

## ON ASYMPTOTICALLY ALMOST PERIODIC SOLUTIONS OF A CONVOLUTION EQUATION

BY

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**ABSTRACT.** We study questions related to asymptotic almost periodicity of solutions of the linear convolution equation  $(*) \mu * x = f$ . Here  $\mu$  is a complex measure, and  $x$  and  $f$  are bounded functions. Basically we are interested in conditions which imply that bounded solutions of  $(*)$  are asymptotically almost periodic. In particular, we show that a certain necessary condition on  $f$  for this to happen is also sufficient, thereby strengthening earlier results. We also include a result on existence of bounded solutions, and indicate a generalization to a distribution equation.

**1. Introduction.** We study the linear convolution equation

$$\mu * x = f. \tag{1.1}$$

Here  $\mu$  is a complex measure,  $x$  and  $f$  are bounded, complex functions, and  $\mu * x$  is the convolution of  $\mu$  and  $x$ . More specifically, we are interested in the existence of asymptotically almost periodic solutions. If  $x$  is asymptotically almost periodic, then so is  $f$ , and therefore we throughout (except in §5) assume that  $f$  is asymptotically almost periodic.

The zero set  $Z$  of the Fourier transform of  $\mu$  plays a crucial role in the analysis of (1.1). The case when  $Z$  is empty is fairly well known, but when  $Z$  is nonempty there still remain some questions to be answered. See, e.g., the discussions in [3], [4], [5] and [7]. This paper has been motivated by two facts. First of all, the existing results are not as sharp as they could be, and secondly, some of them are only given in a more or less implicit way.

There is, in particular, one fact that has not been fully recognized in the treatments cited above. The question of whether bounded solutions of (1.1) are asymptotically almost periodic or not can be rephrased as a question concerning a comparison of the local behavior of the Fourier transforms of  $\mu$  and  $f$ . No global conditions (like moment conditions) on  $\mu$  and  $f$  are needed. Of course, global conditions may be useful in the sense that they imply local conditions on Fourier transforms, but they are not necessary.

We begin in §§2–3 by discussing the patching technique which is needed when one turns local results into global ones. In §2 we discuss almost periodic functions, and in §3 asymptotically almost periodic functions. We get to the heart of the matter in §4, where we discuss the asymptotic almost periodic behavior of (1.1). We first investigate what can be said about the bounded solutions of (1.1) when no

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other conditions are imposed on  $f$  than asymptotic almost periodicity. We then study necessary and sufficient conditions on  $f$  which imply that bounded solutions are asymptotically almost periodic. It turns out that this question is related to the question of existence of bounded solutions of (1.1). We discuss this in §5. Finally, in §6 we indicate how the results can be extended to a distribution equation.

The results in §§2–3 and the first part of §4 are not really new. Many of them have perhaps not been recorded in this explicit form before, but their proofs are more or less adaptations or combinations of known arguments.

**2. Preliminaries. Almost periodic functions.** We let  $M$  be the space of (bounded) complex measures on  $R$ , and regard the set  $L^1$  of Lebesgue integrable functions on  $R$  as a subset of  $M$ . We denote the set of bounded, Borel measurable functions on  $R$  by  $L^\infty$ . One could work equally well in the class of essentially bounded, Lebesgue measurable functions, but we prefer to use bounded Borel functions, because then the *convolution*

$$\mu * x(t) = \int_R x(t-s) d\mu(s) \quad (t \in R)$$

is defined everywhere rather than almost everywhere for  $\mu \in M$ ,  $x \in L^\infty$ . Both  $M$  and  $L^\infty$  are contained in the space  $\mathcal{S}'$  of tempered distributions, and so the *Fourier transform* operator  $\hat{\phantom{x}}$  is well defined on them. In particular, the Fourier transform  $\hat{\varphi}$  of a function  $\varphi \in L^\infty$  is a distribution. We call the support of  $\hat{\varphi}$  the spectrum of  $\varphi$ , and denote it  $\sigma(\varphi)$ . This definition of a spectrum is equivalent to several other definitions commonly used. See, e.g., the discussions in [6, Chapter VI] and [7, pp. 297–299]. In particular, one has

$$\sigma(\varphi) = \bigcap_{a \in A} \{\omega \in R \mid \hat{a}(\omega) = 0\}, \quad (2.1)$$

where  $A = \{a \in L^1 \mid a * \varphi = 0\}$  [13, p. 243]. This fact is closely related to Wiener's Tauberian theorem. Another version of Wiener's Tauberian theorem which we shall use frequently is the following lemma:

**WIENER'S LEMMA.** *Let  $\mu \in M$ ,  $\omega_0 \in R$ , and  $\hat{\mu}(\omega_0) \neq 0$ . Then there exists  $a \in L^1$  such that  $\hat{a}(\omega)\hat{\mu}(\omega) = 1$  in a neighborhood of  $\omega_0$ .*

**PROOF.** There exists  $\varepsilon > 0$  such that  $\hat{\mu}(\omega) \neq 0$  for  $|\omega - \omega_0| < \varepsilon$ . Choose  $\eta_1 \in L^1$  such that  $\hat{\eta}_1(\omega) = 1$  for  $|\omega - \omega_0| < \varepsilon$ , and  $\eta_2 \in L^1$  such that  $\hat{\eta}_2(\omega) = 1$  for  $|\omega - \omega_0| \leq \varepsilon/2$ , and  $\hat{\eta}_2(\omega) = 0$  for  $|\omega - \omega_0| \geq \varepsilon$ . Then  $\mu * \eta_1 \in L^1$ , and by Lemma 6.1 in [6, p. 227], there exists  $a \in L^1$  such that  $\hat{a}(\omega) = \hat{\eta}_2(\omega)/(\hat{\mu}(\omega)\hat{\eta}_1(\omega))$  for  $|\omega - \omega_0| < \varepsilon$ . Clearly then  $\hat{a}(\omega)\hat{\mu}(\omega) = 1$  for  $|\omega - \omega_0| < \varepsilon/2$ .  $\square$

We call a function  $\varphi \in L^\infty$  *asymptotically uniformly continuous* if it satisfies the Tauberian condition

$$\lim_{t \rightarrow \infty, s \rightarrow 0} [\varphi(t+s) - \varphi(t)] = 0. \quad (2.2)$$

This name is motivated by the fact that every bounded  $\varphi$  satisfying (2.2) is of the form  $p + q$ , where  $p$  is bounded and uniformly continuous, and  $q(t) \rightarrow 0$  ( $t \rightarrow \infty$ ) (see [15, p. 866]). We denote the set of bounded and asymptotically uniformly

continuous functions by BAUC. The set BAUC is a closed subspace of  $L^\infty$ . Moreover, if  $\mu \in M$  and  $\varphi \in \text{BAUC}$ , then  $\mu * \varphi \in \text{BAUC}$ .

The limit set  $\Gamma(\varphi)$  of a function  $\varphi \in \text{BAUC}$  is defined as follows:

$$\Gamma(\varphi) = \{ \psi \in \text{BUC} \mid \tau_{\alpha_k} \varphi \rightarrow \psi \text{ uniformly on compact sets, for some sequence } \alpha_k \rightarrow \infty \}.$$

Here BUC is the set of bounded and uniformly continuous functions, and the translation operator  $\tau_\alpha$  is defined by  $\tau_\alpha \varphi(t) = \varphi(t + \alpha)$  ( $t, \alpha \in R$ ). Some properties of  $\Gamma(\varphi)$  are given in [13] and [14]. The asymptotic spectrum  $\sigma_\infty(\varphi)$  of a function  $\varphi \in \text{BAUC}$  is by definition the closure of the union of the spectra of the functions contained in  $\Gamma(\varphi)$ .

