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DIFFERENTIABLE GROUP ACTIONS ON HOMOTOPY SPHERES. II: ULTRASEMIFREE ACTIONS

BY

REINHARD SCHULTZ

ABSTRACT. A conceptually simple but very useful class of topological or differentiable transformation groups is given by semifree actions, for which the group acts freely off the fixed point set. In this paper, the slightly more general notion of an ultrasemifree action is introduced, and it is shown that the existing machinery for studying semifree actions on spheres may be adapted to study ultrasemifree actions equally well. Some examples and applications are given to illustrate how ultrasemifree actions (i) may be used to study questions not answerable using semifree actions alone, and (ii) provide examples of unusual smooth group actions on spheres with no semifree counterparts.

Introduction. Over the past fifteen years, numerous papers have been written about group actions with two orbit types–fixed points and free orbits (i.e., *semifree* actions). In particular, smooth actions of this sort on homotopy spheres have been classified in many cases [1], [3], [12], [36], [56], [63] and this work has yielded large classes of new smooth group actions. One purpose of the present paper is to begin an extension of these methods and results to actions that are not semifree. Another purpose is to give some indications of the ways in which such ideas can be used to study further problems about group actions on homotopy spheres.

If a compact Lie group G actions on a closed manifold M^n, a basic theorem on transformation groups implies the number of orbit types is finite [9], [10]. This fact suggests that group actions be considered in terms of the number of distinct orbit types that occur. In a very strong sense, the study of actions with one orbit type reduces to the study of certain types of fiber bundles, and accordingly it translates into problems of ordinary topology [10], [13]. Obviously, the next question in this viewpoint deals with two orbit types, and this has been studied in various ways for some time (see [1], [3], [11], [12], [26], [30], [34], [39], [56], [63], [65], [69], [72], [75], [77], [89] for a representative sampling). Various formal and informal considerations suggest that techniques from the two types case plus some inductive formalism allow treatment of many questions for actions with *linearly ordered* orbit types. One aim of this paper is to follow this suggestion for group actions on homotopy spheres and show that it works.

Of course, extensions of the above sort can be viewed as a routine matter, and thus it is probably desirable (perhaps even necessary) to give further motivation for

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such a study. One reason for interest in semifree actions is that *all* actions of $\mathbf{Z}_p$ (p prime) are semifree ($\{1\}$ and $\mathbf{Z}_p$ are the only subgroups). Since the subgroups of $\mathbf{Z}_{p^r}$ (p prime, $r \geqslant 1$) are linearly ordered by inclusion, the isotropy subgroups of arbitrary $\mathbf{Z}_{p^r}$ actions are also always linearly ordered. To motivate some interest in $\mathbf{Z}_{p^r}$ actions for $r \geqslant 2$, we mention that certain $\mathbf{Z}_{p^r}$ actions on homotopy spheres have no semifree counterparts. Particular examples are (i) topologically linear smooth actions on spheres that are nonzero in oriented smooth $\mathbf{Z}_{2^r}$-bordism [**75**], and (ii) smooth $\mathbf{Z}_{p^r}$ actions on certain exotic spheres that do not support semifree actions (this will be postponed to paper IV in this series).

Such (relatively) unusual actions have very strong implications for a natural question that has been around for some time: Does every exotic n-sphere (say $n \geqslant 7$) admit a smooth effective S^1 action? In fact, classification questions of this sort first arose in our attempts to study this problem. As an illustration of how our machinery applies to such problems. We shall use it to prove that the exotic 8-sphere admits no smooth effective actions of the 3-sphere (circle actions exist [**66**]).

Although we are interested in actions with linearly ordered isotropy subgroups, it is useful to treat a slightly more general class of actions that we call *ultrasemifree*. The precise definition is in §4, the main feature being the existence of a *preferred closed normal subgroup* $H \subseteq G$, properly containing the *principal* isotropy subgroup (which is $\{1\}$), where G *acts freely off the fixed point set of* H. In fact, this feature explains the inductive idea for handling the general linearly ordered case in analogy with the semifree case. In the general case, one uses the fixed set of H *with its induced* G/H*-action* in place of the fixed point set of G (the semifree case corresponds to $H = G$). This once again breaks into a free piece and a second piece with a "smaller" group action (at least there are fewer orbit types), so that a formal induction on the number of orbit types becomes feasible in many cases.

We shall now outline the contents of this paper. The first section discusses the conditions needed for a well-defined connected sum of two G-manifolds. Although this material is basically known (e.g., [**64**], [**78**]), we reformulate it in a manner most convenient for us here. The next two sections (2 and 3) deal with G-vector bundles that admit equivariant fiber homotopy trivializations as a second piece of structure. Since the G-vector bundles we study all have free G actions off the zero section, everything is more or less parallel to the corresponding study without group actions. This is strongly related to the free G-vector bundle theory developed in [**75**, §1].

In §4 we apply the ideas of §§2 and 3 to expand the results of paper I in this series [**72**] to ultrasemifree actions. The basic step is to define an analog of the *knot invariant* in [**72**] for the actions considered. It turns out that the groups of equivariant homotopy classes defined in §§2 and 3 are the correct value groups for these invariants in the nonsemifree case. Once this is done, the main results of [**72**], relating the knot invariant of an action to the ambient sphere's differential structure (Theorems 3.4 and 3.6 in [**72**]) can be proved exactly as in the previous paper. Similarly, the results of [**77**] relating the knot invariant to the isotropy class

representation can be generalized in the present context. A short sequel (§5) gives some consequence of the results of §4 that parallel [77, §4].

There is one extra complication in §4, however; the fixed point set of the preferred subgroup generally has much greater influence on the ambient differential structure. Specific and systematic examples will be given in paper IV of this series.

In §§6 and 7 we extend the methods of Browder, Petrie, and Rothenberg for classifying semifree actions [12], [63], [65] to the ultrasemifree case. Most of the formalism is entirely parallel, and all the extensions fit in naturally with those of §4. Given an ultrasemifree action one defines groups Θ_V^G of homotopy G-spheres, all fixed point sets being homotopy spheres, with an extra condition to dispose of codimension 2 embedding problems. §6 gives an exact sequence involving Θ_V^G that generalizes [69, (1.1)], and §7 gives methods for calculating dim $\Theta_V^G \otimes \mathbf{Q}$ (which is finite modulo low-dimensional problems). The calulations of §7 extend unpublished work by Browder and Petrie in the semifree case, where complete calculations were given for $G = Z_2$, S^1, Pin_2, S^3 and partial calculations were given for $G = \mathbf{Z}_n$ up to the G-signature problems studied in [25], [26] and [70]. I am grateful to T. Petrie for showing me an unpublished manuscript containing tables of their results. Unfortunately, there are many special cases depending on the multiplicities of various irreducible representations in V, and for this reason no precise tabulation of dimensions is given here. However, a determined reader would be able to give such a tabulation by combining the results of §7 with a lot of tedious but elementary arithmetic.

§8 contains some applications of our machinery to group actions on the exotic 8-sphere. In particular, we prove that it supports no effective smooth S^3 actions, and the largest groups that might act are T^2 and SO_3. In [71] it was asserted that T^2 also could not act. However, in revising this manuscript a mistake in the calculations became apparent; it turns out that the result of [71, 3.4] for circle actions is false. We have postponed some of the details needed for §8 in order to clarify the outline of our approach. These details appear in the final §§9 and 10.

One of the most troublesome features of the literature concerning group actions on spheres is that the foundations of the subject have appeared in print either only after unusual delays or else not at all (this paper included–its key ideas date back to early 1974). We have tried to add enough expository material to provide a more or less definitive account of the theory of semifree and ultrasemifree actions. The referee's comments on an earlier version of this paper were a decisive influence in our decision to recapitulate several known ideas that must be generalized (perhaps one should add that several of these ideas have not appeared in print up to this time, although of course they have been well-understood to those who have worked in the area). Thus it is to be hoped that this paper will be of some use as a central source to those wishing to learn about work regarding semifree and ultrasemifree actions on spheres.

Further applications of the machinery and concepts of this paper will appear in subsequent papers. For example, the generalizations of [76] obtained in §4 will be

used to complete the proofs of some assertions made in that paper–namely, certain isotopy classes in $\pi_0(\mathrm{Diff}^+ S^n)$ admit no periodic representatives of arbitrary period. In another direction, the machinery we develop is adequate to give fairly complete information about the set of exotic spheres admitting smooth S^1 actions with codimension 4 fixed point sets (the study of such actions has been fairly extensive; e.g., see **[39]**, **[44]**, **[56]** and forthcoming work of R. Fintushel and P. Pao). More importantly, we shall employ the ideas presented here in papers III and VI of this series to complete the proofs of results announced in **[76]**. For example, this yields a purely homotopy theoretic characterization of those exotic spheres admitting smooth $\mathbf{Z}_p$ actions (p an odd prime) with fixed point sets of a given codimension. As one might expect, the final answer is not in a simple form, but it is precise enough to be effectively computable in any specific case; in particular, for any given dimension n one can get complete information after a finite amount of computation.

