CONDITIONAL EXPECTATIONS IN $C^*$-CROSSED PRODUCTS

BY

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ABSTRACT. Let $(A, G, \alpha)$ be a $C^*$-dynamical system. Let $B$ be a $C^*$-subalgebra of $A$ and $P$ be a conditional expectation of $A$ onto $B$ such that $\alpha_t P = P \alpha_t$ for each $t \in G$. Then it is proved that there exists a conditional expectation of $C^*(G, A, \alpha)$ onto $C^*(G, B, \alpha)$. In particular, if $G$ is amenable and $A$ is unital, then there always exists a conditional expectation of $C^*(G, A, \alpha)$ onto $C^*(G)$. Some related results are also obtained.

1. Introduction. Recently Anantharaman-Delaroche [1], [2] investigated the existence of a conditional expectation of a $W^*$-crossed product $W^*(G, M, \alpha)$ onto a $W^*$-crossed product $W^*(G, N, \alpha)$ under appropriate conditions, where $N$ is a von Neumann subalgebra of a von Neumann algebra $M$.

In this paper corresponding results for $C^*$-crossed products are studied and analogous results are obtained. In particular, if a $C^*$-algebra $A$ is unital and $G$ is amenable, then the existence of a conditional expectation of $C^*(G, A, \alpha)$ onto $C^*(G)$ is shown.

2. Projections in $C^*$-algebras. Let $A$ be a $C^*$-algebra, and let $M_n(A)$ be the $C^*$-algebra of $n \times n$ matrices $M = [a_{ij}]$ with entries $a_{ij}$ in $A$ (cf. Paschke [10, Appendix], Takesaki [17, IV. §3]).

Lemma 2.1 (cf. Paschke [10, Proposition 6.1], Takesaki [17, Lemma IV.3.2]). An element $M = [a_{ij}]$ of $M_n(A)$ is positive if and only if $\sum_{i,j} x_i^* a_{ij} x_j > 0$ in $A$ for any $x_1, \ldots, x_n \in A$.

Let $B$ be a $C^*$-subalgebra of $A$. A bounded linear map $P: A \rightarrow B$ is called a conditional expectation if $P$ has the following properties (cf. Umegaki [19]):

(i) $P$ is an onto projection of norm one, that is, $P^2 = P$ and $\|P\| = 1$;
(ii) $P$ is positive, that is, for any $x \in A, P(x^* x) > 0$;
(iii) for any $x \in A, y, z \in B$, $P(yxz) = yP(x)z$;
(iv) for any $x \in A$, $P(x^* P(x) \leq P(x^* x)$.

If $P: A \rightarrow B$ is a conditional expectation, then, by (ii), $P(x^*) = P(x)^*$ for each $x \in A$. Tomiyama [18] proved that if $P: A \rightarrow B$ is an onto projection of norm one (that is $P$ satisfies condition (i)), then $P$ becomes a conditional expectation (cf. Takesaki [17, Theorem III.3.4]). In fact, in this case $P$ satisfies conditions (ii) and (iv) above and

(iii) $P(xy) = P(x)y$ and $P(yx) = yP(x)$ for every $x \in A, y \in B$.

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PROPOSITION 2.2. Let $P: A \rightarrow B$ be a conditional expectation. Then, for any positive integer $n$ and any $x_1, \ldots, x_n \in A$, the $n \times n$ matrix $M = [P(x_i^*x_i) - P(x_i^*)P(x_i)]$ of $M_n(B)$ is positive.

Proof. For each $y_1, \ldots, y_n \in B$, $x = \sum_i x_iy_i$ is in $A$. By condition (iv) we have $P(x^*)P(x) \leq P(x^*x)$; hence

$$\sum_{i,j} y_i^*P(x_i^*)P(x_j)y_j \leq \sum_{i,j} y_i^*P(x_i^*x_j)y_j.$$

By Lemma 2.1, the matrix $M = [P(x_i^*x_i) - P(x_i^*)P(x_i)]$ is positive in $M_n(B)$.

COROLLARY 2.3 (Nakamura, Takesaki and Umegaki [9], cf. Størmer [14, Theorem 4.1]). Let $A, B$ and $P$ be as above. Then, for each positive integer $n$ and each $x_1, \ldots, x_n \in A$, $M = [P(x_i^*x_i)]$ is a positive element of $M_n(B)$, that is, $P$ is a completely positive map (cf. Takesaki [17, IV. §3]).