A function  $\varphi \in \text{BUC}$  is almost periodic ( $\varphi \in \text{AP}$ ), if the set of translates  $\{ \tau_\alpha \varphi \mid \alpha \in R \}$  of  $\varphi$  is relatively compact in  $L^\infty$ . Equivalently,  $\varphi$  is almost periodic if for every  $\varepsilon > 0$ , the  $\varepsilon$ -translation set  $\{ \alpha \mid \sup | \tau_\alpha \varphi - \varphi | < \varepsilon \}$  is relatively dense, i.e. its complement does not contain arbitrarily long intervals. The set AP is a closed subspace of  $L^\infty$ , e.g., sums and uniform limits of functions in AP belong to AP. Moreover, if  $\mu \in M$  and  $\varphi \in \text{AP}$ , then  $\mu * \varphi \in \text{AP}$ . For more details see [2, Chapter 1] and [6, §VI.5].

Almost periodic functions have the nice property that their asymptotic behavior determines their global behavior completely.

LEMMA 2.1. Let  $\varphi \in \text{AP}$ . Then  $\varphi \in \Gamma(\varphi)$ .

The converse of Lemma 2.1 is not true. There exist functions satisfying  $\varphi \in \Gamma(\varphi)$  which are not almost periodic (e.g., the almost automorphic functions).

PROOF. For each  $k \in N$ , the  $(1/k)$ -translation set of  $\varphi$  is relatively dense, and so we can find  $\alpha_k \geq k$  such that  $\sup | \tau_{\alpha_k} \varphi - \varphi | \leq 1/k$ . Thus  $\tau_{\alpha_k} \varphi \rightarrow \varphi$  uniformly,  $\alpha_k \rightarrow \infty$  ( $k \rightarrow \infty$ ), and we conclude that  $\varphi \in \Gamma(\varphi)$ .  $\square$

It follows from Lemma 2.1 that the set AP is closed under convergence which a priori is weaker than uniform convergence.

LEMMA 2.2. Let  $\varphi_n \in \text{AP}$ , and suppose that

$$\lim_{m,n,t \rightarrow \infty} [ \varphi_m(t) - \varphi_n(t) ] = 0. \tag{2.3}$$

Then  $\varphi_n$  converges uniformly to a function  $\varphi \in \text{AP}$ .

Lemma 2.2 is implicitly contained in [3]. For completeness we include a proof.

PROOF. Fix  $m, n \in N$ . Then  $\varphi_m - \varphi_n \in \text{AP}$ , and so by Lemma 2.1,  $\varphi_m - \varphi_n \in \Gamma(\varphi_m - \varphi_n)$ . This implies

$$\sup_{t \in R} | \varphi_m(t) - \varphi_n(t) | = \limsup_{t \rightarrow \infty} | \varphi_m(t) - \varphi_n(t) |.$$

As (2.3) is equivalent to

$$\lim_{m,n \rightarrow \infty} \limsup_{t \rightarrow \infty} | \varphi_m(t) - \varphi_n(t) | = 0,$$

we conclude that  $\varphi_n$  is a uniform Cauchy sequence, and so  $\varphi_n$  converges uniformly to a function  $\varphi \in \text{AP}$ .  $\square$

When one works with asymptotically almost periodic functions (defined in §3), the concept of convergence expressed by (2.3) is quite useful. We say that  $\varphi_n$  converges asymptotically to  $\varphi$ , if

$$\lim_{n,t \rightarrow \infty} [\varphi_n(t) - \varphi(t)] = 0. \quad (2.4)$$

We call a sequence satisfying (2.3) an *asymptotic Cauchy sequence*. If  $\varphi \in L^\infty$ ,  $\varphi_n \in \text{BAUC}$  and  $\varphi_n \rightarrow \varphi$  asymptotically, then  $\varphi \in \text{BAUC}$ . In this sense BAUC is closed even under asymptotic convergence.

The almost periodicity of a bounded function can be regarded as a local property of its Fourier transform. Let  $\varphi \in L^\infty$ . We call  $\hat{\varphi}$  *almost periodic at a point*  $\omega \in R$  ( $\hat{\varphi} \in \text{AP}^\wedge$  at  $\omega$ ), if there exists  $\psi \in \text{AP}$  such that  $\hat{\varphi} = \hat{\psi}$  in a neighborhood of  $\omega$ .

**THEOREM 2.3.** *A function  $\varphi \in \text{BAUC}$  is almost periodic iff  $\hat{\varphi}$  is almost periodic everywhere.*

Observe that we do not a priori require  $\varphi$  to be uniformly continuous. The uniform continuity, or more precisely, as elements of  $L^\infty$  are determined uniquely only modulo the values in a set of measure zero, the fact that  $\varphi$  equals a uniformly continuous function a.e., follows from the asymptotic uniform continuity together with the Fourier transform condition. A similar observation is found in [3, Theorem 1].

In the proof of Theorem 2.3, as well as in several other places, we shall need a sequence of functions approximating the unit point mass at zero (the Dirac  $\delta$ -measure). We call a sequence  $\eta_n$  belonging to the set  $\mathfrak{S}$  of rapidly decreasing test functions a  $\delta$ -sequence, if  $\eta_n(t) = n\eta(nt)$  ( $n \in N$ ,  $t \in R$ ), where  $\eta$  is a fixed function satisfying  $\hat{\eta}(0) = 1$ . Recall that  $\hat{\eta}_n(\omega) = \hat{\eta}(\omega/n)$  [6, p. 121]. If  $\varphi \in \text{BAUC}$ ,  $\eta_n$  is a  $\delta$ -sequence, and  $\varphi_n = \eta_n * \varphi$ , then  $\varphi_n \rightarrow \varphi$  pointwise almost everywhere as well as asymptotically (cf. [3, Lemma 2]). If  $\sigma(\eta)$  is compact, then  $\sigma(\eta_n)$  is compact for every  $n$ , and we say that the  $\delta$ -sequence has a compact spectrum. If  $\hat{\eta}(\omega) \neq 0$  ( $\omega \in R$ ), then every  $\eta_n$  satisfies the same inequality, and we say that the  $\delta$ -sequence has a nonvanishing Fourier transform.

**PROOF OF THEOREM 2.3.** Trivially, if  $\varphi \in \text{AP}$ , then  $\hat{\varphi} \in \text{AP}^\wedge$  everywhere.

Suppose that  $\hat{\varphi} \in \text{AP}^\wedge$  everywhere, and that  $\sigma(\varphi)$  is compact. Then we claim that  $\varphi \in \text{AP}$ . This follows from a lemma in [6, p. 166], but as we shall anyhow need the same argument later we repeat it below. As  $\sigma(\varphi)$  is compact, and  $\hat{\varphi} \in \text{AP}^\wedge$  everywhere, we can find open, bounded intervals  $I_k$  ( $1 \leq k \leq n$ ) and functions  $\psi_k \in \text{AP}$  such that  $\sigma(\varphi) \subset \bigcup I_k$ , and  $\hat{\varphi} = \hat{\psi}_k$  in  $I_k$ . Pick functions  $\eta_k \in \mathfrak{S}$  such that  $\sigma(\eta_k) \subset I_k$ , and  $\sum \hat{\eta}_k = 1$  on  $\sigma(\varphi)$ . Define  $\varphi_k = \eta_k * \varphi$ . Then  $\varphi = (\sum \eta_k) * \varphi = \sum \varphi_k$ . Now  $\sigma(\eta_k) \subset I_k$ , and  $\hat{\varphi} = \hat{\psi}_k$  in  $I_k$ , and so  $\varphi_k = \eta_k * \varphi = \eta_k * \psi_k \in \text{AP}$ . Thus  $\varphi \in \text{AP}$ .