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1. Preliminaries concerning orientations. Strictly speaking, the definitions of knot invariants and equivariant connected sums require a more delicate notion of orientation than the usual one. This was previously presented in work of Rothenberg and Sondow **[65]** and independently by M. Sebastiani **[78]**. Since our notation is somewhat different from theirs, we shall summarize here the points that are important to us.

Let G be a compact Lie group, let U be a G-module, let M be a smooth G-manifold (actually, locally smoothable **[10]** would suffice), and let x be a fixed point of M. Then (M, x) is said to be a (*based*) *U-manifold* if the tangent space at x is equivalent to U (as G-modules); we shall use this notion, essentially due to Pulikowski **[60]** (also studied by Kosniowski and Stong), rather than the roughly equivalent notion of (G, u) manifold used in **[65]**. If (M, x) is a U-manifold and F_0 is the component of M^G containing x, we shall say (M, F_0) is a *semibased U-manifold.*

The existence of a group action yields a reduction of the structure group for the bundle $\pi_M|F_0$ (the restricted tangent bundle) to $C(U)$, the orthogonal centralizer. A (*G-equivariant*) *U-orientation* of the semibased U-manifold (M, F_0) is defined to be a further reduction to the identity component $C_0(U)$. As noted in **[65**, especially Proposition 1], a U-orientation of (M, F_0) is the correct extra structure need to construct well-defined equivariant connected sums. If $x \in F_0$, then a U-orientation prescribes a unique homotopy class of linear isomorphisms to the tangent space at each fixed point and this homotopy class in turn prescribes a unique class of

tubular neighborhoods about x (necessarily diffeomorphic to the unit disk $D(U)$); we shall call such tubular neighborhoods *canonical*.

REMARKS. 1. In [**10**, Chapter VI, §8] Bredon defined an equivariant connected sum for $\mathbf{Z}_2$-manifolds with specified orientations for both the manifold and the fixed set. It is an elementary exercise to check that Bredon's pair of orientations will induce an essentially unique U-orientation for suitable U and the connected sum defined in [**10**] coincides with the appropriate U-oriented connected sum.

2. To see that the connected sum may depend very strongly on the choice of U-orientation, consider some actions of $(\mathbf{Z}_2)^s$ on $\mathbf{C}P^n$ by projective collineations. By varying the U-orientation one gets a family of actions on $\mathbf{C}P^n \# \mathbf{C}P^n$ for which each subgroup H has fixed point set components of the form $\mathbf{C}P^k \# \varepsilon_H \mathbf{C}P^k$, where $\varepsilon_H = \pm 1$ depends on H. Appropriate choices of the original action and U-orientations yield a large number of possibilities for the $\{\varepsilon_H\}$.

3. It is well known that the centralizer $C(U)$ is isomorphic to a product of orthogonal unitary, and symplectic groups, and accordingly $\tau_M | M^G$ splits naturally into a sum of real, complex, and quaternionic vector bundles [**79**]. However, the complex summands are slightly ambiguous; at first glance there is no way of choosing between a bundle and its complex conjugate. To remove this ambiguity, we shall assume that for each irreducible representation α with $\dim_R \operatorname{Hom}_G(\alpha, \alpha) = 2$, we have chosen specific isomorphism from $\operatorname{Hom}_G(\alpha, \alpha)$ to the complex numbers. Of course there are always exactly two choices, and one can make uniform families of choices over certain broad families of groups (e.g., subgroups of S^1), but we shall not go into this any further.

4. We should mention that a U-orientation does not necessarily correspond to an orientation of M in the usual sense, although it does induce ordinary orientations on a neighborhood of M^G and on M^G itself. On the other hand, if M and M^G are connected and M is orientable in the usual sense, then a U-orientation of M induces a unique orientation of M in this sense. A similar statement is true for the fixed point set of each subgroup of G.

5. Given a U-orientation of M, there is an opposite U-orientation $-M$ such that $M \amalg -M = (M \times I)$, where $M \times I$ carries the canonical product orientation as a $U \oplus 1 = U \times \mathbf{R}$-manifold.

2. Retraction structures on G-vector bundles. One of the first major steps in studying smooth embeddings of S^k in S^{k+n} for $n > 3$ is a result of Massey [**19**], which states the normal sphere bundle of such an embedding is always fiber homotopically trivial. This fiber homotopy trivialization is in fact canonical, coming from the fact that the composite $S^{n-1} \subseteq S(\nu_f) \subseteq S^{k+n} = f(S^k)$ is a homotopy equivalence. In [**42**] Levine transformed this into an effective tool for classifying all knotted homotopy k-spheres in S^{k+n}, defining an invariant involving ν_f and the canonical fiber homotopy trivialization. Such pairs of vector bundles with fiber homotopy trivializations are classified by elements of the homotopy group $\pi_k(G_n/O_n)$, where G_n is the space of homotopy self-equivalences of S^{n-1}. There is an important analog of this situation for smooth semifree actions on homotopy $(n+k)$-spheres with homotopy k-spheres as fixed point sets; namely, if f denotes

inclusion of the fixed point set, then the vector bundle ν_f is a G-vector bundle, and one can in fact check that the fiber homotopy trivialization is also equivariant. These generalizations are particularly easy because G acts freely on the invariant submanifolds $S^{n-1} \subset S(\nu_f) \subset M^{k+1} - F^k$. Roughly speaking, ν and the resulting G-fiber homotopy retraction $S(\nu) \to S^{n-1}$ correspond to the *knot invariant* of the action studied in [**72**] and elsewhere. This invariant still makes sense for M and F suitable homology spheres (by Smith theory this is the general case) provided one performs some suitable localizations.

Suppose now that we choose to study actions $\mathbf{Z}_{p^2}$ (p prime) with exactly three orbit types. Then one could proceed identically by letting F be the fixed point set of $\mathbf{Z}_p$. In this case we get a G-vector bundle ν over F, such that G acts freely off the zero section (i.e., a *free* G-vector bundle [**75**, §1]), and a G-equivariant retraction from $S(\nu)$ back to the fiber over a fixed point. The purpose of this section is to describe precisely the sets in which such invariants lie. In later sections we shall use these descriptions to give a formal definition of the knot invariant for actions of groups such as $\mathbf{Z}_{p^r}$, and to show that they behave almost exactly like their semifree counterparts.

Throughout this section G will denote a compact Lie group, and all G-spaces will be assumed to lie in the category of G-equivariant cell complexes defined by S. Illman [**32**] and T. Matumoto [**50**]–[**52**]. By the Illman-Matumoto equivariant triangulation theorem this category contains all G-homotopy types N/M, where N is a compact differentiable G-manifold and M is an invariant smooth submanifold (an alternate proof may be given using invariant Morse functions). As noted by Matumoto [**52**], the equivariant CW category satisfies all the axioms needed to prove the representability theorems of E. H. Brown as presented in [**13**], and thus the results of [**13**], [**67**], may be used as needed.

Note. Since the proof of the Illman-Matumoto equivariant triangulation theorem depends formally on results of Cairns that are known to be incorrect (see [**83**, §3] for counterexamples), it is technically incomplete. However, this can be repaired in many ways, the following approach being the first valid one in the sense of historical priority. A result of W. Lellmann [**41**] shows that the orbit space of a smooth action is a stratified space in the sense of Thom and Mather, each strata being a set of orbits with the same type. (*Caution*: The subsequent triangulation theorem in [**41**] also depends on the incorrect result of Cairns.) But there is a result of F. E. A. Johnson [**35**] which says that a stratified space may be triangulated with each stratum a subcomplex. The latter is exactly what is needed in the proof of the triangulation theorem.[1] It should be noted that for G finite, an alternate and more canonical proof of triangulation has been given by S. Illman [**95**].

