

## THE STABLE GEOMETRIC DIMENSION OF VECTOR BUNDLES OVER REAL PROJECTIVE SPACES

BY

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**ABSTRACT.** An elementary argument shows that the geometric dimension of any vector bundle of order  $2^e$  over  $RP^n$  depends only on  $e$  and the residue of  $n \pmod 8$  for  $n$  sufficiently large. In this paper we calculate this geometric dimension, which is approximately  $2e$ . The nonlifting results are easily obtained using the spectrum  $bJ$ . The lifting results require  $bo$ -resolutions. Half of the paper is devoted to proving Mahowald's theorem that beginning with the second stage  $bo$ -resolutions act almost like  $K(\mathbb{Z}_2)$ -resolutions.

**1. Introduction.** Let  $P_k^l$  denote the stunted real projective  $RP^l/RP^{k-1}$ . In the stable range ( $l + 8 < 2k - 1$ ) there is a map  $P_{k+8}^{l+8} \rightarrow P_k^l$  (when  $k$  is odd and  $l$  even) which induces an isomorphism in  $KO$ -theory. Thus if  $f: P_k^l \rightarrow BO$  classifies a (stable) vector bundle of order  $2^e$  in  $(KO)^\sim(P^l)$ , then  $f\varphi$  classifies a bundle of order  $2^e$  in  $(KO)^\sim(P^{l+8})$ . Since the geometric dimension  $gd(\theta)$  of a stable vector bundle  $\theta$  is the smallest  $r$  such that its classifying map lifts to  $BO_r$ , this implies that for fixed  $e$  the  $gd$  of a bundle of order  $2^e$  over  $P^n$  is a nonincreasing function of  $n$  for  $n$  in a fixed mod 8 congruence class, and hence must achieve a stable value. In this paper we compute this stable value for almost every  $e$ .

**THEOREM 1.1.** (i) *If  $e \geq 20$  and  $n$  is sufficiently large, the geometric dimension of any vector bundle of order  $2^e$  over  $P^n$  is  $2e + \delta(n, e)$ , where  $\delta$  is given by*

		$n \pmod 8$							
		0	1	2	3	4	5	6	7
$e \pmod 4$	0	0	0	0	0	0	0	0	0
	1	2	0	0	0	0	0	2	2
	2	2	0	-1	-1	-1	0	2	2
	3	1	0	-2	-2	-2	0	1	1

(ii) *For any  $e$  and  $n$  the geometric dimension of any vector bundle of order  $2^e$  over  $P^n$  is  $\geq 2e + \delta(n, e)$ .*

1.1(i) can be proved by these methods for some smaller values of  $e$  (see Remark 4.3), but for some very small values of  $e$ , the stable geometric dimension is greater

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than  $2e + \delta(n, e)$ . The condition “ $n$  sufficiently large” can be made precise as  $n \geq 4e + 30 + 4[\log_2(e + 4)]$ .

The nonlifting results ( $\geq$  part) are virtually identical to those of every other approach to the geometric dimension question which utilizes, however indirectly, the Adams operation  $\psi^3 - 1$  ([4], [9], [6], [7]). In certain congruences our results are slightly stronger because the results were overlooked in [4], and [9] did not consider  $n$  odd.

The significance of our work is that these  $bJ$ -primary obstructions are a total obstruction for the lifting questions dealt with in 1.1(i). It shows that our control over homotopy theory as applied to this classical geometrical question is rather complete.

REMARK 1.2. Another interpretation of 1.1(i) is that for these bundles the geometric dimension equals the fibre-homotopy geometric dimension (see Remark 2.6).

The most significant and most novel part of the theorem is the lifting results. This represents the first successful attempt to use obstruction theory with arbitrarily many nontrivial obstruction groups. Since the obstruction groups, i.e. homotopy groups of Stiefel manifolds, are known completely only through at most 29 nontrivial dimensions [13], a lifting procedure which works for purely dimensional reasons is clearly required here. This is provided by Mahowald’s theory of  $bo$ -resolutions [14]. In §3 we expand considerably upon the proof given in [14], adapting the result to spaces other than  $S^\circ$ .

Another novel aspect of our lifting results is that we are usually lifting far beyond the metastable range. For example, the case  $e = 20, n = 8l + 4$ , says  $gd(a \cdot 2^{4l-17}\xi_{8l+4}) = 40$ , when  $a$  is odd and  $l \geq 16$ . Thus a theory of unstable resolutions is required; this is provided by the  $\lambda$ -algebra methods of Curtis et al.

**2. Nonlifting results.** Let  $\nu(m)$  denote the exponent of the largest 2-power dividing  $m$ . In this section we shall prove

THEOREM 2.1. *If  $n \leq 4\nu(m) - 3$ , then  $gd(m\xi_n) \geq n - 2\nu(m) + \epsilon(n, \nu(m))$  where  $\epsilon$  is given by the table*

		$n \bmod 4$			
		0	1	2	3
$\nu(m) \bmod 4$	0	0	1	1	0
	1	1	1	2	1
	2	2	1	2	1
	3	2	1	0	-1

Since the order of  $m\xi_n$  in  $(KO)^\sim(P^n)$  is  $2^{\varphi(n) - \nu(m)}$ , where  $\varphi(n)$  is the number of positive integers  $\leq n$  which are  $\equiv 0, 1, 2,$  or  $4 \pmod 8$ , Theorem 2.1 readily implies that the numbers in 1.1 are lower bounds.

Let  $V_r = SO/SO_r$  and  $P_r = RP^\infty/RP^{r-1}$ . Let  $QX = \Omega^\infty \Sigma^\infty X$ . The functor  $Q$  may be considered as the composite of the functor  $\Sigma^\infty$ : Spaces  $\rightarrow$  Spectra which forms the suspension spectrum with the functor  $\Omega^\infty$ : Spectra  $\rightarrow$  Spaces which takes

the 0th space in the associated  $\Omega$ -spectrum. The inclusion  $P_r \rightarrow V_r$  admits a stable retraction  $V_r \rightarrow QP_r$ . This follows from early work of James. See, for example, [19, ¶4.2].

Let  $bo$  denote the spectrum for connective  $KO$ -theory localized at 2,  $\Sigma^4bsp$  the spectrum obtained from  $bo$  by killing  $\pi_i$  for  $i < 4$ , and  $bJ$  the fibre of the Adams operation  $\theta = \psi^3 - 1: bo \rightarrow \Sigma^4bsp$  [16], [5]. The map  $\Sigma^\infty P_r \rightarrow P_r \wedge bJ$  induces a map  $QP_r \rightarrow \Omega^\infty(P_r \wedge bJ)$ .

If  $n < 4\nu(m) - 3$  (the hypothesis of 2.1),  $m\xi_n$  is trivial on a skeleton of dimension  $b - 1 > n/2$ , so that its classifying map  $f_m$  factors through  $P_b^n$ , which has a desuspension  $\Sigma^{-1}P_b^n$ . Adjoint to  $f$  is a map  $\Sigma^{-1}P_b^n \rightarrow m\xi SO$ . 2.1 will be proved by showing that if  $r = n - 2\nu(m) + \epsilon - 1$ , the composite

$$(2.2) \quad \Sigma^{-1}P_b^n \xrightarrow{m\xi} SO \rightarrow SO/SO_r = V_r \rightarrow QP_r \rightarrow \Omega^\infty(P_r \wedge bJ)$$

is essential.

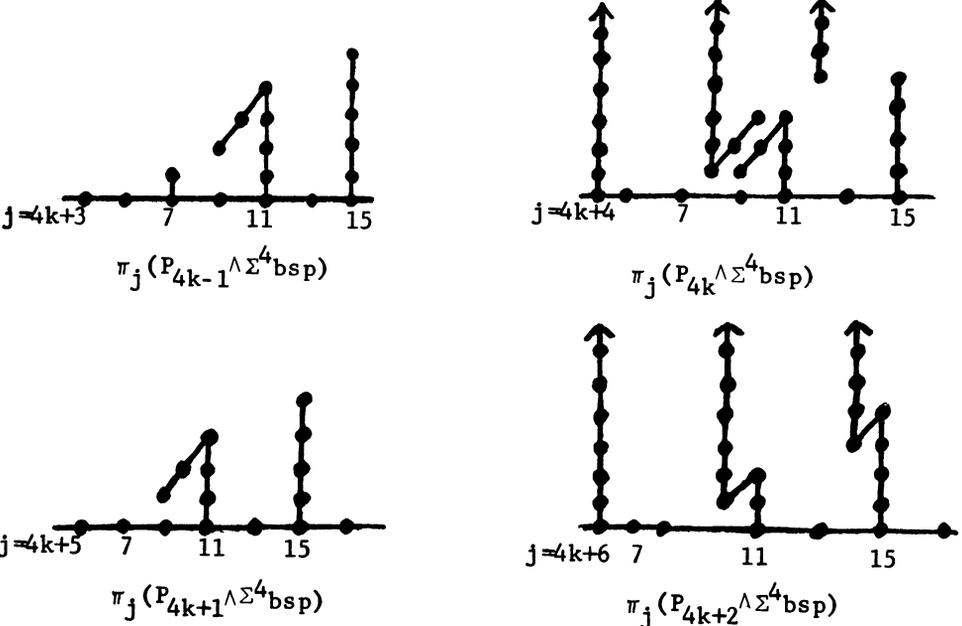
If  $X$  is a space and  $b$  is a spectrum,  $[X, b]$  will denote  $[\Sigma^\infty X, b] \approx [X, \Omega^\infty b]$ . In the exact sequence below, the isomorphism  $[\Sigma^{-1}X, \Sigma^{-1}b] \approx [X, b]$  has been used in the first two groups.

$$(2.3) \quad [P_b^n, P_r \wedge bo] \xrightarrow{\theta_*} [P_b^n, P_r \wedge \Sigma^4bsp] \rightarrow [\Sigma^{-1}P_b^n, P_r \wedge bJ] \rightarrow [\Sigma^{-1}P_b^n, P_r \wedge bo].$$

The groups  $\pi_*(P_r \wedge \Sigma^4bsp)$  have been calculated, e.g. in [6, 3.4], to be given by charts which begin as below and continue similarly. In these charts dots indicate nonzero elements, vertical lines multiplication by 2, diagonal lines multiplication by  $\eta \in \pi_1(S^0)$ , and an arrow an infinite tower. Thus for example

$$\pi_{4k+11}(P_{4k-1} \wedge \Sigma^4bsp) \approx \mathbb{Z}_{32} \quad \text{and} \quad \pi_{4k+4}(P_{4k} \wedge \Sigma^4bsp) \approx \mathbb{Z}_{(2)},$$

the integers localized at 2.





are all homomorphisms, since the homotopy and Whitney sums must agree on  $[ , Q(SO)]$ . Thus divisibility by 2 in  $(KO)^*(P^n)$  corresponds to filtration in  $[P^n, P_r \wedge \Sigma^4bsp]$ .

If  $a$  is any odd integer,  $16a\xi_8$  is trivial and  $16a\xi_9$  is nontrivial but not detected by Stieffel-Whitney classes. Thus the element of  $[P^n, P_1 \wedge \Sigma^4bsp]$  corresponding to  $16a\xi$  is trivial on  $P^8$  and detected on the 9-cell by the filtration 1 class in  $\pi_9(P_1 \wedge \Sigma^4bsp)$ . Thus by additivity  $2^e a\xi$  corresponds to a class of filtration  $e - 3$ .

