sigma1' = \phi(r1)$, so that $\phi$ is $\sigma$-semilinear. Let $\mathcal{O}$ have a free basis $1, x_1, \ldots, x_7$ over $R$. 

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Since \( \phi x_i \) and \( 1' \) are linearly independent, \( t'(\phi x_i) \) is determined by the equation

\[
(\phi x_i)^2 - t'(\phi x_i)\phi x_i + n'(\phi x_i)1' = 0.
\]

Applying \( \phi \) to the equation \( x_i^2 - t(x_i)x_i + n(x_i)1 = 0 \) shows that \( t'(\phi x_i) = t(x_i)^n \).

Since \( t'(\phi x_i) = 2' = t(1)^n \), \( t'(\phi x_i) = t(x)^n \) for \( x \in \mathcal{O} \). Since \( x^d = t(x)1 - x \), \( \phi(x^d) = (\phi x)^d \).

Applying \( \phi \) to the equation \( xx^d = n(x)1 \) yields \( n'(\phi x) = n(x)^n \).

Conversely, let \( (\phi, \sigma) \) be a norm semisimilarity of \( \mathcal{O} \) onto \( \mathcal{O}' \) such that \( \phi 1 = 1' \).

Since \( n(\phi x) = n(x)^n \), the equations \( t(x) = n(x, 1) \) and \( x^d = t(x)1 - x \) imply that \( t(\phi x) = t(x)^n \) and \( \phi(x^d) = (\phi x)^d \) for \( x \in \mathcal{O} \). \( n(x) \) is a nondegenerate quadratic form on \( \mathcal{O} \) [3, Definition 1.6]. Take \( a \in \mathcal{O} \) such that \( C_1 = R + Ra \) is a free module of rank 2 on which \( n \) is nondegenerate. Since \( a^2 - t(a)a + n(a)1 = 0 \), \( \phi \) is a ring isomorphism of \( C_1 \) into \( \mathcal{O}' \). By induction, assume we have found a subalgebra \( C_1 \) of \( \mathcal{O} \) such that \( C_1 \) is a free \( R \)-module of rank 2, the restriction of \( n \) to \( C_1 \) is nondegenerate, and there is a \( \sigma \)-semilinear algebra isomorphism \( \psi_1 \) of \( C_1 \) into \( \mathcal{O}' \) such that \( \psi_1(x^d) = (\psi_1 x)^d \). Applying \( \psi_1 \) to \( xx^d = n(x)1 \) shows that \( n'(\psi_1 x) = n(x)^n \).

Then \( \phi^{-1}\psi_1 \in \text{Hom}_R(C_1, \mathcal{O}) \) satisfies \( n(\phi^{-1}\psi_1 x) = n(x) \) for \( x \in C_1 \). \( \phi^{-1}\psi_1 \) extends to \( \eta \in \text{Hom}_R(\mathcal{O}, \mathcal{O}) \) such that \( n(\eta x) = n(x) \) for \( x \in \mathcal{O} \), by Witt's cancellation theorem for nondegenerate quadratic forms over local rings [2, p. 80]. Then \( \phi \eta \) is a \( \sigma \)-semilinear isomorphism of \( \mathcal{O} \) and \( \mathcal{O}' \) such that \( n'(\phi \eta x) = n(x)^n \) for \( x \in \mathcal{O} \). If \( t < 3 \), take \( p \in C_r^+ \) such that \( n(p) \) is a unit. \( \phi \eta p \in (\psi_1 C_1)^- \) and \( n'(\phi \eta p) = n(p)^n \). Set \( C_{r+1} = C_r + C_1p \) and set

\[
\psi_{r+1}(x + yp) = \psi_r x + (\psi_\sigma y)(\phi \eta p)
\]

for \( x, y \in C_r \). The proof of [3, Lemma 1.9] shows that \( C_{r+1} \) is a free \( R \)-module of rank \( 2^{r+1} \), \( C_{r+1} \) is a subalgebra of \( \mathcal{O} \), \( n \) is nondegenerate on \( C_{r+1} \), and \( \psi_{r+1} \) is a \( \sigma \)-semilinear ring isomorphism of \( C_{r+1} \) into \( \mathcal{O}' \) such that \( \psi_{r+1}(z^d) = (\psi_{r+1} z)^d \) for \( z \in C_{r+1} \), completing the induction. \( \psi_3 \) is a ring isomorphism of \( \mathcal{O} \) onto \( \mathcal{O}' \), by Nakayama's Lemma [5, p. 7].

If \( \text{char } R/m \neq 2 \), the next lemma follows from [4, Theorem 3.5].

**Lemma 1.2.** Let \( \{x_1, \ldots, x_t\} \) span \( J \) as an \( R \)-module and let \( R[\eta_1, \ldots, \eta_t] \) be a polynomial ring. Let \( X = \sum \eta_i x_i \in J \otimes_R R[\eta] \). Then \( X^2, X, \text{ and } 1 \) are linearly independent over \( R[\eta] \).

**Proof.** Let \( h \) be a positive integer and let \( P(h, R) \) be the \( R \)-submodule of \( R[\eta] \) composed of polynomials of total degree at most \( h \). Let \( t_1, \ldots, t_d \) be the monic monomials of total degree at most \( h - 2 \) in \( R[\eta] \). The images of the \( t_iX^2, t_iX, \text{ and } t_i1 \) in

\[
(J \otimes_R P(h, R))/m(J \otimes_R P(h, R)) \cong (J/mJ) \otimes_{R/m} P(h, R/m)
\]

are linearly independent over \( R/m \) [6, p. 10]. Since \( J \otimes_R P(h, R) \) is a finite free \( R \)-module, the \( t_iX^2, t_iX, \text{ and } t_i1 \) are linearly independent over \( R \) as elements of \( J \otimes_R P(h, R) \) [5, p. 24]. Since \( J \otimes_R P(h, R) \) is isomorphic to its image in \( J \otimes_R R[\eta] \), the lemma follows. \( \Box \)
Theorem 1.3. The following conditions are equivalent:
(1) There is a ring isomorphism of \( S \) onto \( S' \).
(2) There is a semilinear algebra isomorphism of an isotope of \( J \) onto \( J' \).
(3) There is a norm semisimilarity of \( J \) onto \( J' \).

Proof. (1) \( \Rightarrow \) (2). If \( \phi \) is a ring isomorphism of \( S \) onto \( S' \), the last sentence of Lemma 1.1 implies that there is \( \sigma : R \rightarrow R' \) such that

\[
\sum \alpha_i e_i + a_i[jk] \rightarrow \sum \alpha_i e_i + \phi a_i[k][jk]
\]

is a \( \sigma \)-semilinear algebra isomorphism of \( H(S_3, 1') \) and \( H(S_3', 1') \). (2) follows, since \( Y \rightarrow Y' \) is an isomorphism of the \( \gamma \)-isotope \( H(S_3, 1') \) onto \( H(S_3, \gamma) \). (2) \( \Rightarrow \) (3). Let \( J^{(u)} \) be the \( u \)-isotope of \( J \), \( u \in J \) invertible, and let \( (\phi, \sigma) \) be a semilinear algebra isomorphism of \( J^{(u)} \) onto \( J' \). Let \( X \in J^{(u)} \otimes_R R[\eta] \) be as in Lemma 1.2 and extend \( (\phi, \sigma) \) to a semilinear algebra isomorphism of \( J^{(u)} \otimes_R R[\eta] \) onto \( J' \otimes_R R'[\eta] \). \( X \) satisfies a monic polynomial over \( R[\eta] \) of degree three with constant term \( N(u)N(X) \) [9, p. 500]. Applying \( \phi \) to this polynomial and taking the corresponding polynomial for \( \phi(X) \) shows that \( \phi(X) \) satisfies monic polynomials of degree three with constant terms \( N(u)N(X) \) and \( N'(\phi X) \). Applying Lemma 1.2 to \( \phi X = \sum \phi(x_i) \eta_i \) gives \( N(u)N(X) = N'(\phi X) \). Specializing \( \eta_i = 1 \) and \( \eta_i = 0 \) for \( i \geq 2 \) gives \( N(u)N(X) = N'(\phi x_1) \). Since \( x_1 \) is an arbitrary element of \( J \), \( \phi \) is a norm semisimilarity. (3) \( \Rightarrow \) (1). Let \( (\phi, \sigma) \) be a norm semisimilarity of \( J \) onto \( J' \). We can assume that \( \phi(R e_i) = R' e_i' \), since \( S(J') \) is transitive on three-points [3, Proposition 2.1]. Then \( \phi(S[jk]) = S'[jk] \), by the proof of [3, Lemma 3.2]. Since \( N(e_1 + x[23]) = -\gamma_2 \gamma_3 x^3 \) for \( x \in S \), define a norm semisimilarity \( \psi \) of \( S \) onto \( S' \) by \( (\psi x)[23] = \phi(x[23]) \). \( \psi 1 \) is invertible, since \( n(\psi 1) \) is a unit. \( X \rightarrow (\psi 1)^{-1} \psi X \) is a norm semisimilarity of \( S \) onto \( S' \) taking \( 1 \) to \( 1' \). We are done by Lemma 1.1. □

2. Involutions of the first and second kinds. Henceforth we assume that \( 2 \) is a unit in \( R \). In this section we study two types of involutions (elements of order two) in \( P G \). Define an involution of the first kind to be an involution in \( P G \) fixing a point \( a_* \) and all points on a line \( b^* \), where \( a_* \sim b^* \). Define an involution of the second kind to be an involution in \( P G \) fixing a four-point.