As mentioned above, the typical sort of object to begin with is a free G-vector bundle on some invariantly pointed G-space (i.e., G acts freely off the zero section and trivially on the basepoint in the base); the fundamental properties of such

[1](Added January, 1981). A proof of a fairly strong triangulation theorem for stratifications has recently appeared in print [**102**].

objects are developed in [75, §1]. If we are interested in adding equivariant fiber retractions, we may formulate the

DEFINITION. Let (X, x_0) be an invariantly pointed G-space (compare [75]), and let M be a free G-module (over R). An *M-pointed free G-vector bundle over X with G-fiber retraction* is a pair (ξ, ρ) where ξ is an M-pointed free vector over X and ρ: $S(\xi) \to S(M)$ is an equivariant map such that $\rho|S(\xi)_{x_0}$ = identity and ρ restricted to each fiber is a (nonequivariant) homotopy equivalence. (*Notation*: $S(\xi)$, $S(M)$ denote the unit sphere bundles for some riemannian metric and $S(\xi)_{x_0}$ is the fiber x_0.) Two such objects (ξ_i, ρ_i) $(i = 0, 1)$ over X are *equivalent* if there is a similar sort of object (η, σ) over $X \times [0, 1]$ and there are G-vector bundle isomorphisms h_i; $\xi_i \to \eta|X \times \{i\}$ such that $\rho_i = \sigma h_i$; this is an equivalence relation, and the equivalence classes form a set denoted by $F/O_{G,\text{free},M}(X, x_0)$. An obvious basepoint for this set is given by $\xi = X \times M$, with ρ = projection onto the second factor. The pullback construction makes these objects into homotopy functors, and it is routine to check that the objects are representable. If X is a trivial G-space, an elementary argument shows that $F/O_{G,\text{free},M}(X, x_0)$ is naturally equivalent to the object $[X, SF_G(M)/SC_G(M)]$ considered in [72, Part I]. The Whitney sum induces well-behaved natural transformations

$$F/O_{G,\text{free},M_0} \times F/O_{G,\text{free},M_1} \to F/O_{G,\text{free},M_0 \oplus M_1}$$

for each pair (M_0, M_1) of free G-modules, and one can take direct limits in the usual way (compare [75]) to form the corresponding stable functor $F/O_{G,\text{free}}$. This functor also satisfies Brown's representability criteria; furthermore, if X is a trivial G-space then $F/O_{G,\text{free}}(X)$ reduces to $[X, SF_G/SC_G]$ in a manner consistent with stabilization. Finally $F/O_{G,\text{free}}$ has a functorial abelian monoid structure induced by the Whitney sum.

Next suppose that $G = \mathbf{Z}_p$, (p prime), and consider free- G-vector bundles together with *equivariant localized fiber retractions* ρ: $S(\xi) \to S(M)_{(p)}$; in other words, $S(M)_{(p)}$ is an equivariant localization of $S(M)$ at p (with the convention of [72, §2] if G does not act orientation-preservingly), and $\rho|S(\xi)_{x_0}$ is (homotopically) the equivariant localization map. We also assume that ρ restricted to each fiber is localization. In this case the appropriate homotopy functor will be called $F_{(p)}/O_{G,M,\text{free}}$, and the corresponding stable version will be denoted by $F_{(p)}/O_{G,\text{free}}$; in analogy with the unlocalized case, these functors when applied to a trivial G-space X yield $[X, SF_G(M)/SC_G(M)]$ and $[X, SF_{G(p)}/SC_G]$, respectively. Compositions with the localization maps $S(M) \to S(M)_{(p)}$ induce a natural transformation from F/O-objects to the corresponding $F_{(p)}/O$-objects that commute with Whitney sums; if X is a trivial G-space, such transformations are given by the maps $SF_G(M)/SC_G(M) \to SF_G(M)_{(p)}/SC_G(M)$ and $SF_G/SC_G \to SF_{G(p)}/SC_G$ induced by localization of $BSF_G(M)$ and BSF_G at p.

Of course, for ordinary homotopy functors defined on pointed cell complexes it is generally desirable to know what they yield when applied to spheres (the basic objects needed to attach cells). If we consider equivariant cell complexes, the corresponding basic complexes have forms such as $G/H \times S^n$ or $G/H \times S^n/(G/H \times \{\text{pt}\})$, where G acts trivially on S^n and H is a closed subgroup of G

(compare [32], [50], [51]). For the functors $F/O_{G,M,\text{free}}$, $F/O_{G,\text{free}}$ $\cdots$ constructed above, the previously mentioned descriptions for trivial G-spaces may be combined with the results below to evaluate the functors on basic G-complexes.

Notation. If Y is a G-space, then Y^+ denotes the (equivariant) disjoint union of Y with a point where G acts trivially on the point and in the original fashion on Y.

PROPOSITION 2.1. *Let $T_{G,\text{free}}$ denote any of the functors $F/O_{G,M,\text{free}}$, $F/O_{G,\text{free}}$, $F_{(p)}/O_{G,M,\text{free}}$, or $F_{(p)}/O_{G,\text{free}}$ described above, let H be a closed subgroup of G, and let X be a G-space (with possibly trivial G-action).*

(i) *There is a natural isomorphism*

$$T_{G,\text{free}}(G/H^+ \wedge X) \cong T_{H,\text{free}}(X)$$

given by taking the induced object over the H-invariant subspace

$$H/H \times X \subseteq G/H \times X/(G/H \times \{\text{pt}\}) = G/H^+ \wedge X.$$

(ii) *Similarly, there is a natural isomorphism*

$$T_{G,\text{free}}((G/H \times X)^+) \cong T_{H,\text{free}}(X^+)$$

given by restricting to the induced H-object over

$$X^+ = (H/H \times X)^+ \subseteq (G/H \times X)^+ .$$

The above isomorphisms and certain other natural maps considered above fit into various commutative diagrams; in particular, these isomorphisms commute with stabilization, localization, and Whitney sums. In each case the verification is an elementary exercise.

PROOF. (i) The inverse is given by a modified balanced product construction (compare [79]). For example, given the H-object (ξ, ρ), with ξ: $E \to X$ and ρ: $S(\xi) \to S(M|H)$, take the balanced product bundle $G \times_H E$ over $G \times_H X = G/H \times X$ together with the map

$$\rho_0\colon S(G \times_H E) \xrightarrow{G \times_H \rho} G \times_H S(M|H) \xrightarrow{\alpha} S(M)_{(p)},$$

where α is defined using the G-action on $S(M)$. Transform $G \times_H E$ into a G-vector bundle over $G/H^+ \wedge X$ by identifying elements in $G \times_H E_{x_0} \cong G \times_H S(M|H)$ via the G-action on M_0. If this is done, the map ρ_0 will pass to a fiber retraction on the new bundle. It is an easy exercise to verify that this construction is inverse to the one given in the theorem. Modifications of this argument yield the result for all the functors $T_{G,\text{free}}$ under consideration.

(ii) This is essentially a "basepoint free" version of (i), and its proof is similar but somewhat easier (compare [75, §1]). ■

3. Generalizations of the sequence $F \to F/O \to BO$. One basic property of the homotopy functor $[\ \ldots, F/O]$ on ordinary complexes is the existence of an exact sequence relating it to stable cohomotopy and reduced real K-theory; this of course follows from the fibration sequence $F/O \to BO \to BF$, but it also may be established by direct geometrical arguments. The latter is perhaps the most tangible method for generalizing the exact sequence to the equivariant setting.

To define a substitute for the homotopy functor induced by F, proceed as follows: Given a free G-module M and a pointed G-space, define $F_G(X; M)$ to be all G-homotopy classes of maps $X \times S(M) \to S(M)$ rel$\{x_0\} \times S(M)$ whose restriction to each $\{x'\} \times S(M)$ has degree $+1$. This is a homotopy functor, and passage to the limit over all free G-modules yields a representable abelian group valued functor $F_{G,\text{free}}(X, x_0)$.

Notational convention. If T is a homotopy functor on a category of (invariantly) pointed (G)-spaces, then $T^{-i}(X)$ denotes $T(S^i(X))$ (= ith reduced suspension) for $i < 0$.

THEOREM 3.1. *Suppose X is a finite G-equivariant cell complex. Then the following sequence is exact and functorial in X (assuming $i < 0$):*

$$\begin{aligned} &\cdots \to KO^{i-1}_{G,\text{free}}(X) \to F^i_{G,\text{free}}(X) \to F/O^i_{G,\text{free}}(X) \to KO^i_{G,\text{free}}(X) \\ &\to F^{i+1}_{G,\text{free}}(X) \to \cdots \to F_{G,\text{free}}(X) \to F/O_{G,\text{free}}(X) \to KO_{G,\text{free}}(X). \end{aligned} \tag{3.2}$$

The functor $KO_{G,\text{free}}$ is defined as in [75, §1]. *Furthermore, there is a similar exact sequence* $(3.2)_p$ *when $G = \mathbf{Z}_{p^r}$, in which $[F/O]^*$ and F^* are replaced by $[F_{(p)}/O]^*$ and $F^* \otimes \mathbf{Z}_{(p)}$, respectively. Finally, there is a map from* (3.2) *to* $(3.2)_p$ *in which $[F/O]^*$ maps to $[F_{(p)}/O]^*$ by fiber localization, KO^*_G to itself by the identity, and F^* to $F^* \otimes \mathbf{Z}_{(p)}$ by algebraic localization.*

As in the nonequivariant case, the map $KO^{*-1} \to F^*$ is an equivariant J-homomorphism, the map $F^* \to [F/O]^*$ is given by viewing maps $X \times S(M) \to S(M)$ as fiber retractions for trivial bundles and the map $[F/O]^* \to KO^*$ specifies the underlying vector bundle.

