TWIST MAPS, COVERINGS AND
BROUWER'S TRANSLATION THEOREM

BY

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ABSTRACT. We apply the Brouwer Translation Theorem to a class of twist maps of
the annulus (which contains \( C^1 \) area preserving maps) to show that, if \( h \) belongs to
this class, then a certain set \( \mathscr{P}_0 \) of periodic points of \( h \) cannot be dense. The
definition of \( \mathscr{P}_0 \) does not impose any a priori restrictions on the periods of the
points of \( \mathscr{P}_0 \).

Introduction. Let \( h: \mathbb{R}^2 \to \mathbb{R}^2 \) be an orientation-preserving, fixed point free,
self-homeomorphism of the two-dimensional plane.

The weakest version of Brouwer’s Translation Theorem (see e.g. [1]) states: if
\( A \subset \mathbb{R}^2 \) is an arc-wise connected set such that \( A \cap h(A) = \emptyset \), then \( A \cap h^n(A) = \emptyset \)
for all \( n \neq 0 \). Here \( h^n \) denotes the \( n \)-fold iterate of \( h \).

In order to see whether this strong result on the behavior of all iterates of \( h \) (i.e.,
a result of dynamics) has any relevance for the dynamics on other surfaces
(specially compact surfaces, where all this is more interesting) it is natural to
proceed as follows: Let \( h: \mathcal{M} \to \mathcal{M} \) be an orientation-preserving self-homeomor-
phism of a connected, orientable two-manifold \( \mathcal{M} \) (with or without boundary \( \partial \mathcal{M} \)) such that
its fixed point set, \( \text{Fix } h \), does not separate \( \mathcal{M} \); let \( \mathcal{M}_0 = \mathcal{M} - (\partial \mathcal{M} \cup 
\text{Fix } h) \) and \( h_0 = h|_{\mathcal{M}_0} \); then, since the universal cover \( \tilde{\mathcal{M}}_0 \) of \( \mathcal{M}_0 \) is homeomorphic
to \( \mathbb{R}^2 \) and any lifting to \( \tilde{\mathcal{M}}_0, \tilde{h}_0: \tilde{\mathcal{M}}_0 \to \tilde{\mathcal{M}}_0, \) of \( h_0 \) is orientation-preserving and fixed
point free, we can apply Brouwer’s Translation Theorem to \( \tilde{h}_0 \) and ask whether, in
this way, we obtain any relations downstairs on the dynamics of \( h_0: \mathcal{M}_0 \to \mathcal{M}_0 \),
which is the same as that of the original homeomorphism \( h: \mathcal{M} \to \mathcal{M} \).

We prove that the answer is yes, by using this idea to find a relation on a certain
set \( \mathscr{P}_0 \) of periodic points of \( h \), when \( h: \mathcal{A} \to \mathcal{A} \) is a twist homeomorphism, which
satisfies a mild purely homotopy-theoretical condition (see Definition 1.1, below).
The relation is that \( \mathscr{P}_0 \) cannot be dense and so, in particular, if \( h \) is also topologi-
cally transitive, \( \mathscr{P}_0 \) is nowhere dense.

If \( h \) is at least \( C^1 \), then the same proof shows that \( \bigcup H_0(P) \), the union taken over
all hyperbolic \( P \in \mathscr{P}_0 \), is not dense either. Here \( H_0(P) \) denotes a certain set of
homoclinic points of \( P \) (see Definition 1.3, below).

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1. Statement of the Theorem. Let \( A = \{(x, y) \in \mathbb{R}^2 | 0 < a^2 < x^2 + y^2 < b^2\} \) denote a planar annulus and \( h: A \to A \) an orientation-preserving homeomorphism which maps each component of the boundary of \( A \) onto itself. Let \( \tilde{A} \) denote the band \( \tilde{A} = \{(x, y) \in \mathbb{R}^2 | a < y < b\} \), which we regard as the universal cover of \( A \) and \( \pi: \tilde{A} \to A \) the projection.

Recall that \( h: A \to A \) is called a twist map if \( h \) has a lifting \( \tilde{h}: \tilde{A} \to \tilde{A} \) which maps the points of the parallel lines \( y = a, y = b \) in opposite directions.

