THE SECOND CONJUGATE ALGEBRA
OF THE FOURIER ALGEBRA OF A LOCALLY COMPACT GROUP

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ABSTRACT. Let $G$ be a locally compact group and let $VN(G)$ denote the von Neumann algebra generated by the left translations of $G$ on $L_2(G)$. Then $VN(G)^*$, when regarded as the second conjugate space of the Fourier algebra of $G$, is a Banach algebra with the Arens product. We prove among other things that when $G$ is amenable, $VN(G)^*$ is neither commutative nor semisimple unless $G$ is finite. We study in detail the class of maximal regular left ideals in $VN(G)^*$. We also show that if $G_1$ and $G_2$ are discrete groups, then $G_1$ and $G_2$ are isomorphic if and only if $VN(G_1)^*$ and $VN(G_2)^*$ are isometric order isomorphic.

1. Introduction. Let $G$ be a locally compact group. Then $A(G)$, the Fourier algebra of $G$, is the linear subspace of $C_0(G)$ (bounded continuous complex-valued functions on $G$ vanishing at infinity) consisting of all functions of the form $(h * k)^*$ where $h, k \in L_2(G)$, $\check{g}(x) = g(x^{-1})$ and $\check{k} = k(x^{-1})$. If $f \in L_2(G)$ and $g \in G$, define $\lambda(g)f(t) = f(g^{-1}t)$, $t \in G$. Let $VN(G)$ denote the closure in the weak operator topology of the linear span of $\{\lambda(g); g \in G\}$ in $\mathfrak{B}(L_2(G))$, the algebra of bounded linear operators from $L_2(G)$ into $L_2(G)$. Then $\phi = (h * k)^*$ in $A(G)$ can be regarded as an ultraweakly continuous linear functional on the von Neumann algebra $VN(G)$ defined by $\phi(T) = (Th, k)$ (inner product in $L_2(G)$) for each $T \in VN(G)$. Furthermore, each ultraweakly continuous linear functional on $VN(G)$ is of this form [6, pp. 210 and 218]. In particular, $A(G)$ can be identified as the unique predual of $VN(G)$. Also, $A(G)$ with pointwise multiplication and the norm $\|\phi\| = \sup\{|\phi(x)|; x \in VN(G) \text{ and } \|x\| < 1\}$ is a commutative Banach algebra with spectrum $G$ [6, p. 222].

It is the purpose of this paper to study the Banach algebra $VN(G)^*$, the second conjugate algebra of $A(G)$ equipped with the Arens product [1]. We prove, in §3, that $VN(G)^*$ is semisimple if and only if $G$ is finite. We also prove that $A(G)$ is an ideal in $VN(G)^*$ if and only if $G$ is discrete. When $G$ is amenable, then the Fourier-Stieltjes algebra of $G$ is a quotient of $VN(G)^*$. We characterize, in §4, the regular maximal left ideals in $VN(G)^*$. We show that any such ideal must contain $I_g = \{\phi \in A(G); \phi(g) = 0\}$ for some $g \in G$. Furthermore, if $G$ is amenable, then every regular maximal left ideal in $VN(G)^*$ contains a unique $I_g$, $g \in G$, if and
only if $G$ is compact. We also show that $G$ is discrete if and only if each regular left ideal in $VN(G)^*$ either contains $A(G)$ or must be the kernel of some $\lambda(g) \in VN(G)$, $g \in G$. In §5 this result is used to prove the following isomorphism theorem for discrete groups: if $G_1$ and $G_2$ are discrete groups, then the Banach algebras $VN(G_1)^*$ and $VN(G_2)^*$ are isometric order isomorphic if and only if $G_1$ and $G_2$ are isomorphic.

Most of our results in §§3 and 4 are known and proved by Civin and Yood [4] and Civin [3] when $G$ is an abelian locally compact group. In this case, $A(G)$ is isometric isomorphic to $L_1(\hat{G})$ via the Fourier transform. Our proofs depend heavily on some deep analysis of the Fourier algebra $A(G)$ and various subspaces of $VN(G)$ of P. Eymard [6] and E. Granirer [7]–[9].

2. Preliminaries. Let $E$ be a linear space, and $\phi$ be a linear functional on $E$, then the value of $\phi$ at an element $x$ in $E$ will be written as $\phi(x)$ or $\langle \phi, x \rangle$.

Throughout this paper, $G$ denotes a locally compact group with a fixed left Haar measure. Let $C(G)$ denote the Banach space of bounded continuous complex-valued functions on $G$ with the supremum norm. Then $G$ is amenable if there exists a positive linear functional $\phi$ on $C(G)$ of norm one such that $\phi(af) = \phi(f)$ for each $a \in G$ and each $f \in C(G)$, where $a(t) = f(at)$ for each $t \in G$. Amenable groups include all solvable groups and all compact groups. However, the free group on two generators is not amenable.

Let $P(G)$ denote the subspace of $C(G)$ consisting of all continuous positive definite functions on $G$, and let $B(G)$ be its linear span. Then $B(G)$ can be identified with the dual of $C^*(G)$, the group $C^*$-algebra of $G$ (see [6, p. 192]). Also, $B(G)$ with pointwise multiplication and the dual norm is a commutative Banach algebra called the Fourier-Stieltjes algebra of $G$ [6, Proposition 2.16].