3. Projections in $C^*$-crossed products. Let $G$ be a locally compact group and $dt$ be the left Haar measure on $G$. Let $A^*$ be the dual space of a $C^*$-algebra $A$ and $\langle \cdot, \cdot \rangle$ be the duality pairing between $A$ and $A^*$. Denote by $\text{Aut}(A)$ the *-automorphism group of $A$. We suppose that $\alpha: G \rightarrow \text{Aut}(A)$ is a strongly continuous homomorphism. Then $(A, G, \alpha)$ is called a $C^*$-dynamical system.

Let $L^1(G, A)$ be the set of all (equivalence classes of) $A$-valued Bochner integrable functions on $G$ with respect to $dt$. $L^1(G, A)$ is a Banach*-algebra with an approximate identity whose multiplication, involution and norm are respectively defined by

$$(xy)(t) = \int x(s)\alpha_s(y(s^{-1}t)) \, ds, \quad (x^*)(t) = \Delta(t)^{-1}\alpha_t(x(t^{-1}))^*,$$

$$||x|| = \int ||x(t)|| \, dt$$

for each $x, y \in L^1(G, A)$ and $t \in G$, where $\Delta$ is the modular function of $G$ (Doplicher, Kastler and Robinson [5, §§II, III]; cf. Bratteli and Robinson [3, §2.7.1], Pedersen [11, §7.6]). We denote by $C^*(G, A, \alpha)$ the enveloping $C^*$-algebra of $L^1(G, A)$ (Doplicher, Kastler and Robinson [5, §IV]; cf. Bratteli and Robinson [3, §2.7.1], Pedersen [11, 7.6.5]). $C^*(G, A, \alpha)$ is called the $C^*$-crossed product (or the covariance algebra) of $(A, G, \alpha)$. We also denote by $C^*(G)$ the group $C^*$-algebra of $G$ (cf. Dixmier [4, 13.9.1], Pedersen [11, 7.1.5]). $C^*(G)$ is nothing but $C^*(G, C, \alpha_0)$, where $C$ is the complex numbers and $\alpha_0: G \rightarrow \text{Aut}(C)$ is the unique trivial homomorphism.

Now let $P$ be a conditional expectation of $A$ onto a $C^*$-subalgebra $B$ of $A$. Assume that $\alpha_t P = P \alpha_t$ for every $t \in G$. Then for each $t \in G$, $\alpha_t(B) \subset B$, hence $\alpha_t$ may also be considered as an *-automorphism of $B$.

PROPOSITION 3.1. Let $(A, G, \alpha)$ be a $C^*$-dynamical system, $B$ be a $C^*$-subalgebra of $A$, and $P$ be a conditional expectation of $A$ onto $B$. Suppose that for any $t \in G$, $\alpha_t P = P \alpha_t$. Then $C^*(G, B, \alpha)$ is a $C^*$-subalgebra of $C^*(G, A, \alpha)$. 

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Proof. If \( y \in L^1(G, B) \), then \( y \in L^1(G, A) \); thus it is sufficient to show that the norm \( \|y\|_B \) of \( y \) in \( C^*(G, B, \alpha) \) is equal to the norm \( \|y\|_A \) of \( y \) in \( C^*(G, A, \alpha) \). The inequality \( \|y\|_A < \|y\|_B \) is clear from the definition. We must prove that \( \|y\|_A > \|y\|_B \). Let \( \Psi \) be a (continuous) positive linear form on \( L^1(G, B) \) with \( \|\Psi\| < 1 \). Then \( \Psi \) extends to a positive linear form on \( C^*(G, B, \alpha) \) (cf. Dixmier [4, 2.7.5]). We use the same symbol \( \Psi \) for the extended linear form. To this \( \Psi \), there corresponds a norm continuous positive definite function \( \psi: G \to B^* \) by Pedersen [11, 7.6.7, 7.6.8]. Define \( \phi: G \to A^* \) by \( \langle \alpha, \phi(t) \rangle = \langle P(\alpha), \psi(t)(\alpha) \rangle \) \((\alpha \in A, t \in G)\). Then \( \phi \) is a norm continuous positive definite function. In fact, for any positive integer \( n \), any \( a_1, \ldots, a_n \in A \) and any \( t_1, \ldots, t_n \in G \), the \( n \times n \) matrix \( M = [P(a_i^* a_j)] \) is a positive element in \( M_n(B) \) by Corollary 2.3. Thus \( M = N^* N \) for some \( N = [b_{ij}] \in M_n(B) \), that is, \( P(a_i^* a_j) = \sum_k b_{ik}^* b_{kj} \) \((1 \leq i, j \leq n)\). Since \( \psi \) is positive definite, it follows that

