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THE STRUCTURE OF TENSOR PRODUCTS OF SEMILATTICES WITH ZERO

BY

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ABSTRACT. If A and B are finite lattices, then the tensor product C of A and B in the category of join semilattices with zero is a lattice again. The main result of this paper is the description of the congruence lattice of C as the free product (in the category of bounded distributive lattices) of the congruence lattice of A and the congruence lattice of B. This provides us with a method of constructing finite subdirectly irreducible (resp., simple) lattices: if A and B are finite subdirectly irreducible (resp., simple) lattices then so is their tensor product. Another application is a result of E. T. Schmidt describing the congruence lattice of a bounded distributive extension of M_3.

1. Introduction. Tensor products of semilattices have been studied by J. Anderson and N. Kimura [1], G. Fraser [2]–[5], and Z. Shmuely [9]. In this paper we deal with join semilattices with zero. In certain cases the tensor product of two semilattices with zero is a lattice; thus, we can investigate the lattice of lattice congruences of the tensor product. Our main result is a representation theorem: The congruence lattice of such a tensor product is a tensor product of the congruence lattices.

More precisely, we are working in the category S_0, whose objects are join semilattices with zero and whose morphisms are the zero-preserving join homomorphisms. Similarly, S_1 is the category of meet semilattices with unit whose morphisms are the unit-preserving meet homomorphisms. If $A \in S_0$ is a lattice, then $\mathrm{Con}_L(A)$ denotes the lattice of lattice congruences of A. Let $\otimes$ denote the tensor product in S_0. Our theorem then states that under suitable conditions on $A, B \in S_0$,

$$\mathrm{Con}_L(A \otimes B) \simeq \mathrm{Con}_L(A) \otimes \mathrm{Con}_L(B).$$

This result has two interesting consequences. $A \otimes B$ is a finite simple (respectively, subdirectly irreducible) lattice iff both A and B are finite simple (respectively, subdirectly irreducible) lattices.

Note that, in general, the tensor product of A and B in S_0 is not isomorphic to the tensor product of A and B in S_1 even though their congruence lattices are isomorphic.

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We next describe how we came to the main result. In Schmidt [8], the lattice $M_3[D]$ is discussed. For a finite distributive lattice d, $M_3[D]$ denotes the lattice of all isotone maps from the poset of join irreducible elements of D into M_3. This is isomorphic to $\mathrm{Hom}_0(D, M_3)$ the lattice of all zero preserving join homomorphisms of D into M_3. E. T. Schmidt shows that $M_3[D^d]$ (where D^d is the dual of D) is isomorphic to the subposet of D^3 consisting of all ordered triples (a, b, c) such that $a \wedge b = a \wedge c = b \wedge c$. But this latter poset is isomorphic to $\mathrm{Hom}_1(M_3, D)$, the poset (in fact, lattice) of all unit-preserving meet homomorphisms of M_3 into D. Thus Schmidt's result can be stated as follows:

$$\mathrm{Hom}_0(D^d, M_3) \simeq \mathrm{Hom}_1(M_3, D).$$

Moreover, he proves that the congruence lattice of $M_3[D]$ is isomorphic to the congruence lattice of D.

We will generalize Schmidt's result and show that it readily follows from our results. However, we must proceed with some care due to the numerous dualities present. For $A \in S_i$, let A^d be the dual of A. Note that $M_3 \simeq M_3^d$ and that although in general $D \not\simeq D^d$, they have dually isomorphic posets of join irreducible elements. We will show that

$$A \otimes B \simeq [\mathrm{Hom}_0(A, B)]^d.$$

To obtain Schmidt's result, let $J(D)$ (respectively, $M(D)$) be the poset of join (respectively, meet) irreducible elements of a finite $D \in D_{0,1}$; it is well known that $J(D) \simeq M(D)$. For posets P and Q, let $Q[P]$ be the poset of isotone maps from P to Q. Since D is a finite distributive lattice,

$$\mathrm{Hom}_0(D^d, M_3) \simeq M_3[J(D^d)] \simeq M_3[M(D^d)] \simeq \mathrm{Hom}_1(D^d, M_3).$$

But the commutativity of the tensor product in S_1 now tells us that

$$\mathrm{Hom}_1(D^d, M_3) \simeq \mathrm{Hom}_1(M_3, D^d) \simeq \mathrm{Hom}_1(M_3, D),$$

which is Schmidt's result.

For notation and concepts not defined in this paper, see [7].

2. The tensor product. Let $A, B, C \in S_0$; a map $f\colon A \times B \to C$ is a *bimorphism* if for each $a \in A$ and $b \in B$, we have $f(a, -) \in \mathrm{Hom}_0(B, C)$ and $f(-, b) \in \mathrm{Hom}_0(A, C)$. Notice that if $f\colon A \times B \to C$ is a bimorphism, then for all $a \in A$ and $b \in B$, $f(a, 0) = f(0, b) = 0$.

We say that $A \otimes B \in S_0$ is a *tensor product* of A and B in S_0 if there is a bimorphism $f\colon A \times B \to A \otimes B$ such that

(i) $f(A \times B)$ generates $A \otimes B$;

(ii) for any bimorphism $g\colon A \times B \to C$, there is an $h \in \mathrm{Hom}_0(A \otimes b, C)$ such that $g = hf$.

Let $A \circ B$ be the direct product of the posets $A - \{0\}$ and $B - \{0\}$; we make $A \circ B$ into a partial join semilattice as follows: If (a, b) and (a', b') are comparable or if $a = a'$ or if $b = b'$, then $(a, b) \vee (a', b')$ is defined and equals $(a \vee a', b \vee b')$; otherwise $(a, b) \vee (a', b')$ is undefined. Let $F_0(A \circ B)$ be the join semilattice with zero freely generated by $A \circ B$ (in particular, a zero is added even if

$A \circ B$ has a least element). It is easily seen that $F_0(A \circ B)$ is a tensor product of A and B. This implies that the tensor product is commutative and associative. We will denote the elements of $A \circ B$ by $a \otimes b$; if $a = 0$ or $b = 0$, then $a \otimes b$ denotes 0.

