A PHÄRGMÉN-LINDELÖF THEOREM CONJECTURED BY D. J. NEWMAN

BY

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In memory of H. C. Wang

ABSTRACT. Let \( D \) be a region of the complex plane, \( \infty \in \partial D \). If \( f(z) \) is holomorphic in \( D \), write \( M(r) = \sup_{|z| \leq r, z \in D} |f(z)| \).

Theorem 1. If \( f(z) \) is holomorphic in \( D \) and \( \limsup_{\xi \to \infty, \xi \in D} |f(z)| < 1 \) for \( \xi \in \partial D \), \( \xi \neq \infty \), then one of the following holds:

(a) \( |f(z)| < 1 \) (\( z \in D \)).
(b) \( f(z) \) has a pole at \( \infty \).
(c) \( \log M(r)/\log r \to \infty \) as \( r \to \infty \). If \( M(r)/r \to 0 \) (\( r \to \infty \)), then (a) must hold.

1. The main result of this paper is a proof of a conjecture of D. J. Newman.
Throughout the paper \( D \) is an unbounded region (\( = \) connected open set) of the \( z \)-plane with at least one boundary point \( \neq \infty \), \( f(z) \) is holomorphic in \( D \) and for every boundary point \( \xi \neq \infty \)

\[
\limsup_{z \to \xi} |f(z)| < 1. \tag{1.1}
\]

Let \( M(r) = \sup_{z \in D, |z| = r} |f(z)| \).

Theorem 1. Under the conditions on \( D \) and \( f(z) \) stated above one of the following three mutually exclusive possibilities must occur:

(a) \( |f(z)| < 1 \) (\( z \in D \)).
(b) \( f(z) \) has a pole at \( \infty \).
(c) \( \log M(r)/\log r \to \infty \) (\( r \to \infty \)).

Theorem 2 (Newman's conjecture). Under the assumptions of Theorem 1

\[
\liminf_{r \to \infty} M(r)/r = 0 \quad (r \to \infty) \tag{1.2}
\]

implies

\[
|f(z)| < 1 \quad (z \in D). \]

This is an immediate consequence of Theorem 1, since (1.2) excludes the possibilities (b) and (c).

An interesting feature of Theorem 2 is that the growth condition (1.2) is entirely independent of the geometry of \( D \). Another feature that distinguishes Theorem 2 from other theorems of Phragmén-Lindelöf type is the following: The standard

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method of proving such theorems is to translate them into a statement about the
subharmonic function \( \log|f(z)| \) and then to prove this statement for subharmonic
functions by an application of the theory of harmonic measure. This is not possible
in our case: The translation to subharmonic functions is: "If \( u(z) \) is subharmonic in
the region \( D \),
\[
\limsup_{z \to \xi, z \in D} u(z) < 0 \quad (\xi \in \partial D, \xi \neq \infty)
\]
and
\[
\liminf_{r \to \infty} \left\{ \sup_{|z|=r, z \in D} u(z) - \log r \right\} = -\infty,
\]
then \( u(z) \leq 0 \) in \( D \)."

But this statement is not always true, as the case \( D = \{z: |z| > 1\} \), \( u(z) = \frac{1}{2} \log|z| \) shows.

To get a conclusion for the subharmonic case it is convenient to bring the
"exceptional boundary point" \( z = \infty \) into the finite plane. Without any loss of
generality we may assume that \( 0 \in D \). Inversion \( z \to 1/z \) changes \( D \) into a region
\( B \) which contains a full neighborhood of \( \infty \). \( z = \infty \) is thrown to the origin.

**Theorem 3.** Let \( B \) be a region containing \( z = \infty \) and let \( z = 0 \) belong to the
boundary \( \partial B \) of \( B \).

Let \( u(z) \) be subharmonic in \( B \) and let \( \limsup_{z \to \xi, z \in B} u(z) < 0 \) \( (\xi \in \partial B, \xi \neq 0) \).

Then
\[
u(z) \leq C \log |z|^{-1} \left( z \in B, |z| < \frac{1}{2} \right) \text{ implies } u(z) \leq 0 \ (z \in B),
\]
if and only if \( \partial B \) has positive logarithmic capacity and \( z = 0 \) is a regular boundary
point for the Dirichlet problem in \( B \).