PROOF. It follows from [75, Theorem 1.7] that elements of $KO_{G,\text{free}}(SY)$ correspond via clutching functions to stabilized homotopy classes of vector bundle automorphisms of trivial bundles $Y \times M$, where M is some free G-module (recall we are using *reduced* suspensions). Furthermore, elements of $F/O_{G,\text{free}}(SY)$ correspond to clutching functions together with extensions of the induced maps $Y \times S(M) \to S(M)$ to Cone$(Y) \to S(M)$. Thanks to these identifications the proof of (3.2) becomes a sequence of routine verifications.

Derivation of $(3.2)_{(p)}$ requires the use of a new functor $F_{(p)G,\text{free}}$ defined as follows: Let $T^{(0)} = F_{G,\text{free}}$, and let $T^{(k)}(X)$ denote stabilized equivariant homotopy classes of maps $X \times S(M) \to S(M)$ rel f_k: $\{x_0\} \times S(M) \to S(M)$, where f_k is equivariant,

$$\deg f_k = \prod_{i=0}^{k} (i|G| + 1)$$

(recall $|G| = p^r$), and restriction to each $\{x'\} \times S(M)$ has the same degree. Then $T^{(k)}$ is again a representable abelian group valued functor, and an equivariant map $S(M) \to S(M)$ of degree $(k+1)|G| + 1$ (there is a unique one up to equivariant homotopy) induces a natural homomorphism $T^{(k)} \to T^{(k+1)}$. Since there is *only* one equivariant map of given degree up to equivariant homotopy, the latter homomorphism is uniquely defined. Define $F_{(p)G,\text{free}}$ to be inj $\lim_K T^{(k)}$. Then $F_{(p)G,\text{free}}$ is

abelian monoid (in fact, group) valued via direct sum, and the natural map from $F_{G,\text{free}}$ is a homomorphism.

It follows that an exact sequence of type $(3.2)_{(p)}$ exists with $F^*_{(p)}$ in place of $F^* \otimes \mathbf{Z}_{(p)}$; furthermore, there is a canonical map of (3.2) into this sequence. Thus it suffices to prove that the two natural transformations

(i) $F^*_{(p)} \to F^*_{(p)} \otimes \mathbf{Z}_{(p)}$,

(ii) $F^* \otimes Z_{(p)} \to F^*_{(p)} \otimes \mathbf{Z}_{(p)}$

are isomorphisms; corresponding results in the nonequivariant case have been known for some time (e.g., see **[86]**).

First consider the case $X = G/H^+ \wedge S^n$ or $(G/H \times S^n)^+$. It is easy to check that all functors considered satisfy the properties described in Proposition 2.1; hence it suffices to prove

$$F^*_{(p)H,\text{free}}(X) \to F^*_{(p)H,\text{free}}(X) \otimes \mathbf{Z}_{(p)},$$
$$F^*_{H,\text{free}}(X) \otimes \mathbf{Z}_{(p)} \to F^*_{(p)H,\text{free}}\ (X) \otimes \mathbf{Z}_{(p)}$$

are isomorphisms when $H \subseteq G$ and $X = S^n$ or S^{n+} has a trivial H-action. For such spaces $F_{H,\text{free}}(X) \cong [X, SF_H]$, where SF_H is the identity component of the spaces studied in [7]; also,

$$F_{(p)H,\text{free}}(X) = \operatorname{inj}\lim_k \left[X, SF_H^{(k)}\right],$$

where $SF_H^{(k)}$ is the space of stable equivariant self-maps of degree $\prod_{i=1}^k (i|H| + 1)$. Thus the problem reduces to considering the canonical homomorphism

$$\pi_*(SF_H) \to \operatorname{inj}\lim_k \pi_*\left(SF_H^{(k)}\right).$$

The results of [7] show that $\pi_n(SF_H) \cong \pi_n(SF_H(M_0))$ for any free H-module M_0 of real dimension $> n + 1$, and a similar argument works for $SF_H^{(k)}$; consequently, we shall study the unstable case. But here there are spectral sequences

$$E^2_{i,j} = H^{-i}\left(S(M_0)/H;\ \pi_j(S(M_0))\right) \Rightarrow \pi_{i+1}\left(SF_H^{(k)}(M_0)\right)$$

given by **[67**, §2], and from the naturality properties of these spectral sequences it is immediate that the homomorphism induced from $SF_H^{(k)}$ to $SF_H^{(k+1)}$ by a map of degree $d_{k+1} = (k + 1)|H| + 1$ corresponds on E^2 to the coefficient endomorphism of $\pi_j(S(M_0))$ induced by a map of degree d_{k+1}. Therefore, if we take limits over maps of degree d_{k+1} for all k, we obtain a new spectral sequence

$$H^{-i}\left(S(M_0)/H; \operatorname{inj}\lim_k (d_{k+1})_*\right) \Rightarrow \operatorname{inj}\lim \pi_{i+j}\left(SF_H^{(k)}(M_0)\right).$$

But $\operatorname{inj}\lim (d_{k+1})_* \cong \pi_*(S(M_0)) \otimes \mathbf{Z}_{(p)}$ (compare the localization construction in §2), and from this it is clear that the E^∞ term of the spectral sequence is naturally a $\mathbf{Z}_{(p)}$-module, proving (i). On the other hand, it is also clear that the spectral sequence is naturally equivariant to the localization of one for $SF_H(M_0)$ at p, and (ii) readily follows from this.

Finally, the proof for arbitrary finite equivariant cell complexes follows from the familiar technique of induction on the number of (equivariant) cells combined with the (equivariant) Puppe sequence and the five lemma. ■

REMARK. The fundamental theorems of infinite loop space theory imply that (3.2) and its local analogs extend infinitely far to the right [99], [101]. Forthcoming results (in [100]) on equivariant infinite loop space theory will imply that (3.2) and related sequences also extend infinitely far to the right; however, we do not need this information here.

COROLLARY 3.3. *The functors $F/O_{G,\text{free}}$ and $F_{(p)}/O_{G,\text{free}}$ are abelian group-valued, and the natural map $F/O_G^* \to F_{(p)}/O_G^*$ is an isomorphism when localized at p (for finite equivariant cell complexes).*

PROOF. In dimensions $\leqslant -1$ this follows upon localizing the exact sequence map at p and applying the five lemma; however, the proof for dimension zero requires a more direct inspection. By the methods used in Theorem 3.1, it suffices to prove the result for $T^*_{H,\text{free}}(X)$, where $X = S^n$ or S^{n+} with trivial H-action and $T = F/O$ or $F_{(p)}/O$. But in this case $F/O_{H,\text{free}}(X) = \pi_n(SF_{H(p)}/SC_H)$, $F_{(p)}/O_{H,\text{free}}(X) = \pi_n(SF_{H(p)}/SC_H)$, and the localization of the canonical map at p is an isomorphism by the observation of [71, §2]. ■

To complete the analogy between Theorem 3.1 and the nonequivariant case, it is necessary to give a stable homotopy-theoretic interpretation of F_G^*. We shall do this using the methods of [8]; the results will be needed in §§5 and 8.

LEMMA 3.4. *If X is a finite pointed equivariant CW complex and* $\dim V \gg \dim X$, *then G-homotopy classes of maps*

(a) $$X \times S(V) \to S(V)[\operatorname{rel}\{x_0\} \times S(V) = S(V)]$$

are in one-one correspondence with elements of the cohomotopy group

(e) $$[\operatorname{Th}(\pi^* \nu_M - \pi^* \zeta)/\operatorname{Th}(\operatorname{same}|\{x_0\} \times M(V)), S^N].$$

Here π: $X \times_G S(V) \to M(V)$ is reduced projection and ζ is defined as in [7]; *$M(V)$ denotes $S(V)/G$, and* Th *denotes the Thom space of a vector bundle. If X is a compact smooth G-manifold, then these sets also correspond to the stable homotopy group*

(f) $$\{S^0, \operatorname{Th}(\nu_{\text{fib}} \oplus \zeta)/\operatorname{Th}(\zeta)\},$$

where ν_{fib} is the bundle of normals along the fiber for $X \to X \times_G (V) \to M(V)$.