We claim that \( P S \) is transitive on pairs \( a_* \sim b^* \). There are \( c_* \) and \( d_* \) on \( b^* \) such that \( c_* \sim d_* \), since \( P S \) is transitive on lines [3, Proposition 2.1]. \( a_* \sim b^* = (c \times d)^* \), so \( \{a_*, c_*, d_* \} \) is a three-point. The claim follows, since \( P S \) is transitive on three-points [3, Proposition 2.1].

Define \( \xi_{a_*,b^*} \in S \) to be \( 1 \) on \( Re_i + J_0(e_i) \) and \( -1 \) on \( J_{1/2}(e_i) \).

Proposition 2.1. If \( a_* \sim b^* \), there is a unique involution \( P^\xi_{a_*,b^*} \in PG \) fixing \( a_* \) and all points on \( b^* \). If \( P_{\phi} \in P_{\Gamma} \),

\[
P_{\phi}P^\xi_{a_*,b^*}P_{\phi}^{-1} = P^\xi_{P_\phi a_*,P_\phi b^*},
\]

Involutions of the first kind form a conjugacy class of \( P S \).

Proof. Since \( P S \) is transitive on pairs \( a_* \sim b^* \) and \( \xi_{a_*,b^*} \in S \), it suffices to prove that \( P^\xi_{e_1,e_2^*} \) is the unique involution in \( P G \) fixing \( e_1 \) and all points on \( e_2^* \). Let \( P_{\psi} \in P G \) be another such involution. Since \( P_{\psi} \) fixes \( e_2^* \), we can replace \( \psi \) by a
scalar multiple and assume that $\psi$ fixes $e_2$. $P\psi^2 = 1$, so $\psi^2$ is scalar multiplication [3, Lemma 3.3]. Then $\psi^2 = 1$, since $\psi$ fixes $e_2$. Since $\psi$ fixes each $Re_i$, it fixes $\mathcal{O}[23]$ [3, Lemma 3.2]. Because $\psi$ fixes $e_2$ and $R(e_2 + a[23] + \gamma_2\gamma_3n(a)e_3)$ for all $a \in \mathcal{O}$, $\psi$ is the identity map on $J_0(e_1)$. It follows that $\psi$ is multiplication by $\tau \in R - m$ on $J_1/2(e_1)$, where $\psi e_1 = \tau e_1$ [3, Lemma 5.1]. $\psi^2 = 1$ implies that $\tau^2 = 1$. $\tau = \pm 1$, since $\psi \neq 1$ and $\psi = \xi_1 \in \mathcal{O}$.

**Proposition 2.2** If $P\phi \in PT$ and $a \sim b*$, then $P\phi$ commutes with $P\xi_{a,b^*}$ if and only if $P\phi$ fixes $a$ and $b^*$.

**Proof.** If $P\phi$ fixes $a$ and $b^*$, Proposition 2.1 implies that $P\phi$ commutes with $P\zeta_{a,b^*}$. Conversely, assume that $P\phi$ commutes with $P\zeta_{a,b^*}$. We can assume that $a = e_1e_1$ and $b^* = e_3e_3$. By Proposition 2.1, $\phi\zeta_{1,e_1e_1}^{-1} = \alpha\zeta_{1,e_1e_1}$ for $\alpha \in R - m$. $J$ is the direct sum of eigenspaces for $\zeta_{1,e_1e_1}$ of ranks 11 and 16, so $\alpha = 1$ and $\phi$ commutes with $\zeta_{1,e_1e_1}$. Thus $\phi$ preserves the 1-eigenspace $Re_1 + J_0(e_1)$ of $\zeta_{1,e_1e_1}^{-1}$. If $x \in Re_1 + J_0(e_1)$, $x \in J - mJ$, and $x^* = 0$, it follows that either $x \in Re_1$ or $x \in J_0(e_1)$. Let $x_1 = e_2 + 1[23]$ and $x_2 = e_2$, and $x_3 = e_3$. Since at most one of the $x_i$ can satisfy $\phi x_i \in Re_1$, there are $j \neq k$ such that $\phi x_j, \phi x_k \in J_0(e_1)$. Then

$$P\phi e_j^* = P\phi(x_j \times x_k)^* = (\phi x_j \times \phi x_k)^* = e_j^*.$$ 

Since $J_0(e_1) = \sum Rx$ such that $x_0[e_1^*]$, $\phi$ preserves $J_0(e_1)$. As above, either $\phi e_1 \in Re_1$ or $\phi e_1 \in J_0(e_1)$, since $\phi$ preserves $Re_1 + J_0(e_1)$. Since $\phi$ preserves $J_0(e_1)$, $\phi e_1 \in Re_1$. Thus $P\phi$ fixes $e_1$ and $e_3$.

A subalgebra $Q$ of $\mathcal{O}$ is called a quaternion subalgebra if $Q$ is a free $R$-module of rank 4 and the restriction of $n(x)$ to $Q$ is nondegenerate. $\mathcal{O} = Q \oplus Q^\perp$ and we define $\tau_Q \in \text{End}_R(\mathcal{O})$ to be 1 on $Q$ and $-1$ on $Q^\perp$. $\tau_Q$ is an algebra automorphism of period two, by the proof of [3, Lemma 1.9]. Define an algebra automorphism $\zeta_Q$ of $J$ by

$$\zeta_Q(\Sigma a_ie_i + \Sigma a_i[jk]) = \Sigma a_ie_i + \Sigma \tau_Q(a_i)[jk].$$

As in [7, p. 66], we note that every algebra automorphism $\tau$ of $\mathcal{O}$ of order two has the form $\tau_Q$ for a quaternion subalgebra $Q$. To see this, let $Q$ be the 1-eigenspace of $\tau$ and let $P$ be the $-1$-eigenspace. Since 2 is a unit in $R$, $\mathcal{O} = Q \oplus P$, whence $Q$ and $P$ are free $R$-modules [5, p. 24]. $n(\tau x) = n(x)$ for $x \in J$ [Lemma 1.1]. It follows that $n(P, Q) = 0$, so $n$ is nondegenerate on $Q$ and $P$. Since $P \neq 0$, it contains an element whose norm is a unit. Multiplication by this element interchanges $Q$ and $P$, so $Q$ has rank 4. Then $Q$ is a quaternion subalgebra and $\tau = \tau_Q$.

**Proposition 2.3**. Every involution of the second kind is conjugate in $PG$ to $P\zeta_Q$ for some quaternion algebra $Q$ of $\mathcal{O}$.

**Proof.** Let $P\zeta$ be an involution of the second kind. Let $J_1 = H(\mathcal{O})$, and define a norm similarity $\psi: J \to J_1$ by $\psi(X) = X^{-1}$. $P\psi\psi^{-1}$ is an involution of the second kind in $PG(J_1)$. There is $\phi \in G(J_1)$ such that $P\phi\psi\psi^{-1}\phi^{-1}$ fixes $e_1e_1, e_2e_2, e_3e_3$, and $(\Sigma e_i + \Sigma 1[3k])_\mathcal{O}$, since $PG(J_1)$ is transitive on four-points [3, Lemma 8.1]. The proof of [3, Theorem 8.4] shows that there is a ring automorphism $\tau$ of $\mathcal{O}$ such that $P\phi\psi\psi^{-1}\phi^{-1} = P\eta$, where $\eta$ is the norm semisimilarity of $J_1$ applying $\tau$ to each element.
coordinate. \( \eta \) is a scalar multiple of \( \phi \psi \psi^{-1} \phi^{-1} \) [3, Lemma 3.3], so \( \eta \) is linear and \( \tau \) is an algebra automorphism. \( Pn_t^2 = 1 \), so \( \eta_t^2 \) is scalar multiplication. Since \( \tau \) fixes \( 1 \in \mathcal{O}, \tau^2 = 1 \). As above, \( \tau = \tau_Q \) for a quaternion subalgebra \( Q \) of \( \mathcal{O} \). \( \eta = \xi_Q \) and

\[
P(\psi^{-1}\phi\psi)\xi(\psi^{-1}\phi^{-1})\psi = P\psi^{-1}\xi_Q\psi,
\]

where \( \psi^{-1}\phi\psi \in \mathcal{G}(J) \) and \( P\psi^{-1}\xi_Q\psi \) is the involution in \( PG(J) \) applying \( \tau_Q \) coordinatewise. \( \square \)

3. Classification of involutions in \( PG \). We prove in this section that every involution in \( PG \) is of the first or second kind. (It is immediate that every involution in \( PG \) belongs to \( PS \), but it is convenient to work in \( PG \) since \( G \) is closed under scalar multiplication.)