If A and B are finite, $F_0(A \circ B)$ is isomorphic to the lattice of all ideals of $A \circ B$; recall that a subset I of $A \circ B$ is an *ideal* provided:

(i) if $(a, b) \in I$ and $(a', b') \leqslant (a, b)$, then $(a', b') \in I$;

(ii) if $(a, b), (a', b') \in I$ and $(a, b) \vee (a', b')$ is defined, then $(a, b) \vee (a', b') \in I$.

Note that the empty set is an ideal.

G. Fraser [4] solved the word problem for the tensor product in the category of join semilattices; we state the analogous result for lattices in S_0.

THEOREM 2.1 (G. FRASER [4]). *Let $A, B \in S_0$ be lattices. Let $a, a_1, \ldots, a_n \in A - \{0\}$ and let $b, b_1, \ldots, b_n \in B - \{0\}$. Then $a \otimes b \leqslant \bigvee \{a_i \otimes b_i | 1 \leqslant i \leqslant n\}$ iff there is an n-ary lattice polynomial p such that $a \leqslant p(a_1, \ldots, a_n)$ and $b \leqslant p^d(b_1, \ldots, b_n)$, where p^d is the dual of p.*

COROLLARY 2.2 (G. FRASER [4]). *Let $A, B \in S_0$ be lattices. Let $a, a_1, a_2 \in A - \{0\}$ and let $b, b_1, b_2 \in B - \{0\}$. Then $a \otimes b \leqslant (a_1 \otimes b_1) \vee (a_2 \otimes b_2)$ iff one of the following four conditions hold.*

(i) $a \leqslant a_1$ *and* $b \leqslant b_1$;

(ii) $a \leqslant a_2$ *and* $b \leqslant b_2$;

(iii) $a \leqslant a_1 \vee a_2$ *and* $b \leqslant b_1 \wedge b_2$;

(iv) $a \leqslant a_1 \wedge a_2$ *and* $b \leqslant b_1 \vee b_2$.

COROLLARY 2.3. *Let $A, B \in S_0$ be lattices. Let $a, a_1, a_2 \in A - \{0\}$ and let $b, b_1, b_2 \in B - \{0\}$. Let $a_1 \leqslant a_2$ and $b_2 \leqslant b_1$, or let $a_2 \leqslant a_1$ and $b_1 \leqslant b_2$; then $a \otimes b \leqslant (a_1 \otimes b_1) \vee (a_2 \otimes b_2)$ iff $a \otimes b \leqslant a_1 \otimes b_1$ or $a \otimes b \leqslant a_2 \otimes b_2$.*

This is the first of two key observations; it tells us that, under the conditions stated the ideal of $A \circ B$ generated by $\{a_1 \otimes b_1, a_2 \otimes b_2\}$ is the union of the principal ideals generated by $a_1 \otimes b_1$ and $a_2 \otimes b_2$. The second key observation follows.

COROLLARY 2.4. *Let A and B be lattices. For $a, a_i, c_j \in A - \{0\}$ and $b, b_i, d_j \in B - \{0\}$, $a \otimes b \leqslant \bigvee \{a_i \otimes b_i | 1 \leqslant i \leqslant n\}$ and $a \otimes b \leqslant \bigvee \{c_j \otimes d_j | 1 \leqslant j \leqslant m\}$ iff there are polynomials p and q such that $a \leqslant p(a_1, \ldots, a_n) \wedge q(c_1, \ldots, c_m)$ and $b \leqslant p^d(b_1, \ldots, b_n) \wedge q^d(d_1, \ldots, d_m)$.*

Corollary 2.4 will be used to describe meets in $A \otimes B$, provided that it is a lattice.

Let C_2 denote the 2-element semilattice. Let $A, B \in S_0$ be finite. It is easily seen that $\mathrm{Hom}_0(A, C_2) \simeq A^d$. Since $\mathrm{Hom}_0(A, B) \in S_0$,

$$\mathrm{Hom}_0(A \otimes B, C) \simeq \mathrm{Hom}_0(A, \mathrm{Hom}_0(B, C)).$$

Thus

$$(A \otimes B)^d \simeq \mathrm{Hom}_0(A \otimes B, C_2) \simeq \mathrm{Hom}_0(A, B^d).$$

THEOREM 2.5 (J. ANDERSON AND N. KIMURA [1]). *Let $A, B \in S_0$ be finite; then* $A \otimes B \simeq (\mathrm{Hom}_0(A, B^d))^d$.

Of course, we can define the tensor product in S_1 analogously and obtain analogous results.

Let $A, B \in S_0$ be lattices; when is $A \otimes B$ a lattice? By 2.4, if $A \otimes B$ is a lattice, then

$$\bigvee \{a_i \otimes b_i | 1 \leqslant i \leqslant n\} \wedge \bigvee \{c_j \otimes d_j | 1 \leqslant j \leqslant m\}$$

is the join of all elements of the form

$$(p(a_1, \ldots, a_n) \wedge q(c_1, \ldots, c_m)) \otimes (p^d(b_1, \ldots, b_n) \wedge q^d(d_1, \ldots, d_m)),$$

where p and q are lattice polynomials. This join will exist exactly when it is equal to the join of finitely many of its components. We can guarantee that the join exists by requiring that A be locally finite and that B be A-lower bounded. To define this latter condition, let F_C be the free lattice over C on countably many free generators $x_1, \ldots,$ and let F be the free lattice on countably many free generators $x_1, \ldots;$ let ρ_C: $F \to F_C$ be induced by mapping x_i to x_i for all i. We say that B is *A-lower bounded* if for every $p \in F_A$, $\rho_B(\rho_A^{-1}(p))$ contains a least element (denoted by p_1).