When  $n \equiv 3 (4)$ , the class of  $2^e a\xi$  in  $[P^n, P_r \wedge \Sigma^4bsp]$  is the sum of the filtration  $e - 3$  classes in both summands. This follows since the map extends over  $P^{n+1}$ , and  $[P^{n+1}, P_r \wedge \Sigma^4bsp] \rightarrow [P^n, P_r \wedge \Sigma^4bsp]$  maps to the sums. This is important because the  $\pi_n(P_r \wedge \Sigma^4bsp)$ -summand extends one filtration higher than  $[P^{n-1}, P_r \wedge \Sigma^4bsp]$ , giving stronger results if the top class is not in the image of  $\theta_*$ .

Similarly if  $r \equiv 0 (4)$ ,  $[P^n, P_r \wedge \Sigma^4bsp]$  has a summand due to the  $r$ -cell which extends one filtration higher than its  $[P^n, P_{r+1} \wedge \Sigma^4bsp]$ -summand. This is utilized effectively if it is not in the image of  $\theta_*$  because  $[P^n, P_1 \wedge \Sigma^4bsp] \rightarrow [P^n, P_r \wedge \Sigma^4bsp]$  maps onto the sum of the two summands when  $r \equiv 0 (4)$ .  $\square$

From the charts of  $\pi_*(P_r \wedge \Sigma^4bsp)$  and the discussion of how to determine  $[P^n, P_r \wedge \Sigma^4bsp]$  from them, we tabulate

**PROPOSITION 2.5.** *If  $n \equiv 3 (4)$ , the largest  $r \equiv 0 (4)$  such that  $[P^n, P_r \wedge \Sigma^4bsp]$  contains a nonzero class in filtration  $e - 3$  is  $n - 2e + \epsilon(n, e) - 1$ , where  $\epsilon$  is as in 2.1.*

2.1 follows immediately from 2.4 and 2.5.

**REMARK 2.6.** Let  $SG_r$  denote the space of degree 1 maps  $S^{r-1} \rightarrow S^{r-1}$  and  $SG = \cup_r SG_r$ .  $BSG_r$  is the classifying space for fiber homotopy equivalence classes of  $S^{r-1}$ -fibrations. There is a map of fibrations

$$\begin{array}{ccccccccc} SO_r & \rightarrow & SO & \rightarrow & V_r & \rightarrow & BSO_r & \rightarrow & BSO \\ \downarrow & & \downarrow & & \downarrow & & \downarrow j_r & & \downarrow j \\ SG_r & \rightarrow & SG & \rightarrow & SG/SG_r & \rightarrow & BSG_r & \rightarrow & BSG \end{array}$$

in which  $j$  corresponds to taking the sphere bundle of a vector bundle, and the fiber-homotopy geometric dimension of a stable spherical fibration is the smallest  $r$  such that the classifying map lifts to  $BSG_r$ . Since the map  $V_r \rightarrow QP_r$ , utilized in (2.2) factors through  $SG/SG_r$ , Remark 1.2 follows immediately from the fact that in the cases covered by 1.1(i) a map  $\Sigma^{-1}P^n \rightarrow V_r$  is null-homotopic if and only if its composite into  $\Omega^\infty(P_r \wedge bJ)$  is null-homotopic.

**3. Resolutions.** In this section we will prove some results about (unstable) Adams resolutions and stable  $bo$ -resolutions which will be used in §4 to prove the lifting part of Theorem 1.1.

**DEFINITION 3.1.** Let  $X$  be a space of finite type. A resolution of  $X$  is a diagram

$$\begin{array}{ccccccc} X & \xleftarrow{p_0} & X^{<1>} & \xleftarrow{p_1} & X^{<2>} & \leftarrow & \dots \\ \downarrow i_0 & & \downarrow i_1 & & \downarrow i_2 & & \\ K_0 & & K_1 & & K_2 & & \end{array}$$

where  $K_i$  is a product of Eilenberg-Mac Lane spaces  $K(\mathbb{Z}_2, n)$  for  $i > 0$ , and  $K_0$  is a product of  $K(\mathbb{Z}_2, n)$ 's and possibly also  $K(\mathbb{Z}, n)$ 's, and  $\Omega K_{s-1} \rightarrow X^{(s)} \xrightarrow{p_s} X^{(s-1)}$  is the fibration classified by  $i_{s-1}$ . If also  $i_s^*: H^*(K_s) \rightarrow H^*(X^{(s)})$  is surjective for all  $s$ , this is an Adams resolution. (Here and throughout the paper all cohomology groups have  $\mathbb{Z}_2$ -coefficients unless indicated otherwise.) An Adams resolution through dimension  $d$  is a resolution in which the  $K(\mathbb{Z}_2, n)$ 's have  $n \leq d$  and  $i_s^*$  is surjective through dimension  $d$ . An Adams resolution of a spectrum is defined similarly. If  $X$  is a space, then applying  $\Omega^\infty(\ )$  to an Adams resolution of the spectrum  $\Sigma^\infty X$  yields a resolution

$$\begin{array}{ccccccc} QX & \leftarrow & \underline{X^{(1)}} & \leftarrow & \underline{X^{(2)}} & \leftarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ K_0 & & K_1 & & K_2 & & \end{array}$$

which may not be an Adams resolution but satisfies  $E_2^{s,t} \approx \text{Ext}_{\mathbb{Q}}^{s,t}(H^*X, \mathbb{Z}_2)$ .

If  $\pi_*(\ )$  is applied to a resolution of  $X$ , a spectral sequence is obtained with  $E_1^{s,t+s} = \pi_t(K_s)$  and  $E_\infty^{r,t+r}$  an associated graded group to  $\pi_t(X)$  if  $X^{(\infty)}$  is acyclic.

**PROPOSITION 3.2.** (i) *If  $f: X \rightarrow Y$  is a map,  $\{X^{(s)}\}$  is an Adams resolution, and  $\{Y^{(s)}\}$  is a resolution, there exists a map of resolutions  $\{X^{(s)} \rightarrow Y^{(s)}\}$ .*

(ii) *If  $\{X^{(s)} \rightarrow Y^{(s)}\}$  is a map of resolutions covering a map  $f: X \rightarrow Y$ , then there is a resolution of the fiber  $F$  of  $f$  such that there is an exact sequence*

$$\dots \rightarrow E_2^{s,t}(F) \rightarrow E_2^{s,t}(X) \rightarrow E_2^{s,t}(Y) \rightarrow E_2^{s+1,t}(F) \rightarrow \dots$$

**PROOF OF (ii).** Let  $F^{(s)}$  be the fiber of  $X^{(s)} \rightarrow Y^{(s-1)}$ . Then  $F^{(s+1)} \rightarrow F^{(s)}$  is classified by  $K_s(X) \times \Omega K_{s-1}(Y)$ , so that  $0 \rightarrow E_1^{s-1,t}(Y) \rightarrow E_1^{s,t}(F) \rightarrow E_1^{s,t}(X) \rightarrow 0$  is exact.  $\square$

**THEOREM 3.3.** *Suppose  $\{g_s\}$  is the map of resolutions induced by the map  $V_r \rightarrow QP_r$  considered in §2. In the diagram*

$$\begin{array}{ccc} & & V_r^{(s)} \xrightarrow{g_s} \underline{P_r^{(s)}} \\ & \nearrow l & \downarrow \\ \Sigma^{-1}P^n & & V_r^{(s-1)} \\ & \searrow f & \end{array}$$

if  $n \leq 2r + 2s - 4$  and  $[g_s \circ l] = 0$ , then  $[f] = 0$ .

**PROOF.** Let  $X_{r+i,i}$  denote the fibre of  $V_{r+i,i} \rightarrow Q(P_r^{r+i-1})$ ,

$$X_r = \text{fibre}(V_r \rightarrow Q(P_r)),$$

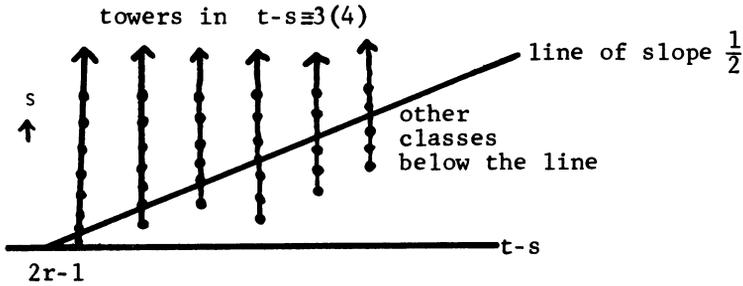
and  $F_{r+i} = \text{fibre}(S^{r+i} \rightarrow Q(S^{r+i}))$ . There is a diagram

$$\begin{array}{ccccc} X_{r+i,i} & \rightarrow & X_{r+i+1,i+1} & \rightarrow & F_{r+i} \\ \downarrow & & \downarrow & & \downarrow \\ V_{r+i,i} & \rightarrow & V_{r+i+1,i+1} & \rightarrow & S^{r+i} \\ \downarrow & & \downarrow & & \downarrow \\ Q(P_r^{r+i-1}) & \rightarrow & Q(P_r^{r+i}) & \rightarrow & Q(S^{r+i}) \end{array}$$

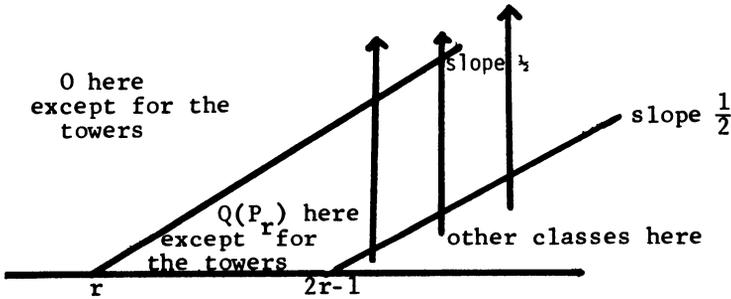
in which the rows and columns are fibrations. By 3.2(ii) resolutions can be chosen so that these fibrations induce long exact sequences in  $E_2^{*,*}$ . By [3],  $E_2^{s,s+j}(F_m) = 0$  if  $j < 2m + 2s - 2$ , except for  $E_2^{s,s+2m-1}(F_m) = \mathbf{Z}_2$  if  $m$  is even. Since  $X_r = \lim_{\rightarrow i} X_{r+i,i}$  and  $X_{r+1,1} = F_r$ , the exact sequences in  $E_2^{*,*}$  obtained from the first row of the above diagram for all  $i > 1$  show that  $E_2(X_r)$  is built up from  $E_2(F_m)$  for  $m > r$ . Hence by Curtis's vanishing theorem above, if  $j < 2r + 2s - 2$ ,

$$E_2^{s,s+j}(X_r) = \begin{cases} \mathbf{Z}_2, & j \equiv 3 \pmod{4}, j \geq 2r - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Pictorially, the  $E_2$ -chart for  $X_r$  with the usual  $(t - s, s)$  grading is



The exact sequence in  $E_2$  induced by  $X_r \rightarrow V_r \rightarrow Q(P_r)$  now implies that  $E_2(V_r)$  has the following rough form:



If the composite  $\Sigma^{-1}P^n \rightarrow^l V_r^{(s)} \rightarrow^{g_r} P_r^{(s)}$  is null-homotopic, then  $l$  factors through  $\Sigma^{-1}P^n \rightarrow X_r^{(s)} \rightarrow^i V_r^{(s)}$ . Through dimension  $n - 1$ ,

$$X_r^{(s)} = \prod_{j > 2r-1; j \equiv 3 \pmod{4}} K(\mathbf{Z}, j),$$

and the map  $X_r^{(s)} \rightarrow X_r^{(s-1)}$  is multiplication by 2 on each  $K(\mathbf{Z}, j)$ . Thus the composite  $\Sigma^{-1}P^n \rightarrow X_r^{(s)} \rightarrow X_r^{(s-1)}$  is the image of

$$\bigoplus_{j > 2r; j \equiv 0 \pmod{4}} H^j(P^n; \mathbf{Z}) \rightarrow^2 \bigoplus_{j > 2r; j \equiv 0 \pmod{4}} H^j(P^n; \mathbf{Z}),$$

which is 0.  $\square$

REMARK 3.3.1. Theorem 3.3 can be strengthened slightly by using the fact that  $F_m$  is  $\Sigma^{m-1}P_m$  through a range, so that in 3.3: if  $\text{Ext}_{\mathbb{Q}}^i(H^*(\Sigma^{r-1}P_r), \mathbf{Z}_2) = 0$  for  $2r - 1 < t - s \leq n - 1$  and  $[g_s \circ l] = 0$ , then  $[f] = 0$ . This upper edge of  $\text{Ext}$  is known from [1] or [13].