Finally, suppose that  $\hat{\varphi} \in \text{AP}^\wedge$  everywhere, but that  $\sigma(\varphi)$  is not compact. Define  $\varphi_n = \eta_n * \varphi$ , where  $\eta_n$  is a  $\delta$ -sequence with compact spectrum. Let  $\omega \in R$ . Then there exists  $\psi \in \text{AP}$  and a neighborhood  $I$  of  $\omega$  such that  $\hat{\varphi} = \hat{\psi}$  in  $I$ . But  $\eta_n * \psi \in \text{AP}$ , and  $\hat{\varphi}_n = \hat{\eta}_n \hat{\varphi} = \hat{\eta}_n \hat{\psi} = (\eta_n * \psi)^\wedge$  in  $I$ , and so  $\hat{\varphi}_n \in \text{AP}^\wedge$  everywhere. Moreover,  $\sigma(\varphi_n)$  is compact, and so  $\varphi_n \in \text{AP}$ . Let  $n \rightarrow \infty$ . Then  $\varphi_n \rightarrow \varphi$  almost

everywhere and asymptotically. But this implies that  $\varphi_n$  is an asymptotic Cauchy sequence, and so by Lemma 2.2,  $\varphi_n$  converges uniformly. Thus  $\varphi \in AP$ .  $\square$

There is an alternative way of proving Theorem 2.3 without using asymptotic convergence. One could first prove that  $\varphi$  is an almost periodic distribution (cf. §6), and then use the inclusion  $\varphi \in \Gamma(\varphi)$  to get the uniform continuity. Of course, this means that one has to restate and reprove Lemma 2.1 for almost periodic distributions.

We recall the following result on almost periodicity:

**THEOREM 2.4.** *Let  $\varphi \in L^\infty$ . Then the set where  $\hat{\varphi}$  is not almost periodic is perfect.*

This theorem is implicit in [6].

**PROOF.** Trivially, the set where  $\hat{\varphi} \notin AP^*$  is closed.

Suppose that  $\hat{\varphi} \notin AP^*$  at  $\omega_0$ , but that for some neighborhood  $I$  of  $\omega_0$ ,  $\hat{\varphi} \in AP^*$  in  $I \setminus \{\omega_0\}$ . Choose  $\eta \in \mathcal{S}$  such that  $\sigma(\eta)$  is compact,  $\sigma(\eta) \subset I$ , and  $\hat{\eta} = 1$  in a neighborhood of  $\omega_0$ . Define  $\psi = \eta * \varphi$ . Then  $\hat{\psi} \in AP^*$  everywhere, except possibly at  $\omega_0$ , and  $\sigma(\psi)$  is compact. By Theorem 5.20 in [6, p. 167], applied to the function  $e^{i\omega_0 t} \psi(t)$ ,  $\psi \in AP$ . Thus  $\hat{\varphi} \in AP^*$  at  $\omega_0$ , and this contradiction shows that the set where  $\hat{\varphi} \notin AP^*$  has no isolated point.  $\square$

Combining Theorems 2.3 and 2.4 we get the following result:

**COROLLARY 2.5.** *A function  $\varphi \in BAUC$  is almost periodic iff the set where  $\hat{\varphi}$  is not almost periodic is countable.*

This is true because every nonempty perfect set is uncountable.

In particular, if  $\varphi \in BAUC$  has a countable spectrum, then  $\varphi \in AP$ , because in the complement of the spectrum,  $\hat{\varphi}$  equals zero locally, hence is almost periodic.

**3. Asymptotically almost periodic functions.** A function  $\varphi \in L^\infty$  is *asymptotically almost periodic* ( $\varphi \in AAP$ ), if it is of the form  $\varphi = p + q$ , where  $p \in AP$  and  $q(t) \rightarrow 0$  ( $t \rightarrow \infty$ ). By using Lemma 2.1 one can easily show that the splitting of  $\varphi$  into  $p + q$  is unique. Clearly,  $AAP \subset BAUC$ . The set  $AAP$  is a closed subspace of  $L^\infty$ , and if  $\mu \in M$ ,  $\varphi \in AAP$ , then  $\mu * \varphi \in AAP$ . For more details, see [2, Chapter 9].

As a matter of fact,  $AAP$  is even closed under asymptotic convergence.

**LEMMA 3.1.** *Let  $\varphi \in L^\infty$ ,  $\varphi_n \in AAP$ , and suppose that  $\varphi_n \rightarrow \varphi$  asymptotically. Then  $\varphi \in AAP$ .*

The proof of Lemma 3.1 is an adaption of an argument in [3].

**PROOF.** As  $\varphi_n$  converges asymptotically, it is an asymptotic Cauchy sequence, i.e. given  $\epsilon > 0$  we can find  $M > 0$  such that

$$|\varphi_m(t) - \varphi_n(t)| < \epsilon \quad (m, n, t \geq M).$$

Split each  $\varphi_n$  into  $\varphi_n = p_n + q_n$ , where  $p_n \in AP$  and  $q_n(t) \rightarrow 0$  ( $t \rightarrow \infty$ ). Then, for  $m, n \geq M$ ,

$$\limsup_{t \rightarrow \infty} |p_m(t) - p_n(t)| \leq \epsilon,$$

i.e.  $p_n$  is an asymptotic Cauchy sequence. By Lemma 2.2, there exists  $p \in AP$  such that  $p_n \rightarrow p$  uniformly. Define  $q = \varphi - p$ . Then  $q_n \rightarrow q$  asymptotically, and as each  $q_n$  satisfies  $q_n(t) \rightarrow 0$  ( $t \rightarrow \infty$ ), we find that  $q(t) \rightarrow 0$  ( $t \rightarrow \infty$ ).  $\square$

Let  $\varphi \in L^\infty$ . We call  $\hat{\varphi}$  asymptotically almost periodic at a point  $\omega$  ( $\hat{\varphi} \in AAP^\wedge$  at  $\omega$ ), if there exists  $\psi \in AAP$  such that  $\hat{\varphi} = \hat{\psi}$  in a neighborhood of  $\omega$ .

The following analogue of Theorem 2.3 holds.

**THEOREM 3.2.** *A function  $\varphi \in BAUC$  is asymptotically almost periodic iff  $\hat{\varphi}$  is asymptotically almost periodic everywhere.*

As far as we know, Theorem 3.2 is new.

**PROOF.** Trivially, if  $\varphi \in AAP$ , then  $\hat{\varphi} \in AAP^\wedge$  everywhere.

Suppose that  $\hat{\varphi} \in AAP^\wedge$  everywhere, and that  $\sigma(\varphi)$  is compact. Then exactly the same argument as in the corresponding part of the proof of Theorem 2.3 (with AP replaced by AAP) shows that  $\varphi \in AAP$ .