The simplest examples of twist maps are obtained by integrating the flow shown in Figure 1.1.

![Figure 1.1](image)

Notice that \( h \) itself can have fixed points on the boundary of \( A \) and that the set of twist maps is open in the space of all homeomorphisms of \( A \), furnished with the \( C^0 \) topology.

**Definition 1.1.** Let \( h: A \to A \), \( \tilde{h}: \tilde{A} \to \tilde{A} \) be a twist map and assume the fixed point set of \( \tilde{h} \), \( \text{Fix} \tilde{h} \), is not empty and does not separate \( \tilde{A} \); let \( \tilde{A}_0 = \tilde{A} - \text{Fix} \tilde{h} \), \( \tilde{h}_0 = \tilde{h}|\tilde{A}_0 \) and \( A_0 \) be the universal cover of \( A_0 \). We say \( h: A \to A \) satisfies condition L if \( \tilde{h}_0: A_0 \to \tilde{A}_0 \) has a lifting \( \tilde{h}_0: A_0 \to \tilde{A}_0 \) which sends every boundary component of \( \partial \tilde{A}_0 \) onto itself.

**Example.** It is clear that any diffeomorphism obtained by integrating the flow of Figure 1.1 satisfies condition L.

We remark that condition L is a purely homotopy-theoretical condition: it holds if and only if the maps induced by \( \tilde{h} \) on \( \Pi_1(\tilde{A}_0, \tilde{\partial}^+) \) and \( \Pi_1(\tilde{A}_0, \tilde{\partial}^-) \) are the identity; here \( \tilde{\partial}^+, \tilde{\partial}^- \) denote the upper and lower boundary components of \( \tilde{A}_0 \) and \( \Pi_1(X, Y) \) denotes the homotopy classes of arcs in \( X \) with endpoints in \( Y \); the homotopies should keep endpoints in \( Y \).

We now assume without loss of generality that \( h, \tilde{h} \) is a twist map such that \( \tilde{h} \) sends points of the lower boundary \( \tilde{\partial}^- \) of \( \tilde{A} \) to the left. Let \( T: \tilde{A} \to \tilde{A} \) be the generator of the covering transformation group which sends all points of \( \tilde{A} \) to the right. If \( P \in A \) is a periodic point of \( h \) of period \( n \) and \( \tilde{P} \in \tilde{A} \) lies above \( P \), then there
exists a unique integer \( m = m(P) \) such that \( T^m(\widetilde{h}^n(\bar{P})) = \bar{P} \) and \( m \) does not depend on which point \( \bar{P} \) lying over \( P \) we chose.

Let \( p: \bar{A} \to A \) be the covering projection and \( h: A \to A, \widetilde{h}: \bar{A} \to \bar{A} \) a twist map (such that \( \widetilde{h} \) sends points of \( \bar{\partial}^{-} \) to the left).

**Definition 1.2.** We say the periodic point \( P \) of \( h \) (of period \( n \)) belongs to the set \( \mathcal{P}_0 \) if in \( A_0 = A - p(\text{Fix } \bar{h}) \) there exists an arc \( \gamma \) from \( P \) to \( \partial^{-} \) such that

1. \( m(P) > 0 \),
2. \( \gamma \) and \( \bar{h}^n(\gamma) \) are homotopic in \( A_0 \) by a homotopy, which keeps \( P \) fixed and the endpoints of \( \gamma \) and \( \bar{h}^n(\gamma) \) on \( \partial^{-} \),
3. \( \gamma \) has a lifting, \( \bar{\gamma} \), to \( \bar{A} \), such that
   \[
   \bar{h}(\bar{\gamma}) \cap \bar{\gamma} = \emptyset.
   \]

For example, in Figure 1.1 one can easily find a diffeomorphism such that \( P \) is a periodic point \( \in \mathcal{P}_0 \) with \( m(P) = 1 \).

If the boundary component \( \partial^{-} \) were a fixed point then (b) means that \( P \) and \( \partial^{-} \) lie in the same Nielsen fixed point class with respect to the map \( h_0: A_0 \to A_0 \).