**Lemma 3.1.** If \( R = F \) is a field, there is no involution \( P\phi \in PG \) such that \( P\phi x_\ast \sim x_\ast \) for all \( x_\ast \in PJ \).

**Proof.** Assume such \( P\phi \) exists. We claim that \( P\phi \) fixes either a point or a line. Suppose that \( P\phi \) does not fix \( e_1 \ast \). Since \( F \) is a field, \( PS \) is transitive on pairs of connected points [6, p. 38]. Replacing \( P\phi \) by a conjugate, we can assume that \( P\phi e_1 = (a[12])_\ast \). Applying \( P\phi \) to the equation \( (a[12])_\ast \sim e_2 \ast \) gives \( e_1 \ast \sim P\phi e_2 \ast \).

Since \( e_2 \ast \sim P\phi e_2 \ast \) as well, \( P\phi e_2 \ast = (b[12])_\ast \). Then

\[
P\phi e_3 = P\phi(e_1 \times e_2) \ast = (a[12] \times b[12]) \ast = (-\gamma_1 \gamma_2 n(a, b)e_3) \ast,
\]

so \( P\phi \) fixes \( e_3 \ast \). Thus \( P\phi \) fixes a point or a line.

We can assume that either \( \phi^2 = 1 \) or \( (\phi^{-1})^2 = 1 \), by the proof of Proposition 2.1. Since \( \phi \to \phi^{-1} \) is a group isomorphism of \( PT \), \( \phi^2 = 1 \) in either case. Since \( P\phi e_1 \sim e_1 \ast \), \( \phi e_1 = ae_1 + c[12] + d[31] \), where \( n(c) = 0 = n(d) \) and \( dc = 0 \). Set \( f_1 = e_1 + \phi e_1 \) and \( f_2 = e_1 - \phi e_1 \). \( f_1 \ast = 0 \) and \( \phi f_1 = \pm f_1 \). Since \( F \) is not of characteristic two, the \( e_1 \) coefficient of at least one of the \( f_i \) is nonzero. Thus \( P\phi \) fixes a point \( f_i \ast \) such that the \( e_i \) coefficient of \( f_i \) is nonzero.

Replacing \( P\phi \) by a conjugate, we can assume that \( P\phi \) fixes \( e_2 \ast \) [3, Proposition 2.1]. Applying the last paragraph again shows that \( P\phi \) fixes a point \( f_\ast \sim e_2 \ast \). Conjugating \( P\phi \), we can assume that it fixes \( e_2 \ast \) and \( e_3 \ast \). Again by the preceding paragraph, \( P\phi \) fixes a point \( f_i \ast \) such that \( (f_i \ast, e_2 \ast, e_3 \ast) \) is a three-point.

Hence we can assume that \( P\phi \) fixes \( (e_1 \ast, e_2 \ast, e_3 \ast) \) and \( \phi^2 = 1 \). Since \( \phi e_1 = \pm e_1 \), we can replace \( \phi \) by \( -\phi \) if necessary and assume that \( \phi \) fixes at least two of the \( e_i \).

Replacing \( \phi \) by a conjugate, we can assume that it fixes \( e_1, e_2 \) and \( Fe_3 \).

\( \phi \) induces \( \sigma \in \text{End}_F(\mathcal{O}) \) by \( \phi(a[12]) = a^\sigma[12] \) [3, Lemma 3.2]. \( \sigma^2 = 1 \), so \( \mathcal{O} = \mathcal{O}_1 + \mathcal{O}_- \), where \( \mathcal{O}_i \) is the \( i \)-eigenspace of \( \sigma \). \( n(a^\sigma) = n(a) \) for \( a \in \mathcal{O} \), since \( (\phi x)^\sigma = 0 \) for \( x = e_1 + a[12] + \gamma_1 \gamma_2 n(a)e_2 \) and \( \phi \) fixes \( e_1 \) and \( e_2 \). It follows that the \( \mathcal{O}_i \) are orthogonal and the restriction of \( n \) to \( \mathcal{O}_- \) is nondegenerate. If \( \mathcal{O}_- \neq 0 \), there is \( a \in \mathcal{O}_- \) such that \( n(a) \) is nonzero; taking \( x = e_1 + a[12] + \gamma_1 \gamma_2 n(a)e_2 \) gives \( x \times \phi x \neq 0 \), a contradiction. Thus \( \mathcal{O}_- = 0 \) and \( \phi \) is the identity on \( J_\phi(e_3) \). Since \( \phi \) fixes \( F e_3 \), \( P\phi = P\phi e_\ast \ast e_3 \) [Proposition 2.1]. Then \( x_\ast \sim P\phi x_\ast \) for \( x = e_2 + 1[23] + \gamma_2 \gamma_3 e_3 \), a contradiction. \( \square \)
**Lemma 3.2.** There is no involution \( P\phi \in PG(J) \) inducing the identity in \( PG(J/mJ) \).

**Proof.** Assume such \( P\phi \) exists. Only a finite number of elements of \( R \) are required to express the multiplication in \( J \) and the action of \( \phi \) in terms of a given free basis \( \{x_i\} \) of \( J \). These elements generate a Noetherian subring \( R' \) of \( R \). Replacing \( R' \) by its localization at \( R' \cap m \), we can assume that \( R' \) is local. Replacing \( R \) by \( R' \), \( J \) by \( \sum R'x_i \), and \( \phi \) by its restriction, we can assume that \( R \) is Noetherian. Tensoring \( J \) with the completion of \( R \), we can assume that \( R \) is complete local Noetherian. The fact that \( P\phi \) is an involution is preserved under tensoring, since \( P\phi^2 = 1 \) if and only if \( \phi^2 \) is multiplication by \( \beta \in R - m \) [3, Lemma 3.3]. Let \( \phi \) induce multiplication by \( \alpha_1 \in R - m \) on \( J/mJ \). Setting \( \alpha_{i+1} = \alpha_i + (2\alpha_i)^{-1}(\beta - \alpha_i^2) \) gives \( \alpha_i^2 \equiv \beta \) and \( \alpha_{i+1} \equiv \alpha_i \pmod{m^i} \) by induction. \( \alpha = \lim \alpha_i \) satisfies \( \alpha^2 = \beta \). \( J \) is the direct sum of eigenspaces for \( \pm \alpha \). Since \( \phi \) induces multiplication by \( \alpha_1 \) on \( J/mJ \), Nakayama's Lemma implies that the \( \alpha \)-eigenspace equals \( J [5, p. 7] \). Thus \( P\phi = 1 \), a contradiction. \( \square \)

**Lemma 3.3.** If \( P\phi \in PG \) is an involution, there are \( a_\bullet \) and \( d_\bullet \) in \( PJ \) such that \( (a_\bullet P\phi a_\bullet, d_\bullet, P\phi d_\bullet) \) is a four-point.

**Proof.** \( P\phi \) induces an involution \( P\phi_1 \in PG(J/mJ) \) [Lemma 3.2]. There is \( a_\bullet \in P(J/mJ) \) such that \( P\phi_1 a_\bullet \sim a_\bullet \) [Lemma 3.1]. Since \( PS(J/mJ) \) is transitive on points and the canonical map \( PS(J) \to PS(J/mJ) \) is surjective [3, Corollary 6.5], there is \( a_\bullet \in PJ \) whose image in \( P(J/mJ) \) is \( a_\bullet \sim P\phi a_\bullet \), since \( a_\bullet \sim P\phi a_\bullet \). There is a line \( b^* \) on \( a_\bullet \) such that \( P\phi a_\bullet \sim b^* \) (since we can assume that \( a_\bullet = e_1 \) and \( P\phi a_\bullet = e_2 \)). We repeatedly apply [3, Lemma 8.2] and its dual. \( P\phi b^* \sim b^* \), since \( P\phi a_\bullet \sim P\phi b^* \) and \( P\phi a_\bullet \sim b^* \). Let \( c_\bullet = (b \times \phi b) \), \( c_\bullet \sim a_\bullet \), since \( a_\bullet \sim P\phi b^* \). There is \( d_\bullet b^* \) such that \( d_\bullet \sim c_\bullet \) and \( d_\bullet \sim c_\bullet \) (since we can assume that \( d_\bullet = e_1 \) and \( c_\bullet = e_2 \), so \( b^* = e_3 \)). \( d_\bullet \sim P\phi b^*\), else \( P\phi b^* \sim (d \times c)^* = b^* \). \( P\phi a_\bullet \sim b^* = (a \times d)^* \) and \( d_\bullet \sim P\phi b^* = (\phi a \times \phi d)^* \), and applying \( P\phi \) gives \( a_\bullet \sim (\phi a \times \phi d)^* \) and \( P\phi d_\bullet \sim (a \times d)^* \). \( \square \)