If  $\hat{\varphi} \in AAP^\wedge$  everywhere, but  $\sigma(\varphi)$  is not compact, then one again argues as in the last part of the proof of Theorem 2.3, but replaces AP by AAP and Lemma 2.2 by Lemma 3.1.  $\square$

The analogue of Theorem 2.4 is not true for asymptotically almost periodic functions. Take the specific example  $\varphi(t) = \sin(|t|^{1/2})$  ( $t \in R$ ). Then  $\varphi'(t) \rightarrow 0$  ( $t \rightarrow \infty$ ), and so every  $\psi \in \Gamma(\varphi)$  is a constant. Try to write  $\varphi$  in the form  $\varphi = p + q$ , where  $p \in AP$  and  $q(t) \rightarrow 0$  ( $t \rightarrow \infty$ ). Then  $\Gamma(\varphi) = \Gamma(p)$ , and so by Lemma 2.1,  $p$  is a constant. But that implies that  $\varphi(t)$  tends to a constant as  $t \rightarrow \infty$ , and this contradiction shows that  $\varphi \notin AAP$ . On the other hand, if  $\eta \in \mathfrak{S}$  and  $\hat{\eta}(0) = 0$ , then, e.g., an integration by parts shows that  $\eta * \varphi(t) \rightarrow 0$  ( $t \rightarrow \infty$ ), (hence  $\eta * \varphi \in AAP$ ). This implies that  $\hat{\varphi} \in AAP^\wedge$  in  $R \setminus \{0\}$ . Combined with Theorem 3.2 this argument shows that the set where  $\hat{\varphi} \notin AAP^\wedge$  consists of exactly one point, i.e. the origin.

**4. The equation  $\mu * x = f$ .** We now turn to equation (1.1). Recall that  $Z$  stands for the zero set,

$$Z = \{\omega \in R \mid \hat{\mu}(\omega) = 0\}. \tag{4.1}$$

We begin by collecting some more or less known results on the asymptotic behavior of a bounded solution  $x$  of (1.1) in the case when nothing more is assumed of  $f$  than asymptotic almost periodicity.

First of all, the case when  $f$  is asymptotically almost periodic separates into two simpler cases.

**PROPOSITION 4.1.** *Let  $\mu \in M$ ,  $x \in BAUC$ ,  $f \in AAP$  satisfy (1.1), and write  $f$  in the form  $f = p + q$ , where  $p \in AP$  and  $q(t) \rightarrow 0$  ( $t \rightarrow \infty$ ). Then  $x$  is of the form  $x = y + z$ , where  $y \in BUC$ ,  $z \in BAUC$ , and*

$$\mu * y = p, \quad \mu * z = q. \tag{4.2}$$

This result is essentially the same as Lemma 1 in [4]. For completeness we give a proof.

**PROOF.** By Lemma 2.1, we can find a sequence  $\alpha_k \rightarrow \infty$  such that  $\tau_{\alpha_k} p \rightarrow p$  uniformly on compact sets. By passing to a subsequence we may assume that  $\tau_{\alpha_k} x$

converges uniformly on compact sets to some function  $y \in \Gamma(x) \subset \text{BUC}$ . By (1.1),

$$\mu * (\tau_{\alpha_k} x) = \tau_{\alpha_k} p + \tau_{\alpha_k} q.$$

Let  $k \rightarrow \infty$ . Then  $\tau_{\alpha_k} p(t) \rightarrow p(t)$ ,  $\tau_{\alpha_k} q(t) \rightarrow 0$  ( $t \in R$ ), and by Lebesgue's dominated convergence theorem,  $\mu * (\tau_{\alpha_k} x)(t) \rightarrow \mu * y(t)$  ( $t \in R$ ). Thus  $\mu * y = p$ . Defining  $z = x - y$  one gets the conclusion of Proposition 4.1.  $\square$

The splitting of  $x$  into  $y + z$  is not unique if  $Z \neq \emptyset$ . Any solution  $x_0 \in \text{BUC}$  of the homogeneous equation

$$\mu * x = 0 \tag{4.3}$$

can obviously be subtracted from  $y$  and added to  $z$ . At least for some measures  $\mu$  these solutions can be completely characterized by their spectral properties.

**PROPOSITION 4.2.** *Let  $\mu \in M$ ,  $x \in L^\infty$  satisfy (4.3). Then  $\sigma(x) \subset Z$ . Conversely, suppose that  $\mu \in M$ ,  $x \in L^\infty$ ,  $\sigma(x) \subset Z$ , and that either  $Z$  has a countable boundary, or that  $\hat{\mu}$  is Hölder continuous with exponent  $\frac{1}{2}$ . Then (4.3) holds.*

**PROOF.** Let (4.3) hold, and let  $\eta \in \mathcal{S}$  have a nonvanishing Fourier transform. Then  $(\mu * \eta) * x = 0$ , and by (2.1),  $\sigma(x) \subset \{\omega \in R \mid \hat{\eta}(\omega)\hat{\mu}(\omega) = 0\} = Z$ .

Conversely, if  $\mu$  is absolutely continuous and  $\sigma(x) \subset Z$ , then it follows from a result essentially due to Mandelbrojt that  $\sigma(\mu * x)$  is a perfect subset of the intersection of  $\sigma(x)$  and the closure of the set  $R \setminus Z = \{\omega \in R \mid \hat{\mu}(\omega) \neq 0\}$  (see [9, Lemmas 9.1–9.2]; Lemma 9.1 in [9] is stated in a weaker way, but its proof yields exactly the preceding claim). Thus,  $\sigma(\mu * x)$  is a perfect subset of the boundary of  $Z$  (intersected with the boundary of  $\sigma(x)$ ). As every nonempty perfect set is uncountable, we conclude that  $\sigma(\mu * x) = \emptyset$ , hence  $\mu * x = 0$ . The general case follows if one convolves (4.3) with a  $\delta$ -sequence with a nonvanishing Fourier transform. Finally, if  $\mu$  is absolutely continuous and  $\hat{\mu}$  is Hölder continuous with exponent  $\frac{1}{2}$ , then (4.3) follows from [9, Theorem 11.1]. Again, one gets the general case by convolving (4.3) with a  $\delta$ -sequence.  $\square$

We now return to our main line of development and study one of the two extreme cases in Proposition 4.1.

**PROPOSITION 4.3.** *Let  $\mu \in M$ ,  $x \in \text{BAUC}$ ,  $f \in \text{AP}$  satisfy (1.1), and suppose that  $Z$  is countable. Then  $x \in \text{AP}$ .*

The converse is trivial: If  $x \in \text{AP}$  then  $f \in \text{AP}$ . The claim  $x \in \text{AP}$  means that  $x$  a.e. equals an almost periodic function. In the case when  $f = 0$  and  $Z$  has no accumulation point, Proposition 4.3 is equivalent to Theorem 1 in [3]. The version one gets by replacing  $x \in \text{BAUC}$  by  $x \in \text{BUC}$  can be deduced, e.g., from [7, Proposition 8.1] combined with [1, Lemma 2]. For completeness we include a proof.

**PROOF.** It follows from (1.1) that  $\hat{\mu}\hat{x} = \hat{f}$ . By Wiener's Lemma, if  $\omega_0 \notin Z$ , then there exists a function  $a \in L^1$  such that  $\hat{a}(\omega)\hat{\mu}(\omega) = 1$  in a neighborhood of  $\omega_0$ . Thus  $\hat{x} = \hat{a}\hat{f}$  in a neighborhood of  $\omega_0$ , and as  $a * f \in \text{AP}$ , we find that  $\hat{x} \in \text{AP}^*$  at  $\omega_0$ . By Corollary 2.5,  $x \in \text{AP}$ .  $\square$

It is interesting to observe that the countability assumption on  $Z$  is necessary: If  $Z$  is uncountable and  $f \in AP$ , then either (1.1) has no bounded solution, or it has solutions which are not almost periodic. This follows from a construction in [7, pp. 300–301].