2. Proof of Theorem 1. We must prove that either (a) or (b) holds, if (c) is false,
i.e. if
\[
\liminf \log M(r)/\log r < p < \infty.
\]

If we suppose that (a) is false, then
\[
1 < \sup_{z \in D} |f(z)| = \mu < \infty.
\]

Since \( f'(z) \) has at most a denumerable set of zeros in \( D \), there is at most a
denumerable set \( F \) of \( A \), \( 1 < A < \mu \), such that one of the level curves,
\[
|f(z)| = A \quad (z \in D),
\]
contains a zero of \( f' \). If \( A \notin F \), each curve on which (2.3) holds is a simple curve,
either closed or without a finite endpoint. We choose an \( A \notin F \) and consider the
nonempty open set
\[
D_A = \{ z: z \in D, |f(z)| > A \}.
\]

Each component of \( D_A \) must be unbounded, by the maximum modulus principle.
We consider two cases.

**Case 1.** There is an unbounded level curve (2.3). We shall show that in this case
(a) must hold. Choose a point \( b \) on this level curve at minimum distance from the
origin and let $C$ be one of the two unbounded portions into which $b$ divides the level curve. For $R > |b|$ let $\overline{C}_R$ be the portion of $C$ between $b$ and the first point of intersection of $C$ with $|z| = R$.

Let $\omega_R(z)$ be the harmonic measure of $|z| = R$ with respect to the region $\Delta_R = \{|z| < R\} \setminus \overline{C}_R$.

The Carleman-Milloux theorem [1, p. 108] asserts that

$$\omega_R(z) < \varphi_R(-|z|) \quad (z < R), \tag{2.5}$$

where $\varphi_R(z)$ is the harmonic measure of $\{|z| = R\}$ with respect to

$$\{|z| < R\} \setminus \{z = x + iy : y = 0, |b| < x < R\}.$$

A direct calculation of $\varphi_R$ shows that

$$\varphi_R(-|z|) < K(z) R^{-1/2} \quad (z \text{ fixed}, R \to \infty). \tag{2.6}$$

The function $u(z) = \log |f(z)| - \log M(R) \omega_R(z) - \log A$ is harmonic in $\{|z| < R\} \cap D_A \subset \Delta_R$. At every boundary point of $\{|z| < R\} \cap D_A$, $u(z) < 0$ and therefore

$$u(z) < 0, \quad z \in \{|z| < R\} \cap D_A. \tag{2.7}$$

Keep $z \in D_A$ fixed and let $R \to \infty$ through a sequence of values for which $\log M(R) < \rho \log R$. This is possible, by (2.1). Then it follows from (2.7), (2.5) and (2.6) that

$$\log |f(z)| < \log A \quad (z \in D_A).$$

But this contradicts (2.4) and shows that (2.2) is not tenable; we have the alternative (a) of Theorem 1.

Case 2. All level curves on which (2.3) holds are bounded. We shall prove that there is a positive $\rho$ such that

$$\{|\rho < |z| < \infty\} \subset D_A.$$ 

It will then follow easily that under the hypothesis (2.1) either (a) or (b) must be true.

We show first that only a finite number of components $C_n$ of (2.3) can meet a circle $|z| = r$ which intersects $D_A$:

The intersection of $|z| = r$ and $D$ is the union of disjoint, open, circular arcs $I$.

The function

$$g(z) = f(z) \sqrt{r^2/\overline{z}}$$

is holomorphic in a neighborhood of any closed subarc $J$ of an arc $I$ and, on $|z| = r$, $g(z) = |f(z)|^2$.

Suppose that an infinity of distinct level curves $C_n$ given by (2.3) intersect $|z| = r$. Let $z_n$ be a point of intersection of $C_n$ with $|z| = r$. Without loss of generality we may assume that $z_n \to Z$, $|Z| = r$, $g(Z) = A^2 = g(z_n)$. Either $Z \in D$ or $Z \in \partial D$. The latter possibility can be excluded, since every boundary point of $D$ other than $\infty$ has a neighborhood in which $|f(z)| < A^2 = g(z_n)$. Therefore $Z$ is an (interior) point of one of the intervals I described above and $g(z)$ is holomorphic in a neighborhood of $Z$. For sufficiently large $n$, $z_n$ belongs to this neighborhood.
and, because of $g(z_n) = g(Z) = A^2, g(z) = |f(z)|^2 = A^2$ on $I$. But this contradicts the assumption that distinct components of (2.3) pass through $z_1, z_2, \ldots$; only a finite number of components of (2.3) can meet $|z| = r$.