**Theorem 3.4.** Every involution \( P\phi \in PG \) is of the first or second kind.

**Proof.** Let \( P\phi \in PG \) be an involution. Let

\[
\begin{align*}
    z_1 &= \gamma_1 e_1 + \gamma_2 e_2 + 1[12], \\
    z_2 &= \gamma_1 e_1 + \gamma_2 e_2 - 1[12], \\
    z_3 &= \gamma_1 e_1 + \gamma_3 e_3 + 1[31], \\
    z_4 &= \gamma_1 e_1 + \gamma_3 e_3 - 1[31], \\
    z_5 &= \gamma_2 e_2 + \gamma_3 e_3 + 1[23], \\
    z_6 &= \gamma_2 e_2 + \gamma_3 e_3 - 1[23].
\end{align*}
\]

\((z_1 \times z_2)_e = e_{3\bullet}, (z_3 \times z_4)_e = e_{2\bullet}, \) and \((z_5 \times z_6)_e = e_{1\bullet}\), so \((z_1^*, z_2^*, z_3^*, z_4^*)\) is the dual of a four-point. [3, Lemma 8.2] implies that \((z_1 \times z_3)_e, (z_1 \times z_4)_e, (z_2 \times z_3)_e, (z_2 \times z_4)_e\) is a four-point. There are \( a_\bullet \) and \( d_\bullet \) in \( PJ \) such that
(a*, Pφa*, d*, Pφd*) is a four-point [Lemma 3.3]. Since PG is transitive on four-points, we can replace Pφ by a conjugate and assume that these two four-points are equal [3, Lemma 8.1]. Computation shows that \((z_1 \times z_3) \times e_1 = \gamma_1 z_6_*, \text{so} \ (z_1 \times z_3)_* \) is on \(z_6^*, \) and \(z_1^*, z_2^*, \text{and} \ z_6^* \) are concurrent. Applying \(P\) shows that \(z_1^*, z_2^*, \text{and} \ z_6^* \) are concurrent, applying \(P\) shows that \(z_2^*, z_3^*, \text{and} \ z_5^* \) are concurrent, and applying \(P\) shows that \(z_2^*, z_4^*, \text{and} \ z_6^* \) are concurrent, as in Figure 1.

Let

\[ y_* = (e_3 \times z_2)_* = (y_2 e_1 + \gamma_1 e_2 + 1[12])_*. \]

\(P\) interchanges \(a_* \) and \(d_* \) with \(P\)a* and \(P\)d* respectively, so \(P\) interchanges \(z_3^* \) and \(z_4^* \) with \(z_4^* \) and \(z_3^* \) respectively and fixes \(z_1^* \) and \(z_5^* \). Then \(P\) fixes all \(e_* \) and \(e_1^* \) and hence \(y_* \). We can assume that \(P\) fixes \(e_1 \) and that \(P^2 = 1 \), by the proof of Proposition 2.1.

![Figure 1](image)

Since \(P\) fixes each \(R e_i\), it fixes each \(\mathcal{O}[j k] \) [3, Lemma 3.2]. Define \(\sigma \in \text{End}_a(\mathcal{O}) \) by \(a^*[31] = \phi(a[31]) \). \(\sigma^2 = 1 \), since \(\phi^2 = 1 \). Write \(\mathcal{O} = \mathcal{O}_1 \oplus \mathcal{O}_1 \), where \(\mathcal{O}_i \) is the \(i\)-eigenspace of \(\sigma \). \(P\) interchanges \(a_* = (z_1 \times z_3)_* \) with \(P\)a* = \((z_1 \times z_3)_* \) with \(z_1 \times z_3 \) and \(z_1 \times z_4 \) both have the form \(\gamma_2 \gamma_3 e_1 + \gamma_1 \gamma_2 e_2 + \gamma_1 \gamma_2 e_3 + \cdots \). It follows that \(\phi\) fixes \(e_2\) and \(e_3\) as well as \(e_1\). Then \(\phi\) fixes \(1 \in J\), so \(\phi \in S \). The equation \(N(e_2 + a[31]) = -\gamma_1 \gamma_3 n(a) \) implies that \(n(a^*) = n(a) \) for \(a \in \mathcal{O} \). Thus the restrictions of \(n\) to \(\mathcal{O}_1 \) and \(\mathcal{O}_1 \) are nondegenerate. If \(\mathcal{O} = \mathcal{O}_1 \), \(P\) and \(P\) are agree on \(e_2^*, y^*, \) and all points on \(e_3^* \). It follows that \(P\) equals \(P\) [3, Lemma 8.3] and \(P\) is an involution of the first kind. If \(\mathcal{O} \neq \mathcal{O}_1 \), there is \(x \in \mathcal{O}_1 \) such that \(n(x) \) is a unit. Then \((e_2^*, e_3^*, y^*, (e_1 + x[31]) + \gamma_1 \gamma_3 n(x) e_3)_* \) is a four-point fixed by \(P\), and \(P\) is of the second kind. \(\square\)

4. Group-theoretic classification of involutions. We prove now that involutions of the first kind can be distinguished from those of the second kind by their group-theoretic properties within the little projective group. Together with Theorem
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3.4, this implies that an isomorphism of little projective groups preserves each kind of involution.

For \( \eta \in S \), let \( C(\eta) \) be the centralizer of \( \eta \) in \( S \) and let \( C(P_\eta) \) be the centralizer of \( P_\eta \) in \( PS \). Let \( P_\xi \) be an involution of the first or second kind and let \( P_\psi \in C(P_\xi) \). Then \( \psi \xi \psi^{-1} = \alpha \xi \) for \( \alpha \in R - m \). Since \( J \) decomposes into eigenspaces of distinct ranks for \( \xi \), \( \alpha = 1 \) and \( \psi \in C(\xi) \). Hence \( C(P_\xi) = PC(\xi) \cong C(\xi)/(R^\times \cap S) \), where \( R^\times \) is the group of units of \( R \).

**Lemma 4.1.** If \( R = F \) is a field and \( P_\xi \) is an involution of the first kind, then \( C(P_\xi) \) has a normal series where all factor groups are abelian except for one which is simple.

**Proof.** We can assume that \( \xi = \xi_{e_1e_2} \) [Proposition 2.1]. Since \( C(P_\xi) \cong C(\xi)/(F^\times \cap S) \), it suffices to prove that \( C(\xi) \) has such a normal series. If \( \psi \in S \), \( \psi \in C(\xi) \) if and only if \( \psi \) preserves \( F_{e_1} \) and \( J_\psi(e_1) \), by the proof of Proposition 2.2. Let \( \lambda \) be the kernel of the homomorphism from \( C(\xi) \) to \( F^\times \), taking \( \psi \) to \( \sigma \) such that \( \psi e_1 = \sigma e_1 \). \( C(\xi)/\lambda \) is abelian. Let \( O(J_\psi) \) be the orthogonal group of \( J_\psi(e_1) \) relative to the quadratic form \( x \to T(x^*, e_1) \). Since elements of \( S \) preserve \( T(x^*, y) \) [6, p. 10], we can define a homomorphism \( \lambda : N \to O(J_\psi) \) by restriction. The kernel of \( \lambda \) is \( \{1, \xi\} \) [3, Lemma 5.1]. The image of \( \lambda \) is the reduced orthogonal group \( O'(J_\psi) \) [6, p. 31]. \( O'(J_\psi) \) modulo its center is simple, since \( J_\psi(e_1) \) is an isotropic space of dimension 10 [1, pp. 195 and 209]. □

**Lemma 4.2.** If \( R = F \) is a field with more than three elements and \( P_\xi \) is an involution of the second kind, then \( C(P_\xi) \) has a normal series with two nonsolvable factor groups.

**Proof.** We can assume that \( \gamma = 1 \) and that \( P_\xi = P_\xi Q \) for a quaternion subalgebra \( Q \) of \( O \) [Theorem 1.3 and Proposition 2.3]. Since \( C(P_\xi Q) \cong C(\xi_{Q})/(F^\times \cap S) \), it suffices to prove that \( C(\xi_{Q}) \) has a normal series with two nonsolvable factor groups. \( C(\xi_{Q}) \) consists of the elements of \( S \) preserving the \( \pm 1 \)-eigenspaces of \( \xi_{Q} \). Identify the 1-eigenspace of \( \xi_{Q} \) with \( H_1(Q_3, \pi) \), where \( \pi \) is the standard involution conjugate transpose of \( Q_3 \). Let \( \tau \) be the restriction homomorphism from \( C(\xi_{Q}) \) to \( \text{End}_F(H(Q_3, \pi)) \). We claim that neither the kernel nor the image of \( \tau \) is solvable. If \( Q \) is a division algebra, we are done by the proof of [10, pp. 333–334]. Assume that \( Q \) is split, so \( Q = F_2 \) [8, p. 169].

The proof of [10, p. 334] shows that the kernel of \( \tau \) is isomorphic to the group of elements of \( Q \) of norm 1. This group is isomorphic to \( \text{SL}_2(F) \), which is not solvable if \( F \) has more than three elements [1, p. 169].