In the other extreme case, i.e. when  $f(t) \rightarrow 0 (t \rightarrow \infty)$ , we get the following result:

**PROPOSITION 4.4.** *Let  $\mu \in M, x \in BAUC, f \in L^\infty, f(t) \rightarrow 0 (t \rightarrow \infty)$ , and suppose that (1.1) holds. Then  $\sigma_\infty(x) \subset Z$ . Conversely, suppose that  $\mu \in M, x \in BAUC, \sigma_\infty(x) \subset Z$ , and suppose that either  $Z$  has a countable boundary, or that  $\hat{\mu}$  is Hölder continuous with exponent  $\frac{1}{2}$ . Define  $f$  by (1.1). Then  $f(t) \rightarrow 0 (t \rightarrow \infty)$ .*

The asymptotic spectrum  $\sigma_\infty(x)$  of  $x$  was defined in §2.

**PROOF.** Take any  $y \in \Gamma(x)$ , and a sequence  $\alpha_k \rightarrow \infty$  such that  $\tau_{\alpha_k} x \rightarrow y$  uniformly on compact sets. Then, by (1.1), Lebesgue’s dominated convergence theorem and the fact that  $f(t) \rightarrow 0 (t \rightarrow \infty)$ ,  $\mu * y = 0$ . Proposition 4.2 yields  $\sigma(y) \subset Z$ . Thus  $\sigma_\infty(x) \subset Z$ .

For the converse part, first observe that  $f \in BAUC$ . Take any  $p \in \Gamma(f)$ , and a sequence  $\alpha_k \rightarrow \infty$  such that  $\tau_{\alpha_k} f \rightarrow p$  uniformly on compact sets. By passing to a subsequence we may assume that also  $\tau_{\alpha_k} x$  converges to some function  $y \in \Gamma(x)$  uniformly on compact sets. Then  $\mu * y = p$ . However, by Proposition 4.2,  $\mu * y = 0$ . This shows that  $\Gamma(f) = \{0\}$ , and so  $f(t) \rightarrow 0 (t \rightarrow \infty)$ .  $\square$

In particular, we have the following well-known result (cf. [3, Theorem 2]):

**COROLLARY 4.5.** *Let  $\mu \in M, x, f \in BAUC$  satisfy (1.1), and suppose that  $\hat{\mu}(\omega) \neq 0 (\omega \in R)$ . Then  $x(t) \rightarrow 0 (t \rightarrow \infty)$  iff  $f(t) \rightarrow 0 (t \rightarrow \infty)$ .*

Corollary 4.5 (which in the absolutely continuous case reduces to Pitt’s form of Wiener’s Tauberian theorem [10, Theorem 9.7(b)]) follows directly from Proposition 4.4, because a function tends to zero iff its asymptotic spectrum is empty.

We now turn to the question of which type of conditions are needed on  $\mu$  and  $f$ , if one wants to assure that all BAUC solutions of (1.1) are asymptotically almost periodic. Trivially,  $f$  is necessarily asymptotically almost periodic. It is also true that necessarily  $Z$  is countable (cf. the remark following the proof of Proposition 4.3). In the sequel we assume throughout these necessary conditions. It follows from Propositions 4.1 and 4.3 that the question of whether or not bounded solutions of (1.1) are asymptotically almost periodic has nothing to do with the almost periodic part of  $f$ . However, to get a full understanding of this case we need a refined version of Proposition 4.1.

**PROPOSITION 4.6.** *Let  $\mu \in M, f = p + q, p \in AP, q \in L^\infty, q(t) \rightarrow 0 (t \rightarrow \infty)$ , and suppose that  $Z$  is countable. If (1.1) has an asymptotically almost periodic solution, then the equation*

$$\mu * z = q \tag{4.4}$$

*has a solution  $z \in L^\infty$  satisfying  $z(t) \rightarrow 0 (t \rightarrow \infty)$ . Conversely, if (4.4) has a solution of this type, then every BAUC solution of (1.1) is asymptotically almost periodic.*

In particular, either (1.1) has no asymptotically almost periodic solution, or all BAUC solutions are asymptotically almost periodic. The additional condition  $z(t) \rightarrow 0 (t \rightarrow \infty)$  makes the splitting of  $x$  into  $y + z$  in Proposition 4.1 unique.

PROOF. If (1.1) has a solution  $x \in \text{AAP}$ , then write  $x = y + z$ , where  $y \in \text{AP}$  and  $z(t) \rightarrow 0 (t \rightarrow \infty)$ . Then  $\mu * y \in \text{AP}$ ,  $\mu * z(t) \rightarrow 0 (t \rightarrow \infty)$ , and  $f = \mu * y + \mu * z$ . By the uniqueness of the vanishing part of a function in AAP,  $\mu * z = q$ . Thus (4.4) has a solution  $z$  satisfying  $z(t) \rightarrow 0 (t \rightarrow \infty)$ .

Conversely, let  $z_0$  be a solution of (4.4) satisfying  $z_0(t) \rightarrow 0 (t \rightarrow \infty)$ , and let  $x$  be a BAUC solution of (1.1). Apply Proposition 4.1 to split  $x$  into  $y + z$  satisfying (4.2). By Proposition 4.3,  $y \in \text{AP}$ . Define  $z_1 = z - z_0$ . Then  $\mu * z_1 = 0$ , and so by Proposition 4.3,  $z_1 \in \text{AP}$ . As  $x = (y + z_1) + z_0$ , where  $y + z_1 \in \text{AP}$ ,  $z_0(t) \rightarrow 0 (t \rightarrow \infty)$ , we conclude that  $x \in \text{AAP}$ .  $\square$

Thanks to Proposition 4.6, if one wants to know whether or not all BAUC solutions of (1.1) for a specific function  $f \in \text{AAP}$  are asymptotically almost periodic, it suffices to study the case when  $f(t) \rightarrow 0 (t \rightarrow \infty)$ , and to look for solutions satisfying  $x(t) \rightarrow 0 (t \rightarrow \infty)$ . The difficulties are caused by the zeros of  $\hat{\mu}$ , and if  $\hat{\mu}$  has no zeros, then there are no difficulties.

**COROLLARY 4.7.** *Let  $\mu \in M$ ,  $x, f \in \text{BAUC}$  satisfy (1.1), and suppose that  $\hat{\mu}(\omega) \neq 0 (\omega \in R)$ . Then  $x \in \text{AAP}$  iff  $f \in \text{AAP}$ .*

Corollary 4.7 follows directly from Corollary 4.5 and Proposition 4.6 (or alternatively, Proposition 4.1).

Let us illustrate the difficulties by an example. Suppose that  $\mu \in M$ ,  $\hat{\mu}(\omega) = 0$  if  $\omega = 0$ ,  $x(t) = \sin(|t|^{1/2})$ , and define  $f$  by (1.1). Then by Proposition 4.4,  $f(t) \rightarrow 0 (t \rightarrow \infty)$ , but by the argument at the end of §3,  $x \notin \text{AAP}$ . Thus, we need a condition which excludes this type of behavior. For the moment, let us follow [3] and assume that  $\mu$  satisfies certain additional conditions. We call  $\omega_0 \in Z$  a zero of (finite integral) order  $m$  of  $\hat{\mu}$  ( $m$  is a positive integer), if the function  $\hat{\nu}(\omega) = (\omega - \omega_0)^{-m} \hat{\mu}(\omega) (\omega \in R)$  is the Fourier transform of a bounded measure  $\nu$ , and  $\hat{\nu}(\omega_0) \neq 0$  ( $\nu$  will automatically be absolutely continuous). Fink and Madych [3] assume that all zeros of  $\hat{\mu}$  are of finite integral order. Observe that this implies that the zeros are isolated, in particular,  $Z$  is countable.