\[
\sum_{i,j} \left\langle \alpha_i^{-1}(a_i^* a_j), \phi(t_i^{-1} t_j) \right\rangle = \sum_{i,j} \left\langle P(\alpha_i^{-1}(a_i^* a_j)), \psi(t_i^{-1} t_j) \right\rangle \\
= \sum_{i,j} \left\langle \alpha_i^{-1}(P(a_i^* a_j)), \psi(t_i^{-1} t_j) \right\rangle \\
= \sum_{i,j} \left\langle \alpha_i^{-1}\left( \sum_k b_{ki}^* b_{kj} \right), \psi(t_i^{-1} t_j) \right\rangle \\
= \sum_k \sum_{i,j} \left\langle \alpha_i^{-1}(b_{ki}^* b_{kj}), \psi(t_i^{-1} t_j) \right\rangle > 0.
\]

Hence \( \phi \) is positive definite. Let \( \Phi \) be a positive linear form on \( C^*(G, A, \alpha) \) corresponding to \( \phi \) by Pedersen [11, 7.6.7, 7.6.8]. Then \( \|\Phi\| = \|\phi(e)\| = \|\psi(e)\| = \|\Psi\| < 1 \), where \( e \) is the identity of \( G \) (cf. Pedersen [11, 7.6.7]), and

\[
\Phi(y^* y) = \int \langle (y^* y)(t), \phi(t) \rangle \, dt = \int \langle (y^* y)(t), \psi(t) \rangle \, dt = \Psi(y^* y).
\]

This implies that \( \|y\|_B \leq \|y\|_A \). Therefore, \( \|y\|_B = \|y\|_A \) for any \( y \in L^1(G, B) \), and \( C^*(G, B, \alpha) \) is a \( C^* \)-subalgebra of \( C^*(G, A, \alpha) \). (In the sequel we denote by \( \|x\| \) the norm of \( x \in L^1(G, A) \) in \( C^*(G, A, \alpha) \).)

Theorem 3.2. Let \((A, G, \alpha)\), \( B \) and \( P \) be as in Proposition 3.1. Then there exists a conditional expectation of \( C^*(G, A, \alpha) \) onto \( C^*(G, B, \alpha) \).

Proof. Define \( Q: L^1(G, A) \to L^1(G, B) \) by \( (Q(x))(t) = P(x(t)) \) \((x \in L^1(G, A), t \in G)\). Then \( Q \) is a linear map and \( Q(y) = y \) for every \( y \in L^1(G, B) \). Take a positive linear form \( \Psi \) on \( L^1(G, B) \) with \( \|\Psi\| < 1 \). As in the proof of Proposition 3.1, to this \( \Psi \) there correspond a norm continuous positive definite function \( \psi: G \to B^* \), a norm continuous positive definite function \( \phi: G \to A^* \) and a positive linear form \( \Phi \) on \( C^*(G, A, \alpha) \) with \( \|\Phi\| < 1 \). Let \( K(G, A) \) be the set of all \( A \)-valued
continuous functions on $G$ with compact support. Then for $x \in K(G, A)$,

$$
\Psi(Q(x)\ast Q(x)) = \int \langle (Q(x)\ast Q(x))(t), \psi(t) \rangle \, dt \\
= \int \int \Delta(s^{-1}) \langle \alpha_s(P(x(s^{-1})\ast P(x(s^{-1}t))), \psi(t) \rangle \, ds \, dt \\
= \int \int \langle \alpha_{s^{-1}}(P(x(s)\ast P(x(t))), \psi(s^{-1}t)) \rangle \, ds \, dt
$$

and

$$
\Phi(x\ast x) = \int \langle (x\ast x)(t), \phi(t) \rangle \, dt \\
= \int \int \Delta(s^{-1}) \langle \alpha_s(P(x(s^{-1})\ast x(s^{-1}t))), \psi(t) \rangle \, ds \, dt \\
= \int \int \langle \alpha_{s^{-1}}(P(x(s)\ast x(t))), \psi(s^{-1}t) \rangle \, ds \, dt.
$$