To see that the image of \( \tau \) is not solvable, let \( T = T_{x[i]y[j]} \) for \( x \in Q, i \neq j \). \( T \in S(J) \) [6, p. 18]. \( T \in C(\xi_{Q}) \), since \( \xi_{Q} \) is an automorphism fixing \( x[i] \) and \( y[j] \). Define a homomorphism \( \lambda \) from the group of invertible elements of \( Q_3 \) to the group of invertible linear transformations of \( H(Q_3, \pi) \) by \( \lambda(A) = AXA^\pi \), \( A \in Q_3 \). \( X \in H(Q_3, \pi) \). \( \tau(T) = \lambda(1 + xe_{ij}) \), where the \( e_{ij} \) are matrix units decomposing \( Q_3 \) over \( Q \). Let \( W \) be the multiplicative subgroup of \( Q_3 \) generated by \( 1 + xe_{ij}, x \in Q, i \neq j \). Identify \( Q \cong F_2 \) and let the matrix units of \( F_2 \) be written as \( f_{ij} \). If \( \alpha \in F \), \( s \neq t \) and \( i \neq j \), \( W \) contains

\[
(1 + e_{ij})(1 - f_{st}e_{ij})(1 + \alpha f_{st}e_{ij})(1 + f_{st}e_{ij})(1 - e_{ij}) = 1 + \alpha f_{st}e_{ij}.
\]
Thus $W$ contains $1 + a
i \tau e_j$, $a \in F$, if either $s \neq t$ or $i \neq j$. Let $\eta$ be the natural isomorphism of $(F_2)_3$ onto $F_6$ and let the matrix units of $F_6$ be written $g_{uv}$. Then $\eta(W)$ contains $1 + ag_{uv}$ for all $a \in F$ and $u \neq v$. These elements generate $\text{SL}_6(F)$ [1, p. 156], so $W = \eta^{-1}(\text{SL}_6(F))$. One verifies that the kernel of $\lambda$ is $\pm 1$. Since $\text{PSL}_6(F)$ is simple [1, p. 169], $(W)$ is not solvable. Since $\lambda(1 + xey) = \tau(T)$, $\lambda(W)$ is contained in $\tau(C(\xi_Q))$ and $\tau(C(\xi_Q))$ is not solvable. □

Let $PH_i(J)$ be the set of involutions of the $i$th kind in $PS(J)$.

**Lemma 4.3.** Let $\theta$ be an isomorphism of $PS(J)$ onto $PS(J')$, where $R = F$ and $R' = F'$ are fields. Then $\theta(PH_i(J)) = PH_i(J')$.

**Proof.** Since $PH_i(J)$ is a conjugacy class of $PS(J)$, it suffices to prove that either $\theta(PH_i(J)) \subset PH_i(J')$ or $\theta^{-1}(PH_i(J')) \subset PH_i(J)$. We are done by Theorem 3.4 and Lemmas 4.1 and 4.2, unless both $F$ and $F'$ have three elements. Assume that is the case. If $P_\xi(J) \in PH_i(J)$, the proof of Lemma 4.1 shows that $C(\xi_J)$ has a normal subgroup $N$ of index at most two such that $N \cap [1, \xi_J] = \Sigma(J_0)$ and $\Sigma(J_0)$ modulo its center is simple. The center of $\Sigma(J_0)$ has order at most two [1, p. 133].

Let $Q'$ be a quaternion subalgebra of $\Sigma$. Since $F'$ is finite, $Q'$ is split [1, p. 144]. The second paragraph of the proof of Lemma 4.2 shows that $C(\xi_Q)$ has a normal subgroup isomorphic to $\text{SL}_2(F')$. $\text{SL}_2(F')$ is solvable of order 24 [1, p. 170], so $C(\xi_Q)$ is not isomorphic to $C(\xi_Q')$. $C(P_\xi(Q)) \cong C(\xi_Q)$ and $C(P_\xi(Q')) \cong C(\xi_Q')$, since $F' \cap S = \{a \in F|a^3 = 1\} = \{1\}$. Thus $C(P_\xi(Q))$ is not isomorphic to $C(P_\xi(Q'))$. If $P_\xi(Q) \in PH_2(J)$, there is $\psi \in G$ such that $P_\psi^{-1}(Q) = P_\xi(Q)$. Thus $C(P_\xi(Q))$ and $C(P_\xi(Q'))$ are not isomorphic. We are done by Theorem 3.4. □

**Theorem 4.4.** If $\theta$ is an isomorphism of $PS(J)$ onto $PS(J')$, then $\theta(PH_i(J)) = PH_i(J')$.

**Proof.** The canonical homomorphism from $PS(J)$ to $PS(J/mJ)$ is surjective [3, Corollary 6.5]. Let $PS_m(J)$ be its kernel, so $PS(J)/PS_m(J) \cong PS(J/mJ)$. $\theta(PS_m(J)) = PS_m(J')$, since $PS_m(J)$ is the unique maximal normal subgroup of $PS(J)$ [3, Corollary 7.5]. Thus $\theta$ induces an isomorphism $\theta_m$ of $PS(J/mJ)$ onto $PS(J'/mJ')$. By Lemma 4.3, $\theta_m(PH_i(J/mJ)) = PH_i(J'/mJ')$. It follows from Theorem 3.4 that $\theta(PH_i(J)) = PH_i(J')$. □

5. Commuting involutions of the first kind. We prove in this section $P_{\xi_a,b \ast}$ and $P_{\xi_e,e_1 \ast}$ commute and their product is an involution of the first kind if and only if $a_e\ast d_\ast$ and $e_{1_e}b_\ast$. The key step is to characterize the action of involutions of the first kind in terms of the harmonic properties of the plane.

**Lemma 5.1.** $P_{\xi_e,e_1 \ast}$ commutes with $P_{\xi_a,b \ast}$ if and only if one of the following conditions holds:

1. $a_\ast = e_1 \ast$ and $b_\ast = e_1 \ast$,
2. $a_{1_e}e_\ast$ and $e_{1_e}b_\ast$,
3. $a, b \in J_{1/2}(e_1)$. 

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Proof. Write $e_1\ast$ as $e_\ast$. $P_{S_{a,b}, e_\ast}$ commutes with $P_{S_{a,b}, e_\ast}^\perp$ if and only if $P_{S_{a,b}, e_\ast}$ fixes $a_\ast$ and $b_\ast$ [Proposition 2.2]. This holds if and only if $\xi_{e_\ast}$ fixes $Ra$ and $Rb$, since one checks that $\xi_{e_\ast}^{-1} = \xi_{e_\ast}$. This is equivalent to assuming that $a$ and $b$ are elements of $Re + J_0(e)$ or $J_{1/2}(e)$, the eigenspaces of $\xi_{e_\ast}$. This holds if and only if $a$ and $b$ are elements of $Re$, $J_0(e)$, and $J_{1/2}(e)$, by the proof of Proposition 2.2. The lemma follows, since $a_\ast \sim b_\ast$ and the spaces $Re$, $J_0(e)$, and $J_{1/2}(e)$ are orthogonal with respect to $T(x,y)$.

Lemma 5.2. If $a_\ast \sim b_\ast$, $c_\ast \sim d_\ast$, $a_\ast|d_\ast$, and $c_\ast|b_\ast$, then

$$P_{S_{a,b}, d_\ast}^\perp P_{S_{c,d}, e_\ast}^\perp = P_{S_{(b \times d), e_\ast}}^\perp (a \times c)^\ast$$

and $(a_\ast, c_\ast, (b \times d)_\ast)$ is a three-point.

Proof. $a_\ast \sim b_\ast$ implies that $b_\ast \sim d_\ast$ and $a_\ast \sim (b \times d)_\ast$, by [3, Lemma 8.2] and its dual. Then $(a \times (b \times d))^\ast = d_\ast$, so $c_\ast \sim d_\ast$ implies that $(a_\ast, c_\ast, (b \times d)_\ast)$ is a three-point. We can assume that this three-point equals $(e_1, e_2, e_3)_\ast$ [3, Proposition 2.1]. Then $b_\ast = (c \times (b \times d))^\ast = e_1^\ast$ and $d_\ast = (a \times (b \times d))^\ast = e_2^\ast$. Since $\xi_{e_\ast e_\ast}^{-1}$ is 1 on $Re_i + J_0(e_i)$ and $-1$ on $J_{1/2}(e_i)$, $P_{S_{e_1,e_2}^\perp, f_{S_{e_1,e_2}^\perp}} = P_{S_{e_1,e_2}^\perp}$. The lemma follows, since $a_\ast = e_1\ast$, $b_\ast = e_2\ast$, $c_\ast = e_2\ast$, and $d_\ast = e_2^\ast$.

When $R = F$ is a field, $x_\ast \sim y_\ast$ and $w_\ast(\langle x \times y \rangle)^\ast$, Faulkner has defined the harmonic conjugate of $w_\ast$ with respect to $x_\ast$ and $y_\ast$ [6, p. 42]. We remark that his results and the following lemma extend directly to octonion planes over local rings, but we do not need this extension.