**THEOREM 3.4.** *If  $f: \Sigma^{-1}P_{(n+1)/2}^n \rightarrow SO$  classifies  $m\xi$ , then  $f$  lifts to  $SO^{\langle \nu(m) - [\log_2 n] \rangle}$ .*

**PROOF.** If  $0 \leq b \leq 3$ , let  $\rho(4a + b) = 8a + 2^b$ ; thus  $\rho(n)$  is the  $(n + 1)$ st positive integer  $\equiv 0, 1, 2$ , or  $4 \pmod{8}$ . If  $X$  is any space or spectrum, let  $X[j] \rightarrow X$  denote a map from a  $(j - 1)$ -connected space which induces an isomorphism in  $\pi_i(\ )$  for  $i \geq j$ . We shall need

**LEMMA 3.5 [11].**  $H^i(SO[\rho(k) - 1]) \rightarrow H^i(SO[\rho(k + 1) - 1])$  is 0 for  $i < 2^{k+1} - 1$ .

We form the unstable Adams resolution of  $SO$  through dimension  $n - 1$ . There is a commutative diagram (where  $l = [\log_2 n]$ )

$$\begin{array}{ccc}
 SO[\rho(\nu(m)) - 1] & \xrightarrow{r_{\nu(m)-l}} & SO^{\langle \nu(m)-l \rangle} \\
 \downarrow q_{\nu(m)} & & \downarrow \\
 \vdots & & \vdots \\
 \downarrow & & \downarrow \\
 SO[\rho(l + 1) - 1] & \xrightarrow{r_1} & SO^{\langle 1 \rangle} \\
 \downarrow q_{l+1} & & \downarrow p_0 \\
 SO[\rho(l) - 1] & \xrightarrow{q_1 \circ \cdots \circ q_l} & SO
 \end{array}$$

which exists because, by 3.5,  $H^i(q_j) = 0$  for  $i \leq n - 1$  and  $j > l$ , and each  $p_j$  is induced by a map into Eilenberg-Mac Lane spaces with generators in degree  $\leq n - 1$ . Since  $m\xi$  is trivial on  $P^{\rho(\nu(m))-1}$ ,  $f$  lifts to  $SO[\rho(\nu(m)) - 1]$ . Following by  $r_{\nu(m)-l}$  yields the result.  $\square$

If  $X$  is a space, the  $bo$ -resolution of  $X$  is the diagram of spectra

$$\begin{array}{ccccccc}
 \Sigma^\infty X & \xleftarrow{q_0} & X_1 & \xleftarrow{q_1} & X_2 & \xleftarrow{q_2} & \cdots \\
 \downarrow j_0 & & \downarrow j_1 & & \downarrow j_2 & & \\
 X \wedge bo & & X_1 \wedge bo & & X_2 \wedge bo & &
 \end{array}$$

where  $q_s$  is the fibration induced by  $j_s$ . Let  $I$  denote the fiber of the unit map  $\mathbb{S} \rightarrow bo$ , and  $I^{\wedge s} = I \wedge \cdots \wedge I$  with  $s$  factors. Then  $X_s = X \wedge I^{\wedge s}$ . Splicing the homotopy exact sequences of these fibrations yields a cochain complex  $C_i(X)$ :

$$\pi_i(X \wedge bo) \xrightarrow{d_0} \pi_i(X \wedge \overline{bo} \wedge bo) \xrightarrow{d_1} \pi_i(X \wedge \overline{bo}^{\wedge 2} \wedge bo) \xrightarrow{d_2} \cdots$$

where  $\overline{bo} = \Sigma I = \text{cofibre}(\mathbb{S} \rightarrow bo)$  and  $d_i = (X \wedge \overline{bo}^{\wedge i}) \wedge (bo \rightarrow \overline{bo}) \wedge i$ .

The main theorem of  $bo$ -resolutions, adapted to the space  $X$ , states

**THEOREM 3.6.** *Suppose  $X$  is  $(m - 1)$ -connected and there are no nontrivial differentials in the Adams spectral sequence for  $X \wedge bo$  through dimension  $i$ . Then the cohomology of the cochain complex  $C_i(X)$  satisfies*

$$H^s(C_i(X)) = \begin{cases} 0, & i < 4s + m, \\ V_{s,i}, & s \geq 2, i \geq 4s + m, \\ Q_i \oplus V_{1,i}, & s = 1, i \geq 4 + m, \end{cases}$$

where  $V_{s,i}$  is a  $\mathbf{Z}_2$ -vector space all elements of which are detected in  $\mathbf{Z}_2$ -cohomology, i.e. they have Adams filtration 0, and

$$Q_i = \text{coker}(\pi_i(X \wedge bo) \xrightarrow{\theta_*} \pi_i(X \wedge \Sigma^4bsp)),$$

where  $\theta_*$  is as in §2.

The significance of this result is that *bo*-resolutions have the *bJ*-primary obstructions at levels 0 and 1, and above this they work *almost* like a  $K(\mathbf{Z}_2)$ -resolution, but they have the advantage over  $K(\mathbf{Z}_2)$ -resolutions of increasing the connectivity faster. To be more precise, we have

**COROLLARY 3.6.1.** *Suppose there are no nontrivial differentials through dimension  $N$  in the ASS for  $D_N Y \wedge X \wedge bo$  where  $D_N Y$  denotes a stable  $N$ -dual of a finite complex  $Y$ . Let  $X_s$  denote the  $s$ th spectrum in the *bo*-resolution of  $X$ . If  $s \geq 2$  and  $l: Y \rightarrow X_s$  is cohomologically trivial, then there exists a map  $l': Y \rightarrow X_s$  such that  $q_{s-1}l = q_{s-1}l': Y \rightarrow X_{s-1}$  and  $l'$  lifts to  $X_{s+1}$ . (Here and later  $Y \rightarrow X$  means  $\Sigma^\infty Y \rightarrow X$  when  $Y$  is a space and  $X$  is a spectrum.)*

**PROOF.** In the diagram

$$\begin{array}{ccccccc}
 \Sigma^{-1}X_{s-1} \wedge bo & & X_{s+1} & & & & \\
 & \searrow i & \downarrow & & & & \\
 Y & \xrightarrow{l} & X_s & \xrightarrow{j} & X_s \wedge bo & \xrightarrow{d_s} & \Sigma X_{s+1} \wedge bo \\
 & & \downarrow q_{s-1} & & & & \\
 & & X_{s-1} & & & & 
 \end{array}$$

one smash factor of  $d_s \cdot j$  is  $\mathcal{S} \rightarrow bo \rightarrow \overline{bo}$  so  $d_s \cdot j$  is trivial. By applying 3.6 to  $C_N(D_N Y \wedge X)$  and using  $S$ -duality,

$$\frac{\ker((d_s)_*: [Y, X_s \wedge bo] \rightarrow [Y, \Sigma X_{s+1} \wedge bo])}{\text{im}(ji)_*: [Y, \Sigma^{-1}X_{s-1} \wedge bo] \rightarrow [Y, X_s \wedge bo]}$$

is a  $\mathbf{Z}_2$ -vector space all elements of which are detected in  $\mathbf{Z}_2$ -cohomology. Since  $[j] \in \ker(d_s)_*$  and  $H^*(jl) = 0$ , there exists  $\phi: Y \rightarrow \Sigma^{-1}X_{s-1} \wedge bo$  such that  $[ji\phi] = [j]$ . Then  $l' = l - i\phi$  is the desired map.  $\square$

The remainder of this section will be devoted to the proof of 3.6. It follows quite directly from the methods of [14], but we hope to clarify some aspects of that proof. No aspects of this proof will be required in §4.

Let  $W$  denote the fiber of  $\Omega^2 S^3 \rightarrow S^1$ . Then  $\Omega^2 S^3 \simeq W \times S^1$ . The May filtration on  $\Omega^2 S^3$  induces a filtration  $F_n(W)$ . Let  $\overline{B}(n)$  denote the Thom spectrum defined by the composite  $F_{2n}(W) \hookrightarrow \Omega^2 S^3 \xrightarrow{\Omega^2} BO$ . Let  $B_n = \Sigma^{4n} \overline{B}(n)$  and if  $\bar{n} = (n_1, \dots, n_s)$  let  $B_{\bar{n}} = B_{n_1} \wedge \dots \wedge B_{n_s}$ . Let  $R_s$  denote the set of  $s$ -tuples of positive integers. If  $(n_1, \dots, n_s) \in R_s$  and  $\epsilon = 0$  or  $1$ , we say that  $(n_1, \dots, n_{i+1}, k, n_i - k, n_{i+1}, \dots, n_s)$  is an  $\epsilon$ -successor of  $(n_1, \dots, n_s)$  if  $\nu_k^{(n_i)} = \epsilon$ .

A map of spectra  $f: X \rightarrow Y$  is a *homotopy equivalence mod  $K(\mathbf{Z}_2)$ 's* if there are wedges of  $K(\mathbf{Z}_2)$ 's,  $K_1$  and  $K_2$ , and homotopy equivalences  $h_1$  and  $h_2$  so that the  $X_1 \rightarrow Y_1$  component of

$$X_1 \vee K_1 \xrightarrow[h_1]{\cong} X \xrightarrow[f]{} Y \xrightarrow[h_2]{\cong} Y_1 \vee K_2$$

is a homotopy equivalence. The map of spaces  $\Omega^\infty X \rightarrow \Omega^\infty Y$  is also called a homotopy equivalence mod  $K(\mathbf{Z}_2)$ 's.

The bulk of the work in proving 3.6 is in

**THEOREM 3.7.** *There are homotopy equivalences*

$$h_s: \bigvee_{\bar{n} \in R_s} B_{\bar{n}} \wedge bo \rightarrow \overline{bo}^{\wedge s} \wedge bo$$

such that, for the map  $d_s$  which induces the chain complexes  $C(S^0)$  of 3.6

$$\bigvee_{\bar{n} \in R_s} B_{\bar{n}} \wedge bo \xrightarrow{h_s} \overline{bo}^{\wedge s} \wedge bo \xrightarrow{d_s} \overline{bo}^{\wedge (s+1)} \wedge bo \xrightarrow{h_{s+1}^{-1}} \bigvee_{\bar{m} \in R_{s+1}} B_{\bar{m}} \wedge bo,$$

the component  $B_{\bar{n}} \wedge bo \rightarrow B_{\bar{m}} \wedge bo$

- (i) is a homotopy equivalence mod  $K(\mathbf{Z}_2)$ 's if  $\bar{m}$  is a 0-successor of  $\bar{n}$ ,
- (ii) lifts to a homotopy equivalence mod  $K(\mathbf{Z}_2)$ 's  $B_{\bar{n}} \wedge bo \rightarrow (B_{\bar{m}} \wedge bo)^{\langle 1 \rangle}$  if  $\bar{m}$  is a 1-successor of  $\bar{n}$ ,
- (iii) has filtration  $> 1$  if  $|\bar{m}| < |\bar{n}|$ , or  $|\bar{m}| = |\bar{n}|$  and  $\bar{m}$  is neither a 0- nor a 1-successor of  $\bar{n}$ . (Here and elsewhere  $|\bar{n}| = \sum n_i$ .)