Let $P_p(G)$ denote the closure of $P(G) \cap C_{00}(G)$ in the compact open topology, where $C_{00}(G)$ is the set of all functions in $C(G)$ with compact support, and let $B_p(G)$ denote the linear span of $P_p(G)$. Then $B_p(G)$ is a closed ideal in $B(G)$ and $B_p(G)$ is precisely the dual of $C_p^*(G)$ [6, Propositions 1.21 and 2.16], where $C_p^*(G)$ is the norm closure of $\{\lambda(f); f \in L_1(G)\}$ in $L_2(G)$, and $\lambda(f)(h) = f \ast h$ for $h \in L_2(G)$. As is known [10, p. 61], $B_p(G) = B(G)$ if and only if $G$ is amenable.

The Fourier algebra $A(G)$ as defined in §1 is the closed linear span of $P(G) \cap C_{00}(G)$. It is a closed ideal in $B(G)$ (see [6, p. 208]). Also $A(G) \subseteq B_p(G)$.

There is a natural module action of $A(G)$ on $VN(G)$ given by $\langle \phi, x \rangle = \langle \phi \gamma, x \rangle$ for each $\phi, \gamma \in A(G)$ and each $x \in VN(G)$. Recently E. Granirer [7, p. 373] has defined the subspace $UCB(\hat{G})$ of $VN(G)$ as the norm closure of $A(G) \cdot VN(G)$. Then $UCB(\hat{G})$ is a $C^*$-subalgebra of $VN(G)$ [8, Proposition 2(a)] and elements in $UCB(\hat{G})$ are called uniformly continuous functionals on $A(G)$. It follows from [8, Propositions 1 and 3] and [9, Theorem 3] that $UCB(\hat{G}) = VN(G)$ if and only if $G$ is compact.

A positive linear functional $m \in VN(G)^*$ of norm one is called a topological invariant mean on $VN(G)$ if $\langle m, \phi \cdot x \rangle = \langle m, x \rangle$ for each $x \in VN(G)$ and each $\phi \in P(G) \cap A(G)$ such that $\phi(e) = 1$ where $e$ is the identity of $G$. As known [16, Theorem 4], $VN(G)$ always has a topological invariant mean.
A subset $X$ of $VN(G)$ is **topologically invariant** if $\phi \cdot x \in X$ for each $\phi \in A(G)$ and each $x \in X$. If $X$ is a topological invariant linear subspace of $VN(G)$, we say that $X$ is **topologically introverted** if for each $m \in VN(G)^*$ and each $x \in X$, the functional $\gamma \to \langle m, \gamma \cdot x \rangle$ on $A(G)$, denoted by $m \circ x$, also defines an element in $X$. Examples of topologically introverted subspaces of $VN(G)$ include $UCB(\hat{G})$ and $C_p^*(G)$ (see [13, Proposition 5.2]). Also, any weak*-closed topologically invariant subspace of $VN(G)^*$ is topologically introverted [13, Lemma 5.1].

3. The Banach algebra $VN(G)^*$. In [1] Arens shows that given a Banach algebra $B$, it is possible to define a multiplication on $B^{**}$ which extends multiplication on $B$. In case $B = A(G)$, $m, n \in VN(G)^*$, the Arens product $m \circ n$ is defined by the formula: $\langle m \circ n, x \rangle = \langle m, n \circ x \rangle$ for each $x \in VN(G)$, where $\langle n \circ x, \phi \rangle = \langle n, \phi \cdot x \rangle$, $\phi \in A(G)$. This same formula certainly makes sense when $VN(G)$ is replaced by a topologically invariant and introverted subspace $X$ of $VN(G)$. The following observations will be useful in the sequel.

**Lemma 3.1.** (i) $A(G)$ is in the centre of $VN(G)^*$.
(ii) For each $n \in VN(G)^*$, the map $m \to m \circ n$ from $VN(G)^*$ into $VN(G)^*$ is weakly*-weak* continuous.
(iii) Any weak*-closed right ideal in $VN(G)^*$ is an ideal.
(iv) If $g \in G$, then the functional on $VN(G)^*$ defined by $m \to \langle m, \lambda(g) \rangle$ is multiplicative.

**Proof.** Both (ii) and (iii) are immediate. Also (iii) follows from (i) and [13, Lemma 5.1]. To prove (iv), we simply observe that if $n \in VN(G)^*$, then $n \circ \lambda(g) = \langle n, \lambda(g) \rangle \lambda(g)$. Hence $\langle m \circ n, \lambda(g) \rangle = \langle m, \lambda(g) \rangle \langle n, \lambda(g) \rangle$ for any $m \in VN(G)^*$.

**Proposition 3.2.** (a) $VN(G)^*$ has a right identity if and only if $G$ is amenable.
(b) $VN(G)^*$ has a left identity if and only if $G$ is compact.

**Proof.** (a) is a direct consequence of Leptin's theorem [14] and Proposition 7 in [2, p. 146].

(b) If $G$ is compact, then $A(G) = B(G)$. Hence $A(G)$ has an identity which is also the identity for $VN(G)^*$. Conversely, if $G$ is not compact, then $UCB(\hat{G})$ is a proper $C^*$-subalgebra of $VN(G)$ by Granirer's results [8, Proposition 1] and [9, Theorem 3]. Hence there exists $m \in VN(G)^*$ such that $m \neq 0$ and $m(y) = 0$ for all $y \in UCB(\hat{G})$. Then $m \circ x = 0$ for each $x \in VN(G)$. Consequently $n \circ m = 0$ for each $n \in VN(G)^*$. In particular $VN(G)^*$ cannot have a left identity.