Lemma 5.3. If $R = F$ is a field, $a_\ast \sim b_\ast$, and $a \in J_{1/2}(e_i)$. Then $P_{S_{a,b}, e_\ast}$ takes every point on $d_\ast$ to its harmonic conjugate with respect to $a_\ast$ and $(b \times d)_\ast$.

Proof. $d_\ast \sim b_\ast$ and $a_\ast \sim (b \times d)_\ast$, since $a_\ast \sim b_\ast$ [6, p. 36]. There is $f_{a|b}^\ast$ such that $f_{a|b}^\ast \sim (b \times d)_\ast$ [6, p. 36]. $a_\ast \sim b_\ast = ((b \times d) \times f)^\ast$, so $(a_\ast, (b \times d)_\ast, f_{a|b})$ is a three-point, and we can assume that it equals $(e_1, e_2, e_3)_\ast$ [6, p. 33]. It follows that $b_\ast = e_1^\ast$ and $d_\ast = e_3^\ast$. The harmonic conjugate of $(a_1 e_1 + s[12] + a_2 e_2)_\ast$ with respect to $e_1\ast$ and $e_2\ast$ is $(a_1 e_1 - s[12] + a_2 e_2)_\ast$ [6, p. 42]. Then $P_{S_{e_1,e_2}^\perp}$ takes every point on $e_3^\ast$ to its harmonic conjugate with respect to $e_1\ast$ and $e_2\ast$ as required.

Lemma 5.4. Assume that $R = F$ is a field, $a_\ast \sim b_\ast$, and $a, b \in J_{1/2}(e_i)$. Then $P_{S_{a,b}, e_\ast}$ is not an involution of the first kind.

Proof. Since $F$ is a field, there is a line $f^\ast$ on $e_1\ast$ and $a_\ast$ [6, p. 35]. $f^\ast \sim e_\ast$, since $e_1\ast, f^\ast$ and $e_1\ast \sim e_1^\ast$ [6, p. 36]. Take $g_\ast|e_\ast^\ast$ such that $g_\ast \sim (f \times e_1)_\ast$, $e_1\ast \sim e_1^\ast = ((f \times e_1) \times g)^\ast$, so there is $P_{\phi} \in PS$ taking the three-point $(e_1, (f \times e_1)_\ast, g_\ast)$ to $(e_1, e_2, e_3)_\ast$. Since $((f \times e_1) \times g)^\ast = e_1^\ast$, $P_{\phi}$ fixes $e_1\ast$ and $e_1^\ast$. Lemma 5.1 implies that $\phi a, \phi b \in J_{1/2}(e_i)$. Thus we can replace $P_{S_{a,b}, e_\ast}$ with its conjugate by $P_{\phi}$. Then $f^\ast = (e_1 \times (f \times e_1)) \ast = e_3^\ast$, so $a_\ast|e_3^\ast$.

$e_3^\ast \sim b_\ast$, since $a_\ast|e_3^\ast$ and $a_\ast \sim b_\ast$. $a$ and $e_3 \times b$ are in $J_{1/2}(e_i) \cap J_0(e_3)$, so $a_\ast = (s[12])_\ast$, and $(e_3 \times b)_\ast = (s[12])_\ast$ for $n(s) = 0 = n(t)$. $a_\ast \sim b_\ast$ implies that $a_\ast \sim (e_3 \times b)_\ast$, so $n(s, t)$ is a unit. Replace $t$ by a scalar multiple to make $n(s, t) = -y_1^\ast y_2^\ast$.

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Let \( W \) be the orthogonal complement of \( Fs + Ft \) in \( \mathcal{O} \) with respect to \( n(x, y) \). Define a linear transformation \( \psi \) on \( J_0(e_2) \) to interchange \( e_1 \) and \( e_2 \) with \( s[12] \) and \( t[12] \) respectively and to be the identity on \( W[12] \). \( \psi^2 = 1 \) and \( \psi \) belongs to the orthogonal group of the quadratic form \( T(x^2) \) on \( J_0(e_2) \). Since its determinant is 1, \( \psi \) is the product of an even number of hyperplane reflections [1, p. 129]. Then \( \psi \) is induced by an element \( \eta \) of \( G \) preserving \( Fs_3 \) [3, Theorem 5.3]. \( P_{\xi_{a,b}}^x \) takes every point on \( e_3^* \) to its harmonic conjugate with respect to \( a_0 \) and \( (e_3 \times b)^* \) [Lemma 5.3]. Likewise \( P_{\xi_{c,d}}^x \) takes every point on \( e_3^* \) to its harmonic conjugate with respect to \( e_1^* \) and \( e_2^* \). It follows that \( P_{\xi_{a,b}}^x \) agrees with \( P_{\eta_{\xi_{c,d}}^x} \eta^{-1} \) on all points on \( e_3^* \). Let

\[
x_* = (ae_1 + \beta e_2 + (\delta s + \lambda t + w)[12])_*,
\]

where \( \alpha, \beta, \delta, \lambda \in F, w \in W \), and \( x^2 = 0 \). Then

\[
P_{\xi_{a,b}}^x(x_*) = (\psi_{\xi_{e_3 \times t}} \psi x)_*,
\]

(1)

\[
P_{\xi_{c,d}}^x P_{\xi_{a,b}}^x(x_*) = (-\alpha e_1 - \beta e_2 + (-\delta s - \lambda t + w)[12])*_.
\]

(2)

Take a preimage of \( P_{\xi_{c,d}}^x P_{\xi_{a,b}}^x \) in \( G \) and let \( \tau \) be its restriction to \( J_0(e_3) \). (2) implies that \( \tau \) is a scalar multiple of the map which is 1 on \( W[12] \) and \(-1\) on \( W[12]^* \). Thus \( \tau \) has eigenspaces of dimensions 4 and 6.

Assume that \( P_{\xi_{c,d}}^x P_{\xi_{a,b}}^x \) in \( G \) and let \( \xi \) be its restriction to \( J_0(e_3) \). (1) implies that \( \xi \) is an involution of the first kind. \( \xi \) is a scalar multiple of \( \tau \).

Theorem 5.5. If \( a_* \sim b^* \) and \( c_* \sim d^* \), the following conditions are equivalent:

1. \( P_{\xi_{a,b}}^x \) and \( P_{\xi_{c,d}}^x \) commute and their product is an involution of the first kind.
2. \( a_* | d^* \) and \( c_* | b^* \).

Proof. (1) \( \Rightarrow \) (2). By conjugation, we can assume that \( c_* = e_1^* \) and \( d^* = e_3^* \). By Lemma 5.1, either \( a_* | e_1^* \) and \( e_1^* | b^* \) or \( a, b \in J_{1/2}(e_1) \). Taking images in \( PS(J/mJ) \) and applying Lemma 5.4 shows that the latter condition does not hold. (2) \( \Rightarrow \) (1), by Lemmas 5.1 and 5.2.

We say that \( P_{\xi_{a,b}}^x \) and \( P_{\xi_{c,d}}^x \) commute exactly if the conditions of Theorem 5.5 are satisfied.

Corollary 5.6. \( a_* \sim c_* \) if and only if there exist \( b^* \) and \( d^* \) such that \( P_{\xi_{a,b}}^x \) and \( P_{\xi_{c,d}}^x \) commute exactly.
Proof. If $P_{S_{a,b}}$ and $P_{S_{c,d}}$ commute exactly, then $a_0 \sim c_0$ [Lemma 5.2]. Conversely, if $a_0 \sim c_0$, we can assume that $a_0 = e_1$ and $c_0 = e_2$ [3, Proposition 2.1], so we can take $b^* = e_1^*$ and $d^* = e_2^*$. \(\square\)

Corollary 5.7. $a_0 | d^*$ if and only if there exist $b^*$ and $c_0$ such that $P_{S_{a,b}}$ and $P_{S_{c,d}}$ commute exactly.

Proof. If $a_0 | d^*$, there is $g_0 | d^*$ such that $a_0 \sim g_0$ [3, dual of Lemma 2.3]. We can assume that $a_0 = e_1$ and $g_0 = e_2$, so $d^* = e_2^*$. It suffices to take $b^* = e_1^*$ and $c_0 = e_2$. The converse is trivial. \(\square\)

6. Classification of isomorphisms of collineation groups. Let $PH$ be a subgroup of $PG(J)$ containing $PS(J)$ and let $PH'$ be a subgroup of $PG(J')$ containing $PS(J')$. We prove that every isomorphism of $PH$ onto $PH'$ has the form $P \phi \to P \phi' P^{-1}$ where $P \phi: PJ \to PJ'$ is a collineation or a correlation.

Lemma 6.1. Let $a_0, b^*, c^* \in PJ$ be such that $a_0 \sim b^*, a_0 \sim c^*$, and $b^* \sim c^*$. Then there is $d^* \in PJ$ such that $a_0 \sim d^*, b^* \sim d^*$, and $c^* \sim d^*$.