If  $\hat{\mu}$  has a zero of finite integral order,  $x(t) \rightarrow 0 (t \rightarrow \infty)$ , and  $f$  is defined by (1.1), then we get the following necessary condition on  $f$ .

**THEOREM 4.8.** *Let  $\mu \in M$ ,  $x \in L^\infty$ ,  $x(t) \rightarrow 0 (t \rightarrow \infty)$ , and define  $f$  by (1.1). Moreover, suppose that  $\omega_0 \in R$  is a zero of  $\hat{\mu}$  of finite integral order  $m$ . Then  $e^{-i\omega_0 t} f(t)$  is the  $m$ th derivative of a bounded function  $F$  satisfying  $F(t) \rightarrow 0 (t \rightarrow \infty)$ .*

Here, as well as in the following theorems, one may interpret the differentiation in the distribution sense, or equivalently, require  $F$  to be  $m - 2$  times continuously differentiable (if  $m \geq 2$ ),  $F^{(m-1)}$  to be locally absolutely continuous, and  $F^{(m)}(t) = e^{-i\omega_0 t} f(t)$  almost everywhere. Theorem 4.8 shows that in some sense the convolution operator  $\mu*$  acts like a differentiation operator.

PROOF. Define  $\hat{\nu}(\omega) = i^{-m}(\omega - \omega_0)^{-m}\hat{\mu}(\omega)$ . Then  $\hat{\nu}$  is the Fourier transform of a measure  $\nu \in M$ . Put  $F(t) = e^{-i\omega_0 t} \nu * x(t)$ . Then  $F$  is bounded,  $F(t) \rightarrow 0$  ( $t \rightarrow \infty$ ), and

$$\hat{F}(\omega) = \hat{\nu}(\omega + \omega_0)\hat{x}(\omega + \omega_0) = (i\omega)^{-m}\hat{\mu}(\omega + \omega_0)\hat{x}(\omega + \omega_0).$$

Thus the  $m$ th derivative of  $F$  equals  $e^{-i\omega_0 t} \mu * x(t) = e^{-i\omega_0 t} f(t)$ .  $\square$

As a matter of fact, the necessary condition of  $f$  which Theorem 4.8 yields is also sufficient, provided all the zeros of  $\hat{\mu}$  are of finite integral order.

THEOREM 4.9. Let  $\mu \in M$ ,  $x \in \text{BAUC}$ ,  $f \in L^\infty$  satisfy (1.1). Let  $f(t) \rightarrow 0$  ( $t \rightarrow \infty$ ), and let the zeros of  $\hat{\mu}$  be of finite integral order. Moreover, assume that for each zero  $\omega_j \in Z$  of order  $m_j$ , the function  $e^{-i\omega_j t} f(t)$  is the  $m_j$ th derivative of a bounded function  $F_j$  satisfying  $F_j(t) \rightarrow 0$  ( $t \rightarrow \infty$ ). Then  $x \in \text{AAP}$ .

We omit the proof of Theorem 4.9, as it is contained in Theorem 4.10 below. Fink and Madych prove Theorem 4.9 under the (additional) assumption that  $f$  satisfies the moment condition

$$\int_0^\infty t^{m-1} |f(t)| dt < \infty, \tag{4.5}$$

where  $m$  is the maximum of the order of the zeros of  $\hat{\mu}$  ( $m = \infty$  is allowed, then (4.5) should hold for all finite  $m$ ). A still earlier version has been proved by Jordan and Wheeler [5], where  $Z$  is supposed to be finite, and the hypothesis that the zeros of  $\hat{\mu}$  be of finite integral order is guaranteed by the moment condition

$$\int_{-\infty}^\infty |t|^m d|\mu|(t) < \infty \tag{4.6}$$

(see [3, Lemma 5]). Here  $m$  is the same integer as in (4.5). The moment condition (4.6) is a global condition on  $\mu$ , whereas the requirement that  $\hat{\mu}$  has a zero of order  $m$  at  $\omega_0$  is a local condition on  $\hat{\mu}$ : If  $\mu_1, \mu_2 \in M$  and  $\hat{\mu}_1 = \hat{\mu}_2$  in a neighborhood of  $\omega_0$ , then  $\hat{\mu}_1$  has a zero of order  $m$  iff  $\hat{\mu}_2$  has a zero of order  $m$  at  $\omega_0$ . Similarly, the moment condition (4.5) on  $f$  is a global condition on  $f$ , whereas the requirement that  $e^{-i\omega_0 t} f(t)$  is the  $m$ th derivative of a bounded function  $F$  satisfying  $F(t) \rightarrow 0$  ( $t \rightarrow \infty$ ) is a local condition on  $\hat{f}$ : If  $f_1, f_2 \in L^\infty$  satisfy  $f_1(t) \rightarrow 0, f_2(t) \rightarrow 0$  ( $t \rightarrow \infty$ ), and  $\hat{f}_1 = \hat{f}_2$  in a neighborhood of  $\omega_0$ , then  $e^{-i\omega_0 t} f_1(t)$  is the  $m$ th derivative of a bounded function tending to zero iff  $f_2$  is so. In particular, this condition on  $f$  is satisfied at every point  $\omega_0 \notin \sigma(f)$ , whenever  $f \in L^\infty$  and  $f(t) \rightarrow 0$  ( $t \rightarrow \infty$ ). In this respect Fink and Madych only go half the way. They replace the moment condition (4.6) on  $\mu$  by a local condition on  $\hat{\mu}$ , but they do not replace (4.5) by a similar condition on  $\hat{f}$ .

In Theorem 4.9 it is possible to relax the requirement that the zeros of  $\hat{\mu}$  be of integral order. We say that  $\hat{\mu}$  has a (local) inverse  $\hat{\nu}$  of order  $m$  at  $\omega_0$ , if there exists a bounded measure  $\nu$  such that

$$\hat{\nu}(\omega)\hat{\mu}(\omega) = (\omega - \omega_0)^m \tag{4.7}$$

in a neighborhood of  $\omega_0$ . Here  $m$  is a nonnegative integer. By Wiener's Lemma, if  $\hat{\mu}$  has a zero of order  $m$  at  $\omega_0$  ( $m = 0, 1, 2, \dots$ ), then  $\hat{\mu}$  has an inverse of order  $m$  at  $\omega_0$ . In this case  $\hat{\nu}$  in (4.7) satisfies  $\hat{\nu}(\omega_0) \neq 0$ , but the crucial point in our definition

of an inverse is that we do not in general require  $\hat{\nu}(\omega_0) \neq 0$ . Thanks to this fact,  $\hat{\mu}$  may have an inverse of order  $m$  in spite of the fact that it does not have a zero of order  $m$  at  $\omega_0$ . This follows trivially from the fact that if  $\hat{\mu}$  has an inverse of order  $m$  at  $\omega_0$ , then it has inverses of order  $n$  for every  $n > m$  (convolve  $\nu$  by a measure whose Fourier transform equals  $(\omega - \omega_0)^{n-m}$  in a neighborhood of  $\omega_0$ ). A less trivial example is the following. Let  $\mu$  be the difference between the unit point mass at zero and the Fejer kernel, so that

$$\hat{\mu}(\omega) = \begin{cases} |\omega|, & |\omega| \leq 1, \\ 1, & |\omega| > 1. \end{cases} \tag{4.8}$$

Then  $\omega^{-1}\hat{\mu}(\omega)$  is bounded but discontinuous, and this implies that the zero at the origin is not of integral order. In spite of this fact  $\hat{\mu}$  can be inverted, because (4.7) holds with  $\nu = \mu$ ,  $m = 2$ ,  $\omega_0 = 0$ , hence  $\hat{\mu}$  serves as its own inverse of order two at the origin. Other examples of invertible kernels could be constructed with the aid of a result in [12].