We shall return to the proof of 3.7 at the end of this section. 3.7 is easily adapted to spaces  $X$  satisfying the hypotheses of 3.6.

**COROLLARY 3.8.** *Suppose the Adams spectral sequences*

$$\text{Ext}_{\mathcal{Q}_1}(H^*X, \mathbf{Z}_2) \Rightarrow \pi_*(X \wedge bo)$$

has no nontrivial differentials. If  $X \wedge$  is applied to the maps of 3.7, the component  $X \wedge B_{\bar{n}} \wedge bo \rightarrow X \wedge B_{\bar{m}} \wedge bo$

- (i) is a homotopy equivalence mod  $K(\mathbf{Z}_2)$ 's if  $\bar{m}$  is a 0-successor of  $\bar{n}$ ,
- (ii) lifts to a map  $X \wedge B_{\bar{n}} \wedge bo \rightarrow (X \wedge B_{\bar{m}} \wedge bo)^{\langle 1 \rangle}$  which is surjective in  $\pi_*(\ )$  and whose kernel consists at most of filtration zero  $\mathbf{Z}_2$ 's, if  $\bar{m}$  is a 1-successor of  $\bar{n}$ ,
- (iii) has filtration  $> 1$  if  $|\bar{m}| < |\bar{n}|$ , or  $|\bar{m}| = |\bar{n}|$  and  $\bar{m}$  is neither a 0- nor 1-successor of  $\bar{n}$ .

If there are no differentials through dimension  $i$ , then the results are true through dimension  $i$ .

**PROOF.** (i) is clear, and (iii) follows immediately from 3.7(iii) and the map  $X \wedge (B_{\bar{m}} \wedge bo)^{\langle 2 \rangle} \rightarrow (X \wedge B_{\bar{m}} \wedge bo)^{\langle 2 \rangle}$ . To prove (ii) we use 3.7(ii) to obtain

$$X \wedge B_{\bar{n}} \wedge bo \xrightarrow{\cong} X \wedge (B_{\bar{m}} \wedge bo)^{\langle 1 \rangle} \xrightarrow{l} (X \wedge B_{\bar{m}} \wedge bo)^{\langle 1 \rangle}.$$

A standard result in homological algebra (e.g. [17, Exercise 3, p. 102]) shows that applying  $\text{Ext}_{\mathcal{Q}}^s(H^*(\ ), \mathbf{Z}_2)$  to  $l$  induces an isomorphism when  $s > 0$  and an epimorphism when  $s = 0$ . We will be done once we note that the ASS for  $X \wedge B_{\bar{n}} \wedge bo$

has no nontrivial differentials. This follows from the following lemma which will be proved later in this section.

LEMMA 3.9. *If  $\bar{n} = (n_1, \dots, n_s)$ , let  $|\bar{n}| = \sum n_i$  and  $\alpha(\bar{n}) = \sum \alpha(n_i)$ .*

$$B_{\bar{n}} \wedge bo \simeq K \vee \begin{cases} \Sigma^{4|\bar{n}|} bo^{\langle 2|\bar{n}| - \alpha(\bar{n}) \rangle} & \text{if } |\bar{n}| \text{ is even,} \\ \Sigma^{4|\bar{n}|} bsp^{\langle 2|\bar{n}| - 1 - \alpha(\bar{n}) \rangle} & \text{if } |\bar{n}| \text{ is odd,} \end{cases}$$

where  $K$  is a wedge of suspensions of  $K(\mathbb{Z}_2)$ .

Since there is a map  $X \wedge (bo^{\langle t \rangle}) \rightarrow (X \wedge bo)^{\langle t \rangle}$  inducing an isomorphism in  $\text{Ext}^t(\ )$  for  $s > 0$ ,  $X \wedge B_{\bar{n}} \wedge bo$  has no nontrivial differentials when  $|\bar{n}|$  is even. A similar argument works when  $|\bar{n}|$  is odd once we note that there are no differentials in the ASS for  $X \wedge bsp$ . This follows from the fact that there are maps  $\Sigma^8 bsp \rightarrow \Sigma^4 bo \rightarrow bsp$  such that the composite induces an injection in  $\text{Ext}_{\mathcal{Q}}(H^*(X \wedge \ ), \mathbb{Z}_2)$ .

□

In order to see that 3.8 implies 3.6, we shall use the following combinatorial result, which orders the summands in  $\bigvee_{\bar{n} \in R_s} B_{\bar{n}} \wedge bo$ . It may be viewed as an elaboration upon the argument used in [14, proof of 5.11].

LEMMA 3.10. *There are partitions  $R_s = T_s \amalg U_s$ , total orderings  $\ll$  on  $T_s$  and  $U_s$ , and bijections  $\Phi: T_s \rightarrow U_{s+1}$  so that  $\Phi(\bar{n})$  is the first 0-successor of  $\bar{n}$  if  $\bar{n}$  contains some non-2-powers, and  $\phi(\bar{n})$  is the first 1-successor of  $\bar{n}$  if  $\bar{n}$  consists of all 2-powers. Also,  $|\bar{m}| < |\bar{n}|$  implies  $\bar{m} \ll \bar{n}$ .*

We shall return to the rather detailed proof of 3.10 after using it to deduce 3.6.

PROOF OF 3.6. The case  $s = 1$  follows from the methods below together with those of [14] and [16], so we shall assume  $s \geq 2$ . The first case of 3.6 follows immediately from the fact that  $bo$  is 3-connected.

We may ignore the split  $K(\mathbb{Z}_2)$ 's in the equivalences mod  $K(\mathbb{Z}_2)$ 's of 3.8. We choose homotopy classes  $g_\alpha$  corresponding to as basis of  $\text{Ext}_{\mathcal{Q}}^{0,*}(H^*(B_{\Phi(\bar{n})} \wedge bo))$  for all  $\bar{n} \in T_{s-1}$  which consist only of 2-powers. We first show that any element  $y$  of  $\pi_*(X \wedge \bar{bo}^{\wedge s} \wedge bo)$  plus perhaps some  $g_\alpha$ 's is equivalent mod  $\text{im}(d_{s-1})$  to an element of  $\pi_*(X \wedge \bigvee_{\bar{n} \in T_s} B_{\bar{n}} \wedge bo)$ . It suffices to show that this is true up to any filtration.

If  $z \in \pi_*(X \wedge \bar{bo}^{\wedge s} \wedge bo)$ , let  $z_{\bar{n}}$  denote the component of  $z$  in  $\pi_*(X \wedge B_{\bar{n}} \wedge bo)$ . Suppose we have shown that for  $\hat{y} = y + d_{s-1}(w) + \sum g_\alpha$ ,  $\text{filtr}(\hat{y}_{\bar{n}}) \geq j$  for all  $\bar{n} \in U_s$ ,  $\text{filtr}(\hat{y}_{\bar{n}}) > j$  for all  $\bar{n} \ll \bar{l}$ , and  $\text{filtr}(\hat{y}_{\bar{l}}) = j$ . If  $j = 0$  and  $\bar{l} = \phi(\bar{e})$  where  $\bar{e}$  has all 2-powers, then by adding an appropriate  $g_\gamma \in \pi_*(X \wedge B_{\bar{l}} \wedge bo)$ , we can make  $\text{filtr}(\hat{y} + g_\gamma)_{\bar{l}} \geq 1$ . Otherwise, by 3.10 and 3.8 there exists  $v \in \pi_*(X \wedge B_{\bar{e}} \wedge bo)$  where  $\bar{l} = \phi(\bar{e})$ , such that  $(d_{s-1}v)_{\bar{l}} = \hat{y}_{\bar{l}}$ . Moreover, we still have

$$\text{filtr}(\hat{y} - d_{s-1}v)_{\bar{n}} > j \quad \text{for all } \bar{n} \ll \bar{l}$$

and

$$\text{filtr}(\hat{y} - d_{s-1}v)_{\bar{n}} \geq j \quad \text{for all } \bar{n} \in U_s.$$

This inductive procedure enables us to eliminate all  $U_s$ -components.

We will now show that  $d_s$  is injective on  $\pi_*(X \wedge \bigvee_{\bar{n} \in T_s} B_{\bar{n}} \wedge bo)$ ; by the preceding paragraphs this will imply 3.6. We partition  $T_s = P_s \amalg Q_s$ , where  $P_s$  are those tuples consisting of all 2-powers. If  $y \neq 0 \in T_s$ , let

$$j = \min(\{\text{filtr}(y_\alpha) : \alpha \in Q_s\} \cup \{\text{filtr}(y_\alpha) + 1 : \alpha \in P_s\})$$

and let  $\alpha_0$  be the smallest  $\alpha$  realizing  $j$ . By 3.10 and 3.8 the  $\phi(\alpha_0)$ -component of  $d_s y$  is nonzero.  $\square$

PROOF OF 3.10. Let  $T_s = \{(2^{e_1}, \dots, 2^{e_s}) : e_1 > e_2 \geq e_3 \geq \dots \geq e_s\} \cup \{(n_1, \dots, n_{j-1}, 2^e + k, 2^{e_{j+1}}, \dots, 2^{e_s}) : 0 < k < 2^e, e_j \geq e_{j+1} \geq \dots \geq e_s\}$ ,  $\phi(2^{e_1}, \dots, 2^{e_s}) = (2^{e_1-1}, 2^{e_1-1}, 2^{e_2}, \dots, 2^{e_s})$ , and

$$\phi(n_1, \dots, n_{j-1}, 2^e + k, 2^{e_{j+1}}, \dots, 2^{e_s}) = (n_1, \dots, n_{j-1}, k, 2^e, 2^{e_{j+1}}, \dots, 2^{e_s}).$$

$\phi$  is easily checked to be a bijection  $T_s \rightarrow R_{s+1} - T_{s+1}$  which preserves  $||$ .

We shall call the elements of  $T_s$  which contain all 2-powers “type B” and the other elements “type A”. A *successor* is a 0-successor of a type A element or a 1-successor of a type B element. Note that  $\Phi(\bar{n})$  is always a successor of  $\bar{n}$ . Define a relation  $\ll$  on  $T_s$  by

$$(*) \quad \text{if } \bar{n} \neq \bar{m} \in T_s \text{ and } \phi(\bar{n}) \text{ is a successor of } \bar{m}, \text{ then } \bar{m} \ll \bar{n}.$$

We now show that  $(*)$  causes no loops  $\bar{m} \ll \bar{n}_1 \ll \dots \ll \bar{n}_{q-1} \ll \bar{m}$ . Suppose such a loop exists.

Case 1. There are no type B’s in the sequence. If  $e_1 < \dots < e_r$ , let  $p(\sum_{i=1}^r 2^{e_i}) = (2^{e_1}, \dots, 2^{e_r})$  and let  $p(a_1, \dots, a_s) = (p(a_1), \dots, p(a_s))$ . Then  $p(\bar{m}) < p(\bar{n}_1) < \dots < p(\bar{n}_{q-1}) < p(\bar{m})$  under the lexicographical ordering from the left. But this cannot happen since the lexicographic order is a total order.

Case 2. There is at least one type B. By considering the various type A or B possibilities, one shows that  $\bar{m} \ll \bar{n}$  under  $(*)$  implies  $\nu(\text{first component of } \bar{m}) < \nu(\text{first component of } \bar{n})$  and if  $\bar{n}$  has type B then this inequality is strict. This implies that there can be no looping under  $(*)$  when a type B occurs.

Since there are no loops,  $\ll$  can be extended to a total order on  $T_s$  (compatible with  $||$ ). If  $U_{s+1}$  is ordered by  $\phi(\ll)$ , then  $(*)$  guarantees that  $\phi(\bar{m})$  is the first successor of  $\bar{m}$ .  $\square$

PROOF OF 3.9. Except for the construction of maps near the end, this argument was essentially given in [14], but again some clarification seems worthwhile.