Now it is easy to see that $D_A$ is a region: Since every component of $D_A$ is unbounded, it is enough to show that two points $z_1, z_2 \in D_A$ with $|z_1| = |z_2| = r$ can be connected by a curve lying in $D_A$. If one of the two arcs of $|z| = r$ with endpoints $z_1, z_2$ lies in $D_A$ we have nothing to prove. If this is not the case, choose one of these two arcs. It intersects the components $C_1, C_2, \ldots, C_m$ of (2.3) all of which are smooth simple curves with a minimum distance $\delta > 0$ from each other; $z_1$ and $z_2$ can be connected by a path consisting of parts of $C_1, C_2, \ldots, C_m$ and circular arcs of $|z| = r$ which lie in $D_A$. A slight displacement of the path into $D_A$ will produce the desired curve joining $z_1$ and $z_2$ in $D_A$.

Let $D(r) = D_A \cap \{ c < |z| < r \}$.

For sufficiently large $c$ the boundary $\partial D(r)$ of $D(r)$ will consist of arcs of $|z| = c$ and $|z| = r$ which lie in $D_A$ and of portions of level curves (2.3).

We apply Gauss's Theorem to the harmonic function $\log|f(z)|$ in $D(r)$:

$$\int_{\partial D(r)} \frac{\partial}{\partial n} \log|f(z)| \, |ds| = 0. \quad (2.8)$$

Here $\partial / \partial n$ denotes the differentiation in the direction of the outward normal to $\partial D(r)$ and $ds$ is the element of arc length. On the level curves (2.3)

$$(\partial / \partial n) \log|f(z)| < 0. \quad (2.9)$$

Equality in (2.9) cannot hold for $z$ on a level curve $C$ satisfying (2.3), because on such a curve $(\partial / \partial s) \log|f(z)| = 0$ ($\partial / \partial s = \text{differentiation in a direction tangential to } C$) while

$$|f'(z)| = \left| \frac{\partial}{\partial n} \log|f(z)| - i \frac{\partial}{\partial s} \log|f(z)| \right| |f(z)| \neq 0.$$

Therefore on any such level curve $C$

$$0 > \int_C \frac{\partial}{\partial n} \log|f(z)| \, |ds|. \quad (2.10)$$

By the Cauchy-Riemann equations

$$\int_C \frac{\partial}{\partial n} \log|f(z)| \, |ds| = \text{change of arg} f(z) \text{ as } C \text{ is described}$$

once in the clockwise direction.

By the single-valuedness of $f(z)$ the expression on the right-hand side of the last equation must be an integer multiple of $2\pi$. Hence, by (2.10), on any closed level curve,

$$\int_C \frac{\partial}{\partial n} \log|f(z)| \, |ds| < -2\pi. \quad (2.11)$$
Let $\nu(r)$ be the number of components of (2.3) entirely situated in $c < |z| < r$. By (2.8), (2.9) and (2.11)
\[
0 < -2\pi\nu(r) + \int_{re^\theta \in D_A} \frac{\partial}{\partial r} \log|f(re^{i\theta})| \; d\theta
\]
\[
- \int_{ce^\theta \in D_A} \left[ \frac{\partial}{\partial r} \log|f(re^{i\theta})| \right]_{r=c} \; d\theta
\]
\[
< -2\pi\nu(r) + \int_{re^\theta \in D_A} \frac{\partial}{\partial r} \log|f(re^{i\theta})| \; d\theta + K \quad (K = K(c)).
\]

Divide by $r$, integrate from $c$ to $\rho$:
\[
2\pi \int_c^\rho \frac{\nu(r)}{r} \; dr < K \log(\rho/c) + \int_{re^\theta \in D(\rho)} \frac{\partial}{\partial r} \log|f(re^{i\theta})| \; d\theta \; dr. \quad (2.12)
\]

We estimate the integral on the right-hand side by first performing the $r$-integration. For fixed $\theta$ the set of $r$ over which we have to integrate consists of disjoint intervals. If $pe^{i\theta} \in D_A$, then the value of $\log|f(re^{i\theta})|$ at the right-hand endpoint of one interval is $\log|f(pe^{i\theta})|$. If $ce^{i\theta} \in D_A$, then the value of $\log|f|$ at one left-hand endpoint is $\log|f|$. At all other endpoints $\log|f|$ has the value $\log A$. Hence
\[
\int_{re^\theta \in D(\rho)} \frac{\partial}{\partial r} \log|f(re^{i\theta})| \; d\theta < \log|f(re^{i\theta})/A|/A
\]
(where we set $|f(pe^{i\theta})| = A$ outside $D_A$).