If $G$ is finite, then $A(G) = VN(G)^*$. In particular, $VN(G)^*$ is commutative. The following shows that the converse of this also holds when $G$ is amenable.

**Proposition 3.3.** Let $G$ be amenable. Then $VN(G)^*$ is commutative if and only if $G$ is finite.

**Proof.** If $VN(G)^*$ is commutative, then $VN(G)$ has a unique topological invariant mean (see [13, Theorem 5.6] and [7, Proposition 5]). Hence $G$ is discrete by [16, Theorem 11]. Also since $G$ is amenable, it follows from [13, Theorem 5.6]
and [8, Proposition 3] that \( VN(G) = UCB(\hat{G}) \). In particular, \( G \) is finite by [8, Proposition 1 and Theorem 4].

Let \( \text{Rad}(VN(G)^*) \) denote the radical of \( VN(G)^* \).

**Theorem 3.4.** The Banach algebra \( VN(G)^* \) is semisimple if and only if \( G \) is finite.

**Proof.** If \( G \) is finite, then \( A(G) = VN(G)^* \) which is semisimple.

If \( VN(G)^* \) is semisimple, we first show that \( VN(G) \) must have a unique topological invariant mean. In particular \( G \) is discrete [16, Theorem 11]. Indeed, if \( m_1, m_2 \) are distinct topological invariant means on \( VN(G) \), let \( J = \{ m \in VN(G)^*; \langle m, \lambda(e) \rangle = 0 \text{ and } m \odot \phi = m \text{ for each state } \phi \text{ in } A(G) \} \). Then \( m_1 - m_2 \) is a nonzero element in \( J \). \( J \) is a right ideal of \( VN(G)^* \). In fact, if \( m \in J, n \in VN(G)^* \), then \( \langle m \odot n, \lambda(e) \rangle = \langle m, \lambda(e) \rangle \langle n, \lambda(e) \rangle = 0 \) by Lemma 3.1(iv). Also if \( \phi \) is a state, in \( A(G) \), then \( \langle m \odot n, \phi \rangle = \langle m \odot \phi, n \rangle = m \odot n \) by Lemma 3.1(i). A similar argument shows that \( J \) is a left ideal. Also \( J^2 = \{ 0 \} \). In particular, \( J \) is nil and must be included in \( \text{Rad}(VN(G)^*) \) (see Corollary in [11, p. 9]). Hence \( VN(G)^* \) is not semisimple.

If \( G \) is not finite, then \( UCB(\hat{G}) \) is a proper \( C^* \)-subalgebra of \( VN(G) \) ([8, Proposition 1] and [9, Theorem 3]). Let

\[
K = \{ m \in VN(G)^*; \langle m, x \rangle = 0 \text{ for each } x \in UCB(\hat{G}) \}.
\]

Then \( K \neq 0 \). Also, an application of Lemma 3.1(i) and (ii) shows that \( K \) is a left ideal and \( K^2 = \{ 0 \} \). In particular \( K \subseteq \text{Rad}(VN(G)^*) \) and hence \( VN(G)^* \) is not semisimple.

Let \( K_p = \{ m \in VN(G)^*; \langle m, \lambda(f) \rangle = 0 \text{ for each } f \in L_1(G) \} \).

**Theorem 3.5.** \( K_p \) is a weak*-closed two-sided ideal in \( VN(G)^* \) containing \( \text{Rad}(VN(G)^*) \). The quotient Banach algebra \( VN(G)^*/K_p \) is isometrically isomorphic to \( B_p(G) \). Also \( K_p = \text{Rad}(VN(G)^*) \) if and only if \( G \) is discrete.

**Proof.** Suppose \( n \in VN(G)^* \), then \( n \odot x \in C_p^*(G) \) for each \( x \in C_p^*(G) \) by [13, Proposition 5.2]. Hence \( m \odot n \in K_p \) whenever \( m \in K_p \). Since \( K_p \), being the intersection of weak*-closed sets, is weak*-closed, it follows from Lemma 3.1(iii) that \( K_p \) is an ideal.

Since \( K_p = C_p^*(G)^\perp \), the annihilator of \( C_p^*(G) \), there is a natural linear isometry \( \pi \) from the quotient space \( VN(G)^*/K_p \) onto \( C_p^*(G)^* \). A simple computation shows that \( \pi \) is even an algebra isomorphism when \( C_p^*(G)^* \) is equipped with the induced Arens product. Since the Arens product on \( C_p^*(G)^* = B_p(G) \) is precisely the pointwise multiplication, [13, Proposition 5.3], the second assertion follows.

Finally, since \( B_p(G) \) is semisimple, \( K_p \) must contain \( \text{Rad}(VN(G)^*) \).

If \( G \) is discrete, then \( C_p^*(G) = UCB(\hat{G}) \) [8, Proposition 3]. Hence if \( m \in K_p \), then Lemma 3.1(i) and (ii) implies that \( n \odot m = 0 \) for all \( n \in VN(G)^* \). In particular, \( K_p \) is nil, and hence must be included in \( \text{Rad}(VN(G)^*) \).