Let  $\text{filtr}(\xi_1^i \xi_2^i \dots) = \sum i_j 2^{j-1}$ . Then  $\chi(H^* \bar{B}(n))^*$  is the subspace  $S_n$  of  $\mathbf{Z}_2[\xi_1^2, \xi_2, \dots]$  spanned by monomials of filtration  $< 2n$ . This is proved by noting that it is  $\ker(R_{S_{q^1}}, R_{S_{q^{2n+2}}}, \dots : \mathcal{Q}^* \rightarrow \bigoplus \mathcal{Q}^*)$ .

For  $j \in \mathbf{Z}_4$  and  $n \geq 0$ , we define  $\mathcal{Q}_j$ -modules  $Q_{j,n}$  by induction on  $n$  by the nontrivial extension of  $\mathcal{Q}_1$ -modules

$$0 \rightarrow \{1, \text{Sq}^2, \text{Sq}^3, \text{Sq}^2 \text{Sq}^3\} \rightarrow Q_{j,n} \rightarrow \Sigma^4 Q_{j,n-1} \rightarrow 0$$

with  $Q_{0,0} = \mathbf{Z}_2$ ,  $Q_{1,0} = \{1, \text{Sq}^2, \text{Sq}^3\}$ ,  $Q_{2,0} = \Sigma^{-2}\{1, \text{Sq}^1, \text{Sq}^2, \text{Sq}^2 \text{Sq}^1, \text{Sq}^3 \text{Sq}^1\}$ ,  $Q_{3,0} = \Sigma^{-3}\{1, \text{Sq}^1, \text{Sq}^2 \text{Sq}^1\}$ . Note that  $Q_{j,n}$  is  $P_{ijn}/N_i$  in the notation of [4, 3.6] (with  $j = 1$  and 3 reversed) and are the modules  $T, S, Y$ , and  $Z$  of [18, pp. 133–135]. The next lemma is a restatement of [18, 1.15]. It can also be proved by the method of [4, 3.9].

LEMMA 3.11. *There is an isomorphism of  $\mathcal{Q}_1$ -modules*

$$\mathcal{Q}_{j_1, n_1} \otimes \mathcal{Q}_{j_2, n_2} \approx F \oplus \mathcal{Q}_{j_1 + j_2, n_1 + n_2 + \delta}$$

where  $F$  is a free  $\mathcal{Q}_1$ -module and

$$\delta = \begin{cases} -1 & \text{if } \{3\} \subset \{j_1, j_2\} \subset \{2, 3\}, \\ 1 & \text{if } \{1\} \subset \{j_1, j_2\} \subset \{1, 2\}, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 3.12. *There is an isomorphism of  $\mathcal{Q}_1$ -modules*

$$H^*(\bar{B}(n_1) \wedge \cdots \wedge \bar{B}(n_s)) \approx F \oplus \mathcal{Q}_{\sum \alpha(n_j), \sum n_j - D}$$

where  $D = 2l$  if  $\sum \alpha(n_j) = 4l$ , and  $D = 2l + 1$  if  $4l + 1 < \sum \alpha(n_j) < 4l + 3$ , where  $F$  is a free  $\mathcal{Q}_1$ -module.

PROOF. The  $\mathcal{Q}_1$ -submodule  $T_{2^i}$  of  $S_{2^i}$  generated over  $\mathcal{Q}_1$  by  $\xi_{i+2}$  and  $\xi_j^{8(2^{i-1-j}-1-k)} \xi_{j+1}^{4k+2} \xi_{j+2}$ ,  $0 \leq k \leq 2^{i-1-j} - 1$ ,  $0 < j < i - 1$ , has its dual isomorphic to  $\mathcal{Q}_{1, 2^i - 1}$ . Multiplying out  $\otimes_{2^i \in n} T_{2^i} \rightarrow S_n$  induces an isomorphism in  $\mathcal{Q}_0$ - and  $\mathcal{Q}_1$ -homology. The result now follows from several applications of 3.11.  $\square$

The following result is our restatement of [18, 2.4].

LEMMA 3.13. *There are isomorphisms of  $\mathcal{Q}$ -modules*

$$\begin{aligned} H^*(bo\langle n \rangle) &\approx \mathcal{Q} \otimes_{\mathcal{Q}_1} \mathcal{Q}_{-n, (n+\varepsilon_1)/2}, \\ H^*(bsp\langle n \rangle) &\approx \mathcal{Q} \otimes_{\mathcal{Q}_1} \mathcal{Q}_{1-n, (n+\varepsilon_2)/2} \end{aligned}$$

where

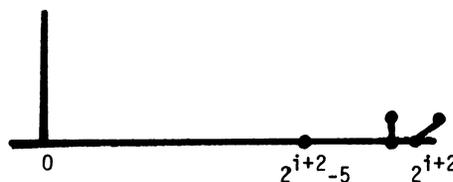
$$\varepsilon_1 = \begin{cases} 1, & n \equiv 1 \pmod{4}, \\ -1, & n \equiv 3 \pmod{4}, \\ 0, & n \text{ even,} \end{cases}$$

and

$$\varepsilon_2 = \begin{cases} 2, & n \equiv 2 \pmod{4}, \\ 0, & n \equiv 0 \pmod{4}, \\ 1, & n \text{ odd.} \end{cases}$$

3.12 and 3.13 imply that the spaces in 3.9 have isomorphic cohomology modules. It remains to construct the maps realizing the isomorphisms. The main step is the map  $\bar{B}(2^i) \rightarrow bo\langle 2^{i+1} - 1 \rangle$ .

By the Adams edge theorem [1], a chart for the ASS of  $\mathbb{S}\langle 2^{i+1} - 1 \rangle$  begins



Thus the inclusion of the bottom cell of  $\bar{B}(2^i)$  into  $\mathbb{S}^{\langle 2^{i+1}-1 \rangle}$  extends over the  $(2^{i+2} - 5)$ -skeleton. The attaching map for the next cell has positive filtration in  $\pi_{2^{i+2}-5}(\bar{B}(2^i)^{\langle 2^{i+2}-5 \rangle})$  and hence maps to 0 in  $\pi_{2^{i+2}-5}(\mathbb{S}^{\langle 2^{i+1}-1 \rangle})$ . Since  $H^j(\bar{B}(2^i)) = 0$  for  $j \geq 2^{i+2}$ , there are no obstructions to extending the map to  $\bar{B}(2^i) \rightarrow \mathbb{S}^{\langle 2^{i+1}-1 \rangle}$ . Following into  $bo^{\langle 2^{i+1}-1 \rangle}$  yields our desired map.

After applying  $\wedge bo$ , the map  $\bigwedge_{2^i \in n} \bar{B}(2^i) \rightarrow \bar{B}(n)$  discussed in [12] and [14] becomes a homotopy equivalence mod  $K(\mathbb{Z}_2)$ 's. (This follows from 3.12 or from [14].) A homotopy inverse on the irreducible part followed by the smash product of the maps constructed above and the map  $\bigwedge_{2^i \in n} bo^{\langle 2^{i+1}-1 \rangle} \rightarrow bo^{\langle 2n-\alpha(n) \rangle}$  yields the desired map  $\bar{B}(n) \rightarrow bo^{\langle 2n-\alpha(n) \rangle}$  if  $n$  is even. The case  $n$  odd and the case of products of  $\bar{B}(n)$ 's follow from these methods using also the maps  $bsp \wedge bo \rightarrow bsp$  and  $bsp \wedge bsp \rightarrow bo^{\langle 2 \rangle}$ .  $\square$

We now proceed toward the proof of 3.7, the final piece of unfinished business.

LEMMA 3.14. *There is an isomorphism of  $\mathcal{Q}_1$ -modules*

$$\bigoplus_{n > 0} \chi H_* B_n \xrightarrow{\bigoplus \chi f_n} \chi H_* bo$$

given by  $\chi f_n(\xi_1^{i_1} \xi_2^{i_2} \dots) = \xi_1^{4n - \sum 2^{i_j} \xi_2^{i_1} \xi_3^{i_2} \dots}$ .

PROOF. By [2],  $\chi H_* bo = \mathbb{Z}_2[\xi_1^4, \xi_2^2, \xi_3, \dots]$ .  $\chi f_n$  is an  $\mathcal{Q}_1$ -morphism, injective, and onto the filtration  $4n$  part of  $\chi H_* bo$ .  $\square$

LEMMA 3.15. *Let  $m = (\mu \wedge \mu) \circ (1 \wedge T \wedge 1)$ :  $(bo \wedge bo) \wedge (bo \wedge bo) \rightarrow bo \wedge bo$ . Let  $\Phi_{n,m}: B_n \wedge B_m \rightarrow B_{n+m}$  denote the multiplication. There are maps  $f_n: B_n \rightarrow bo \wedge bo$  such that  $f_n(i_n) = \xi_1^{4n} \otimes 1$ , and if  $n = 2^i + k$ ,  $1 \leq k \leq 2^i$ , then  $f_n \circ \phi_{2^i, k} - m \circ (f_{2^i} \wedge f_k)$  has filtration  $\geq 4$ .*

PROOF. The maps will be constructed inductively to have the multiplicative property. The homology property will follow automatically. Indeed we will have  $\chi f_n(\xi_1^{i_1} \xi_2^{i_2} \dots) = \xi_1^{4n - \sum 2^{i_j} \xi_2^{i_1} \xi_3^{i_2} \dots} \otimes 1$ .

If  $k < 2^i$  the element of filtration  $\geq 4$  is not required in the multiplicative property. We show that

$$[m(f_{2^i} \wedge f_k)] \in \text{im}(\phi_{2^i, k}^*: [B_{2^i+k}, bo \wedge bo] \rightarrow [B_{2^i} \wedge B_k, bo \wedge bo]).$$

These groups are calculated from the ASS where  $E_2$ -terms are

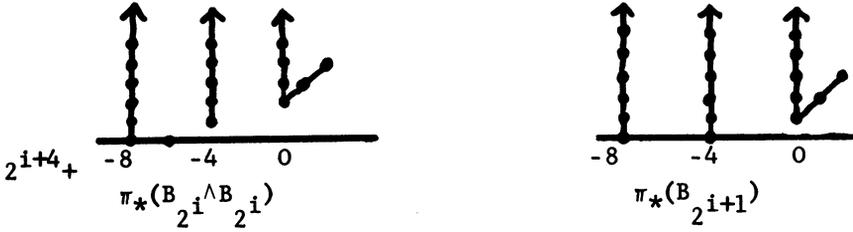
$$\text{Ext}_{\mathcal{Q}_1}^s(\mathcal{Q}/\mathcal{Q}_1, H^* B_{2^i+k}) \rightarrow \text{Ext}_{\mathcal{Q}_1}^s(\mathcal{Q}/\mathcal{Q}_1, H^*(B_{2^i} \wedge B_k)).$$

By 3.12 these homomorphisms fit into a long exact sequence with  $\text{Ext}_{\mathcal{Q}_1}(\mathcal{Q}/\mathcal{Q}_1, \text{free } \mathcal{Q}_1\text{-module})$ , and hence are isomorphisms for  $s > 0$ . Thus it suffices to show that the cohomology homomorphism  $(m(f_{2^i} \wedge f_k))^* \in \text{im}(\phi_{2^i, k}^*)$ , or equivalently  $\chi(m(f_{2^i} \wedge f_k))_*(K) = 0$ , where  $K = \ker(\chi(\phi_{2^i, k})_*)$ . But  $K$  is a free  $\mathcal{Q}_1$ -module spanned over  $\mathbb{Z}_2$  by  $\xi^{I_1} \otimes \xi^{J_1} + \xi^{I_2} \otimes \xi^{J_2}$  with  $I_1 + J_1 = I_2 + J_2$ , which is annihilated by the multiplication map  $\chi(\phi_{2^i, k})_*$ .