THEOREM 2.6. *Let $A, B \in S_0$ be lattices. If A is locally finite and if B is A-lower bounded, then $A \otimes B$ is a lattice.*

PROOF. From 2.4 and our assumption that B is A-lower bounded we see that

$$\bigvee \{a_i \otimes b_i | 1 \leqslant i \leqslant n\} \wedge \bigvee \{c_j \otimes d_j | 1 \leqslant j \leqslant m\}$$

is the join of all elements of the form

$$(p(a_1, \ldots, a_n) \wedge q(c_1, \ldots, c_m)) \otimes (p_1^d(b_1, \ldots, b_n) \wedge q_1^d(d_1, \ldots, d_m))$$

where p and $q \in F_A$; since A is locally finite, this is indeed a finite join.

COROLLARY 2.7. *If $A, B \in S_0$ are locally finite lattices, then $A \otimes B$ is a locally finite lattice.*

COROLLARY 2.8. *If A is a distributive lattice, then $A \otimes B$ is a lattice.*

PROOF. Let A be distributive; thus A is locally finite. If $p \in F_A$, then the least element in $\rho_A^{-1}(p)$ is the disjunctive normal form of the lattice polynomials in $\rho_A^{-1}(p)$. Thus for any B, $\rho_B(\rho_A^{-1}(p))$ has a least element; that is, B is A-lower bounded.

3. Proof of the main result. In this section let A be a finite lattice and let B be an A-lower bounded lattice with 0. As a first step, we embed $\mathrm{Con}_L(A)$ and $\mathrm{Con}_L(B)$ into $\mathrm{Con}_L(A \otimes B)$.

Let C and D be lattices with 0 such that $C \otimes D$ is a lattice. For $\phi \in \mathrm{Con}_L(C)$ and $\psi \in \mathrm{Con}_L(D)$, let ϕ^* be the lattice congruence on $C \otimes D$ generated by $\{(c_1 \otimes d, c_2 \otimes d) | (c_1, c_2) \in \phi, d \in D\}$, and let ψ^* be the lattice congruence on $C \otimes D$ generated by $\{(c \otimes d_1, c \otimes d_2) | c \in C, (d_1, d_2) \in \psi\}$.

LEMMA 3.1. *The following isomorphism holds.*

$$(C \otimes D)/\phi^* \simeq (C/\phi) \otimes D.$$

Furthermore, ϕ^* is the semilattice congruence generated by $\{(c_1 \otimes d, c_2 \otimes d) | (c_1, c_2) \in \phi, d \in D\}$. Also, the corresponding statement holds for ψ^*.

PROOF. Let $f_\phi(c \otimes d) = [c]\phi \otimes d$; f_ϕ extends to a homomorphism from $C \otimes D$ onto $(C/\phi) \otimes D$; this homomorphism will also be denoted by f_ϕ. Clearly, $\mathrm{Ker}(f_\phi)$ is the semilattice congruence generated by $\{(c_1 \otimes d, c_2 \otimes d) | (c_1, c_2) \in \phi, d \in D\}$. To show that this establishes the required isomorphism, it suffices to show that f_ϕ preserves meets. Let

$$x = \bigvee \{a_i \otimes b_i | 1 \leqslant i \leqslant n\}, \quad y = \bigvee \{c_j \otimes d_j | 1 \leqslant j \leqslant m\}.$$

Since f_ϕ is isotone, we need only show that

$$f_\phi(x) \wedge f_\phi(y) \leqslant f_\phi(x \wedge y).$$

In order to do this, let

$$[c]\phi \otimes d \leqslant f_\phi(x) \wedge f_\phi(y) = \bigvee \{[a_i]\phi \otimes b_i | 1 \leqslant i \leqslant n\} \wedge \bigvee \{[c_j]\phi \otimes d_j | 1 \leqslant j \leqslant m\}.$$

By 2.4, there are lattice polynomials p and q such that

$$[c]\phi \leqslant p([a_1]\phi, \ldots, [a_n]\phi) \wedge q([c_1]\phi, \ldots, [c_m]\phi)$$

and

$$d \leqslant p^d(b_1, \ldots, b_n) \wedge q^d(d_1, \ldots, d_m).$$

Since ϕ is a lattice congruence, the former condition reduces to

$$[c]\phi \leqslant [p(a_1, \ldots, a_n) \wedge q(c_1, \ldots, c_m)]\phi.$$

Thus, without loss of generality, we may assume that

$$c \leqslant p(a_1, \ldots, a_n) \wedge q(c_1, \ldots, c_m).$$

Hence $c \otimes d \leqslant x \wedge y$, and so

$$[c]\phi \otimes d = f_\phi(c \otimes d) \leqslant f_\phi(x \wedge y)$$

as desired.

Our next step is to describe principal lattice congruences on $A \otimes B$. As usual, let $\theta(f, g)$ denote the smallest lattice congruence on $A \otimes B$ containing (f, g); we may assume that $f \geqslant g$. For $a \in A$, $h \in A \otimes B$, let

$$a_h = \bigvee \{b | b \in B \text{ and } a \otimes b \leqslant h\};$$

since A is finite and B is A-lower bounded, a_h exists.

LEMMA 3.2. $(a \vee a')_h = a_h \wedge a'_h$.