Conversely if \( G \) is not discrete, then \( 1 \notin C_p^*(G) \) by [13, Corollary 4.3]. Choose \( n \in VN(G)^* \) such that \( n(1) \neq 0 \) and \( n(x) = 0 \) for all \( x \in C_p^*(G) \). If \( K_p = \text{Rad}(VN(G)^*) \), then \( n \in \text{Rad}(VN(G)^*) \), which contradicts the second statement of Theorem 3.5 (whose proof does not depend on this result).
Let $C^*_8(G)$ denote the $C^*$-subalgebra of $VN(G)$ generated by the left translations 
\( \{ (\lambda(g); g \in G) \} \) on $L^2(G)$. Then for each $g \in G$, $\phi \in A(G)$, $\phi \cdot \lambda(g) = \phi(g)\lambda(g)$. 

For $n \in VN(G)^*$, $\phi \in A(G)$ and $g \in G$, we have 
\[
\langle n \odot \lambda(g), \phi \rangle = \langle n, \phi \cdot \lambda(g) \rangle = \langle n, \lambda(g) \rangle \phi(g) = \langle \langle n, \lambda(g) \rangle \lambda(g), \phi \rangle
\]
which shows that $C^*_8(G)$ is topologically introverted.

Let $B_8(G)$ denote the linear span of $P_8(G)$, where $P_8(G)$ is the pointwise closure of $A(G) \cap P(G)$. Then $B_8(G)$ is a subalgebra of $B(G_d)$, where $G_d$ denotes the group $G$ with the discrete topology. Furthermore, $B_8(G)$ can be identified with $C^*_8(G)^*$ by the map $\pi(\phi)(f) = \Sigma \{ \phi(t)f(t); t \in G \}$ for each $f \in l_1(G)$ and $\phi \in B_8(G)$ (see \[6, Proposition 1.21\]). Then, as readily checked, $\pi(\phi \cdot \psi) = \pi(\phi) \circ \pi(\psi)$ for all $\phi, \psi \in B_8(G)$.

Let $K_8 = \{ m \in VN(G)^*; \langle m, \lambda(g) \rangle = 0 \text{ for all } g \in G \}$. Then $K_8$ is a weak*-closed two-sided ideal in $VN(G)^*$.

**Theorem 3.6.** $B_8(G)$ with pointwise multiplication and the dual norm is a semisimple commutative Banach algebra isometric algebra isomorphic to $VN(G)/K_8$. In particular $K_8$ contains $Rad(VN(G)^*)$. The natural embedding $\pi$ of $A(G)$ into $VN(G)^*/K_8$ is a linear isometry. Furthermore, if $G$ is amenable, then $\pi$ is onto if and only if $G$ is finite.

**Proof.** That $\pi$ is a linear isometry follows from the Kaplansky density theorem. If $G$ is finite then clearly $\pi$ is onto. Conversely, if $G$ is amenable, and $\pi$ is onto, then $A(G)$ can be identified with $C^*_8(G)^*$. Since $C^*_8(G)$ has a topological invariant mean \[16, Theorem 4\], there exists $\phi_0 \in P(G) \cap A(G)$ such that $\phi_0 \phi_0 = \phi_0$ for all $\phi \in P(G)$. In particular $G$ is discrete \[16, Proposition 5\]. Hence $A(G) = B_8(G) = B(G)$ by amenability of $G$. So $G$ must be finite. Proofs of the remaining assertions are clear.

**Theorem 3.7.** $G$ is discrete if and only if $A(G)$ is an ideal in $VN(G)^*$.

**Proof.** If $G$ is discrete, then $C^*_8(G) = UCB(\hat{G})$ \[8, Proposition 3\]. Hence if $\phi \in A(G)$ and $m \in VN(G)^*$, then 
\[
\langle m \odot \phi, x \rangle = \langle m, \phi \cdot x \rangle = \langle \psi, \phi \cdot x \rangle \text{ for some } \psi \in B_p(G) = \langle \langle \psi \cdot x \rangle
\]
for each $x \in VN(G)$. Hence $m \odot \phi = \psi \phi$ which is in $A(G)$. It now follows from Lemma 3.1(i) that $A(G)$ is an ideal in $VN(G)^*$.

Conversely if $A(G)$ is an ideal in $VN(G)^*$, let $\phi_0 \in A(G) \cap P_1(G)$, where $P_1(G) = \{ \phi \in P(G); \phi(e) = 1 \}$. Let 
\[
K = \{ m \odot \phi_0; m \in VN(G)^*, m > 0 \text{ and } ||m|| = 1 \}.
\]

Then the Banach-Alaoglu Theorem and Lemma 3.1(ii) imply that $K$ is a weakly compact convex subset of $A(G)$. Also, if $\phi \in A(G) \cap P_1(G)$, let $T_\phi(\psi) = \phi \psi$ for each $\psi \in K$. Then $\{ T_\phi; \phi \in A(G) \cap P_1(G) \}$ is a commuting family of continuous affine maps from $K$ into $K$. An application of the Markov-Kakutani fixed point theorem \[5, p. 456\] shows that there exists $\psi_0 \in K$ such that
$T_\phi(\psi_0) = \psi_0$ for all $\phi \in A(G) \cap P_1(G)$. In particular $\psi_0$ is a topological invariant mean on $VN(G)$. By [16, Proposition 5], $G$ must be discrete.