PROOF (SKETCH). Since equivariant maps from a free G-space E into another G-space X correspond bijectively to sections of the associated bundle $F \times_G X \to E/G$ (compare [7]), the homotopy classes of maps (a) are in 1-1 correspondence with isotopy classes of cross sections of the bundle

(b) $$X \times S(V) \times_G S(V) \to X \times_G S(V) \operatorname{rel}\{x_0\} \times M(V).$$

This bundle has a canonical cross section corresponding to $(x, v) \to (x, v, v)$; but sectins of a bundle $E \to B$ with based cross sections are the same as fiberwise basepoint preserving maps $B \times S^0 \to E$ (distinguished section on one copy, the remaining one on the other). Hence the isotopy classes of sections in (b) are merely homotopy classes of ex-maps [7]

(c) $$X \times_G S(V) \times S^0 \to \pi^*(\tau_M \oplus \zeta) \operatorname{rel}\{x_0\} \times M(V) \times S^0.$$

Since $\dim X \ll \dim V$, as in [7] we have an ex-space version of the Freudenthal theorem. Thus if we perform fiberwise suspension by the bundle $\pi^*(S(\nu_M - \zeta))$, we get that the homotopy classes in (c) correspond bijectively to the classes of ex-maps

(d)

$$\pi^*(S(\nu_M - \zeta)) \to X \times_G S(V) \times S^N \text{ rel the inverse image of } \{x_0\} \times M(V).$$

But here the projection maps onto the first factor are always the standard projection, and hence the projections onto S^N carries all the significant information. In addition, since the projection is held constant over the inverse image of $\{x_0\} \times M(V)$, one gets that (d) corresponds to homotopy classes of maps from $\pi^* S(\nu_M - \zeta)$ mod the inverse image of $\{x_0\} \times M(V)$ into S^N. But this set of homotopy classes is just (e).

Now assume that X is a compact smooth manifold. Notice that $\pi|\{x_0\} \times M(V) =$ identity implies $\mathrm{Th}(\nu_M - \zeta)$ is a retract of $\mathrm{Th}(\pi^*\nu_M - \pi^*\zeta)$. Taking S-duals, we see that $\mathrm{Th}(\zeta)$ is a retract of $\mathrm{Th}(\nu_{\mathrm{fib}} \oplus \pi^*\zeta)$, where ν_{fib} is the bundle of normals along the fibers for the smooth bundle $X \to X \times_G S(V) \to M(V)$, defined as in [8] with virtual $\dim \nu = -\dim X$. Thus by S-duality we have a 1-1 correspondence between elements of (a) and (f). ■

If X has a trivial G-action, then $\nu_{\mathrm{fib}} \oplus \zeta \cong \nu_X \times \zeta$ as a bundle over $X \times_G S(V) \cong X \to M(V)$, and it follows immediately from S-duality that the results of [7] are essentially special cases of the correspondence (a) $\leftrightarrow$ (f). Of course, (f) is likely to be quite complicated in general. Subsequently our main interest will lie in the special case $X = S^W$ for some orientable G-module W, and in this case (f) takes a reasonably simple form:

PROPOSITION 3.5. *If $n + \dim W < \dim M(V)$, then $F^{-n}_{G,\mathrm{free}}(S^W)$ is isomorphic to the stable homotopy group $\pi_{n+\dim W}(M(V)^{\zeta+\dim W - W})$, where $-W$ is an inverse vector bundle to $S(V) \times_G W$ with virtual dimension* $-\dim W$.

We use the Atiyah notation [4] of M^α for the Thom complex $\mathrm{Th}(\alpha)$ here in order to emphasize the analogy with [7], which of course treats the case where W is a trivial G-module.

PROOF. The restriction of $\nu_{\mathrm{fib}} \oplus \zeta$ to $\{x_0\} \times M(V)$ is stably isomorphic to $\zeta - W$ because the tangent bundle along the fibers is clearly stably equivariant to $S^W \times S(V) \times_G W$. Thus there is an inclusion of $M(V)^{\zeta - W}$ in $Y = S^W \times_G S(V)^{\nu(\mathrm{fib})\oplus\zeta}$, and it suffices to show that its composite with the map collapsing of $M(V)^\zeta$ to a point induces homology isomorphisms.

Since the (orientable) sphere bundle $S^W \times_G S(V) \to M(V)$ has a cross section, its total space has the integral homology of $S^{\dim W} \times M(V)$, and thus by the Thom isomorphism theorem

$$H_*(Y) \cong H_{*+\dim G - \dim W}(M(V)) \oplus H_{*+\dim G}(M(V)).$$

By S-duality it is clear that the map $Y \to M(V)^\zeta$ corresponds to projection onto the second factor, and by naturality of the Thom isomorphism it is clear that the inclusion $M(V)^{\zeta - W}$ corresponds to injection into the first factor. Thus the composite $M(V)^{\zeta - W} \to Y \to Y/M(V)^\zeta$ is an isomorphism in homology, and consequently the proposition follows from Whitehead's theorem. ■

COMPLEMENT TO PROPOSITION 3.5. *Under the above isomorphisms the forgetful map* $F^*_{G,\text{free}}(S^W) \to F^*_{H,\text{free}}(S^W)$ *(where* $H \subseteq G$*) is induced by an umkehr map* $M(V_G)^{\zeta_G + \dim W - W} \to M(V_H)^{\zeta_H + \dim W - W}$ *analogous to* [7, (4.8), *p*. 7]. ■

The above description of $F_{G,\text{free}}(X)$ is a generalization of the nonequivariant isomorphism $[X, F] \simeq \{X, S^0\}$. There is another generalization of this isomorphism using equivariant stable homotopy theory; following **[21]**, **[22]** and **[28]**, define $\pi^*_{G,\text{free}}(X)$ to be the *free* equivariant stable cohomotopy of X, where "free" signified that the stabilization is done over all free G-modules. As in the nonequivariant cases, the Hopf construction defines a natural map $F^*_{G,\text{free}}(X) \to \pi^*_{G,\text{free}}(X)$ that is an additive bijection if X is an equivariant suspension; further remarks and results in this direction may be found in **[72]**.

4. Knot invariants for ultrasemifree actions. We begin by defining a class of smooth actions that contains all semifree and all $\mathbf{Z}_{p^r}$ actions (p prime).

DEFINITION. An effective action of a group G on a space X is *ultrasemifree* if there is a (closed) normal subgroup $H \neq 1$ with the following properties:

(i) There are no isotropy subgroups L satisfying $1 \subsetneq L \subsetneq H$.

(ii) Every isotropy subgroup L except (perhaps) the identity contains H.

It follows from (i)–(ii) and effectiveness that $\{1\}$ is the minimal isotropy subgroup of the action, and G acts freely off the fixed point set of H.

If G acts semifreely, then the above conditions hold for $H = G$. On the other hand, if $G = \mathbf{Z}_{p^r}$ then $H =$ minimal nontrivial isotropy subgroup satisfies the desired conditions (a unique minimum exists because the finite set of subgroups is linearly ordered). Clearly many S^1 actions are ultrasemifree (e.g., if the isotropy subgroups are a finite set, linearly ordered by inclusion), as are effective S^3 actions with isotropy subgroups $\{\{1\}, S^1, S^3\}$ (e.g., trivial rep.$\oplus$copies of 3-dimensional rep.$\oplus$copies of 4-dimensional rep.).

Further examples of ultrasemifree actions are given by generalized quaternion groups with H being the unique subgroup of order 2. Of course, a similar statement is true for all p-groups of order p, but cyclic and generalized quaterionion groups are well known to exhaust all the possibilities **[90**, pp. 161–162]. On the other hand, actions of Pin_2 (= normalizer of S^1 in S^3) with all finite isotropy subgroups 2-primary also are ultrasemifree.

We shall be particularly interested in smooth ultrasemifree actions of $G = \mathbf{Z}_{p^r}$, S^1, or S^3 on $\mathbf{Z}_p$-homology spheres (the usual conventions for S^1 and S^3); such an action will be called *special* if:

(i) The fixed point set of G is nonempty.

(ii) If $G = S^1$ or S^3, the fixed point set of H is an integral homology sphere.

Notation. The subgroup H appearing above will be called the *subprincipal isotropy bound* of the action.