Recall that \( h: A \to A \) is called topologically transitive if some point has a dense orbit with respect to \( h \). It is an elementary fact that a nonempty, open, invariant set \( \Omega \) (i.e. \( h(\Omega) \subset \Omega \)) of a transitive map is dense in \( A \).

We can now state

**Theorem.** Let \( h: A \to A, \widetilde{h}: \bar{A} \to \bar{A} \) be a twist map which satisfies condition \( L \) and such that \( \widetilde{h} \) sends points of \( \bar{\partial}^{-} \) to the left. Then there exists an open neighborhood \( \Omega \) of \( \partial^{+} \) such that \( \mathcal{P}_0 \cap \Omega = \emptyset \). In particular, if \( h \) is also transitive, then \( \mathcal{P}_0 \) is nowhere dense in \( A \) (i.e., its complement contains an open dense subset of \( A \)).

**Remark 1.** \( \Omega \) can be taken to be the subset of all \( x \in A_0 \subset A \) which can be joined to \( \partial^{+} \) by a free arc \( \alpha \), i.e., an arc \( \alpha \) such that \( \alpha \cap h(\alpha) = \emptyset \).

**Remark 2.** If \( \mathcal{P}_0(N) = \{ P \in \mathcal{P}_0 | \text{period of } P < N \} \) then for each \( N \) the theorem is obvious with respect to \( \mathcal{P}_0(N) \). Thus the point is that an \( \Omega \) independent of \( N \) exists.

**Remark 3.** The theorem is a purely topological theorem; no differentiability or measure-preserving assumptions of any kind are required for \( h \).

Let \( h \) now be at least \( C^1 \) and let \( P \) be a hyperbolic periodic point (of period \( n \)) of \( h \); then the sets

\[
W^s = W^s(P) = \left\{ x \in A \mid \lim_{k \to +\infty} h^{nk}(x) = P \right\}
\]

and

\[
W^u = W^u(P) = \left\{ x \in A \mid \lim_{k \to -\infty} h^{nk}(x) = P \right\}
\]

are called, respectively, the stable and unstable manifolds of \( P \). It is well known that \( W^s \) and \( W^u \) are the images of \( C^1 \) injective immersions \( R \to A \). A point of intersection of \( W^s(P) \) and \( W^u(P) \), other than \( P \), is called a homoclinic point of \( P \).

Let \( H \) be a homoclinic point of \( P \) (with respect to \( h: A \to A \)) and let \( \beta^s (\beta^u) \) be the arcs, in \( W^s \) (\( W^u \)), from \( H \) to \( P \). Assume \( P \in \mathcal{P}_0 \) and let \( \gamma \) be as in Definition 1.2.
DEFINITION 1.3. We say $H$ is inessential (we write $H \sim 0$) if the arc $\gamma \beta \beta^{-1} \gamma^{-1}$ has a lifting to $\tilde{A}_0$ which is homotopic to 0 in $\pi_1(\tilde{A}_0, \tilde{\partial}^-)$ keeping endpoints on $\tilde{\partial}^-$. 

COROLLARY TO THE THEOREM. If $h$ is $C^1$ and satisfies the hypothesis of the Theorem, then there exists an open neighborhood $\Omega$ of $\partial^+$ such that $\mathcal{K}_0 \cap \Omega = \emptyset$; here $\mathcal{K}_0 = \bigcup \mathcal{K}_0(P)$, the union taken over all hyperbolic $P \in \mathcal{P}_0$. (We can take $\Omega$ as in the theorem; see Remark 1.)

2. Proof of the Theorem. Since $\text{Fix } \tilde{h} \neq \emptyset$, the boundary of $\tilde{A}_0$, $\partial \tilde{A}_0$, consists of an infinite number of components, each homeomorphic to the real line. We denote a component of $\partial \tilde{A}_0$ lying over $\partial^+$ or $\partial^-$ by $\tilde{\partial}^+$ or $\tilde{\partial}^-$, and call them opposite boundaries if they lie above opposite boundaries of $A$. We know there exists a lifting $\tilde{h}_0$: $\tilde{A}_0 \to \tilde{A}_0$ of $h_0$: $A_0 \to A_0$ which is orientation-preserving, fixed point free and takes each boundary component of $\partial \tilde{A}_0$ onto itself; therefore we can draw on each such component $\tilde{\partial}$ an arrow which indicates in which direction $\tilde{h}_0|\tilde{\partial}$ moves the points of $\tilde{\partial}$.