The function

$$(s, t) \rightarrow \langle \alpha_{s^{-1}}(P(x(s)\ast P(x(t))), \psi(s^{-1}t) \rangle
$$
on $G \times G$ is continuous and of compact support $S_1$. Similarly the function

$$(s, t) \rightarrow \langle \alpha_{s^{-1}}(P(x(s)\ast x(t))), \psi(s^{-1}t) \rangle
$$
on $G \times G$ is continuous and of compact support $S_2$. Then $S_1$ and $S_2$ are contained in a set $K \times K$ for some compact subset $K$ of $G$. The measure on $K$ induced by the left Haar measure $dt$ is a finite measure, hence it is the weak*-limit of positive measures $m_i$ of finite support. Then $m_i = \sum c_i \delta_i$ for some positive numbers $c_1, \ldots, c_n$ and $t_1, \ldots, t_n \in G$, where $\delta_i$ is the point measure at $i \in G$. We have

$$
\int \int \langle \alpha_{s^{-1}}(P(x(s)\ast P(x(t))), \psi(s^{-1}t) \rangle \, dm_i(s) \, dm_i(t) \\
= \sum_{i,j} \langle \alpha_{s^{-1}}(P(x(t_i)\ast P(x(t_j))), \psi(t_i^{-1}t_j) \rangle c_i c_j
$$

and

$$
\int \int \langle \alpha_{s^{-1}}(P(x(s)\ast x(t))), \psi(s^{-1}t) \rangle \, dm_i(s) \, dm_i(t) \\
= \sum_{i,j} \langle \alpha_{s^{-1}}(P(x(t_i)\ast x(t_j))), \psi(t_i^{-1}t_j) \rangle c_i c_j.
$$

By Proposition 2.2, $M = [c_i c_j (P(x(t_i)\ast x(t_j)) - c_i c_j P(x(t_i)\ast P(x(t_j)))]$ is positive in $M_n(B)$. Thus, $M = N^* N$ for some $N = [b_{ij}] \in M_n(B)$. It follows that

$$
\sum_{i,j} \langle \alpha_{s^{-1}}(P(x(t_i)\ast x(t_j))), \psi(t_i^{-1}t_j) \rangle c_i c_j - \sum_{i,j} \langle \alpha_{s^{-1}}(P(x(t_i)\ast P(x(t_j))), \psi(t_i^{-1}t_j) \rangle c_i c_j \\
= \sum_{i,j} \langle \alpha_{s^{-1}}(b_k^* b_{kj}), \psi(t_i^{-1}t_j) \rangle = \sum_k \sum_{i,j} \langle \alpha_{s^{-1}}(b_k^* b_{kj}), \psi(t_i^{-1}t_j) \rangle > 0.
$$
Therefore
\[
\int \int \langle \alpha_{s^{-1}}(P(x(s)^*P(x(t)))) , \psi(s^{-1}) \rangle \ dm_t(s) \ dm_t(t) < \int \int \langle \alpha_{s^{-1}}(P(x(s)^*x(t))) , \psi(s^{-1}) \rangle \ dm_t(s) \ dm_t(t)
\]
and, in the limit,
\[
\int \int \langle \alpha_{s^{-1}}(P(x(s)^*P(x(t)))) , \psi(s^{-1}) \rangle \ ds \ dt < \int \int \langle \alpha_{s^{-1}}(P(x(s)^*x(t))) , \psi(s^{-1}) \rangle \ ds \ dt.
\]

We have \( \Psi(Q(x)^*Q(x)) \leq \Phi(x^*x) \). Since \( K(G, A) \) is dense in \( L^1(G, A) \), the above inequality holds for any \( x \in L^1(G, A) \). This implies that \( \|Q(x)\| < \|x\| \) for every \( x \in L^1(G, A) \), and \( Q \) extends to a conditional expectation of \( C^*(G, A, \alpha) \) onto \( C^*(G, B, \alpha) \).