PROOF. It is easily seen that $(a \vee a')_h \leqslant a_h \wedge a'_h$. Conversely, if $(a \otimes a_h) \leqslant h$ and $(a' \otimes a'_h) \leqslant h$, then $(a \otimes a_h) \vee (a' \otimes a'_h) \leqslant h$. By 2.2,

$$(a \vee a' \otimes a_h \wedge a'_h) \leqslant (a \otimes a_h) \vee (a' \otimes a'_h),$$

implying that $a_h \wedge a'_h \leqslant (a \vee a')_h$.

Thus, every $h \in A \otimes B$ can be written as $h = \bigvee \{a \otimes a_h | a \in J(A)\}$. Whenever $a \in J(A)$, let a_+ denote the unique element of A covered by a.

LEMMA 3.3. *For any $a \in J(A)$ and any $h \in A \otimes B$,*

$$(h \wedge (a \otimes 1)) \vee (a_+ \otimes 1) = (a \otimes a_h) \vee (a_+ \otimes 1).$$

PROOF. Clearly,

$$(h \wedge (a \otimes 1)) \vee (a_+ \otimes 1) \geqslant (a \otimes a_h) \vee (a_+ \otimes 1)$$

and $a_+ \otimes 1 \leqslant (a \otimes a_h) \vee (a_+ \otimes 1)$. Thus we need only show that

$$h \wedge (a \otimes 1) \leqslant (a \otimes a_h) \vee (a_+ \otimes 1).$$

Let $a' \otimes b' \leqslant h \wedge (a \otimes 1)$; then $a' \otimes b' \leqslant h$ and $a' \otimes b' \leqslant a \otimes 1$. This latter condition means that $a' \leqslant a$. If $a' < a$, then $a' \leqslant a_+$ and, hence, $a' \otimes b' \leqslant a_+ \otimes 1$. If $a' = a$, then $a' \otimes b' \leqslant h$ means that $a' \otimes b' \leqslant a \otimes a_h$. Thus

$$h \wedge (a \otimes 1) \leqslant (a \otimes a_h) \vee (a_+ \otimes 1).$$

COROLLARY 3.4. *Let $a \in J(A)$; then the following congruence holds.*

$$(a \otimes a_f) \vee (a_+ \otimes 1) \equiv (a \otimes a_g) \vee (a_+ \otimes 1) \ (\theta(f, g)).$$

LEMMA 3.5. *Let $a \in J(A)$ and let*

$$\theta_{f,g} = \bigvee \{\theta((a \otimes a_f) \vee (a_+ \otimes 1), (a \otimes a_g) \vee (a_+ \otimes 1)) | a \in J(A)\};$$

then $f \equiv g(\theta_{f,g})$.

PROOF. For $a \in J(A)$, let

$$f_a = (f \wedge ((a \otimes a_f) \vee (a_+ \otimes 1))) \vee g$$

and

$$g_a = (f \wedge ((a \otimes a_g) \vee (a_+ \otimes 1))) \vee g.$$

Thus $f_a \equiv g_a(\theta_{f,g})$. If a is minimal in $J(A)$, then $a_+ = 0$, so that $g_a = g$. Note that f_a equals

$$(a \otimes a_f) \vee \bigvee \{a' \otimes a'_f | a' < a, a' \in J(A)\}$$
$$\vee \bigvee \{a'' \otimes a''_g | a'' \nleqslant a, a'' \in J(A)\}$$

and g_a equals

$$(a \otimes a_g) \vee \bigvee \{a' \otimes a'_f | a' < a, a' \in J(A)\}$$
$$\vee \bigvee \{a'' \otimes a''_g | a'' \nleqslant a, a'' \in J(A)\}$$

which, in turn, equals $\bigvee \{f_{a'} | a$ covers a' in $J(A)\}$. Hence by induction on the height of the poset $J(A)$, $f_a \equiv g(\theta_{f,g})$ for all $a \in J(A)$. Since $f = \bigvee \{f_a | a \in J(A)\}$, we obtain $f \equiv g(\theta_{f,g})$.

THEOREM 3.6. *Every principal congruence in $\mathrm{Con}_L(A \otimes B)$ can be represented as a finite join of congruences of the form*

$$\theta((a \otimes b) \vee (a_+ \otimes 1), (a \otimes b_+) \vee (a_+ \otimes 1))$$

where $a \in J(A)$ and $b > b_+$ in B.

We now fix $f = (a \otimes b) \vee (a_+ \otimes 1)$ and $g = (a \otimes b_+) \vee (a_+ \otimes 1)$ where $a \in J(A)$ and $b > b_+$ in B. We will show that

$$\theta(f, g) = \theta(a, a_+)^* \wedge \theta(b, b_+)^*$$

where $\theta(a, a_+)$ and $\theta(b, b_+)$ are the principal congruences in A and B generated by (a, a_+) and (b, b_+), respectively.

LEMMA 3.7. *Let a_1 cover a_2 in A. Define ρ: $B \to A \otimes B$ by*

$$\rho(b') = (a_1 \otimes b') \vee (a_2 \otimes 1).$$

Then ρ is an isomorphism onto $[a_2 \otimes 1, a_1 \otimes 1]$.