Proof. We can assume that $a_0 = e_1$ and $b^* = e_1^*$, as in §2. Since $b^* \sim c^*$, $c \equiv a_1 + x[12] + y[31] (\mod MJ)$. Since $a_0 \sim c^*$, $a$ is a unit. There is $z \in \mathbb{C}$ such that $n(x, z) \equiv m$ and $n(z) \equiv 1$. It suffices to set $d = e_1 + z[12] + y[31] e_2$, since the coefficient of $e_3$ in $c \times d$ is a unit. \(\square\)

Proposition 6.2. Let $\theta$ be an isomorphism of $PS(J)$ onto $PS(J')$. Assume that there are $a_0, b^*, c^* \in PJ$ such that

1. $a_0 \sim b^*, a_0 \sim c^*$ and there is a point $d_0$ on both $b^*$ and $c^*$,

2. $P_{S_{a,b}} = P_{S_{a,c}} = P_{S_{b,d}}$ where $b^* \sim d^*$.

Then there is a collineation $P \phi: PJ \to PJ'$ such that $P \phi P_{S_{x,y}} = P \phi' P_{S_{x,y}} P^{-1}$ for all $x_0 \sim y^*$.

Proof. Let $f_*, g_*, h_0 \in PJ$ satisfy $f_0 \sim g_0 \sim h_0$, and $g_0 \sim h_0$. Let $P_{S_{a,b}} = P_{S_{a,c}} = P_{S_{b,d}}$. [Theorem 4.4]. We claim that $n_0 = q_0 f_0 \sim (g \times h)_0$, since $f_0 \sim g_0$. $a_0 \sim d_0$, since $a_0 \sim b^*$. Thus there is $P \phi \in PS(J)$ such that $P \phi a_* = (g \times h)_0$ and $P \phi d_* = f_*$. $P_{S_{f,a}}$ and $P_{S_{f,b}}$ commute exactly with $P_{S_{b,d}}$. Applying $\theta$ shows that $P_{S_{n,q}}$ and $P_{S_{q,n}}$ commute exactly with $P_{S_{b,d}}$. [Theorems 4.4 and 5.5]. Setting $P \psi = \theta P \phi \in PS(J')$ gives $P \phi P_{S_{b,d}} = \theta (P \phi P_{S_{a,b}} P \phi^{-1}) = P \psi P_{S_{a,c}} P \psi^{-1} = P_{S_{b,d}}$ and likewise $P \phi P_{S_{b,d}} = P_{S_{b,d}}$. Thus $P_{S_{n,q}}$ and $P_{S_{q,n}}$ commute exactly with $P_{S_{b,d}}$. By hypothesis, $t^* \sim v^*$, so $n_0 = (\psi \times \psi v)_0 = q_0$, as asserted.

We have shown that if $f_0 \sim g_0 \sim h_0$, and $g_0 \sim h_0$, then $P_{S_{f,x}} = P_{S_{g,x}}$ and $P_{S_{f,h}} = P_{S_{g,h}}$. Lemma 6.1 implies that this remains true without the hypothesis that $g_0 \sim h_0$. Thus there is a bijection $P \tau$ from the points of $PJ$ onto the points of $PJ'$ such that $P_{S_{x,y}} = P_{S_{x,y}}$ for all $x_0 \sim y^*$, $x_1 \sim x_2$, if and only if $P \tau x_1 \sim P \tau x_2$ [Corollary 5.6].
Take $a^*, b_*, c_* \in PJ$ such that $b_* \sim a^*, c_* \sim a^*$, and $b_* \sim c_*$. Thus $\theta P_{S_b,a}^* = P_{S_{Pr,b_*},a^*}$ and $\theta P_{S_{c,a}^*}^* = P_{S_{Pr,c_*},a^*}$, where $Pr_1 b_* \sim Pr_1 c_*$. Thus $\theta$ satisfies the duals of conditions (1) and (2). By duality, there is a bijection $Pr_2$ from the lines of $PJ$ onto the lines of $PJ'$ such that

$$\theta P_{S_{x,a}}^* = P_{S_{Pr_2,x},Pr_2,a}^*$$

for all $x_* \sim y^*$; also $y^*_t \sim y^*_s$ if and only if $Pr_2 y^*_t \sim Pr_2 y^*_s$. By Corollary 5.7, $x_*|y^*$ if and only if $Pr_1 x_* \sim Pr_2 y^*$. Thus $x_* \rightarrow Pr_1 x_*$ and $y^* \rightarrow Pr_2 y^*$ define a collineation $Pr$. We are done by (3).

A correlation of two octonion planes consists of bijections between the points of each plane and the lines of the other preserving the relations “on” and “connected to”. Let $P\Psi$ be the canonical correlation of $PJ'$ interchanging $x_*$ and $x^*$. If $P\Phi \in P\Gamma(J')$, $P\Psi P\Phi P\Psi = P\Phi^{-1}$. If $x_* \sim y^*$ in $PJ'$, $P\Psi P_{S_{x,a}^*}^* P\Psi = P_{S_{y^*,a}^*}^*$. Every correlation of $PJ$ onto $PJ'$ has the form $P\Psi P\eta$ where $\eta$ is a norm semisimilarity of $J$ onto $J'$ [3, Theorem 8.4].

**Proposition 6.3.** Let $\theta$ be an isomorphism of $PS(J)$ onto $PS(J')$. Assume that there are $a_*, b^*, c^* \in PJ$ such that

1. $a_* \sim b^*$, $a_* \sim c^*$, and there is a point $d_*$ on both $b^*$ and $c^*$, and
2. $\theta P_{S_{a,d}}^* = P_{S_{a},d^*}^*$ and $\theta P_{S_{a,c}}^* = P_{S_{a},c^*}^*$, where $s_* \sim u_*^*$.

Then there is a correlation $Pr: PJ \rightarrow PJ'$ such that $\theta P_{S_{x,a}}^* = Pr P_{S_{x,a}}^* Pr^{-1}$ for all $x_* \sim y^*$.

**Proof.** Conjugation by the canonical correlation $P\Psi$ of $PJ'$ induces an automorphism $\Psi\phi$ of $PS(J')$. ($\phi \in S$ implies $\phi^{-1} \in S$, by the proof of [3, Lemma 1.7].) $\Psi\phi$ satisfies the hypotheses of Proposition 6.2, so there is a collineation $P\eta: PJ \rightarrow PJ'$ such that $\Psi\phi P_{S_{x,a}^*}^* = P_{S_{x,a}^*}^* P\eta P_{S_{x,a}^*}^* P\eta^{-1}$ for all $x_* \sim y^*$. Set $Pr = P\Psi P\eta$.

**Proposition 6.4.** Let $\theta$ be an isomorphism of $PS(J)$ onto $PS(J')$, where $R = F$ and $R' = F'$ are fields. Then there is a collineation or a correlation $Pr: PJ \rightarrow PJ'$ such that $\theta P_{S_{x,a}}^* = Pr P_{S_{x,a}}^* Pr^{-1}$ for all $x_* \sim y^*$.

**Proof.** Take $a_*, b^* \in PJ$ such that $\theta P_{S_{a,b}}^* = P_{S_{x,a^*},e^*}$ [Theorem 4.4]. There is $c^* \in PJ$ such that $a_* \sim c^*$ and $b^* \sim c^*$ (since $PS(J)$ is transitive on pairs $a_* \sim b^*$). Let $\theta P_{S_{a,c}}^* = P_{S_{f,a^*},g^*}$. If $f_* \sim e_1$, we are done by Proposition 6.3. If $f_* \sim e_1$, we are done by applying Proposition 6.2 to $\theta$. (The hypothesis of Proposition 6.2 that there is a point on both $e_1$ and $g^*$ is satisfied, since $F'$ is a field [6, p. 35].) Thus we can assume that $f_* \sim e_1$ and $f_* \neq e_1$. Let $f = ate + s[12] + r[31]$, where $n(s) = 0 = n(r)$ and at least one of $s$ or $r$ is nonzero. By symmetry, assume that $s \neq 0$. Take $t \in O'$ such that $n(s, t) \neq 0$ and $n'(t) = 0$.