Let \( \langle \phi \rangle \) be a state of \( A \). If \( \langle \alpha_t(a) , \phi \rangle = \langle a , \phi \rangle \) for any \( a \in A \), \( t \in G \), then \( \phi \) is said to be an \( \alpha \)-invariant state. Let \( CB(G) \) be the Banach algebra of all complex-valued bounded continuous functions on \( G \) with supremum norm. For each state \( \phi \) of \( A \) and \( a \in A \), define \( \phi_a \in CB(G) \) by \( \langle \phi_a(t) = \langle \alpha_t(a) , \phi \rangle \). If \( A \) is unital and \( G \) is amenable, then for any (left and right) invariant mean \( m \) (cf. Greenleaf [7, p. 29], Pedersen [11, 7.3.3]), the state \( \phi_m \) of \( A \) defined by \( \langle a , \phi_m \rangle = m(\phi_a) \) is \( \alpha \)-invariant (cf. Emch [6, p. 173]). Notice that, if \( G \) is amenable, the reduced \( C^* \)-crossed product \( C^*_r(G, A, \alpha) \) (Zeller-Meier [20, Définition 4.6 (for \( G \) discrete)], Takai [15]; cf. Pedersen [11, 7.7.4]) is equal to the \( C^* \)-crossed product \( C^*(G, A, \alpha) \) for any \( C^* \)-dynamical system \( (A, G, \alpha) \) (Zeller-Meier [20, Théorème 5.1 (for \( G \) discrete)], Takai [15, Proposition 2.2]; cf. Pedersen [11, 7.7.7]). It is known that \( G \) is amenable if and only if \( C^*(G) = C^*_r(G) \) (cf. Dixmier [4, §18.3], Greenleaf [7, §3.5], Pedersen [11, §7.3]).

**Corollary 3.3.** Let \((A, G, \alpha)\) be a \( C^* \)-dynamical system. Suppose that \( A \) is unital and has an \( \alpha \)-invariant state. Then there exists a conditional expectation of \( C^*(G, A, \alpha) \) onto its \( C^* \)-subalgebra \( C^*(G) \).

**Proof.** Let \( 1 \) be the identity of \( A \) and identify \( C \) with \( C1 \). Let \( \phi \) be an \( \alpha \)-invariant state of \( A \). Define \( P: A \to C1 \) by \( P(a) = \phi(a)1 \) \((a \in A)\). Then \( P \) is a conditional expectation and, since \( \phi \) is \( \alpha \)-invariant, \( P\alpha_t = \alpha_tP \) for every \( t \in G \). By Theorem 3.2 there exists a conditional expectation of \( C^*(G, A, \alpha) \) onto the \( C^* \)-subalgebra \( C^*(G) \) of \( C^*(G, A, \alpha) \).

**Corollary 3.4.** Let \((A, G, \alpha)\) be a \( C^* \)-dynamical system with \( A \) unital. If \( G \) is amenable, then there exists a conditional expectation of \( C^*(G, A, \alpha) \) onto \( C^*(G) \).

Now we consider the case where \( G \) is abelian. Denote by \( \hat{G} \) the dual group of \( G \). For each \( \sigma \in \hat{G} \), define \( \hat{\alpha}_\sigma: L^1(G, A) \to L^1(G, A) \) by
\[
(\hat{\alpha}_\sigma(x))(t) = (\bar{t}, \sigma) x(t) \quad (x \in L^1(G, A), t \in G),
\]
where \((t, \sigma)\) is the value of the character \(\sigma\) at \(t\). Then \(\hat{\alpha}_\sigma\) can be extended to a *-automorphism of \(C^*(G, A, \alpha)\) and, denoting it by the same symbol \(\hat{\alpha}_\sigma\), \(\hat{\alpha}\colon G \to \text{Aut}(C^*(G, A, \alpha))\) is shown to be a strongly continuous homomorphism (Takai [15, pp. 30–31]; cf. Pedersen [11, 7.8.3]). \((C^*(G, A, \alpha), \hat{G}, \hat{\alpha})\) is called the dual \(C^*\)-dynamical system of \((A, G, \alpha)\).