PROOF. Clearly ρ is a one-to-one join homomorphism. To see that ρ preserves meets, it suffices to show that

$$((a_1 \otimes b_1) \vee (a_2 \otimes 1)) \wedge ((a_1 \otimes b_2) \vee (a_2 \otimes 1)) \leqslant (a_1 \otimes (b_1 \wedge b_2)) \vee (a_2 \otimes 1).$$

This readily follows from 2.3 and 2.4. To show that ρ is onto $[a_2 \otimes 1, a_1 \otimes 1]$, let $a_2 \otimes 1 \leqslant h \leqslant a_1 \otimes 1$. Then

$$h = \bigvee \{(c_j \otimes d_j) \vee (a_2 \otimes 1) | 1 \leqslant j \leqslant n\}$$

which, by 2.1, equals

$$\bigvee \{((c_j \vee a_2) \otimes d_j) \vee (a_2 \otimes 1) | 1 \leqslant j \leqslant n\},$$

and so we may assume $c_j \geqslant a_2$. But $h \leqslant a_1 \otimes 1$ implies that $c_j \leqslant a_1$; if $c_j < a_1$, then $c_j = a_2$ and so $(c_j \otimes d_j) \vee (a_2 \otimes 1) = a_2 \otimes 1$. Hence

$$\begin{aligned} h &= \left(\bigvee \{(a_1 \otimes d_j) \vee (a_2 \otimes 1) | 1 \leqslant j \leqslant m\}\right) \vee (a_2 \otimes 1) \\ &= (a_1 \otimes d) \vee (a_2 \otimes 1) \end{aligned}$$

where $d = \bigvee \{d_j | 1 \leqslant j \leqslant m\}$. Thus ρ is onto.

LEMMA 3.8. *Let $a \in J(A)$. For $\phi \in \mathrm{Con}_L(A)$, $\psi \in \mathrm{Con}_L(B)$, and $b_1 \neq b_2$ in B,*

$$(a \otimes b_1) \vee (a_+ \otimes 1) \equiv (a \otimes b_2) \vee (a_+ \otimes 1) \ (\phi^*)$$

iff $a \equiv a_+(\phi)$, and

$$(a \otimes b_1) \vee (a_+ \otimes 1) \equiv (a \otimes b_2) \vee (a_+ \otimes 1) \ (\psi^*)$$

iff $b_1 \equiv b_2(\psi)$.

PROOF. By 3.1, $(A \otimes B)/\phi^* \simeq (A/\phi) \otimes B$. By 3.7, if $[a]\phi \neq [a_+]\phi$, then $B \simeq [[a_+]\phi \otimes 1, [a]\phi \otimes 1]$. Thus the first claim follows. By 3.1, $(A \otimes B)/\psi^* \simeq A \otimes (B/\psi)$. By 3.7,

$$B/\psi \simeq [a_+ \otimes [1]\psi, \ a \otimes [1]\psi].$$

Thus the second claim follows.

LEMMA 3.9. *Let $a_0, a_1, a_2 \in A$ with $a_0 \leqslant a_1$ and $b_0, b_1 \in B$. Then the following hold.*

(i) $((a_1 \otimes b_0) \vee (a_2 \otimes 1)) \vee (a_1 \otimes b_1) = (a_1 \otimes (b_0 \vee b_1)) \vee (a_0 \otimes 1)$,

(ii) $((a_1 \otimes b_0) \vee (a_0 \otimes 1)) \wedge ((a_1 \otimes b_1) \vee (a_0 \otimes 1)) = (a_1 \otimes (b_0 \wedge b_1)) \vee (a_0 \otimes 1)$,

(iii) $((a_1 \otimes b_0) \vee (a_0 \otimes 1)) \vee (a_2 \otimes 1) = ((a_1 \vee a_2) \otimes b_0) \vee ((a_0 \vee a_1) \otimes 1)$,
(iv) $((a_1 \otimes b_0) \vee (a_0 \otimes 1)) \wedge (a_2 \otimes 1) = ((a_1 \wedge a_2) \otimes b_0) \vee ((a_0 \wedge a_2) \otimes 1)$.

PROOF. These readily follow from 2.3 and 2.4.

LEMMA 3.10. *Let $a_0, a_1, a_2, a_3 \in A$ with $a_0 \leqslant a_1$ and $a_2 \leqslant a_3$. If $a_2 \equiv a_3\ (\theta(a_0, a_1))$, then for any $b_0 \leqslant b_1$ in B,*

$$(a_3 \otimes b_1) \vee (a_2 \otimes 1) \equiv (a_3 \otimes b_0) \vee (a_2 \otimes 1)$$

modulo the congruence

$$\theta((a_1 \otimes b_1) \vee (a_0 \otimes 1), (a_1 \otimes b_0) \vee (a_0 \otimes 1)).$$

PROOF. Without loss of generality, we may assume that the quotient a_3/a_2 is weakly projective into a_1/a_0. Then in $A \otimes B$ the quotient

$$((a_1 \otimes b_1) \vee (a_0 \otimes 1))/((a_1 \otimes b_0) \vee (a_0 \otimes 1))$$

is weakly projective into the quotient

$$((a_3 \otimes b_1) \vee (a_2 \otimes 1))/((a_3 \otimes b_0) \vee (a_2 \otimes 1))$$

as follows: If at the ith step in A we go from u_i/v_i to $(u_i \vee c_i)/(v_i \vee c_i)$, then in $A \otimes B$ we go from

$$((u_i \otimes b_1) \vee (v_i \otimes 1))/((u_i \otimes b_0) \vee (v_i \otimes 1))$$

to

$$\frac{((u_i \vee c_i) \otimes b_1) \vee ((v_i \vee c_i) \otimes 1)}{((u_i \vee c_i) \otimes b_0) \vee ((v_i \vee c_i) \otimes 1)}$$

by using 3.9(iii). We make analogous use of 3.9(iv) if the ith step is a meet.

LEMMA 3.11. *Let $b_0, b_1, b_2, b_3 \in B$ with $b_0 \leqslant b_1$ and $b_2 \leqslant b_3$. If $b_3 \equiv b_2(\theta(b_0, b_1))$, then for any $a_0 \leqslant a_1$ in A,*

$$(a_1 \otimes b_3) \vee (a_0 \otimes 1) \equiv (a_1 \otimes b_2) \vee (a_0 \otimes 1)$$

modulo the congruence

$$\theta((a_1 \otimes b_1) \vee (a_0 \otimes 1), (a_1 \otimes b_0) \vee (a_0 \otimes 1)).$$

PROOF. The proof is similar to that of 3.10, using 3.9(i), (ii).