The following strengthened version of Theorem 4.9 holds.

**THEOREM 4.10.** *Let  $\mu \in M$ ,  $x \in \text{BAUC}$ ,  $f \in L^\infty$  satisfy (1.1), and let  $f(t) \rightarrow 0$  ( $t \rightarrow \infty$ ). Moreover, for each  $\omega_j \in Z$  suppose that  $\hat{\mu}$  has an inverse of order  $m_j$  at  $\omega_j$ , and that  $e^{-i\omega_j t}f(t)$  is the  $m_j$ th derivative of a bounded function  $F_j$  satisfying  $F_j(t) \rightarrow 0$  ( $t \rightarrow \infty$ ). Then  $x \in \text{AAP}$ .*

**PROOF.** By Theorem 3.2, it suffices to show that  $\hat{x} \in \text{AAP}^*$  everywhere.

If  $\omega_0 \notin Z$ , then by Wiener's Lemma, there exists  $a \in L^1$  such that  $\hat{a}(\omega)\hat{\mu}(\omega) = 1$  in a neighborhood  $I$  of  $\omega_0$ . Thus,  $\hat{x}(\omega) = \hat{a}(\omega)\hat{f}(\omega) = (a * f)^\wedge(\omega)$  in  $I$ . But  $a * f(t) \rightarrow 0$  ( $t \rightarrow \infty$ ), so in particular,  $a * f \in \text{AAP}$ . Thus  $\hat{x} \in \text{AAP}^*$  at  $\omega_0$ , and so the set where  $\hat{x} \notin \text{AAP}^*$  is contained in  $Z$ .

Let  $\omega_j \in Z$ , and let  $\nu$  be as in (4.7). Define

$$F_0(t) = i^{m_j}e^{i\omega_j t}F_j(t) \quad (t \in R),$$

and put  $\varphi = \nu * F_0$ . Then  $\varphi \in \text{AAP}$  (because  $\varphi(t) \rightarrow 0$  ( $t \rightarrow \infty$ )). Compute

$$\hat{\varphi}(\omega) = \hat{\nu}(\omega)F_0(\omega) = i^{m_j}\hat{\nu}(\omega)\hat{F}_j(\omega - \omega_j) \quad (\omega \in R).$$

Thus,  $(\omega - \omega_j)^{m_j}\hat{\varphi}(\omega) = \hat{\nu}(\omega)\hat{f}(\omega)$  ( $\omega \in R$ ). On the other hand, by (1.1) and (4.7), in some neighborhood  $I$  of  $\omega_j$ ,

$$(\omega - \omega_j)^{m_j}\hat{x}(\omega) = \hat{\nu}(\omega)\hat{\mu}(\omega)\hat{x}(\omega) = \hat{\nu}(\omega)\hat{f}(\omega).$$

Thus, the distributions  $\hat{x}$  and  $\hat{\varphi}$  are equal in  $I \setminus \{\omega_j\}$ . Let  $\eta \in \mathcal{S}$  satisfy  $\sigma(\eta) \subset I$ ,  $\hat{\eta}(\omega) = 1$  in a neighborhood of  $\omega_j$ , and put  $r = \eta * (x - \varphi)$ . Then  $\sigma(r) \subset \{\omega_j\}$ , and so by, e.g., Corollary 2.5,  $r \in \text{AP}$ . Thus,  $\hat{x}$  equals  $(r + \varphi)^\wedge$  in a neighborhood of  $\omega_j$ , and so  $\hat{x} \in \text{AAP}^*$  at  $\omega_j$ .  $\square$

**5. On existence of solutions.** The method of proof used in Theorem 4.10 actually allows us to get existence of solutions of (1.1) under conditions similar to those in Theorem 4.10. For the existence we need to have  $\hat{\mu}$  invertible everywhere, including at the point  $\infty$ . We say that  $\hat{\mu}$  has an inverse of order  $m$  at infinity ( $m = 0, 1, \dots$ ),

if there exists  $\nu \in M$  such that

$$\hat{\mu}(\omega)\hat{\nu}(\omega) = \omega^{-m} \tag{5.1}$$

in a neighborhood of infinity. Observe that this is completely analogous to the definition of an inverse at a finite point. Similar examples to those at finite points show that an inverse at infinity may exist even though  $\hat{\mu}$  does not have “a zero of order  $m$ ” at infinity.

**THEOREM 5.1.** *Let  $\mu \in M, f \in L^\infty$ . Let  $\hat{\mu}$  have an inverse of order  $m$  at infinity, and let  $f^{(m)} \in L^\infty$ . In addition, for each  $\omega_j \in Z$ , suppose that  $\hat{\mu}$  has an inverse of order  $m_j$  at  $\omega_j$ , and that  $e^{-i\omega_j t}f(t)$  is the  $m_j$ th derivative of a function  $F_j \in L^\infty$ . Then (1.1) has a solution  $x \in L^\infty$ . If, in addition,  $f^{(m)} \in \text{BAUC}$ , then  $x \in \text{BAUC}$ .*

Some comments on the hypotheses are needed. If one replaces the word “inverse” by “zero” throughout, then the condition of  $f$  is necessary. The proof of this fact is similar to the proof of Theorem 4.8. The derivatives may be interpreted in the same way as in Theorems 4.8–4.10. The hypothesis on  $\hat{\mu}$  implies that  $Z$  contains only finitely many points. Observe that we here assume nothing (except boundedness) on the asymptotic behavior of  $f$ .

**PROOF.** We use the same patching technique as was used in §§2–3. First of all, we claim that for each  $\omega_0 \in R$ , the equation  $\hat{\mu}\hat{x} = \hat{f}$  has a local solution  $x$  at  $\omega_0$ , i.e., there exists a function  $x \in L^\infty$  (which depends on  $\omega_0$ ) such that  $\hat{\mu}\hat{x} = \hat{f}$  in a neighborhood of  $\omega_0$ . Fix  $\omega_0 \in R$ . If  $\omega_0 \notin Z$ , then by Wiener’s Lemma, there exists  $a \in L^1$  such that  $\hat{a}(\omega)\hat{\mu}(\omega) = 1$  in a neighborhood of  $\omega_0$ . Defining  $x = a * f$  we get  $\hat{\mu}\hat{x} = \hat{a}\hat{\mu}\hat{f} = \hat{f}$  in a neighborhood of  $\omega_0$ , and so  $x$  is a local solution at  $\omega_0$  of the equation  $\hat{\mu}\hat{x} = \hat{f}$ . If  $\omega_0 = \omega_j \in Z$ , then let  $\nu_j$  be the measure given by (4.7), define  $F_0(t) = i^{m_j}e^{i\omega_j t}F_j(t)$ , and put  $x = \nu * F_0$ . Then

$$\hat{\mu}(\omega)\hat{x}(\omega) = \hat{\nu}(\omega)\hat{\mu}(\omega)\hat{F}_0(\omega) = [i(\omega - \omega_j)]^{m_j}\hat{F}_j(\omega - \omega_j) = \hat{f}(\omega)$$

in a neighborhood of  $\omega_0$ , and so we again get a local solution of the equation  $\hat{\mu}\hat{x} = \hat{f}$ .