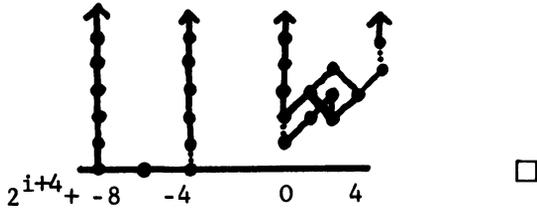
For the case  $k = 2^i$ , we will need the following.

PROPOSITION 3.16. *Let  $F_i = \Sigma^{2^{i+4}-5} M_2 \wedge \bar{B}(1)$ . There is a map  $F_i \rightarrow B_{2^i} \wedge B_{2^i}$  whose cofiber is equivalent mod  $K(\mathbb{Z}_2)$ 's to  $B_{2^{i+1}}$ .*

PROOF. By 3.12,  $\phi_{2^i, 2^i}$  is a  $(2^{i+4} - 5)$ -equivalence, and the homotopy charts are



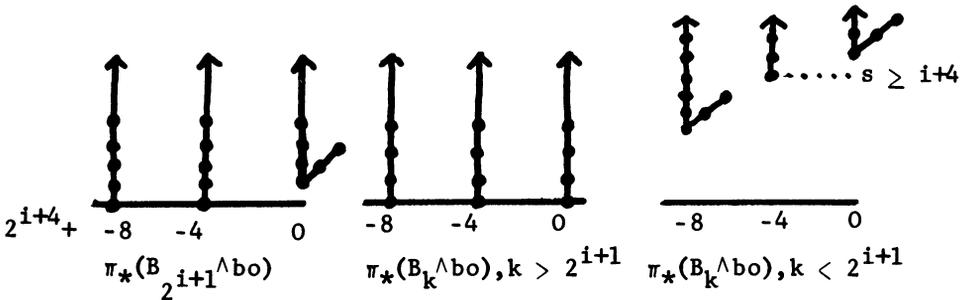
There are no obstructions to extending the map  $\Sigma^{2^{i+4}-5}M \rightarrow^p S^{2^{i+4}-4} \rightarrow^g B_{2^i} \wedge B_{2^i}$  over  $F_i$ . The composite  $F_i \rightarrow^j B_{2^i} \wedge B_{2^i} \rightarrow B_{2^{i+1}}$  is trivial, so that there exists  $MC(j) \rightarrow B_{2^{i+1}}$ , which is easily seen to induce an isomorphism above filtration zero. Indeed the chart for  $\pi_*(MC(j))$  is obtained from those of  $B_{2^i} \wedge B_{2^i}$  and  $F$  as below:



3.15 now follows immediately from the following calculation.

LEMMA 3.17. *If  $m \in [B_{2^i} \wedge B_{2^i}, bo \wedge bo]$  satisfies  $m_*(i_{2^i} \otimes i_{2^i}) = \xi_1^{2^{i+3}}$  there exists  $q$  of filtration  $\geq i + 4$  such that  $j^*(m + q) = 0 \in [F_i, bo \wedge bo]$ .*

PROOF. From 3.14 and 3.9,  $\pi_*(bo \wedge bo)$  in the range  $2^{i+4} - 8 < * < 2^{i+4} + 2$  is a sum of charts of the following three types:



Above filtration zero  $[F_i, bo \wedge bo] \approx \pi_{2^{i+4}-4}(bo \wedge bo) \otimes \mathbb{Z}_2$ . Since  $j^*m$  has positive filtration, it corresponds to some of the  $\mathbb{Z}_2$ 's in filtration  $\geq i + 4$  (or else is zero). Since we may assume by induction that  $bo \wedge bo$  has been split through a range into  $\bigvee B_k \wedge bo$  (or by just looking at the splitting of Ext-groups), it suffices to show:

if  $k < 2^{i+1}$  and the generator of the  $\mathbb{Z}$  in  $\pi_{2^{i+4}-4}(B_k \wedge bo)$  has filtration  $l$ , then  $[B_{2^i} \wedge B_{2^i}, B_k \wedge bo]$  contains a filtration  $l$  class  $f$  such that  $j^*f \neq 0 \in [F_i, B_k \wedge bo]$ .

This may be verified by direct calculation of Ext-groups or (as in [14]) by letting  $f$  be the restriction of the usual map

$$\begin{array}{ccc} \Sigma^{2^{i+3}} bo^{\langle 2^{i+2}-2 \rangle} & \rightarrow & \Sigma^{2^{i+3}} bo \\ \parallel & & \parallel \\ \Sigma^{2^{i+3}} B_{2^i} \wedge B_{2^i} \wedge bo & & (B_k \wedge bo)[2^{i+3}], \end{array}$$

where the first equality is by 3.9 and the second is easily proved by starting with the map  $S^{2^{i+3}} \rightarrow B_k \wedge bo$  corresponding to a homotopy generator.  $\square$

Now we can prove the next result, which is a major step toward 3.7. Let  $\Omega S_+^5 \wedge X_5 \xrightarrow{h} X_5 \wedge X_5$  be the homotopy equivalence constructed in [15]. Let  $o: X_5 \rightarrow bo$  be the rational equivalence given by the  $bo$ -orientation. Let  $g: \bigvee S^{4i} \rightarrow \Omega S^5$  be a fixed stable equivalence.

**THEOREM 3.18.** *There is a homotopy equivalence  $f$  such that the diagram*

$$\begin{array}{ccc} \bigvee S^{4n} \wedge X_5 & \xrightarrow{g \wedge 1} & \Omega S_+^5 \wedge X_5 \\ j \downarrow \wedge o & & \downarrow (o \wedge o)h \\ \bigvee B_n \wedge bo & \xrightarrow{f} & bo \wedge bo \end{array}$$

commutes modulo elements of filtration  $\geq 4$ , where  $j$  is the inclusion of the bottom cell.

**PROOF.** Suppose  $n = 2^i + k$ ,  $0 < k \leq 2^i$ . In the diagram

$$\begin{array}{ccc} S^{4 \cdot 2^i} \wedge S^{4k} & \rightarrow & (\Omega S_+^5 \wedge X_5) \wedge (\Omega S_+^5 \wedge X_5) \xrightarrow{(\mu \wedge \mu) \circ (1 \wedge T \wedge 1)} \Omega S_+^5 \wedge X_5 \\ \downarrow j & & (o \wedge o)h \downarrow (o \wedge o)h \qquad \qquad \qquad \downarrow (o \wedge o)h \\ B_{2^i} \wedge B_k & \xrightarrow{f_{2^i} \wedge f_k} & (bo \wedge bo) \wedge (bo \wedge bo) \xrightarrow{(\mu \wedge \mu) \circ (1 \wedge T \wedge 1)} bo \wedge bo \end{array}$$

where the unlabeled arrow utilizes  $g$  and the inclusion of the bottom cell of  $X_5$ , and  $f_{2^i}$  is the map constructed in 3.15, the right-hand square does not commute, but does when restricted to the bottom cell of  $X_5$ . (Look at the map of base spaces underlying  $h$ .) The left-hand square commutes up to elements of filtration  $\geq 4$  by induction, and hence by the construction of  $f_n$ , we deduce that

$$\begin{array}{ccc} S^{4n} & \xrightarrow{i} & \Omega S_+^5 \wedge X_5 \\ j \downarrow & & \downarrow \\ B_n & \xrightarrow{f_n} & bo \wedge bo \end{array}$$

commutes up to elements of filtration 4.

Hence in the diagram

$$\begin{array}{ccccccc} S^{4n} \wedge X_5 & \xrightarrow{i \wedge 1} & \Omega S_+^5 \wedge X_5 \wedge X_5 & \xrightarrow{1 \wedge \mu} & \Omega S_+^5 \wedge X_5 \\ j \wedge o \downarrow & & \downarrow (o \wedge o)h \wedge o & & \downarrow (o \wedge o)h \\ B_n \wedge bo & \xrightarrow{f_n \wedge 1} & bo \wedge bo \wedge bo & \xrightarrow{1 \wedge \mu} & bo \wedge bo \end{array}$$

the left square commutes up to elements of filtration  $> 4$ , and the right square commutes (because the map  $\Omega S^5 \times \Omega S^5 \rightarrow^q \Omega S^5 \times \Omega S^5$  underlying  $h^{-1}$  sends  $(x, y) \rightarrow (x, x^{-1}y)$  thus satisfying  $(1 \times m)(q \times 1) = q(1 \times m)$ ).

$f = \bigvee (1 \wedge \mu)(f_n \wedge 1)$  is a homotopy equivalence by 3.14.  $\square$

The homotopy equivalences of 3.7 are obtained by iterating 3.18. For example,  $h_2$  is obtained as

$$(\bigvee B_n) \wedge (\bigvee B_n) \wedge bo \xrightarrow{1 \wedge f} (\bigvee B_n) \wedge bo \wedge bo \xrightarrow{f \wedge 1} bo \wedge bo \wedge bo \xrightarrow{1 \wedge \mu} bo \wedge bo.$$

It is easy to deduce from 3.18 that

$$\begin{aligned} \bigvee S^{4\bar{n}} \wedge X_5 &= \bigvee S^{4n_1} \wedge \dots \wedge \bigvee S^{4n_s} \wedge X_5 \rightarrow (\Omega S_+^5)^{\wedge s} \wedge X_5 \\ &\quad \downarrow j_1 \wedge \dots \wedge j_s \downarrow \wedge o \\ \bigvee B_{\bar{n}} \wedge bo &= \bigvee B_{n_1} \wedge \dots \wedge \bigvee B_{n_s} \wedge bo \rightarrow (bo)^{\wedge s} \wedge bo \end{aligned}$$

commutes modulo elements of filtration  $\geq 4$ .

We shall use  $X_5$  to deduce information about the map  $d_s$  of 3.7 by the diagram

$$\begin{array}{ccccc} \rightarrow & (\Omega S^5)^{\wedge s} \wedge X_5 & \rightarrow & (\Omega S^5)^{\wedge (s+1)} \wedge X_5 & \rightarrow \\ & \downarrow & & \downarrow & \\ \rightarrow & (\bar{X}_5)^{\wedge s} \wedge X_5 & \xrightarrow{1 \wedge p \wedge i} & (\bar{X}_5)^{\wedge (s+1)} \wedge X_5 & \rightarrow \\ & \downarrow & & \downarrow & \\ \rightarrow & \bar{bo}^{\wedge s} \wedge bo & \xrightarrow{d_s} & \bar{bo}^{\wedge (s+1)} \wedge bo & \rightarrow \\ & h_s \uparrow \simeq & & h_{s+1} \uparrow \simeq & \\ & \bigvee_{\bar{n} \in R_s} B_{\bar{n}} \wedge bo & & \bigvee_{\bar{m} \in R_{s+1}} B_{\bar{m}} \wedge bo & \end{array}$$

where the first row is the alternating sum of the diagonals. The first square commutes by [15, 3.8] or [14, 5.8]. By the remarks following 3.18 the diagram is equivalent to

$$\begin{array}{ccc} \bigvee_{\bar{n} \in R_s} S^{\bar{n}} \wedge X_5 & \rightarrow & \bigvee_{\bar{m} \in R_{s+1}} S^{\bar{m}} \wedge X_5 \\ \downarrow j \wedge o & & \downarrow j \wedge o \\ \bigvee_{\bar{n} \in R_s} B_{\bar{n}} \wedge bo & \rightarrow & \bigvee_{\bar{m} \in R_{s+1}} B_{\bar{m}} \wedge bo \end{array}$$

where the map in homology induced by the top map is

$$\begin{aligned} &x^{i_1} \otimes \dots \otimes x^{i_{s+1}} \\ \mapsto &\sum_j \sum_a (-1)^j \binom{i_j}{a} x^{i_1} \otimes \dots \otimes x^{i_{j-1}} \otimes x^a \otimes x^{i_j - a} \otimes \dots \otimes x^{i_{s+1}}. \end{aligned}$$