ASSERTION 1. If $\lambda$ is any arc in $A_0$ joining opposite boundaries $\tilde{\partial}^+$, $\tilde{\partial}^-$ with endpoints $\tilde{O}^+$, $\tilde{O}^-$, then $\tilde{h}_0(\tilde{O}^+)$ and $\tilde{h}_0(\tilde{O}^-)$ lie in different (unbounded) components of $A_0 - \lambda$; i.e., the arrows are as shown in Figure 2.1.
Indeed, since \( h \) is a twist map we can orient \( \tilde{A}_0 \) in such a way that the direction in which \( \tilde{h}_0 \) sends the boundaries \( \tilde{\partial}^+, \tilde{\partial}^- \) coincides with the induced orientations on them. Since \( \tilde{A}_0 \) covers \( \tilde{A}_0 \) and \( \tilde{h}_0 \) is an orientation-preserving lifting, the same will hold for any two boundaries \( \tilde{\partial}^+, \tilde{\partial}^- \) lying over \( \partial^+, \partial^- \).

Let \( P \in \mathcal{P}_0 \) via the arc \( \gamma \) and assume without loss of generality\(^3\) that \( m(P) = 1 \) (see Definition 1.2). If \( P \) were sufficiently close to \( \partial^+ \), then there would exist an arc \( \alpha \subset A_0 \) from \( P \) to \( \partial^+ \), and a lifting, \( \tilde{\alpha} \subset \tilde{A}_0 \), to the band \( \tilde{A} \) of \( \alpha \) such that \( \tilde{h}_0(\tilde{\alpha}) \cap \tilde{\alpha} = \emptyset \) (Figure 2.2). This is clear because \( \tilde{h}_0|_{\partial^+} \) is equivariant and fixed point free and \( \partial^+ \) is compact.

![Figure 2.2](image)

Assuming the existence of \( \alpha \), we will arrive at a contradiction:

Let \( Q^- \) be the initial point of \( \gamma \); we pick a boundary component \( \tilde{\partial}_0 \) of \( \partial\tilde{A}_0 \) and on it a point \( \tilde{Q}_0^- \) lying over \( Q^- \). Lift the arc \( \gamma \) to an arc \( \tilde{\gamma}_0 \subset \tilde{A}_0 \) with initial point \( \tilde{Q}_0^- \subset \tilde{\partial}_0^- \); then the endpoint of \( \tilde{\gamma}_0 \), \( \tilde{P}_0 \), will lie over \( P \) (Figure 2.3). Let \( \tilde{Q}_1^- \) be the next point (in the direction indicated) lying over \( Q^- \) and let \( \tilde{P}_1 \) be obtained by again lifting \( \gamma \) to \( \tilde{A}_0 \), but now with initial point \( \tilde{Q}_1^- \), to an arc \( \tilde{\gamma}_1 \) with endpoint \( \tilde{P}_1 \), which again lies over \( P \).

![Figure 2.3](image)

Similarly obtain \( \tilde{Q}_k^- \), \( \tilde{\gamma}_k \), \( \tilde{P}_k \), also for negative \( k \) in the obvious way.

**Assertion 2.** \( \tilde{h}_0^{kn}(\tilde{P}_0) = \tilde{P}_k \) for all integers \( k \).

Indeed, by the covering homotopy property, this is an immediate consequence of the facts that \( P \in \mathcal{P}_0 \) via \( \gamma \) (Definition 1.2(b)) and \( \tilde{h}_0 \) maps \( \tilde{\partial}_0 \) onto itself (Figure 2.3).

\(^3\) In the sense that the proofs for \( m(P) > 1 \) are entirely similar.
Remark. The point of Assertion 2 is: With respect to the induced riemannian metric of $\tilde{A}_0$, the iterate $\tilde{h}_0^{k,n}(P_0)$ is, for all $k$, at a bounded distance from $\tilde{Q}_k^-$, namely bounded by the length of $\tilde{\gamma}_k$, i.e., the length of $\gamma$, which does not depend on $k$.