4. Maximal regular ideals. Let $R$ be a ring. A left ideal $I$ in $R$ is regular if there exists $u \in R$ such that $xu - x \in I$ for each $x \in R$. As is well known, any proper regular left ideal in $R$ is contained in a maximal proper regular left ideal in $R$ (see [15, p. 58]). In this section, we shall study in detail the class of maximal proper left ideals in $VN(G)^*$ of a locally compact group $G$.

If $g \in G$, let $I_g = \{\phi \in A(G); \phi(g) = 0\}$ and let $I_g^w = \{m \in VN(G)^*; \langle m, \lambda(g) \rangle = 0\}$. Then $I_g^w$, being the kernel of the nonzero multiplicative linear functional $\lambda(g)$ on $VN(G)^*$ (Lemma 3.1(iv)), is a maximal weak*-closed ideal in $VN(G)^*$ and $I_g$ is weak*-dense in $I_g^w$. Let $T_g = \{\phi \in A(G); \text{there exist an open set } V \text{ containing } g \text{ and } \phi(x) = 0 \text{ for all } x \in V\}$. Also if $V$ is an open set in $G$, let

$$S_V = \bigcap \{I_g; g \in V\}.$$ 

Then $T_g$ and $S_V$ are also ideals in $A(G)$, and $S_V$ is closed.

**Lemma 4.1.** Let $M$ be a maximal regular left ideal in $VN(G)^*$ and $g \in G$. Then either $M \supseteq T_g$ or there exists an open set $V$ of $G$ containing $g$ such that $M \supseteq S_V$.

**Proof.** If $M$ does not contain $T_g$, let $\phi \in T_g$ and $\phi \notin M$. There exists an open set $K$ of $G$ containing $g$ such that $\phi \in \bigcap \{I_x; x \in V\}$. It follows from the maximality of $M$ that

$$VN(G)^* = M + VN(G)^*\phi.$$ 

Let $u \in VN(G)^*$ such that $n \circ u - n \in M$ for all $n \in VN(G)^*$. Then $u = m + p \circ \phi$ for some $p \in VN(G)^*$ and $m \in M$. Let $\psi \in S_V$, then

$$\psi \circ u = \psi \circ (m + p \circ \phi) = \psi \circ m + p \circ \psi \phi.$$ 

Now $\psi(x) = 0$ for all $x \notin V$ by definition of $S_V$, and $\phi(x) = 0$ if $x \in V$. Hence $(\psi \phi)(x) = \psi(x)\phi(x) = 0$ for all $x \in G$. Consequently, $\psi \phi = 0$ and so

$$\psi - \psi \circ u = \psi - \psi \circ m \in M.$$ 

In particular $\psi \in M$. Thus $S_V \subseteq M$.

**Lemma 4.2.** Let $g \in G$. Then $I_g$ is contained in the norm closure of $T_g$.

**Proof.** This follows directly from Corollary 2(4.11) in [6].

**Theorem 4.3.** Every maximal regular left ideal in $VN(G)^*$ contains $I_g$ for some $g \in G$.

**Proof.** Let $M$ be a maximal regular left ideal in $VN(G)^*$. Then $M$ is closed [15, p. 68]. If $M \supseteq I_g$ for any $g \in G$, then by Lemma 4.2, $M \supseteq T_g$ for any $g \in G$. By Lemma 4.1, we have for each $g \in G$ an open set $V(g)$ containing $g$ such that $M \supseteq S_{V(g)}$. Let $J$ denote the smallest (left) ideal in $A(G)$ containing each $S_{V(g)}$. If $J$ is not dense in $A(G)$, then by the Tauberian property of $A(G)$ [6, p. 223], there exists $g_0 \in G$ such that $J \subseteq I_{g_0}$. Consequently, $\phi(g_0) = 0$ for all $\phi \in J$. However, by [6, Lemma 3.2], there exists $\phi \in A(G)$ such that $\phi(g_0) = 1$ and $\phi \in S_{V(g_0)}$. But
Corollary 4.4. Let $M$ be a maximal regular left ideal in $VN(G)^*$. Then either $M \supseteq A(G)$ or there exists a unique $g \in G$ such that $M \supseteq I_g$.

Proof. By Theorem 4.3, $M \supseteq I_g$ for some $g \in G$. Since $M \cap A(G)$ is an ideal in $A(G)$ and the ideals $I_g$ are all maximal and distinct in $A(G)$, $A(G) \subseteq M$ if $M$ contains more than one $I_g$.

Corollary 4.5. Let $M$ be a maximal regular left ideal in $VN(G)^*$. If there exists $x \in VN(G)$ such that $M \subseteq \{m \in VN(G)^*; m(x) = 0\}$, then there exists $g \in G$ such that $M = I_g^\#$.

Proof. By Theorem 4.3, there exists $g \in G$ such that $M \supseteq I_g$. So $\langle \phi, x \rangle = 0$ for all $\phi \in I_g$. Since $x \neq 0$, $x = \alpha \lambda(g)$ for some nonzero $\alpha \in \mathbb{C}$. In particular $M = I_g^\#$.