Let G act on the homology sphere Σ^{n+2k} by a special ultrasemifree action, and let M^n be the fixed point set of H; assume $k \geqslant 2$ and $M \neq \varnothing$. As in **[72**, §2] the G-invariant submanifold M has an equivariant normal bundle ξ^{2k}, the restriction of the G-action to $\Sigma - M$ is free, and the inclusion of an invariant fiber of ξ^{2k} (over a fixed point of G) into $\Sigma - M$ is a homotopy equivalence of G-spaces when localized at p (by convention unlocalized for S^1 or S^3). Thus the sphere bundle

$S(\xi)$ admits an equivariant map into $S_{(p)}^{2k-1}$ whose restriction to S^{2k-1} is merely (equivariant) localization. In the notation of §§2 and 3, this corresponds to an element $F_{(p)}/O_{G,V,\text{free}}(M^n, \text{pt})$, where V denotes the fiber of the normal bundle over the base point. As in the semifree case, this element is well defined (compare [**72**, §2], and it is called the *primary* (or *first order*) knot invariant of the action.

REMARKS. 1. The isomorphism type of V often does not depend on which fixed point is chosen to be the basepoint. This is immediate if the fixed point set of G is connected and follows from the methods of [**5**], Smith theory, and induction in the disconnected case (the fixed point set is S^0 if it is disconnected), provided $p \neq 2$ or the representation is semifree. If $p = 2$ the situation is generally different as noted in [**15**], but under many circumstances one can still deduce that the representations agree using techniques from [**10**] or [**64**]. Work of Cappell and Shaneson [**18**] shows that the existence of inequivalent representations is linked to the existence of distinct but topologically conjugate representations. In fact, their work suggests that inequivalent representations at isolated fixed points are always topologically conjugate (however, by [**10**] the converse is certainly false); for further information, see the remarks following the derivation of (6.2).

2. If the isotropy subgroups are linearly ordered, say $1 = H_0 \subseteq \cdots \subseteq H_r = G$, and M_i = fixed point set of H_i, then it is natural to define the *ith order knot invariant* to be the primary knot invariant associated to the induced G/H_i-action on M_i. (Of course, this assumes that each M_i is also a $\mathbf{Z}_p$-homology sphere, but the latter must hold if $G = \mathbf{Z}_p$.)

3. Suppose l is empty or some set of primes not containing p, and define $F_l/O_{G,V,\text{free}}$ in analogy with F/O and $F_{(p)}/O$ except that the retraction p: $S(\xi) \to S(V)$ has $\deg \rho|S(V)$ a monomial in elements of l. Then for actions on a $\mathbf{Z}$-homology sphere, a knot invariant is definable in $F_l/O_{G,V,\text{free}}(M)$. A particularly important class of cases with $l = \varnothing$ will be discussed further in §6. ■

Many basic properties of the knot invariants for semifree actions also hold in the ultrasemifree case. For example, the proof of the following result is a word-for-word adaptation of the proof of [**72**, (2.1)].

PROPOSITION 4.1. *Suppose* (Σ, Φ) *and* (Σ', Φ') *are ultrasemifree actions with the same subprincipal isotropy bounds and local representations. Let* K, K' *be the fixed sets of the subprincipal subgroup, and let* f: $K \sharp K' \to K \vee K'$ *be the canonical collapse. Then* $\omega_0(\Phi \sharp \Phi') = f^*(\omega_0(\Phi), \omega_0(\Phi'))$, *where* $\sharp$ *denotes equivariant connected sum.*

In the semifree case, by means of localization theory we were able to transform the knot invariant $F_{(p)}/O_{G,V,\text{free}}(M^n, *)$ into a more tractable, but still useful, invariant in $F/O_{G,\text{free}}(S^n) \otimes \mathbf{Z}_{(p)}$. We need something similar here with S^W replacing S^n. This requires a suitable version of equivariant localization theory; many folk theorems have existed in this area for some time, and everything we need will appear in forthcoming work [**53**]. The functor $F/O_{G,V,\text{free}}$ and its $F_{(p)}/O$-counterpart are representable, as noted before, and they are equivariantly 1-connected (e.g., if $H \subseteq G$, then all $F/O_{H,V,\text{free}}(S^1)$ are trivial). It then follows that one can

construct equivariant p-localizations of these functors, and these localizations $\mathfrak{F} \to \mathfrak{F}_{(p)}$ satisfy familiar properties such as the following:

(4.2) $\mathfrak{F}_{(p)}(S^2X) = \mathfrak{F}(S^2X)_{(p)}$, *the abelian group structure being given by the double suspension.*

(4.3) *If* $f\colon M \to N$ *induces* $\mathbf{Z}_{(p)}$*-homology isomorphisms on all fixed point sets, then* $f^*\colon \mathfrak{F}_{(p)}(N) \to \mathfrak{F}_{(p)}(M)$ *is an isomorphism.*

The first step in our geometric program is to define a map

$$q\theta\colon F_{(p)}/O_{G,V,\mathrm{free}}(M) \to [S(V) \times_G M, F/O]_{(p)} \tag{4.4}$$

by first taking the homology equivalence $S(\nu_{M,\Sigma})/G \to S(V) \times_G M$ induced by the vector bundle and its fiber homotopy trivialization, *then* taking the p-localized normal invariant of the induced homology equivalence as in [**71**, §1]. We may sharpen (4.4) into a map $q_1\theta$ with codomain $[S(V)^+ \wedge_G M, F/O]_{(p)}$ because of the following:

(i) The restriction of $q\theta(\omega_0)$ to $[S(V) \times_G \{m_0\}, F/O]_{(p)} \cong [S(V)/G, F/O]_{(p)}$ is trivial (near $S(V) \times_G \{m_0\}$ the homology equivalence is just a standard self-map $S(V)/G \to S(V)/G$ of positive degree).

(ii) The functor $[S(V) \times_G M, F/O]_{(p)}$ splits naturally as $[S(V) \times_G M, F/O]_{(p)} \oplus [S(V)/G, F/O]_{(p)}$ because $S(V)/G \subseteq S(V) \times_G M$ is a retract (projection onto the first factor yields the obvious one-sided inverse).

We shall find it convenient to restrict attention to the transformation $q_1\theta$ obtained by taking the first coordinate of $q\theta$ with respect to this splitting. Its advantage is that its codomain is naturally a functor on *pointed* G-spaces.

Propositions 2.6 and 2.7 of [**72**] now also generalize word for word.

PROPOSITION 4.5. *The map* $q_1\theta(\omega_0)$ *only depends on the stabilization of* ω_0 *in* $F_{(p)}/O_{G,\mathrm{free}}(M)$. ■

The properties of equivariant localization are also similar enough to those of ordinary localization that one has the following generalization of a result in [**72**, §2]:

PROPOSITION 4.6. *The map*

$$q_1\theta\colon F_{(p)}/O_{G,V,\mathrm{free}}(M) \to [S(V)^+ \wedge_G M, F/O]_{(p)}$$

factors canonically through the localization $\mathfrak{F}(M) \to \mathfrak{F}_{(p)}(M)$, *where* $\mathfrak{F} = F_{(p)}/O_{G,V,\mathrm{free}}$.

The proof is a straightforward adaptation of [**72**], the key changes being: (i) our equivariant localizations all have the appropriate universal mapping properties for maps into localized spaces; (ii) every finite equivariant CW complex has the homotopy type of a compact differentiable G-manifold (one can in fact do this with a handle decomposition mimicking the given cell decomposition). ■

We can now draw a similar conclusion to that drawn in [**72**, Moral to 2.4 and 2.5].

PROPOSITION 4.7. *Given Σ, M, etc. as before, let ω_0 be the knot invariant of the action. Then there is a class $\omega' \in F/O_{G,\mathrm{free},V\otimes}(S^W)$ such that the image of $q_1\theta(\omega')$ in $[S^W \wedge_G S(V)^+, F/O]_{(p)}$ corresponds to that of $q_1\theta(\omega_0)$ under the isomorphism*

$$(1 \wedge_G \kappa)^*[S^W \wedge_G S(V)^+, F/O]_{(p)} \to [M^* \wedge_G S(V)^+, F/O]_{(p)}. \quad \blacksquare$$

The whole pattern of generalizations from [72] in fact carries through to the central results of [72, §3].