**PROOF OF 3.7(i).** If  $\bar{m}$  is a 0-successor of  $\bar{n}$ , the map  $S^{\bar{n}} \wedge X_5 \rightarrow S^{\bar{m}} \wedge X_5$  is a 2-equivalence. The map of  $\mathcal{Q}_0$ -homology induced by  $(j \wedge o)^*$ :  $H^*(B_{\bar{n}} \wedge bo) \rightarrow H^*(S^{\bar{n}} \wedge X_5)$  is an isomorphism since the first is spanned by  $i_{\bar{n}} \otimes \chi \text{Sq}^{4i}$ ,  $i > 0$ . Thus the infinite towers in  $\pi_{4i}(S^{\bar{n}} \wedge X_5)$  are mapped in a filtration-preserving way to  $\pi_{4i}(B_{\bar{n}} \wedge bo)$ . Filling in the diagram,  $\pi_{4i}(B_{\bar{n}} \wedge bo) \rightarrow \pi_{4i}(B_{\bar{m}} \wedge bo)$  must be

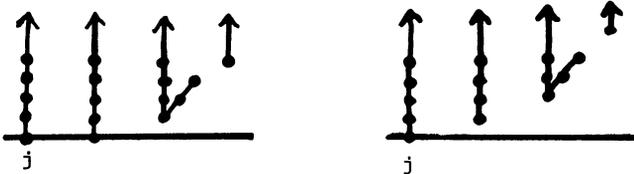
filtration preserving. By 3.9 or 3.12,  $B_{\bar{n}} \wedge bo$  and  $B_{\bar{m}} \wedge bo$  have isomorphic homotopy charts above filtration 0 of the form



where the precise way in which the towers leave filtration 0 depends upon the mod 4 values of  $\alpha(\bar{n})$ .  $\square$

3.7(ii) follows similarly from the observations that if  $\bar{m}$  is a 1-successor of  $\bar{n}$ , then

(1) the homotopy charts of  $B_{\bar{n}} \wedge bo$  and  $B_{\bar{m}} \wedge bo$  are isomorphic (above filtration 0) until they leave filtration 0, at which point the towers of  $B_{\bar{m}} \wedge bo$  always begin one level higher than those of  $B_{\bar{n}} \wedge bo$ ; for example:



(2) the map  $S^{\bar{n}} \wedge X_5 \rightarrow S^{\bar{m}} \wedge X_5$  lifts to a map  $S^{\bar{n}} \wedge X_5 \rightarrow S^{\bar{m}} \wedge X_5^{\langle 1 \rangle}$  which induces an isomorphism in  $H_*(H^*(\ ); Q_0)$  and hence of infinite homotopy towers.  $[H^*(\Sigma X_5^{\langle 1 \rangle}) = \ker(\mathcal{Q} \rightarrow \{\chi Sq^{4i}\})$  so that  $H_*(H^*(\Sigma X_5^{\langle 1 \rangle}); Q_0)$  is spanned by  $\{Sq^1 \cdot \chi Sq^{4i}\}$ .  $H^*(X_5^{\langle 1 \rangle}) \rightarrow H^*(X_5)$  sends  $Sq^1 \rightarrow i$  and hence  $Sq^1 \cdot \chi Sq^{4i} = \chi Sq^{4i} \cdot Sq^1 + \chi Sq^{4i-1} \cdot Sq^2 \mapsto \chi Sq^{4i}$ .]

3.7(iii) follows similarly because if  $|\bar{n}| = |\bar{m}|$  and  $B_{\bar{n}} \wedge bo \rightarrow B_{\bar{m}} \wedge bo$  has filtration 0 or 1, then one of the infinite homotopy towers must be mapped by filtration 0 or 1, and hence the same must be true of  $S^{\bar{n}} \wedge X_5 \rightarrow S^{\bar{m}} \wedge X_5$ . But the above analysis shows this cannot happen unless  $\bar{m}$  is a 0- or 1-successor of  $\bar{n}$ . For example, if  $H^*(X_5^{\langle 1 \rangle}) \rightarrow H^*(X_5)$  sends  $Sq^1 \rightarrow 0$ , then all elements of  $H_*(H^*(X_5^{\langle 1 \rangle}); Q_0)$  are mapped trivially. If  $|\bar{m}| < |\bar{n}|$  any  $\mathcal{Q}$ -homomorphism  $H^*(S^{\bar{m}} \wedge X_5) \rightarrow H^*(S^{\bar{n}} \wedge X_5)$  must be 0, and similarly for the  $Q_0$ -homology elements in  $H^*(S^{\bar{m}} \wedge X_5^{\langle 1 \rangle})$ .

**4. Lifting results.** In this section we prove the lifting part of Theorem 1.1. We will use the maps  $\phi: P_{k+8}^{l+8} \rightarrow P_k^l$  mentioned in the introduction. They were first constructed in [8] and studied in more detail in [10]. Because  $\phi$  is defined only when  $l$  is even, Theorem 1.1 is somewhat more difficult to prove when  $n$  is odd; the case  $n$  odd will be considered in Remark 4.4.

The lifting part of Theorem 1.1 when  $n \equiv 2 \pmod{8}$  and  $e \equiv 0 \pmod{4}$  is an immediate consequence of

**THEOREM 4.1.** *If  $e \equiv 0 \pmod{4}$  and  $e \geq 20$  and  $f$  classifies  $a \cdot 2^{e+6}\xi$  with  $a$  odd, then for some  $d \leq 1 + \lfloor \log_2(e + 2) \rfloor / 2$  the composite*

$$\Sigma^{-1}P_{2e+11+8d}^{4e+10+8d} \xrightarrow{\phi^d} \Sigma^{-1}P_{2e+11}^{4e+10} \xrightarrow{f} SO \rightarrow V_{2e+1}$$

is null-homotopic.

The other cases of Theorem 1.1 when  $n$  is even follow from similar results with slight changes in the parameters. (See Remark 4.3.) The parameters here had to satisfy  $2^e \cdot a2^{e+6}\xi_{4e+10}$  trivial,  $a2^{e+6}\xi_{2e+10}$  trivial, and  $(4e + 10) + 8 < 2(2e + 11) - 1$ .

**PROOF OF 4.1.** Let  $t = e + 6 - \lfloor \log_2(4e + 10) \rfloor$ . By 3.4,  $f$  lifts to  $SO^{\langle t \rangle}$ , and we follow into  $V_{2e+1}^{\langle t \rangle}$ . By 3.3 if  $4e + 10 + 8d \leq 2(2e + 1) + 2t - 4$ , it will suffice to show  $\Sigma^{-1}P_{2e+11+8d}^{4e+10+8d} \xrightarrow{\phi^d} \Sigma^{-1}P_{2e+11}^{4e+10} \xrightarrow{f} P_{2e+1}^{\langle t \rangle} = X$  is null-homotopic. Since  $X = P_{2e+1}^{\langle t \rangle}$  is  $\Omega^\infty$  of a stable Adams resolution, it suffices to show the stable map  $\Sigma^\infty(f' \phi^d)$  is null-homotopic. The condition simplifies to

$$(*) \quad 4d \leq e - \lfloor \log_2(4e + 10) \rfloor.$$

By the Adams edge theorem [1] for  $P_{2e+1}$ ,  $X$  is at least  $(2e - 2 + 2t)$ -connected. If  $X_s$  denotes the  $s$ th stage in the  $bo$ -resolution of  $X$ , then  $X_s$  is at least  $(2e - 2 + 2t + 3s)$ -connected.

We shall return later to the proof of

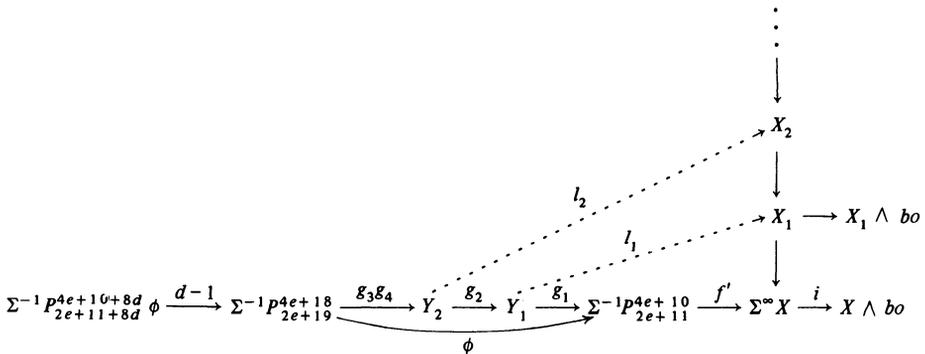
**LEMMA 4.2.** *If  $l$  is even,  $k$  odd, and  $l + 8 < 2k - 1$ ,  $\phi: P_{k+8}^{l+8} \rightarrow P_k^l$  can be written as a composite*

$$P_{k+8}^{l+8} = Y_4 \xrightarrow{g_4} Y_3 \xrightarrow{g_3} Y_2 \xrightarrow{g_2} Y_1 \xrightarrow{g_1} Y_0 = P_k^l$$

such that

- (i)  $H^*(g_i; \mathbb{Z}_2) = 0$  for  $1 \leq i \leq 4$ .
- (ii) For  $0 \leq i \leq 4$  and  $X$  as above, the ASS for  $D_N Y_i \wedge X \wedge bo$  has no nontrivial differentials through dimension  $N$ .

In the diagram



where  $g_1$  and  $g_2$  are as in 4.2, we shall show in the next paragraph that the liftings  $l_1$  and  $l_2$  exist. Then 4.2 and 3.6.1 imply that  $f' \circ \phi^d$  lifts to  $X_{4d}$ , which is at least  $(2e - 2 + 2t + 12d)$ -connected. If  $4d \geq 2[\log_2(4e + 10)] - 1$ , this connectivity is  $\geq 4e + 9 + 8d$ , and so the map  $\Sigma^{-1}P_{2e+11+8d}^{4e+10+8d} \rightarrow X_{4d}$  is trivial, and hence so is  $f' \circ \phi^d$ . Comparing with (\*), such a  $d$  can be chosen if  $4d$  can be chosen in the interval  $[2[\log_2(4e + 10)] - 1, e - [\log_2(4e + 10)]]$ , and thus can be done if  $e > 20$ .

The proof of 4.1 is completed by showing the existence of  $l_1$  and  $l_2$  above.  $l_1$  follows from the observation that any map  $Y_1 \rightarrow X \wedge bo$  which is cohomologically trivial is null-homotopic. [An associated graded for  $[Y_1, X \wedge bo]$  is given by  $\text{Ext}_{\mathbb{Q}_1}^{s,s}(H^*X, H^*Y_1)$ . By the proof below of 4.2,  $DY_1 = (D(\Sigma^{-1}P_{2e+19}^{4e+18}))^{(3)}$  through a sufficiently large skeleton. Thus for  $s > 0$ ,

$$\text{Ext}_{\mathbb{Q}_1}^{s,s}(H^*X, H^*Y_1) \approx \text{Ext}_{\mathbb{Q}_1}^{s+t-3,s+t-3}(H^*P_{2e+1}, H^*(\Sigma^{-1}P_{2e+19}^{4e+18})) = 0$$

by [6, 3.4(iii)].