Corollary 4.6. All maximal regular left ideals of $VN(G)^*$ are weak*-dense except those of the form $I_g^\#$, $g \in G$.

Proof. Let $M$ be a maximal left ideal of $VN(G)^*$. If the weak*-closure of $M$ is not $VN(G)^*$, then there exists $x \in VN(G)$ such that $\langle x, m \rangle = 0$ for all $m \in M$, and $x \neq 0$. By Corollary 4.5, $M = I_g^\#$ for some $g \in G$.

We need the following simple observation.

Lemma 4.7. If $G$ is not discrete, then $\lambda(g) \notin C_p^*(G)$ for any $g \in G$.

Proof. If $\lambda(g) \in C_p^*(G)$, then $\lambda(g^{-1}) \notin C_p^*(G)$ by [13, Corollary 4.3]. Hence there exists $m \in VN(G)^*$ such that $m(\lambda(g^{-1})) \neq 0$ and $m(x) = 0$ for each $x \in C_p^*(G)$. Let $n \in VN(G)^*$ be defined by $n(x) = m(x^*)$, $x \in VN(G)$. Then $n(\lambda(g)) \neq 0$ and $n(x) = 0$ for each $x \in C_p^*(G)$, which is impossible.

Theorem 4.8. $G$ is discrete if and only if each maximal regular left ideal in $VN(G)^*$ either contains $A(G)$ or is of the form $I_g^\#$ for some $g \in G$.

Proof. Assume that $G$ is discrete and that $M$ is a maximal regular left ideal in $VN(G)^*$. If $M$ does not contain $A(G)$, then by Corollary 4.4, there exists a unique $g \in G$ such that $I_g \subseteq M$. Since $A(G) \cap M$ is a proper closed ideal in $A(G)$ containing $I_g$ and $I_g$ is maximal in $A(G)$, we have $A(G) \cap M = I_g$. We shall show that $M = I_g^\#$.

Let $\phi_0 \in A(G)$ be such that $\phi_0(g) = 1$ [6, Lemma 3.2]. Then $\phi_0^2 - \phi_0 \in I_g$ and $\phi_0^2 \notin I_g$. In particular $\phi_0^2 \notin M$ and $VN(G)^* = M + VN(G)^* \cap \phi_0$ by maximality of $M$. Now if $p \in I_g^\#$, then $p = m + n \phi_0$ for some $m \in M$ and $n \in VN(G)^*$. Hence $\phi_0 \circ p = \phi_0 \circ m + \phi_0 \circ n$. Since $\phi_0 \circ p \in A(G)$ (by Theorem 3.7) and $\phi_0 \circ p \in I_g^\#$, we have $\phi_0 \circ p \in I_g$. So $\phi_0 \circ p \in M$. Hence $\phi_0^2 \circ n = \phi_0 \circ p - \phi_0 \circ m$ is also in $M$. However $(\phi_0^2 - \phi_0) \circ n \in I_g^\# \cap A(G) = I_g$ by Theorem 3.7. Hence $\phi_0 \circ n \in M$. Consequently $p \in M$. Since $p$ is arbitrary, we have $I_g^\# \subseteq M$. But $I_g^\#$ is also maximal, hence $I_g^\# = M$. 

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Conversely if \( G \) is not discrete, let \( g_0 \in G \) be fixed. If \( m \in VN(G)^* \), let \( r(m) \) denote the restriction of \( m \) to \( C^*_p(G) \). Then \( r(m) \) may be regarded as an element in \( B_p(G) \). Define \( h(m) = r(m)(g_0) \). Since the Arens product on \( C^*_p(G) \) is precisely the pointwise multiplication on \( B_p(G) \) [13, Proposition 5.3], it follows that \( h \) is a nonzero continuous multiplicative linear functional on \( VN(G) \). Hence \( M = h^{-1}(0) \) is a maximal regular left ideal in \( VN(G)^* \). \( M \) does not contain \( A(G) \) since \( h(\psi) = 1 \) for any \( \psi \in A(G) \) such that \( \psi(g_0) = 1 \). Also \( M \neq I_g \) for any \( g \in G \). Indeed if \( M = I_g \), then \( m(\lambda(g)) = 0 \) for all \( m \in M \). But \( \lambda(g) \in C^*_p(G) \) by Lemma 4.7 and so there exists \( m \in VN(G)^* \) such that \( m(\lambda(g)) \neq 0 \) and \( m(x) = 0 \) for all \( x \in C^*_p(G) \). In particular \( r(m)(g_0) = 0 \) and so \( m \in M \), which is impossible.

**THEOREM 4.9.** Let \( G \) be an amenable locally compact group. Then the following statements are equivalent.

(a) \( G \) is compact.
(b) Each maximal regular left ideal \( M \) in \( VN(G)^* \) contains \( I_g \) for some unique \( g \in G \).
(c) The right identity of \( VN(G)^* \) is contained in the left ideal \( J \) generated by \( A(G) \).

**Proof.** (a) \( \Rightarrow \) (b) If \( G \) is compact, then \( A(G) \) contains the identity of \( VN(G)^* \). Hence there exists no proper left ideal of \( VN(G)^* \) which contains \( A(G) \). Apply now Corollary 4.4.