Let $\delta_0^+$ be the boundary component of $\partial \tilde{A}_0$ determined by lifting the arc $\gamma \cup \alpha$ (Figure 2.2) to an arc $\lambda = \tilde{\gamma}_0 \cup \tilde{\alpha}_0$ (Figure 2.4) with initial point $\tilde{Q}_0^-$ and endpoint $\tilde{Q}_0^+ \in \delta_0^+$. By Assertion 1, the arrows are as shown in Figure 2.4 and $\lambda$ divides $\tilde{A}_0$ into two unbounded components.

Figure 2.4

By Assertion 2 and the compactness of $\lambda$ and the bounded regions formed by $\lambda$, if $k > 0$ is sufficiently large, the points $\tilde{h}_0^{k,n}(P_0) = \tilde{P}_k$ and $\tilde{h}_0^{-k,n}(P_0) = \tilde{P}_{-k}$ will lie in different, unbounded components of $A_0 - \lambda$, $\tilde{P}_k$ lying in that component where $\tilde{h}_0^{k,n}(\tilde{Q}_0^-)$ lies. Moreover, since $\tilde{\alpha}_0$ and $\tilde{\gamma}_0$ are free under $\tilde{h}_0$, by the weak form of Brouwer's Translation Theorem [1, Satz 7, p. 13] they are still free under $\tilde{h}_0^{k,n}$, where $k \neq 0$ is arbitrary.

We now have the following situation (Figure 2.5).
If \( k > 0 \) is sufficiently large, the arc \( \tilde{h}_0^{-kn}(\tilde{a}_0) \) has to intersect the dividing arc \( \lambda = \tilde{\gamma}_0 \cup \tilde{a}_0 \) an odd number of times, because by Assertions 1 and 2, its endpoints \( \tilde{h}_0^{-kn}(\tilde{\gamma}_0^+) \) and \( \tilde{P}_-k \) lie in different unbounded components of \( \tilde{A}_0 - \lambda \). Since \( \tilde{h}_0^{-kn}(\tilde{a}_0) \) cannot intersect \( \tilde{a}_0 \) it intersects \( \tilde{\gamma}_0 \) an odd number of times. Therefore the arc \( \tilde{h}_0^{kn}(\tilde{\gamma}_0) \) has to intersect \( \tilde{a}_0 \) an odd number of times, but since \( \tilde{h}_0^{kn}(\tilde{\gamma}_0) \) cannot intersect \( \tilde{\gamma}_0 \), it intersects the whole dividing arc \( \lambda \) an odd number of times, which is impossible, because its endpoints \( \tilde{P}_k \) and \( \tilde{h}_0^{kn}(\tilde{\gamma}_0^-) \) lie in the same component of \( \tilde{A}_0 - \lambda \) (Figure 2.5).

We have shown \( \tilde{\gamma}_0 \) cannot intersect a small enough neighborhood of \( \partial^+ \) and our theorem is proven.

Proof of the corollary. Let \( H \in H_0(P) \), and assume an arc \( \alpha \) for \( H \) exists. By iterating \( H \), if necessary, we can suppose, without loss of generality, that the arc \( \gamma \cup \beta^s \) is such that a lifting of it to \( \tilde{A}_0 \) does not intersect its image under \( \tilde{h}_0 \). Let \( \tilde{H}_0 \in \tilde{A}_0 \) be the endpoint of the lifting, \( \tilde{\gamma}_0 \cup \tilde{\beta}_o^s \), of \( \gamma \cup \beta^s \) to \( \tilde{A}_0 \) with initial point \( \tilde{\gamma}_0^- \). The facts that \( H \sim 0 \) (see Definition 1.3) and that \( \beta^s, \beta^u \) are mapped into themselves under \( h^n, h^{-n} \) imply that, for large \( k \), \( \tilde{h}_0^{kn}(\tilde{H}_0) \) and \( \tilde{h}_0^{-kn}(\tilde{H}_0) \) lie in different unbounded components of \( \tilde{A}_0 - \tilde{\gamma}_0 \cup \tilde{\beta}_o^s \cup \tilde{a}_0 \) and the argument proceeds as before (Figure 2.6). Here \( \tilde{P}_o \) denotes the endpoint of the lifting, \( \tilde{\beta}_o^u \), to \( \tilde{A}_0 \) of \( \beta^u \) with initial point \( \tilde{H}_0 \); since \( H \sim 0 \), \( \tilde{P}_o = \tilde{P}_k \) for some \( k \).