THEOREM 3.12. $\theta(f, g) = \theta(a, a_+)^* \wedge \theta(b, b_+)^*$.

PROOF. By 3.5, $\theta(f, g) \leqslant \theta(a, a_+)^* \wedge \theta(b, b_+)^*$. For the reverse inclusion, let

$$f' = (a' \otimes b') \vee (a'_+ \otimes 1), \quad g' = (a' \otimes b'_+) \vee (a'_+ \otimes 1),$$

where a' is join irreducible and $b > b'_+$. Then we need only show that $\theta(f', g') \leqslant \theta(a, a_+)^* \wedge \theta(b, b_+)^*$ implies that $\theta(f', g') \leqslant \theta(f, g)$. By 3.8, $\theta(f', g') \leqslant \theta(a, a_+)^* \wedge \theta(b, b_+)^*$ implies that

$$a' \equiv a'_+\ (\theta(a, a_+)) \quad \text{and} \quad b' \equiv b'_+\ (\theta(b, b_+)).$$

But then, by 3.10 and 3.11,

$$f' \equiv g'(\theta((a \otimes b') \vee (a_+ \otimes 1), (a \otimes b'_+) \vee (a_+ \otimes 1)))$$

and

$$(a \otimes b') \vee (a_+ \otimes 1) \equiv (a \otimes b'_+) \vee (a_+ \otimes 1) \ (\theta(f, g)).$$

Hence $\theta(f', g') \leqslant \theta(f, g)$, proving the theorem.

LEMMA 3.13. *Let $a \in J(A)$ and let $b > b_+$ in B; let $\phi \in \mathrm{Con}_L(A)$ and $\psi \in \mathrm{Con}_L(B)$. Then*

$$\theta(a, a_+)^* \wedge \theta(b, b_+)^* \leqslant \phi^* \text{ iff } \theta(a, a_+) \leqslant \phi$$

and

$$\theta(a, a_+)^* \wedge \theta(b, b_+)^* \leqslant \psi^* \text{ iff } \theta(b, b_+) \leqslant \psi.$$

PROOF. If $\theta(a, a_+) \leqslant \phi$, then trivially $\theta(a, a_+)^* \wedge \theta(b, b_+)^* \leqslant \phi^*$, and if $\theta(b, b_+) \leqslant \psi$, then trivially $\theta(a, a_+)^* \wedge \theta(b, b_+)^* \leqslant \psi^*$. Suppose $\theta(a, a_+) \nleqslant \phi$ and $\theta(b, b_+) \leqslant \psi$. By 3.12,

$$\theta(a, a_+)^* \wedge \theta(b, b_+)^* = \theta((a \otimes b) \vee (a_+ \otimes 1), (a \otimes b_+) \vee (a_+ \otimes 1)).$$

Hence it suffices to show that

$$\theta((a \otimes b) \vee (a_+ \otimes 1), (a \otimes b_+) \vee (a_+ \otimes 1))$$

is not contained in either ϕ^* or ψ^*. By 3.1,

$$(A \otimes B)/\phi^* \simeq (A/\phi) \otimes B \quad \text{and} \quad (A \otimes B)/\psi^* \simeq A \otimes (B/\psi).$$

By 3.7,

$$B/\psi \simeq [a_+ \otimes [1]\psi, a \otimes [1]\psi]$$

and since $[a]\phi \neq [a_+]\phi$,

$$B \simeq [[a_+]\phi \otimes 1, [a]\phi \otimes 1].$$

Thus in the first case,

$$((a \otimes b) \vee (a_+ \otimes 1), (a \otimes b_+) \vee (a_+ \otimes 1)) \notin \psi^*$$

and in the second,

$$((a \otimes b) \vee (a_+ \otimes 1), (a \otimes b_+) \vee (a_+ \otimes 1)) \notin \phi^*.$$

THEOREM 3.14. *The set*

$$C(A) = \{\phi^* | \phi \in \mathrm{Con}_L(A)\}$$

is a sublattice of $\mathrm{Con}_L(A \otimes B)$ *isomorphic to* $\mathrm{Con}_L(A)$ *and the set*

$$C(B) = \{\psi^* | \psi \in \mathrm{Con}_L(B)\}$$

is a sublattice of $\mathrm{Con}_L(A \otimes B)$ *isomorphic to* $\mathrm{Con}_L(B)$.

PROOF. It is clear that $\phi \to \phi^*$ and $\psi \to \psi^*$ are join isomorphisms. Thus it remains to show that $C(A)$ and $C(B)$ are closed under meets, that is, $\phi_0^* \wedge \phi_1^* \leqslant (\phi_0 \wedge \phi_1)^*$ and $\psi_0^* \wedge \psi_1^* \leqslant (\psi_0 \wedge \psi_1)^*$. Note that $\phi_0^* \wedge \phi_1^*$ and $\psi_0^* \wedge \psi_1^*$ are lattice congruences. By 3.6 and 3.12, it suffices to show that for $a \in J(A)$ and $b > b_+$ in B, if

$$((a \otimes b) \vee (a_+ \otimes 1), (a \otimes b_+) \vee (a_+ \otimes 1)) \in \phi_0^* \wedge \phi_1^*,$$

then

$$((a \otimes b) \vee (a_+ \otimes 1), (a \otimes b_+) \vee (a_+ \otimes 1)) \in (\phi_0 \wedge \phi_1)^*,$$

and if

$$((a \otimes b) \vee (a_+ \otimes 1), (a \otimes b_+) \vee (a_+ \otimes 1)) \in \psi_0^* \wedge \psi_1^*,$$

then

$$((a \otimes b) \vee (a_+ \otimes 1), (a \otimes b_+) \vee (a_+ \otimes 1)) \in (\psi_0 \wedge \psi_1)^*.$$

But these follow readily from 3.12 and 3.13.