Thus, using (2.12),
\[
2\pi \int_c^\rho \frac{\nu(r)}{r} \; dr < 2\pi \log|f(pe^{i\theta})/A|/A \; d\theta + K \log(\rho/c). \quad (2.13)
\]

If (2.1) holds, then we can find arbitrary large $\rho$ such that (2.13) implies
\[
2\pi \int_c^\rho \frac{\nu(r)}{r} \; dr < (p + K)\log \rho + K_1. \quad (2.14)
\]

Since $\nu(r)$ is a nondecreasing function of $r$, (2.14) is only possible for arbitrarily large $\rho$ if $\nu(r) \leq p + K$, i.e. the number of components of (2.3) is bounded by $p + K$. Since under our present assumptions each component is bounded, one can therefore find a $\rho$ such that
\[
\{\rho < |z| < \infty\} \subset D_A.
\]

In $|z| > \rho$, $f(z)$ can be expanded in a Laurent series $f(z) = \sum_{-\infty}^{\infty} c_n z^n$,
\[
|c_n| = \frac{R^{-n}}{2\pi} \left| \int_{|z|=R} f(Re^{i\theta}) e^{-in\theta} \; d\theta \right| \quad (R > \rho). \quad (2.15)
\]

If (2.1) is true, (2.15) implies $|c_n| < R^{p-n}$ for arbitrarily large $R$ and therefore $c_n = 0 \quad (n > p)$.

Therefore $f(z)$ either has a pole at $\infty$ and (b) of Theorem 1 holds or $\infty$ is a removable singularity of $f(z)$ and (a) holds by the maximum modulus principle.
3. Proof of Theorem 3. We shall use the following three properties of regular and irregular boundary points [2, Theorem III-34, p. 81; Theorem III-36, p. 82; Theorem III-38, p. 83]:

(1) The boundary point $\xi$ of the region $B$ is regular if and only if the Green's function $g(z, z_0)$ of $B$ with pole at $z_0$ tends to $0$ as $z \to \xi, z \in B$.

(2) Regularity is a local property: If $\xi$ is a boundary point of the regions $B$ and $C$, both with boundaries of positive capacity, and if there is a neighborhood $N$ of $\xi$ such that $N \cap B = N \cap C$, then $\xi$ is regular (or irregular) for both $B$ and $C$.

(3) The set of irregular boundary points has capacity zero.

(i) (1.3) does not hold, if the complement $E$ of $B$ in $C$ has logarithmic capacity zero.

**Proof.** The set $E$ has an Evans potential [2, p. 75]

$$v(z) = \int_E \log \frac{1}{|z - \zeta|} \, d\mu(\zeta), \quad \mu(E) = 1, \, d\mu > 0 \quad (3.1)$$

such that, for every $\zeta \in E$, $v(z) \to \infty$ ($z \to \zeta$).

The function $v(z)$ is superharmonic and, near $\infty$, $v(z) + \log|z|$ is harmonic.

Consider

$$u(z) = -\log|z| - v(z) + \log|z_0| + v(z_0) + 1,$$

where $z_0$ is some point in $B$. $u(z)$ is subharmonic in $|z| > 0$ and $u(z_0) = 1 > 0$.

For every boundary point $\xi \neq 0$ of $B$, $\lim_{z \to \xi} u(z) = -\infty$.

Near 0, $u(z) < \log|z|^{-1} + K$. Therefore one can find a constant $A$ such that

$$u(z) < A \log|z|^{-1} \quad (z \in B, \, |z| < \frac{1}{2})$$

(maximum modulus principle applied to $u - A \log|z|^{-1}$ in $(e < |z| < \frac{1}{2}) \cap B$).

Hence $u(z)$ is a counterexample to the implication (1.3).

We can now assume that $E$ has positive capacity and therefore the Green's function $g(z, z_0)$ of $B$ with pole at $z_0$ exists.

Let $B_0 = B \cup \{|z| < \rho\}$.