Consider the nondegenerate quadratic form $Q(x) = T'(x^*)$ on $J_0(e_3)$. If $z \in J_0^*(e_3)$ and $Q(z) \neq 0$, let $S_z \in O(J_0^*(e_3))$ be the hyperplane reflection $x \rightarrow x - Q(z)^{-1} Q(x, z) z$. Let $W$ be the orthogonal complement of the span of $e_1, e_2, s[12]$, and $t[12]$. Since $n(s) = 0$ and $s \neq 0$, $O'$ is split and $J_0^*(e_3)$ has Witt index five [8, p. 169]. Thus there is $w \in W$ such that $Q(w) = Q((s + t)[12])^{-1}$. $S_w S_{(s + t)[12]}$ belongs
to the reduced orthogonal group $\mathcal{O}(J_0(e_3))$, so it extends to $\eta \in S(J')$ fixing $e_3$ [6, p. 31]. $\eta$ fixes $e_1$ and $e_2$ and takes $s[12]$ to $-t[12]$. Since $\eta$ fixes each $e_i$, it preserves each $\mathcal{O}'[jk]$ [3, Lemma 3.2]. Since $\eta$ preserves $e_1$ and $J_0(e_1)$, $P_\eta$ fixes $e_1$ and $e_2$ and thus commutes with $P_{S_{a,e}^{s,e}}$ [Proposition 2.2]. Applying $\theta^{-1}$ shows that $\theta^{-1}P_\eta$ commutes with $P_{S_{a,e}^{s,e}}$. Then $\theta^{-1}P_\eta$ fixes $a$, so

$$P_{S_{a,e}^{s,e}(\theta^{-1}P_\eta)c^{\sigma}} = (\theta^{-1}P_\eta)P_{S_{a,e}^{s,e}}(\theta^{-1}P_\eta)^{-1}.  \tag{4}$$

Applying $\eta$ shows that

$$\eta P_{S_{a,e}^{s,e}(\theta^{-1}P_\eta)c^{\sigma}} = P_\eta \eta P_{S_{a,e}^{s,e}} P_\eta^{-1} = P_{S_{a,e}^{s,e}} P_\eta^{-1} = P_{S_{a,e}^{s,e}} P_\eta^{-1}.$$

Since $\eta$ preserves each $\mathcal{O}'[jk]$, $\eta f = ae_1 - (12) + u[31]$ for $u \in \mathcal{O}'$. The coefficient of $e_3$ in $f \times \eta f$ is nonzero, so $f_\eta \sim P_{\eta f}$. There is a point on both $e^*$ and $(\theta^{-1}P_\eta)c^*$, since $F'$ is a field [6, p. 35]. We are done by applying Proposition 6.3 with equations (4) and $\theta P_{S_{a,e}^{s,e}} = P_{S_{a,e}^{s,e}}$. □

**Theorem 6.5.** Let $PH$ be a subgroup of $PG(J')$ containing $PS(J)$ and let $PH'$ be a subgroup of $PG(J')$ containing $PS(J')$. Let $\theta$ be an isomorphism of $PH$ onto $PH'$ such that $\theta PS(J) = PS(J')$. Then there is a collineation or a correlation $Pr: PJ \to PJ'$ such that $\theta P\phi = PrP\phi Pr^{-1}$ for all $P \phi \in PH$.

**Proof.** Let $PS_m(J)$ be the kernel of the canonical homomorphism from $PS(J)$ to $PS(J/mJ')$. $PS_m(J)$ is the unique largest subgroup of $PS(J)$, and $PS(J)/PS_m(J) \simeq PS(J/mJ)$ [3, Corollaries 6.5 and 7.5]. Thus $\theta$ induces an isomorphism $\theta_m$ of $PS(J/mJ)$ onto $PS(J'/m'J')$. Take $a, b, c \in PJ$ such that $a \sim b, a \sim c$, and $b \sim c$. Let $P_{S_{a,b}^{s,c}} = P_{S_{a,b}^{s,c}}^{s,c}$ and $\theta P_{S_{a,b}^{s,c}} = P_{S_{u,v}^{u,v}}^{s,c}$ [Theorem 4.4]. If $p: PJ \to P(J/mJ)$ and $p': PJ' \to P(J'/m'J')$ are the canonical maps,

$$\theta_m P_{S_{a,b}^{s,c}} P_{b^*} = P_{S_{a,b}^{s,c}} P_{b^*}, \quad \theta_m P_{S_{a,b}^{s,c}} P_{c^*} = P_{S_{u,v}^{u,v}} P_{c^*}.  \tag{5}$$

Since $pb^* \sim pc^*$, Proposition 6.4 implies that either $p'c_\phi \sim p'u_\phi$ or $p'i^* \sim p'v^*$, so either $s_\phi \sim u_\phi$ or $t^* \sim v^*$. Thus Propositions 6.2 and 6.3 imply that there is a collineation or a correlation $Pr: PJ \to PJ'$ such that $\theta P_{S_{a,b}^{s,c}} = PrP_{S_{a,b}^{s,c}} Pr^{-1}$ for all $x \sim y$. If $P \phi \in PH$,

$$(\theta P\phi) PrP_{S_{a,b}^{s,c}} Pr^{-1}(\theta P\phi)^{-1} = (\theta P\phi) P_{S_{a,b}^{s,c}} Pr^{-1} = \theta (P\phi P_{S_{a,b}^{s,c}} Pr^{-1}.  \tag{6}$$

Proposition 2.2 implies that $(\theta P\phi) Pr = PrP\phi$, as required. □

**Corollary 6.6.** Let $H$ be a subgroup of $\Gamma(J)$ containing $S(J)$ and let $H'$ be a subgroup of $\Gamma(J')$ containing $S(J')$. Let $\theta$ be an isomorphism of $H$ onto $H'$ such that $\theta S(J) = S(J')$. Then there is a norm semisimilarity $\tau: J \to J'$ and a map $\chi: H \to R' - m'$ such that either $\phi = (\chi \phi) \tau \phi^{-1}$ or $\phi = (\chi \phi) \tau \phi^{-1} \tau^{-1}$ for all $\phi \in H$. If $\phi_1, \phi_2 \in H$,

$$\chi(\phi_1 \phi_2) = (\chi \phi_1)(\sigma_1 \sigma^{-1}(\chi \phi_2))$$

where $\tau$ is $\sigma$-semilinear and $\phi_1$ is $\sigma_1$-semilinear.
PROOF. Proposition 2.2 implies that the centralizer of PS(J) in PΓ(J) is trivial, so \( R - m \) is the centralizer of \( S(J) \) in \( \Gamma(J) \). Then \( \theta \) maps \( H \cap (R - m) \) onto \( H' \cap (R' - m') \) and \( \theta \) induces an isomorphism \( \Phi \) of \( PH \) onto \( PH' \). By Theorem 6.5, there is a norm semisimilarity \( \tau \) of \( J \) onto \( J' \) such that either \( \Phi \Phi \Phi = \Phi \Phi \Phi^{-1} \) or \( \Phi \Phi \Phi = \Phi \Phi \Phi^{-1} \) for all \( \Phi \in PH \). Thus there is a map \( \chi: H \to R' - m' \) such that either \( \theta \phi = (\chi \phi) \tau \phi \tau^{-1} \) or \( \theta \phi = (\chi \phi) \tau \phi \tau^{-1} \) for \( \phi \in H \). (5) is equivalent to the condition that \( \theta \) is a homomorphism. □

\underline{Corollary 6.7.} (1) The hypothesis that \( \theta S(J) = S(J') \) can be deleted from Theorem 6.5 if \( PH \subset PG(J) \) and \( PH' \subset PG(J') \).

(2) The hypothesis that \( \theta S(J) = S(J') \) can be deleted from Corollary 6.6 if \( H \subset G(J) \) and \( H' \subset G(J') \). In this case, \( \chi \) is a homomorphism whose kernel contains \( S(J) \).

\underline{Proof.} (1) If \( K \) is a group, let \( [K, K] \) denote the subgroup generated by the commutators of elements of \( K \). \([S(J), S(J)] = S(J) \) [3, equation (v) and Corollary 6.5]. It is immediate that \([G(J), G(J)] \subset S(J) \), so \([PH, PH] = PS(J) \) and \([PH', PH'] = PS(J') \). Hence \( \theta PS(J) = PS(J') \). (2) As above, \([H, H] = S(J) \) and \([H', H'] = S(J') \), so \( \theta S(J) = S(J') \). (5) and the assumption that \( H \subset G(J) \) imply that \( \chi \) is a homomorphism. \( \chi(S(J)) = 1 \), since \( S(J) = [H, H] \) and \( R' - m' \) is abelian. □

\underline{Corollary 6.8.} (1) The hypothesis that \( \theta PS(J) = PS(J') \) can be deleted from Theorem 6.5 if \( R = F \) and \( R' = F' \) are fields.

(2) The hypothesis that \( \theta S(J) = S(J') \) can be deleted from Corollary 6.6 if \( R = F \) and \( R' = F' \) are fields.

\underline{Proof.} (1) Since \( PS(J) \) is a normal subgroup of \( PΓ(J) \) and \( \theta PH = PH' \supset PS(J') \), \( \theta PS(J) \) is normalized by \( PS(J') \). Since \( F' \) is a field, any proper subgroup of \( PΓ(J') \) normalized by \( PS(J') \) contains \( PS(J') \) [3, Corollary 7.2]. Thus \( \theta PS(J) \) contains \( PS(J') \), and replacing \( \theta \) by \( \theta^{-1} \) gives the reverse containment. (2) Since \( S(J) \) is a normal subgroup of \( Γ(J) \) and \( \theta H = H' \supset S(J') \), \( \theta S(J) \) is normalized by \( S(J') \). Since \( F' \) is a field, any subgroup of \( Γ(J') \) normalized by \( S(J') \) is either contained in \( F' - 0 \) or contains \( S(J') \) [3, Theorem 7.1]. Since \( S(J) \) is nonabelian, \( \theta S(J) \) contains \( S(J') \). The reverse containment holds by symmetry. □

\underline{References}


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