(b) \( \Rightarrow \) (c) If the right identity of \( VN(G)^* \) is not contained in \( J \), then there exists a maximal regular left ideal \( M \) in \( VN(G)^* \) containing \( J \) (see [15, pp. 58 and 68]). Hence (b) fails.

(c) \( \Rightarrow \) (a) Let \( \theta \) be the right identity of \( VN(G)^* \). If \( \theta \in J \), then \( \theta = \sum_1^n \phi_i \otimes m_i \), \( \phi_i \in A(G) \) and \( m_i \in VN(G)^* \). Then for each \( \psi \in A(G) \) and \( q \in VN(G)^* \), we have

\[
\sum_1^n \psi \phi_i \otimes (q \otimes m_i - m_i \otimes q) = q \otimes \sum_1^n \psi \phi_i \otimes m_i - \left( \sum_1^n \psi \phi_i \otimes m_i \right) \otimes q = q \otimes \psi - \psi \otimes q = 0.
\]

Since \( \theta \) is the weak*-limit of a net in \( A(G) \), it follows that

\[
\sum_1^n \theta \phi_i \otimes (q \otimes m_i - m_i \otimes q) = 0
\]

also. Consequently

\[
q - \theta \otimes q = \sum_1^n q \phi_i \otimes m_i - \sum_1^n \phi_i \otimes m_i \otimes q
\]

\[
= \sum_1^n \phi_i \otimes q \otimes m_i - \sum_1^n \phi_i \otimes m_i \otimes q
\]

\[
= \sum_1^n \phi_i \otimes \theta \otimes q \otimes m_i - \sum_1^n \phi_i \otimes \theta \otimes m_i \otimes q
\]

\[
= 0
\]

since \( A(G) \) is in the centre of \( VN(G)^* \). Hence \( \theta \) is an identity of \( VN(G)^* \), which implies that \( G \) is compact (Proposition 3.2).
Corollary 4.10. Let $G$ be an amenable locally compact group. Then $G$ is finite if and only if each maximal regular left ideal of $VN(G)^\ast$ is of the form $I_g^\ast$ for some $g \in G$.

Proof. This follows readily from Theorems 4.8 and 4.9.

5. An isomorphism theorem. We shall prove in this section that if $G_1$ and $G_2$ are discrete groups, and if the Banach algebras $VN(G_1)^\ast$ and $VN(G_2)^\ast$ are isometric and order isomorphic, then $G_1$ and $G_2$ are isomorphic. We begin with some observations.

Given any group $G$, the second conjugate algebra $VN(G)^{**}$ is again a von Neumann algebra.

Lemma 5.1. Let $G$ be a discrete group and $u$ be an invertible element in $VN(G)^{**}$. If $u$ is multiplicative on $VN(G)^\ast$, then $u = \lambda(g)$ for some $g \in G$.

Proof. Let $M = \{m \in VN(G)^\ast; \langle u, m \rangle = 0\}$. Then $M$ is a regular maximal left ideal in $VN(G)^\ast$. If $M = I_g^\ast$, then $u = \alpha \lambda(g)$ for some $\alpha \in \mathbb{C}$. Choose $\phi \in A(G)$ such that $\phi(g) = 1$. Then $\alpha = \langle u, \phi^2 \rangle = \langle u, \phi \rangle \langle u, \phi \rangle = \alpha^2$. So $\alpha = 1$, i.e. $u = \lambda(g)$. Otherwise, using Theorem 4.8, $M$ must contain $\lambda(G)$. Let

$$A(G)^0 = \{f \in VN(G)^{**}; \langle f, \phi \rangle = 0 \text{ for each } \phi \in A(G)\}.$$ 

Then $A(G)^0$ is a weak*-closed ideal in $VN(G)^{**}$ (see [17, p. 44]). Hence there exists $z$, a central projection in $VN(G)^{**}$, such that $A(G)^0 = z \cdot VN(G)^{**}$. Then $zu = u$ and $z = z(uu^{-1}) = uu^{-1} = e$, where $e$ is the identity in $VN(G)^{**}$. Hence $A(G)^0 = VN(G)^{**}$, which is impossible.

Lemma 5.2. Let $W_1$, $W_2$ be von Neumann algebras with identities $e_1$, $e_2$ respectively. Let $\Phi$ be a linear isometry from $W_1$ onto $W_2$ such that $\Phi(e_1) = e_2$. Let $G_1$ be a group of invertible elements in $W_1$. If $\Phi(G_1) = G_2$ is a group of invertible and linearly independent elements in $W_2$, then $G_1$ and $G_2$ are isomorphic as discrete groups.

Proof. We follow an idea of Martin Walter in the proof of Theorem 3 [18, p. 29]. For each $x \in G_1$, let

$$H_x = \{y \in G_1; \phi(xy) = \phi(x)\phi(y)\}, \quad K_x = \{y \in G_1; \phi(xy) = \phi(y)\phi(x)\}.$$ 

We shall show that for each $x \in G_1$,

(i) $H_x \cup K_x = G_1$,

(ii) $H_x$ and $K_x$ are subgroups of $G_1$.