For sufficiently small $\rho$ the complement of $B_0$ has positive capacity, since otherwise

$$0 < \text{capacity } E = \text{capacity}(E \cap \{|z| < \rho\}) \leq \text{capacity} \{|z| < \rho\} = \rho.$$  

Let $G(z, z_0)$ be the Green's function of $B$ with pole at $z_0$. Since $B_\sigma \subset B_\rho$ $(0 < \sigma < \rho)$,

$$0 < G_\sigma(z, z_0) \leq G_\rho(z, z_0) \quad (\sigma < \rho; \, z, z_0 \in B_\sigma).$$

Therefore $H(z, z_0) = \lim_{\rho \to 0} G(z, z_0)$ exists for $z \in B, z_0 \in B \cup \{0\}$. By Harnack's Theorem $H(z, 0)$ is a nonnegative harmonic function in $B$.

At every regular boundary point $\xi \neq 0$ of $B$

$$0 < \limsup_{z \to \xi} H(z, 0) < \limsup_{z \to \xi} G_\rho(z, 0) = 0,$$

i.e.

$$\limsup_{z \to \xi} H(z, 0) = 0. \quad (3.2)$$
By the definition of the Green's function
\[ G_\rho(z, 0) < \log(1/|z|) + K(\rho) \quad (|z| < \frac{1}{2} \rho, z \in B). \]
Therefore for all sufficiently small \( \delta \)
\[ H(z, 0) < G_\rho(z, 0) < 2\log|z|^{-1} \quad (|z| = \delta, z \in B). \] (3.3)
Also we can find a constant \( C > 2 \) such that
\[ H(z, 0) < G_\rho(z, 0) < C\log|z|^{-1} \quad (|z| = 1, z \in B). \] (3.4)
By the extended maximum principle [2, Theorem III-28, p. 77] applied to
\[ H(z, 0) + C\log|z| \text{ in } \{ \delta < |z| < \frac{1}{2} \} \cap B, \] it follows from (3.2)–(3.4) that
\[ H(z, 0) < C\log|z|^{-1} \quad (|z| < \delta, z \in B). \] (3.5)

(ii) (1.3) holds if and only if \( H(z, 0) \equiv 0 \) (\( z \in B \)).

**Proof.** The condition is necessary: If \( H(z, 0) \) is not identically zero, then
\( H(z, 0) > 0 \) in \( B \). If every boundary point of \( B \) is regular, then \( H(z, 0) \) is a
counterexample to (1.3), by (3.2) and (3.5).
If the set \( F \) of irregular boundary points is not empty, then it has an Evans
potential \( \psi(z) \) given by (3.1) with \( E \) replaced by \( F \). Let \( d \) be the diameter of \( \partial B \).
Then
\[ \psi(z) = \int_F \log \frac{d}{|z - \xi|} \, d\mu(\xi) \]
tends to \( \infty \) as \( z \) tends to a point of \( F \), \( \psi(z) > 0 \) (\( z \in \partial B \)) and \( \psi(z) = \phi(z) + g(z, \infty) \) is harmonic in \( B \)
and
\[ \liminf_{z \to \xi} \psi(z) > 0 \quad (\xi \in \partial B). \]
The function \( H(z, 0) - \epsilon\psi(z) \) now provides a counterexample to (3), if \( \epsilon \) is
chosen so small that, for some \( z_0 \in B, H(z_0, 0) - \epsilon\psi(z_0) > 0 \).
The condition is sufficient: Let \( u(z) \) satisfy the hypotheses of Theorem 3,
including that of (1.3). Then
\[ \psi_\rho(z) = u(z) - 2CG_\rho(z, 0) \]
is subharmonic in \( B \), and at all boundary points \( \xi \) of \( B \) other than 0
\[ \limsup_{z \to \xi} \psi_\rho(z) < \limsup_{z \in \xi} u(z) < 0. \]
In \( |z| < \frac{1}{2} \rho, G_\rho(z, 0) > \log|z|^{-1} - K_1(\rho) \). Hence
\[ \psi_\rho(z) < 2CK_1(\rho) + C\log|z| \quad (|z| < \frac{1}{2}, z \in B) \]
and \( \limsup_{z \to \infty; z \in B} \psi_\rho(z) = -\infty \).
By the maximum principle \( \psi_\rho(z) < 0 \) (\( z \in B \)). The conclusion \( u(z) < 0 \) (\( z \in B \))
follows on letting \( \rho \) tend to 0, since
\[ \lim G_\rho(z, 0) = H(z, 0) \equiv 0 \quad (z \in B). \]