**Corollary 3.5.** Let \((A, G, \alpha)\) be a \(C^*\)-dynamical system with \(A\) unital and \(G\) abelian. Then there exists a conditional expectation of \(C^*(\hat{G}, C^*(G, A, \alpha), \hat{\alpha})\) onto \(C^*(\hat{G}, C^*(G), \hat{\alpha})\).

**Proof.** By Corollary 3.4 there exists a conditional expectation \(P\) of \(C^*(G, A, \alpha)\) onto \(C^*(G)\). In view of the construction of \(P\), it is easy to see that \(\hat{\alpha}_\sigma P = P\hat{\alpha}_\sigma\) for all \(\sigma \in \hat{G}\). Hence, by Theorem 3.2 there exists a conditional expectation of \(C^*(\hat{G}, C^*(G, A, \alpha), \hat{\alpha})\) onto \(C^*(\hat{G}, C^*(G), \hat{\alpha})\).

4. **Projections in \(C^*\)-crossed products with discrete groups.** We now treat the case where \(G\) is discrete. The following theorem is essentially due to Zeller-Meier [20].

**Theorem 4.1.** Let \((A, G, \alpha)\) be a \(C^*\)-dynamical system with \(G\) discrete. Then

(i) there exists a conditional expectation of \(C^*_r(G, A, \alpha)\) onto \(A\), and

(ii) there exists a conditional expectation of \(C^*(G, A, \alpha)\) onto \(A\).

**Proof.** (i) Since \(G\) is discrete, the Haar measure \(dt\) on \(G\) is a counting measure, that is, for any finite subset \(F\) of \(G\), the measure of \(F\) is the number of elements in \(F\). By the correspondence \(a \mapsto \delta_a a\) \((a \in A)\), \(A\) may be considered as a \(C^*\)-subalgebra of \(C^*_r(G, A, \alpha)\) [20, p. 171], where \(\delta_a\) is the characteristic function of \(\{e\}\), that is, \(\delta_a(t) = 1\) if \(t = e\), or \(0\) if \(t \neq e\). Define \(P\colon L^1(G, A) \to A\) by \(P(x) = x(e)\) \((x \in L^1(G, A))\). Then \(P\) can be extended to a conditional expectation of \(C^*_r(G, A, \alpha)\) onto \(A\). In fact, for any state \(\phi\) of \(A\), let \((\pi_\phi, H_\phi, \xi_\phi)\) be the GNS representation of \(A\) induced by \(\phi\) and \(\eta_\phi = \delta_\phi \xi_\phi \in L^2(G, H_\phi)\), where \(L^2(G, H_\phi)\) is the Hilbert space of all \(H_\phi\)-valued functions \(\eta\) on \(G\) such that \(\int ||\eta(t)||^2 dt < \infty\). Denote \(\Pi_\phi = \text{Ind} \pi_\phi\) [20, Définition 4.1]. Then \((\Pi_\phi, L^2(G, H_\phi), \eta_\phi)\) is a cyclic *-representation of \(L^1(G, A)\), and for each \(x \in L^1(G, A)\), \((\Pi_\phi(x)\eta_\phi, \eta_\phi) = \phi(x(e))\) [20, Proposition 4.2(ii)]. It follows that
\[
(\Pi_\phi(x^*x)\eta_\phi, \eta_\phi) = \phi((x^*x)(e)) \geq \phi(x(e)^*x(e)) = \phi(P(x)^*P(x)).
\]
This implies that \(\|P(x)\| < \|x\|\) (the norm of \(x\) in \(C^*_r(G, A, \alpha)\)) and \(P\) extends to a conditional expectation of \(C^*_r(G, A, \alpha)\) onto \(A\).

(ii) By the correspondence \(a \mapsto \delta_a a\) \((a \in A)\), \(A\) is also a \(C^*\)-subalgebra of \(C^*(G, A, \alpha)\) [20, p. 146]. Since \(\|x\|_\tau < \|x\|\) for any \(x \in L^1(G, A)\), \(P\) in the proof of (i) can be extended to a conditional expectation of \(C^*(G, A, \alpha)\) onto \(A\).

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C*-CROSSED PRODUCTS

REFERENCES


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