THEOREM 4.8. *Let G act specially ultrasemifreely on the homotopy sphere Σ with subprincipal isotropy bound H, and let F be the fixed point set of H. Assume that the local representation at a fixed point has the form $\alpha + V$, where α is the fixed set of H and G acts freely on V. Let ω_0 be the knot invariant, let κ: $F \to S^\alpha$ be the equivariant collapse map, and let $\theta(\omega_0)$ be the homology equivalence obtained from the knot invariant as in* (4.4) *and* [72, *Proposition* 2.4, *p.* 111]. *Then the composite*

$$(*) \qquad (1 \times_G \kappa)\theta(\omega_0)\colon S(\xi)/G \to S(V) \times_G S^\alpha$$

has trivial normal invariant. Furthermore, if M is a free G-module of dimension 2 *($G \neq S^3$) or* 4 *($G = S^3$) the composite*

$$(**) \qquad q\theta[\omega_0 \oplus M] + q[1 \times_G \kappa]$$

has normal invariant $(1 \times_G \kappa)^* c^*(-q(\Sigma) \oplus \gamma')$, where $\gamma' \in \pi_{|\alpha|+\dim V+1}(F/O)_{(p)}$ is some undetermined element if G is finite and 0 *if G is infinite, and the map c: $[S^\alpha \times_G L(V \oplus W)/S^\alpha \times_G L(V)] \simeq S^{|\alpha|+\dim V} \vee S^{|\alpha|+\dim V+1}$ (G finite) or $S^{|\alpha|+\dim V}$ (G infinite) is the obvious collapsing map.*

This statement is a word for word generalization of [72, Proposition 3.1 and Theorem 3.4]. We shall explain how the ideas of those proofs adapt to the present context without duplicating various details that merely require systematic substitutions of symbols.

PROOF OF THEOREM 4.8 (SKETCH). To prove that $(*)$ has trivial normal invariant proceed as in [72, Proposition 3.1]. Take a closed equivariant tubular neighborhood $D^\#(\xi)$ of F, and let $S(V)$ be a sphere in a fiber of ξ over a fixed point. Then $S(V)$ has a closed tubular neighborhood of the form $S(V) \times D^{\alpha+1}$ (we write $D^{\alpha+1}$ for $D(\alpha + 1)$ here to stress the analogy with [72]). Choose a smaller tubular neighborhood $D(\xi) \subseteq \operatorname{Int} D^\#(\xi)$ that misses $S(V) \times D^{\alpha+1}$. Then if we let $W = \Sigma - \operatorname{Int} D(\xi) = \operatorname{Int} S(V) \times D^{\alpha+1}$, the orbit manifold W/G is a homological h-cobordism from $(*)$ to the identity.

To deal with $(**)$, we may again restrict ourselves to the case $G = \mathbf{Z}_{p^r}$, where p is prime, without loss of generality; as in [72, p. 116] all other cases are formal consequences of this. The boundary of W/G now takes the form

$$\partial(W/G) = S(\xi)/G \amalg - S(V) \times_G S^\alpha.$$

What we must do is consider the problem of fattening W/G up into a homological h-cobordism between $S(\xi \oplus \mathbf{M})/G$ and $S(V \oplus \mathbf{M}) \times_G S^\alpha$. As in [72] the first thing to do is form

$$\mathfrak{W} = S(V \oplus \mathbf{M}) \times_G S^\alpha \times [0, \varepsilon] \cup W \times_G (\mathbf{M}) \cup s(\xi \oplus \mathbf{M})/G \times [1 - \varepsilon, 1],$$

where we may explain the notation as follows: We may think of $W \times_G D(\mathbf{M})$ as a manifold with corners. In this manifold with corners we have smooth embeddings of closed collar neighborhoods of $\partial_\pm W \times_G D(\mathbf{M})$ (see Figure 1); we assume that the collar portion above $\partial_\pm W \times_G S(\mathbf{M})$ maps into $\partial(W \times_G D(\mathbf{M}))$. But $\partial_\pm W \times_G (\mathbf{M})$ are codimension zero submanifolds of (–) $S(V \oplus \mathbf{M}) \times_G S^\alpha$ and (+) $S(\xi \oplus \mathbf{M})$. We finally identify

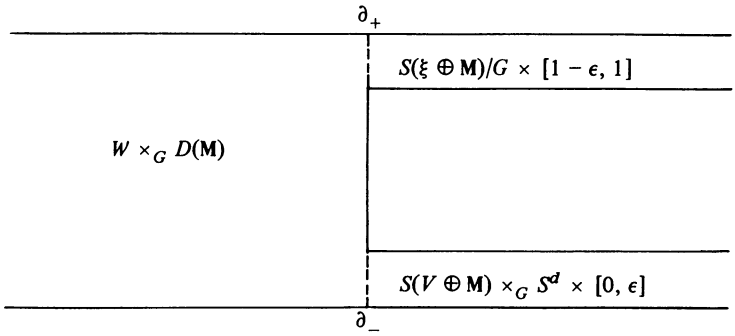

FIGURE 1

the collar neighborhoods of $\partial_\pm W \times_G D(\mathbf{M})$ with the appropriate pieces of the manifolds $S(V \otimes \mathbf{M}) \times_G [0, \epsilon]$ and $S(\xi \oplus \mathbf{M})/G \times [1-\epsilon, 1]$. We then round the corners of $\mathfrak{W}$ as in [72, p. 117] to get a smooth manifold.

The next step is to notice that $\mathfrak{W}$ contains a manifold of the form $P_0 \times I$, where $P_0 = S^\alpha \times D(V) \cup_{\partial_-} W \cup_{\partial_+} D(\xi)$; this proceeds exactly as in [72]. Schematically, $P_0 \times I$ corresponds to the bracket shaped broken line at the right of Figure 1. As in [72], the corner straightened version of P_0 is diffeomorphic to $S^{|\alpha|} \times S^{\dim V} \# \Sigma$, where $|\alpha| = \dim \alpha$.

As in [72], we wish to extend W into a homological h-cobordism from $L(\xi \oplus \mathbf{M})$ to $S(V \oplus \mathbf{M}) \times_G S$, where $L(V \oplus \mathbf{M})$ denotes a smooth thickened dim V-skeleton in the (dim $V + 2$)-dimensional lens space $L(V \oplus \mathbf{M})$, $L(\xi \oplus \mathbf{M})$ denotes the associated fiber bundle, and $S \subseteq S(V \oplus \mathbf{M})$ is the inverse image of L. Geometrically this obstruction corresponds to Σ by the comments of the previous paragraph. To formalize this, one gives the same argument as on pp. 119 and 120 of [72] with the following changes: The manifolds V and V_0 now become $S^\alpha \times_G S(V \oplus \mathbf{M})$ and $S^\alpha \times S(V) \times_G D(\mathbf{M})$, respectively, and from line 10 of p. 119 onwards we take $n = |\alpha| = \dim \alpha$ and $2k = \dim V$. ■

The extra term γ' in Theorem 4.8 may be interpreted as an "isotopy invariant" of the action as in [76]. In particular, we have the following analog of [76, Theorem 2.1]:

THEOREM 4.9. *Let ϕ be a smooth action of $\mathbf{Z}_{p^r}$ (p an odd prime say) on S^{m+2k} with fixed point set of $\mathbf{Z}_p$ given by F^n. Denote the equivariant normal bundle of F^n by ξ, and let $h_M = (1 \times_G \kappa)\theta(\omega_0 \oplus M)$ in the notation of (**), where M is the restriction of an irreducible free S^1-module. Finally, let $\gamma \in \Theta_{n+2k+1} \cong \pi_0(\mathrm{Diff}^+ S^{n+2k})$ correspond to the isotopy class of the Standard Generator $\exp(2\pi i/p^r) \in \mathbf{Z}_{p^r}$. Then h_M is $\mathbf{Z}_{(p)}$-homologically h-cobordant to $(1 \times d)\#\Delta(\gamma)$, where $d \equiv 1 \pmod p$ is the degree of h_M and "$\#\Delta(\gamma)$" denotes taking connected sums with the homotopy sphere γ.* ■

As before, the proof of this result follows the same lines as the $\mathbf{Z}_p$ case once the necessary changes in notation are made. Since we shall not use this theorem in the present paper, we shall omit the details.

Theorem 4.9 is the key result needed to extend the results of [**76**] as asserted in [**76**, 4.5A]–namely, certain clases of order p in $\pi_0(\mathrm{Diff}^+ S^n)$ (suitable n) admit no periodic representatives whatsoever. Unfortunately, a fair amount of extra calculation is needed to complete the proof, so we shall defer this to a later paper (i.e., Part II of [**76**]).

5. Extensions of earlier examples. Using Theorem 4.8 we may give additional examples along the lines of [**72**, §4]. We first give a direct analog of [**54**, Theorem 4.16].