When this is done, then for each $x \in G$, either $H_x = G_1$, or $K_x = G_1$. Let $H = \bigcap \{H_x; x \in G_1\}$ and $K = \bigcap \{K_x; x \in G_1\}$. Then $H$, $K$ are subgroups of $G_1$. Also if $x \in G_1$ and $H_x = G_1$, then by [12, Lemma 6],

$$\Phi(x \cdot y) + \Phi(y \cdot x) = \Phi(x)\Phi(y) + \Phi(y)\Phi(x)$$

(1)

for each $y \in G_1$. So $\Phi(y \cdot x) = \Phi(y)\Phi(x)$ and $x \in H$. Similarly if $K_x = G_1$, then $x \in K$. Consequently $H \cup K = G_1$. Hence either $H = G_1$ or $K = G_1$. In each of the two cases, the groups $G_1$ and $G_2$ are isomorphic. It remains to prove (i) and (ii).

Let $y \in G_1$. If $xy = yx$, then $\Phi(xy) = \Phi(x)\Phi(y)$ by [12, Theorem 5]. Hence $y \in H_x \cap K_x$ by (1). If $xy \neq yx$ and $y \notin H_x \cup K_x$, then $\Phi(xy) = \Phi(yx)$, $\Phi(x)\Phi(y)$
and \( \Phi(y)\Phi(x) \) are distinct elements in \( G_2 \). In fact, if \( \Phi(x)\Phi(y) = \Phi(y)\Phi(x) \), then applying [12, Theorems 5 and 7] to the operator \( \Phi^{-1} \), we get \( xy = yx \). Also, \( \Phi(xy) \neq \Phi(x)\Phi(y) \). For otherwise \( \Phi(xy) = \Phi(y)\Phi(x) \), i.e. \( y \in K_x \), by (1). Similarly, \( \Phi(xy) \neq \Phi(y)\Phi(x) \). Since distinct elements in \( G_2 \) are linearly independent (by assumption), this contradicts (1). Hence (i) holds.

To prove (ii), let \( z_i \in W_i \), \( i = 1, 2 \), be central projections such that \( \Phi \) is a \( * \)-isomorphism from \( W_1z_1 \) onto \( W_2z_2 \) and a \( * \)-anti-isomorphism from \( W_1(e_1 - z_1) \) onto \( W_2(e_2 - z_2) \) (see [12, Theorem 10]). Then \( \Phi(xz_1) = \Phi(x)z_2 \) and \( \Phi(x(e_1 - z_1)) = \Phi(x)(e_2 - z_2) \) for all \( x \in W_1 \). Observe that

\[
y \in K_x \text{ if and only if } (xy - yx)z_1 = 0 \quad (2)
\]

and

\[
y \in H_x \text{ if and only if } (xy - yx)(e_1 - z_1) = 0. \quad (3)
\]

To prove (2), let \( y \in K_x \), then

\[
\Phi((xy - yx)z_1) = \Phi(xy - yx)z_2 = \Phi(y)\Phi(x)z_2 - \Phi(yz_1)\Phi(xz_1)
\]

\[= 0.
\]

Hence \( (xy - yx)z_1 = 0 \). Conversely if \( (xy - yx)z_1 = 0 \) and \( y \notin K_x \), then \( y \in H_x \) by (i). Hence \( (xy - yx)(e_1 - z_1) = 0 \). So \( xy = yx \). Consequently \( y \in K_x \) by [12, Theorem 5] and (1). Statement (3) can be proved similarly. Now if \( y_1, y_2 \in K_x \), then

\[
x(y_1y_2)z_1 = xy_1z_1y_2 = (y_1x)z_1y_2 = y_1(xy_2)z_1 = (y_1y_2)xz_1
\]

using (2). Hence \( y_1y_2 \in K_x \) by (2) again. Also if \( y \in K_x \), then \( (y^{-1}x - xy^{-1})z = y^{-1}[(xy - yx)z]y^{-1} = 0 \). Hence \( y^{-1} \in K_x \) by (2). Hence \( K_x \) is a subgroup of \( G_1 \). Similarly, \( H_x \) is also a subgroup of \( G_x \).

**Theorem 5.3.** Let \( G_1, G_2 \) be discrete groups. If there exists an order preserving isometric algebra isomorphism from \( VN(G_2)^* \) onto \( VN(G_1)^* \), then \( G_1 \) and \( G_2 \) are isomorphic.

**Proof.** We identify \( \lambda(x) \in VN(G_i) \) with \( x \in G_i \), \( i = 1, 2 \). Let \( U \) be an order preserving isometric isomorphism from \( VN(G_2)^* \) onto \( VN(G_1)^* \), and \( \Phi = U^* \). Let \( e_i \) denote the identity of \( G_i \). If \( n \in VN(G)^* \), \( n \geq 0 \), then \( \langle n, \Phi(e_i) \rangle = \langle U(n), e_i \rangle = \|U(n)\| = \|n\| = \langle n, e_i \rangle \). Hence \( \Phi(e_i) = e_2 \). By [12, Theorem 7 and Lemma 12], \( \Phi \) is a Jordan \( * \)-isomorphism mapping the set of unitary elements in \( VN(G_1)^{**} \) onto \( VN(G_2)^{**} \). By Lemma 5.1, \( \Phi \) maps \( G_1 \) onto \( G_2 \). Also, elements in \( G_2 \), being multiplicative on \( VN(G)^* \), are linearly independent (see [2, Lemma 17 and Corollary 18, p. 93] and note that their proofs do not depend on the fact that the Banach algebra is commutative). The conclusion now follows from Lemma 5.2.

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