Let $D_{0,1}$ be the category of bounded distributive lattices with bound-preserving homomorphisms. For $C, D \in D_{0,1}$, let $C * D$ be the free product of C and D in $D_{0,1}$.

LEMMA 3.15. *Let $C, D \in D_{0,1}$. In S_0 form $C \otimes D$, the tensor product of C and D, and in $D_{0,1}$ form $C * D$, the free product of C and D. Then $C \otimes D \simeq C * D$.*

PROOF. Since $D_{0,1}$ is locally finite and direct limits commute with both tensor products and free products, we may assume that both C and D are finite. But then $J(C * D) \simeq J(C) \times J(D)$. Thus $C * D$ is the free join semilattice with zero generated by $J(C) \times J(D)$ (regarded as a partial join semilattice where joins are defined only for comparable elements). Therefore, we can find a join homomorphism from $C * D$ onto $C \otimes D$. On the other hand, f: $C \circ D \to C * D$ defined by $f(c, d) = c \wedge d$ is a bimorphism since $C * D$ is distributive. Hence we can find a homomorphism from $C \otimes D$ onto $C * D$. Since $C \otimes D$ and $C * D$ are finite, they must be isomorphic semilattices, and hence isomorphic lattices.

THEOREM 3.16. *The following isomorphism holds.*

$$\mathrm{Con}_L(A \otimes B) \simeq \mathrm{Con}_L(A) \otimes \mathrm{Con}_L(B).$$

PROOF. Since the congruence lattice of a lattice is distributive, we invoke 3.15 and prove that $\mathrm{Con}_L(A \otimes B) \simeq \mathrm{Con}_L(A) * \mathrm{Con}_L(B)$. We use the solution to the word problem for free products of bounded distributive lattices (see G. Grätzer [6]). By 3.12, $C(A) \cup C(B)$ generates $\mathrm{Con}_L(A \otimes B)$. By 3.14, $C(A) \simeq \mathrm{Con}_L(A)$ and $C(B) \simeq \mathrm{Con}_L(B)$. Let $\phi_0, \phi_1 \in \mathrm{Con}_L(A)$ and $\psi_0, \psi_1 \in \mathrm{Con}_L(B)$. It suffices to show that if $\phi_0^* \wedge \psi_0^* \leqslant \phi_1^* \vee \psi_1^*$, then $\phi_0^* \leqslant \phi_1^*$ or $\psi_0^* \leqslant \psi_1^*$. Suppose $\phi_0^* \nleqslant \phi_1^*$; thus there are a_1 covering a_0 in A with $a_1 \equiv a_0$ (ϕ_0) but not $a_1 \equiv a_0$ (ϕ_1). By 3.7, $[a_0 \otimes 1, a_1 \otimes 1] \simeq B$. Moreover, $\phi_0^* = \iota$ on $[a_0 \otimes 1, a_1 \otimes 1]$ while $\phi_1^* = \omega$ on $[a_0 \otimes 1, a_1 \otimes 1]$. Hence on $[a_0 \otimes 1, a_1 \otimes 1]$ we have $\psi_0^* = \phi_0^* \wedge \psi_0^* \leqslant \phi_1^* \vee \psi_1^*$. But since $[a_0 \otimes 1, a_1 \otimes 1]$ is a convex sublattice of $A \otimes B$, $\phi_1^* \vee \psi_1^* = \psi_1^*$ on $[a_0 \otimes 1, a_1 \otimes 1]$. Because of the canonical isomorphism between $[a_0 \otimes 1, a_1 \otimes 1]$ and B, this means that $\psi_0 \leqslant \psi_1$ and therefore $\psi_0^* \leqslant \psi_1^*$ on $A \otimes B$. This proves the theorem.

LEMMA 3.17. *Let $A, B \in S_0$, and let A_0 and B_0 be sublattices of A and B, respectively. Then the canonical homomorphism from $A_0 \otimes B_0$ to $A \otimes B$ is a lattice embedding.*

PROOF. Let $a_i \in A_0$, $b_i \in B_0$ for $i = 1, \ldots, n$, and let $a \in A$, $b \in B$. Then the solution to the word problem tells us that in $A \otimes B$,

$$a \otimes b \leqslant \bigvee \{a_i \otimes b_i | 1 \leqslant i \leqslant n\}$$

if and only if there are $a' \in A_0$, $b' \in B_0$ such that $a \otimes b \leqslant a' \otimes b'$ and

$$a' \otimes b' \leqslant \bigvee \{a_i \otimes b_i | 1 \leqslant i \leqslant n\}$$

in $A_0 \otimes B_0$. Thus the canonical homomorphism is an embedding. Moreover, using 2.4 we see that the canonical homorphism also preserves meets.

Now we are ready to state and prove our main result.

THEOREM 3.18. *Let $A, B \in S_0$ be lattices; let A be locally finite, and let B be A-lower bounded. Then* $\mathrm{Con}_L(A \otimes B) \simeq \mathrm{Con}_L(A) \otimes \mathrm{Con}_L(B)$.

PROOF. By 3.17, $A \otimes B$ is the direct limit (as lattices) of the lattices $A_f \otimes B_1$, where A_f ranges over the finite sublattices of A and B_1 ranges over the sublattices of B with unit. By 3.16,

$$\mathrm{Con}_L(A_f \otimes B_1) \simeq \mathrm{Con}_L(A_f) \otimes \mathrm{Con}_L(B_1).$$

But $\mathrm{Con}_L(A \otimes B)$ is the direct limit of the lattices $\mathrm{Con}_L(A_f \otimes B_1)$ where A_f ranges over the finite sublattices of A and B_1 ranges over the sublattices of B with unit. Since direct limits and tensor products commute, our result follows.