Next we claim that the equation  $\hat{\mu}\hat{x} = \hat{f}$  also has a local solution at infinity, i.e., there exists  $x \in L^\infty$  and  $M > 0$  such that  $\hat{\mu}\hat{x} = \hat{f}$  in  $(-\infty, -M) \cup (M, \infty)$ . Define  $x = (-i)^m \nu * f^{(m)}$ , where  $\nu$  is the measure in (5.1). Then  $\hat{\mu}(\omega)\hat{x}(\omega) = \omega^m \hat{\nu}(\omega)\hat{\mu}(\omega)\hat{f}(\omega) = \hat{f}(\omega)$  in a neighborhood of infinity, and so the claim is true. Observe that if  $f^{(m)} \in \text{BAUC}$ , then the local solution at infinity satisfies  $x \in \text{BAUC}$ .

By the preceding argument, there exists  $M > 0$  and  $x_0 \in L^\infty$  such that  $\hat{\mu}\hat{x}_0$  equals  $\hat{f}$  in  $I_0 = (-\infty, -M) \cup (M, \infty)$ . The set  $[-M, M]$  is compact, and the equation  $\hat{\mu}\hat{x} = \hat{f}$  has a local solution at each point. Hence there exists a finite number of open intervals  $I_j$  ( $1 \leq j \leq n$ ) covering  $[-M, M]$  and functions  $x_j \in L^\infty$  such that  $\hat{\mu}\hat{x}_j = \hat{f}$  in  $I_j$ . Choose  $\eta_j \in \mathcal{S}$  ( $1 \leq j \leq n$ ) such that  $\sigma(\eta_j) \in I_j$ , and  $\sum \hat{\eta}_j = 1$  on  $[-M, M]$ . Define  $\eta_0$  to be the measure  $\delta - \sum \eta_j$ , where  $\delta$  is the unit point mass at zero. Finally, define  $x = \sum_{j=0}^n \eta_j * x_j$ . Then  $x \in L^\infty$ , and if  $f^{(m)} \in \text{BAUC}$ , then  $x \in \text{BAUC}$ . Moreover,  $\hat{\eta}_j \hat{\mu}\hat{x}_j = \hat{\eta}_j \hat{f}$  everywhere ( $0 < j < n$ ), and by summing over  $j$  one gets  $\hat{\mu}\hat{x} = \hat{f}$ . Thus,  $x$  is a solution of (1.1).  $\square$

**6. Extensions to distribution equations.** Most of the results in §§4–5 can be generalized to the distribution equation

$$\mu * x = f, \quad (6.1)$$

where  $x$  and  $f$  are bounded distributions, and  $\mu$  is a distribution for which convolution with bounded distributions makes sense. We let  $\mathfrak{B}'$  be the space of bounded distributions, i.e. the distributions which are finite sums of (distribution) derivatives of bounded functions. We denote the set of integrable distributions, i.e. those which are finite sums of derivatives of integrable functions, by  $\mathfrak{D}'_L$ . If  $\mu \in \mathfrak{D}'_L$  and  $\varphi \in \mathfrak{B}'$ , then  $\mu * \varphi$  is well defined, and  $\mu * \varphi \in \mathfrak{B}'$  (see [11, pp. 200–205]). Hence, it is possible to study (6.1) for  $\mu \in \mathfrak{D}'_L$ , and  $x, f \in \mathfrak{B}'$ . Jordan, Madych and Wheeler [4] study (6.1) for  $\mu$  belonging to a class of distributions which satisfy what they call “Property H”. By [11, p. 201], this class is exactly  $\mathfrak{D}'_L$ . For instance, the integro-differential equation  $x' + \mu * x = f$ , where prime stands for differentiation,  $\mu \in M$  and  $x, f \in L^\infty$  is of the type (6.1).

There is a standard method for deducing results for distributions from the results on bounded functions in §§2–5. If  $\eta_n$  is a  $\delta$ -sequence, and  $\varphi \in \mathfrak{B}'$ , then  $\eta_n * \varphi \rightarrow \varphi$  in the strong topology of  $\mathfrak{B}'$  [11, p. 204]. Using this fact one easily proves that Theorems 2.3, 2.4 and Corollary 2.5 hold with BAUC and  $L^\infty$  replaced by  $\mathfrak{B}'$ , and AP by the set of almost periodic distributions (see [11, pp. 206–208]). The analogue of Theorem 3.2 also holds. When one studies the asymptotic properties of (6.1) one may choose any  $\delta$ -sequence  $\eta_n$  with nonvanishing Fourier transform, and write (6.1) in the form

$$(\eta_1 * \mu) * (\eta_n * x) = \eta_n * (\eta_1 * f). \quad (6.2)$$

This equation is of the form treated in §4, and so the results in §4 can be applied. We leave the details to the reader.

The existence result, Theorem 5.1, simplifies when it is rephrased for distributions. If  $f \in \mathfrak{B}'$ , then automatically  $f^{(m)} \in \mathfrak{B}'$ , and so this particular assumption may be omitted. The behavior of  $\hat{\mu}$  at infinity still plays a role in the sense that the order of the distribution solution  $x$  constructed in the proof of Theorem 5.1 exceeds the order of  $f$  by  $m$ . If  $\mu$  is of order greater than zero, then one may be able to find an inverse of negative order at infinity. This has the consequence that one may find a solution  $x$  whose order is no greater than or even strictly less than the order of  $f$ . In particular, if  $f \in L^\infty$ , and  $m \leq 0$ , then  $x \in L^\infty$ .

#### REFERENCES

1. R. Doss, *On the almost periodic solutions of a class of integro-differential equations*, Ann. of Math. (2) **81** (1965), 117–123.
2. A. M. Fink, *Almost periodic differential equations*, Lecture Notes in Math., vol. 377, Springer-Verlag, Berlin, 1974.
3. A. M. Fink and W. R. Madych, *On certain bounded solutions of  $g * \mu = f$* , Proc. Amer. Math. Soc. **75** (1979), 235–242.
4. G. S. Jordan, W. R. Madych and R. L. Wheeler, *Linear convolution integral equations with asymptotically almost periodic solutions*, Proc. Amer. Math. Soc. **77** (1980), 337–341.
5. G. S. Jordan and R. L. Wheeler, *Linear integral equations with asymptotically almost periodic solutions*, J. Math. Anal. Appl. **52** (1975), 454–464.

6. Y. Katznelson, *An introduction to harmonic analysis*, Wiley, New York, 1968.
7. J. J. Levin and D. F. Shea, *On the asymptotic behavior of the bounded solutions of some integral equations*. I–III, *J. Math. Anal. Appl.* **37** (1972), 42–82; 288–326; 537–575.
8. L. H. Loomis, *The spectral characterization of a class of almost periodic functions*, *Ann. of Math.* (2) **72** (1960), 362–368.
9. H. Pollard, *The harmonic analysis of bounded functions*, *Duke Math. J.* **20** (1953), 499–512.
10. W. Rudin, *Functional analysis*, McGraw-Hill, New York, 1974.
11. L. Schwartz, *Théorie des distributions*, new ed., Hermann, Paris, 1966.
12. D. F. Shea and S. Wainger, *Variants of the Wiener-Lévy theorem, with applications to stability problems for some Volterra integral equations*, *Amer. J. Math.* **97** (1975), 312–343.
13. O. J. Staffans, *Tauberian theorems for a positive definite form, with applications to a Volterra equation*, *Trans. Amer. Math. Soc.* **218** (1976), 239–259.
14. \_\_\_\_\_, *On the asymptotic spectra of the bounded solutions of a nonlinear Volterra equation*, *J. Differential Equations* **24** (1977), 365–382.
15. \_\_\_\_\_, *An asymptotic problem for a positive definite operator-valued Volterra kernel*, *SIAM J. Math. Anal.* **9** (1978), 855–866.

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