By 3.7 and 3.9  $X_1 \wedge bo$  splits as  $(X \wedge \Sigma^3bsp) \vee W$ , and by 3.6 (and the argument in 3.6.1), since the map  $Y_2 \rightarrow X_1 \wedge bo$  is cohomologically trivial,  $l_1$  can be varied (if necessary) through  $\Omega(X \wedge bo)$  so that the component into  $W$  is trivial. A proof similar to that of the previous paragraph shows

$$[Y_2, X \wedge \Sigma^3bsp] \xrightarrow{(g_3g_4)^*q^*} [\Sigma^{-1}P_{2e+19}^{4e+18}, P_{2e+1} \wedge \Sigma^3bsp]$$

is injective on cohomologically trivial maps, where  $q$  is the usual map  $X = P_{2e+1}^{(t)} \rightarrow P_{2e+1}$ .  $(g_3g_4)^*q_*(l_1)$  is a map which when followed into  $P_{2e+1} \wedge bJ$  is

$$\Sigma^{-1}P_{2e+19}^{4e+18} \xrightarrow{\phi} \Sigma^{-1}P_{2e+11}^{4e+10} \xrightarrow{f} SO \rightarrow \Omega^\infty(P_{2e+1} \wedge bJ),$$

which is trivial since we have chosen these cases so that the  $bJ$ -primary obstructions considered in §2 are zero. Since

$$[\Sigma^{-1}P_{2e+19}^{4e+18}, P_{2e+1} \wedge \Sigma^3bsp] \rightarrow [\Sigma^{-1}P_{2e+19}^{4e+18}, P_{2e+1} \wedge bJ]$$

is injective above filtration 0, this implies that our map  $Y_2 \rightarrow X_1 \wedge bo$  is trivial.  $\square$

REMARK 4.3. In order to prove the analogue of 4.1 for some of the other congruences it is necessary to use the precise upper edge for  $\text{Ext}_{\mathbb{Q}}(H^*P_k, \mathbf{Z}_2)$  rather than the linear approximation to it used in 3.3 and in the estimate of the connectivity of  $X$  in the proof of 4.1 (see Remark 3.3.1). Let  $D(\alpha, \beta)$  denote the smallest  $j > \alpha$  such that  $\pi_j(P_\alpha)$  has a nonzero homotopy class of filtration  $\beta$ . Then  $D(\alpha, \beta) = \alpha + 2\beta - \varepsilon(\alpha, \beta)$  where

$\varepsilon(\alpha, \beta)$	$\beta \pmod{4}$				
	\	0	1	2	3
	0	1	1	2	3
	1	0	1	2	0
$\alpha \pmod{4}$	2	1	1	1	1
	3	0	0	1	2

If we are trying to show  $\Sigma^{-1}P_{2e+c+8d}^{4e+b+8d} \rightarrow^{\phi^d} \Sigma^{-1}P_{2e+c}^{4e+b} \rightarrow^{2^{e+a}\xi} V_{2e+\delta}^{(t)}$  is null-homotopic, where  $t = e + a - \lfloor \log_2(4e + b) \rfloor$ , then the two conditions on  $d$  are

$$4e + b - 1 + 8d < D(2e + \delta, t) + 2e + \delta - 1,$$

$$D(2e + \delta, t) + 12d > 4e + b - 1 + 8d.$$

The parameters  $a, b,$  and  $c$  are chosen to be the minimal values in the proper congruence satisfying the analogues of the three conditions mentioned after the statement of 4.1.

We tabulate below the values of  $a, b,$  and  $c$  used in each congruence, the value of  $\delta$  used (from 1.1), and the smallest value of  $e$  for which there is a  $d$  satisfying the two inequalities

$e$ mod 4	$n$ mod 8	$a$	$b$	$c$	$\delta$	smallest $e$
0	0	4	8	9	1	24
0	2	6	10	11	1	12
0	4	7	12	13	1	16
0	6	7	14	13	1	16
1	0	6	12	11	3	17
1	2	8	14	15	0	17
1	4	5	8	9	0	17
1	6	9	18	17	3	17
2	0	8	16	15	2	18
2	2	6	10	13	-1	18
2	4	7	12	13	-1	22
2	6	7	14	13	2	18
3	0	6	12	11	1	23
3	2	8	14	15	-1	19
3	4	5	8	11	-1	23
3	6	9	18	17	1	19

The proof of the analogue of 4.1 for  $e = 20, n \equiv 0 (8)$  requires a slight modification. We use  $X = P_{2e+1}^{(t-1)}$  but then can lift to  $X_1$  without using the map  $g_1$  in the factorization of  $\phi$ .

PROOF OF 4.2.  $D_N P_k^l \rightarrow^{D_N \phi} D_N P_{k+8}^{l+8}$  is a map of filtration 4; in fact it is stably of the same type as  $\phi - \Sigma^{N-L-k} P_{L+k-l}^L \rightarrow \Sigma^{N-L-k} P_{L+k-l-8}^{L-8}$  for appropriate  $L$ . Letting  $Z = D_N P_{k+8}^{l+8}, D_N \phi$  can be factored as

$$D_N P_k^l \rightarrow Z^{(4)} \rightarrow Z^{(3)} \rightarrow Z^{(2)} \rightarrow Z^{(1)} \rightarrow Z.$$

If  $W_i$  denotes the  $N$ -skeleton  $(Z^{(i)})^{(N)}$ , then  $D_N \phi$  factors as  $D_N P_k^l \rightarrow W_3 \rightarrow W_2 \rightarrow W_1 \rightarrow Z$  with each map cohomologically trivial. Applying  $D_N$  to this sequence yields the sequence stated in 4.3.

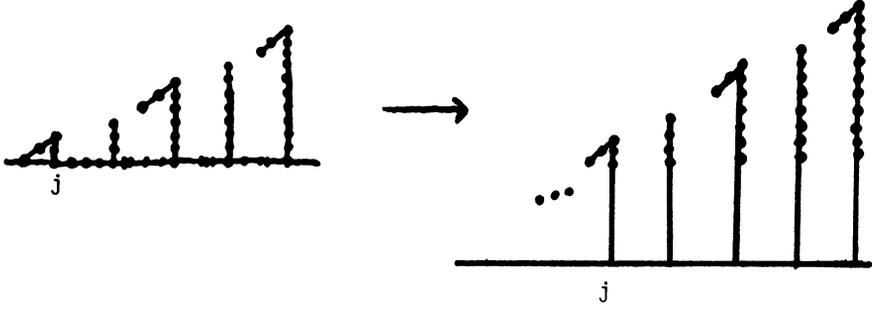
To see that (ii) is satisfied, note that

$$D_N Y_i \wedge X \wedge bo = \left( (\Sigma^{N-L-k} P_{L+k-l-8}^{L-8})^{(i)} \right)^{(N)} \wedge P_{2e+1}^{(t)} \wedge bo = U.$$

For  $s > 0$  and  $* -s \leq N + 2e$ ,

$$\text{Ext}_{\mathcal{Q}}^{s,*}(H^*U, Z_2) \approx \text{Ext}_{\mathcal{Q}_1}^{s+i+t,*+i+t}(H^*(\Sigma^{N-L-k}P_{L+k-l-8}^{L-8} \wedge P_{2e+1}), Z_2).$$

This was calculated in [4, 3.9 and 3.10]. The map in  $\text{Ext}_{\mathcal{Q}}(\ )$  induced by  $U \rightarrow \Sigma^{N-L-k}P_{L+k-l-8}^{L-8} \wedge P_{2e+1} \wedge bo$  is through dimension  $N + 2e$  of the type suggested by the charts



and hence there can be no differentials in  $\text{ASS}(U)$  through dimension  $N + 2e$ .  $\square$

REMARK 4.4. The proof of 1.1 when  $n$  is odd is complicated by the fact that maps  $\phi: P_{k+8}^{n+8} \rightarrow P_k^n$  inducing an isomorphism in  $K$ -theory do not exist when  $n$  is odd. The lifting can be obtained by using the maps  $\phi$  for  $P^{n+1}$ , which is lifted past its  $bJ$ -primary obstruction by varying on the top cell.

Consider for example the case  $e = 4e'$ ,  $n = 8l + 1$ . We must show  $\text{gd}(2^{4l+1-4e'}\xi_{8l+1}) = 8e' + 1$  where

$$\Sigma^{-1}P^{8l+2} \xrightarrow{2^{4l+1-4e'}\xi} P_{8e'+1} \wedge bJ$$

has a nonzero obstruction on the top cell. Let  $X = \underline{P}_{8e'+1}^{\langle l \rangle}$ , where  $t = 4l + 1 - 4e' - [\log_2(8l + 2)] - 2$  (two stages lower than maximal). Then  $f': \Sigma^{-1}P^{8l+2} \rightarrow X$ , a lifting of the map which classifies  $2^{4l+1-4e'}\xi$ , is cohomologically trivial (because  $f'$  lifts to  $\underline{P}_{8e'+1}^{\langle l+1 \rangle}$ ). Thus, by the argument in the next-to-last paragraph of the proof of 4.1,  $\Sigma^{-1}P^{8l+2} \xrightarrow{f'} X \rightarrow X \wedge bo$  is trivial so there is a lifting  $\Sigma^{-1}P^{8l+2} \xrightarrow{l} X_1$ . Since  $f'$  lifts to  $\underline{P}_{8e'+1}^{\langle l+2 \rangle}$ ,  $l$  can be chosen to be 0 in cohomology. As in the last paragraph of the proof of 4.1,  $l$  can be chosen so that when followed into  $X_1 \wedge bo$  only its component  $c$  into  $\underline{P}_{8e'+1}^{\langle l \rangle} \wedge \Sigma^3bsp$  is nonzero. Since the  $bJ$ -primary obstruction for lifting  $2^{4l+1-4e'}\xi_{8l+1}$  to  $8e' + 1$  is zero (by §2),  $c$  is the composite  $\Sigma^{-1}P^{8l+2} \xrightarrow{p} S^{8l+1} \xrightarrow{g} P_{8e'+1} \wedge \Sigma^3bsp$ . Analysis of the upper edge of the Adams charts for

$$\begin{array}{ccc} \pi_*(X_1) & \rightarrow & \pi_*(P_{8e'+1} \wedge \Sigma^3bsp) \\ \downarrow & & \\ \pi_*(\underline{P}_{8e'+1}^{\langle l \rangle}) & \rightarrow & \pi_*(P_{8e'+1} \wedge bo) \end{array}$$

shows  $g$  is the image of a class  $\hat{g} \in \pi_{8l+1}(X_1)$ . Let  $\Delta$  denote the composite  $\Sigma^{-1}P^{8l+2} \xrightarrow{p} S^{8l+1} \xrightarrow{\hat{g}} X_1$ . Then  $l + \Delta$  lifts to  $l_2$  into  $X_2$  and  $l + \Delta|P^{8l+1} = l|P^{8l+1}$ . Now use the method of 4.1 to find  $d$  such that  $\phi^d \circ l_2$  is trivial (for dimensional reasons using the  $bo$ -resolution) and is in the range where triviality

into  $\underline{P}_{8e'+1}^{\langle l \rangle}$  implies triviality into  $\underline{V}_{8e'+1}^{\langle l \rangle}$ . Thus it suffices to show that

$$\Sigma^{-1}P^{8l+1+8d} \rightarrow \Sigma^{-1}P^{8l+2+8d} \xrightarrow{\phi^d} \Sigma^{-1}P_{2m+1}^{8l+2} \xrightarrow{l+\Delta} X_1 \xrightarrow{q} QP_{8e'+1}$$

agrees with  $\Sigma^{-1}P^{8l+1+8d} \rightarrow SO \rightarrow QP_{8e'+1}$  where  $\theta$  classifies a bundle of order  $2^e$ . This follows since  $[\Sigma^{-1}P^{8l+1+8d}, QP_{8e'+1}]$  has no nontrivial elements whose filtration is as large as that of  $\Delta \circ \phi^d$ .

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