4. Consequences of the main result. The following lemma is obvious.

LEMMA 4.1. *Let A and B be bounded distributive lattices, each with a unique atom. Then $A * B$ has a unique atom.*

Throughout this section we assume that $A, B \in S_0$ are lattices such that A is locally finite and B is A-lower bounded.

THEOREM 4.2. *As a lattice, $A \otimes B$ is subdirectly irreducible iff both A and B are subdirectly irreducible lattrices.*

PROOF. Since A is subdirectly irreducible iff $\mathrm{Con}_L(A)$ has a unique atom, the theorem follows from 4.1 and the main result (3.18).

THEOREM 4.3. *Let A be a simple lattice; then* $\mathrm{Con}_L(A \otimes B) \simeq \mathrm{Con}_L(B)$.

PROOF. A is simple iff $\mathrm{Con}_L(A) \simeq C_2$, and $C_2 * D \simeq D$ for any $D \in D_{0,1}$.

COROLLARY 4.4. *As a lattice, $A \otimes B$ is simple iff both A and B are simple lattices.*

COROLLARY 4.5 (E. T. SCHMIDT [8]). *For any $D \in B_{0,1}$,* $\mathrm{Con}_L(M_3[D]) \simeq \mathrm{Con}_L(D)$.

THEOREM 4.6. *Let $\theta \in \mathrm{Con}_L(A \otimes B)$; then $(A \otimes B)/\theta$ is subdirectly irreducible iff $\theta = \phi^* \vee \psi^*$ where A/ϕ and B/ψ are subdirectly irreducible.*

PROOF. $(A \otimes B)/\theta$ is subdirectly irreducible iff θ is completely meet irreducible. Since θ can be written as the meet of elements of the form $\phi^* \vee \psi^*$ (see [6]), and

since θ is meet irreducible, $\theta = \phi^* \vee \psi^*$ for some $\phi \in \mathrm{Con}_L(A)$ and some $\psi \in \mathrm{Con}_L(B)$. Thus $(A \otimes B)/\theta \simeq (A/\phi) \otimes (B/\psi)$ and so 4.2 tells us that A/ϕ and B/ψ are subdirectly irreducible.

For a lattice L, let $D(L)$ be the maximal distributive quotient of L; let $\theta_M(L)$ (respectively, $\theta_D(L)$) be the kernel of the maximal modular (respectively, distributive) quotient of L.

THEOREM 4.7. *The following isomorphism holds.*

$$D(A \otimes B) \simeq D(A) \otimes D(B).$$

Hence, the tensor product in S_0 of two distributive lattices with 0 *is again a distributive lattice with* 0.

See G. Fraser [3].

PROOF. Let $(A \otimes B)/\theta \simeq C_2$; then by 4.6, $\theta = \phi^* \vee \psi^*$ where $A/\phi \simeq C_2 \simeq B/\psi$. From this the theorem readily follows.

LEMMA 4.8. *Let A_0, A_1 both be modular but not distributive; then $A_0 \otimes A_1$ is not modular.*

PROOF. Let $z_i, a_i, b_i, c_i, u_i \in A_i$ form a sublattice isomorphic to M_3 (with z_i the zero and u_i the unit). We claim that

$$(z_0 \otimes u_1) \vee (u_0 \otimes z_1), \quad (a_0 \otimes b_1) \vee (b_0 \otimes a_1), \quad (a_0 \otimes a_1) \vee (b_0 \otimes b_1),$$
$$(a_0 \otimes a_1) \vee (b_0 \otimes b_1) \vee (c_0 \otimes c_1), \quad (u_0 \otimes u_1)$$

form a nonmodular sublattice of $A_0 \otimes A_1$. This follows readily from the solution to the word problem; the only nontrivial part is showing that

$$((a_0 \otimes b_1) \vee (b_0 \otimes a_1)) \wedge ((a_0 \otimes a_1) \vee (b_0 \otimes b_1) \vee (c_0 \otimes c_1))$$
$$= (z_0 \otimes u_1) \vee (u_0 \otimes z_1).$$

But because M_3 is self-dual, we have

$$(x \otimes y) \leqslant (a_0 \otimes a_1) \vee (b_0 \otimes b_1) \vee (c_0 \otimes c_1)$$

iff $(x \otimes y)$ is less than or equal to at least one of $(a_0 \otimes a_1)$, $(b_0 \otimes b_1)$, $(c_0 \otimes c_1)$, $(z_0 \otimes u_1)$, $(u_0 \otimes z_1)$. From this the result is straightforward.

THEOREM 4.9. *The following equality holds.*

$$\theta_M(A \otimes B) = (\theta_M(A)^* \vee \theta_D(B)^*) \wedge (\theta_D(A)^* \vee \theta_M(B)^*).$$

PROOF. Let $(A \otimes B)/\theta$ be subdirectly irreducible and modular. By 4.6, $\theta = \phi^* \vee \psi^*$ where A/ϕ and B/ψ are subdirectly irreducible. By 4.8 and the main result, one of them is modular and the other is distributive. Hence

$$\theta \geqslant (\theta_M(A)^* \vee \theta_D(B)^*) \wedge (\theta_D(A)^* \vee \theta_M(B)^*) \geqslant \theta_M(A \otimes B).$$

But

$$\theta_M(A \otimes B) = \bigwedge \{\theta \mid (A \otimes B)/\theta \text{ is subdirectly irreducible and modular}\};$$

this proves the theorem.

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