(iii) \( H(z, 0) \equiv 0 \) in \( B \) if and only if 0 is a regular boundary point of \( B \).
Proof. Sufficiency. We show first that
\[ H(z, z_0) = g(z, z_0) \quad (z, z_0 \in B). \tag{3.6} \]
Since \( B_\rho \supset B \), \( G_\rho(z, z_0) - g(z, z_0) \) is a nonnegative, bounded harmonic function in \( B \). At every regular boundary point \( \xi \neq 0 \) of \( B \)
\[ \lim_{z \to \xi, z \in B} (G_\rho(z, z_0) - g(z, z_0)) = 0 \quad (\rho < \rho(\xi)). \]
Therefore
\[ w(z) = H(z, z_0) - g(z, z_0) = \lim_{\rho \to 0} (G_\rho(z, z_0) - g(z, z_0)) \]
is a nonnegative, bounded harmonic function in \( B \) and \( \lim_{z \to \xi, z \in B} w(z) = 0 \)
(\( \xi \in \partial B, \xi \neq 0, z \in B \)).

By the extended maximum principle \( w(z) \leq 0 \) (\( z \in B \)). But \( w(z) \) is nonnegative, \( w(z) = 0 \) (\( z \in B \)); (3.6) is proved.

We choose \( z_0 \in B \). Since the irregular boundary points of \( B \) form a set of capacity zero, one can find arbitrarily small \( \delta \) such that \( |z| = \delta \) does not contain any irregular points [2, p. 85].

The functions
\[ f_\rho(z) = \begin{cases} G_\rho(z, z_0) & (z \in B), \\ 0 & (z \notin B) \end{cases} \]
are continuous on such a circle \( |z| = \delta \). They converge to the continuous function \( g(z, z_0) \) (\( g(z, z_0) = 0, z \notin B \)) on \( |z| = \delta \). Since they are decreasing with \( \rho \), they converge uniformly to \( g(z, z_0) \) on \( |z| = \delta \), as \( \rho \to 0 \). Therefore one can find \( \rho = \rho(\delta, \varepsilon) \) such that
\[ g(z, z_0) < G_\rho(z, z_0) < g(z, z_0) + \varepsilon \quad (|z| = \delta, z \in B). \]

If \( z = 0 \) is a regular boundary point, we can choose \( \delta \) so small that, given \( \varepsilon > 0 \),
\[ g(z, z_0) < \varepsilon \quad (|z| = \delta, z \in B), \]
\[ G_\rho(z, z_0) < 2\varepsilon \quad (|z| = \delta, z \in B) \]
for suitable \( \delta = \delta(\varepsilon) < \frac{1}{2}|z_0|, \rho = \rho(\varepsilon) \).

The function \( G_\rho(z, z_0) \) is harmonic and bounded above in \( \{|z| < \delta\} \cap B_\rho \).
At all regular boundary points of \( B_\rho \) in \( |z| < \delta \), \( G_\rho(z, z_0) \) has the boundary value 0.

Therefore, by the extended maximum principle, \( G_\rho(0, z_0) < 2\varepsilon \).
By the symmetry of the Green's function \( G_\rho(z_0, 0) \) < 2\varepsilon for some \( \rho > 0 \). Hence
\[ \lim_{\rho \to 0} G_\rho(z, z_0) = H(z_0, 0) = 0 \quad (z_0 \in B). \]

Necessity. If \( H(z, 0) \equiv 0 \) (\( z \in B \)), then
\[ G_\rho(0, z_0) = G_\rho(z_0, 0) < \varepsilon \quad (\rho = \rho(\varepsilon)). \]
By continuity \( G_\rho(z, z_0) < 2\varepsilon \) (\( |z| < \eta(\varepsilon), \rho = \rho(\varepsilon) \)). Hence
\[ g(z, z_0) < G_\rho(z, z_0) < 2\varepsilon \quad (|z| < \eta(\varepsilon), z \in B). \]

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That is to say \( \lim g(z, z_0) = 0 \) as \( z \to 0, \ z \in B \). Therefore 0 is a regular boundary point.

This completes the proof of Theorem 3.

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