**Diagram 2.6**

Remark. In the proof of the corollary notice that in order to arrive at a contradiction to the existence of \( \alpha \) it is enough that the lifting of \( \gamma \beta^u \beta^s \gamma^{-1} \) to \( \tilde{A}_0 \) of Definition 1.3 determines a boundary component \( \tilde{a}_0^* \) of \( \tilde{A}_0 \) which lies in the right-hand side (unbounded) component of \( \tilde{A}_0 - \tilde{\gamma}_0 \cup \tilde{\beta}_o^s \cup \tilde{a}_0 \) (see Figure 2.7).
3. Remarks. (i) In the (not so interesting) case that the twist map $h: A \to A$ is fixed point free, our theorem takes the form:

$$\mathcal{P} \cap S^+ \cap S^- = \emptyset,$$

where $\mathcal{P}$ denotes the set of all periodic points of $h$ and $S^+, S^-$ are the projections into $A$ of the sets $\tilde{S}^+, \tilde{S}^- \subset \tilde{A}$ defined as follows: $\tilde{X} \in \tilde{A}$ lies in $\tilde{S}^+, \tilde{S}^-$ if there exists an arc $\tilde{a}^+ (\tilde{a}^-)$ from $\tilde{X}$ to $\tilde{a}^+ (\tilde{a}^-)$ such that $h(\tilde{a}^+) \cap \tilde{a}^+ = \emptyset (h(\tilde{a}^-) \cap \tilde{a}^- = \emptyset)$.

In the case that $h$ is fixed point free, twist maps seem to have been first related to Brouwer's Translation Theorem by Kerékjártó, *The plane translation theorem of Brouwer and the last geometric theorem of Poincaré*, Acta Sci. Math. Szeged 4 (1928), 86–102.

(ii) The following is a sufficient metric condition for $h$ to satisfy condition $L$: Assume $h: A \to A$ has only a finite number of fixed points in $A$ and $|h(z) - z| < \alpha$ for all $z \in A$ for some $\alpha$ such that $0 < \alpha < \Pi \alpha$; then if $h$ does not satisfy condition $L$, there exists a fixed point $F$ such that either:

(a) in an $\alpha$-neighborhood of $F$ there exists a point $z$ such that the angle between the segments $\overline{zF}$ and $\overline{h(z)F}$ is $\Pi$, or

(b) a closed neighborhood of $F$ intersects the boundary of $A$ or a closed $2\alpha$-neighborhood of $F$ contains a fixed point different from $F$.

To see this, it is enough to show that in $A_0$ there is a unique shortest arc joining any $z \in A_0$ to $h_0(z)$, because then a good $h_0$ will be obtained by lifting $h_0$ along these arcs. The negation of the conclusions (a), (b) above immediately shows that outside of an $\alpha$-neighborhood of the inner boundary of $A$, we can join $z$ to $h_0(z)$, in $A_0$, by the straight line segment $zh_0(z)$. In an $\alpha$-neighborhood of the inner boundary, since $\alpha \leq \Pi \alpha$, there also is a unique shortest arc from $z$ to $h_0(z)$ (Figure 3.1).
(iii) Let $h: A \to A$ be a $C^1$ twist diffeomorphism obtained by integrating the flow of Figure 1.1 such that the fixed points $F_1, F_2$ are transversal, i.e., in $A \times A$ the graph of $h$ intersects the diagonal transversely at $F_1$ and $F_2$. We already know $h$ satisfies condition $L$; however, by making $h$ coincide with a small rotation near the elliptic fixed point $F_2$ and taking time $t$ small enough, condition $L$ will be satisfied because of the metric condition of (ii) above. It follows that any $C^1 h': A \to A$ sufficiently $C^1$ close to $h$ will also satisfy condition $L$, since its corresponding fixed points $F'_1, F'_2$, will, due to transversality, be near $F_1$ and $F_2$. Hence, at least in this case, condition $L$ is stable under $C^1$ perturbations.

**References**


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