THEOREM 5.1. *Let Σ^m be a homotopy sphere such that $q(\Sigma)\eta \notin 2^r\pi_{m+1}(F/O)_{(2)}$, and let Φ be a smooth orientation-preserving $\mathbf{Z}_{2^r}$ action on Σ with a $2k$-codimensional fixed point set. Then k is even.* ■

EXAMPLES. (I am grateful to M. Mahowald for bringing these examples to my attention.) There are infinitely many classes x such that $x\eta$ is nonzero but divisible by 2. Specifically, let η_j be the stable homotopy class corresponding to h_1h_j in the Adams spectral sequence [**47**], and take $x = \mu_k\eta_j$, where $k = 2^{j-4} - 1$. Work of Mahowald [**48**] shows that $\mu_k\eta_j\eta$ is nonzero and divisible by 4 and not 8. Therefore, Theorem 4.8 restricts the fixed point sets of smooth $\mathbf{Z}_8$ actions on Σ with $q(\Sigma) = \mu_k\eta_j$.

Turning to odd primes, we may prove the following result by the same methods used for [**72**, Theorem 4.18]:

THEOREM 5.2. *Let Σ^m be a homotopy sphere such that $q(\Sigma)\alpha_1 \notin p^r\pi_{m+2(p-1)}(F/O)_{(p)}$ (p an odd prime) and let Φ be a smooth $\mathbf{Z}_{p^r}$ action on Σ with a $2k$-dimensional fixed point set. Then $k \equiv 0 \bmod p$.* ■

As noted in [**72**], if $q(\Sigma) = \beta_k$ and $k \leqslant p - 1$, then Σ satisfies the above hypotheses with $r = 1$. On the other hand, if $q(\Sigma) = \beta_p$, then Σ satisfies the hypotheses with $r = 2$ (compare [**57**, Theorem 7.9] and [**84**]). Moreover, if $p = 3$, then β_{2p} satisfies a similar condition by calculations of M. Tangora and D. Ravenel (see [**8**] for example). Hence 5.2 is a nonempty generalization of [**72**, Theorem 4.18].

(5.3) ADDENDUM TO [**72**, §4]. H. Miller has pointed out Theorem 4.18 of [**72**] applies if $q(\Sigma) = \beta_k$ for arbitrary $k \equiv 0, -1 \bmod p$ (and $k = p - 1$ too). Here is the argument he supplied in a letter from April, 1976. In order to avoid lengthly digressions, we shall merely give references for the technical notation from [**54**] rather than try to give a self-contained explanation; a recommendable introduction to the BP-machinery involved is given in [**61**]. First of all, in π_* we have $\alpha_1\beta_s \neq 0$ if $s \equiv 0, -1(p)$ or $s = p - 1$. If p divided $\alpha_1\beta_s$, then the same would be true in $H^3(BP_*)$ (defined in [**43**, (1.8)]) by the calculations of [**54**]. In terms of the chromatic spectral sequence defined [**54**, §3], this says that p divides $v_2^s t_1/pv_1$ in $H^1(M_0^2)$, which in turn implies $\delta^1(v_t^s t_1/pv_1) = 0$ in $H^2(M_1^1)$, the notation being described in [**54**, pp. 481, 484]. Finally $\delta^1\delta^1 t_1/pv_1$ must also be zero in $H^3(M_2^0)$. But

the cohomology of the module M_2^0 is known (compare [**54**, Theorem 3.15] and [**61**]), and thus one can compute the double boundary of $v_i^s t_1/pv_1$ and determine when it vanishes. This happens only if $s \not\equiv 0, -1(p)$. I am grateful to Miller and D. Ravenel for their remarks on this problem. ■

The final theorem of this section deals with a generalization of [**72**, Theorem 4.17]; the latter says that if a homotopy sphere Σ has an involution and $q(\Sigma)\eta \notin 2\pi_*(F/O)_{(2)}$, $q(\Sigma)\nu \notin 2\pi_*(F/O)_{(2)} + \eta\pi(F/O)_{(2)} + \langle A, 2, \eta\rangle$, ($A$ = 2-torsion in $\pi_*(F/O)_{(2)}$), then the fixed point set has codimension $\equiv 0$ or 1 mod 8. For $\mathbf{Z}_4$ actions one can prove a little more.

THEOREM 5.4. *Let Σ^m be a homotopy sphere satisfying the following hypotheses (compare* [**54**, 4.16, 4.17]):

(i) $q(\Sigma)\eta \notin 2\pi_{m+1}(F/O)_{(2)}$.

(ii) $2q(\Sigma)\nu \notin 4\pi_{m+3}(F/O)_{(2)} + \eta\pi_{m+2}(F/O)_{(2)} + \langle A', 4, \eta\rangle$,

where A' = 4-torsion in $\pi_{m+1}(F/O)_{(2)}$. Assume that $\mathbf{Z}_4$ acts smoothly, and the fixed point set of $\mathbf{Z}_2$ has codimension $8k$ (this happens for some k by [**72**, *Theorem* 4.17]), *and the fixed point set of $\mathbf{Z}_4$ has codimension $8k + 2l$. Then $k + l \equiv 0$* mod 2.

EXAMPLE. Let $\bar{\kappa} \in \pi_{20(2)} \cong \mathbf{Z}_8$ be a generator; then (i) and (ii) hold if $q(\Sigma) = \bar{\kappa}$ (see [**49**]).

PROOF. Let $n = m - 8k - 2l$. The normal invariant associated to the suspended knot invariant is an element of $[S^{n+2l} \times_{\mathbf{Z}_4} (S^{8k+5})^+, F/O]_{(2)}$ having filtration m and extending to $[S^{n+2l} \times_{\mathbf{Z}_4} (S^{8k+5})^+, F/O]_{(2)}$. Since $x = -q^*(\Sigma)$ is the E^∞ element corresponding to this normal invariant by Theorem 4.8, it follows that $d^r(x) = 0$ for $r \leqslant 3$; these differentials follow immediately by a simple extension of [**72**, (4.12)–(4.14)]. However if $r = 4$ the differential is not given by a trivial extension of [**72**, (4.15)] but requires the following extra discussion:

By naturality of the differentials, it will suffice to consider d^4 for $[S^{n+2l} \times_{S^1} (S^{8k+5})^+, F/O]_{(2)}$, where the S^1 actions on S^{n+2l} and S^{8k+5} are the obvious linear extensions of the linear $\mathbf{Z}_4$ actions (i.e., isotropy subgroups $\subseteq \mathbf{Z}_4$); we shall justify this in the final sentence of the paragraph. The S^1 problem reduces to determining the structure of the three cell complex $\mathrm{Th}(l\eta_{4k+2}^2)/\mathrm{Th}(l\eta_{4k-1}^2)$, where Th denotes Thom complex, η_2 is the line bundle $S^{2q+1} \times_{S^1} \mathbf{C}$, and $(\ )^2$ denotes its complex tensor square. But the given complex is homotopy equivalent to $\mathrm{Th}((4k + l)\eta_2)$ since $\eta_2^2 = 4\eta_2$ in $\widetilde{KO}(\mathbf{C}P^2)$ [**2**]. *Now* one can use [**72**, (4.15)] to show that $k + l \equiv 0$ mod 2; for the nontriviality of $2q(\Sigma)\nu$ rules out the possiblity $k + l \equiv 1$ mod (2). Of course, nontrivial means with respect to the $\mathbf{Z}_4$ problem, and for this it is necessary to use the indeterminacy on the right-hand side of (ii). ■

The results on d^4 derived in the course of proving Theorem 5.4 are useful in their own right, and they will be used to study $\mathbf{Z}_4$ actions on certain low-dimensional exotic spheres in §8 and paper VI of this series.

6. Groups of homotopy spheres with group actions. A basic tool for studying semifree differentiable actions on homotopy spheres is the abelian group structure on various equivalence classes of actions that is given by taking connected sums.

These groups fit into various exact sequences that can be studied by the existing machinery of differential topology, particularly surgery and the classification theory of homotopy smoothings. Specific results of such studies (in various forms) have been obtained by Montgomery and Yang [56] (also see [39]), Bredon [10], Rothenberg (both alone [63] and with Sondow [65]), Browder and Petrie [12], Sebastiani [78], Jones [36], Alexander, Hamrick, and Vick [3], Abe [1], K. Wang [89], Löffler [45] and the author [69] (this list is certainly not complete, but to the best of the author's knowledge it appears to be representative). It is very easy to verify that similar abelian group structures may be introduced even for actions that are not semifree; it is considerably less apparent (and indeed not known in general) that such groups can be studied effectively using exact sequences and more or less standard differential-topological machinery. In this section we shall show that the knot invariant defined earlier is the key notion required to extend the Browder-Petrie-Rothenberg exact sequences to ultrasemifree actions. Our extensions works particularly well for actions whose isotropy subgroups are normal and linearly ordered by inclusion; of course, if p is a prime, every action of $\mathbf{Z}_{p^r}$ satisfies this condition.

For the sake of completeness, we begin by generalizing the groups of actions originally defined by Browder, Petrie, and Rothenberg [12], [63], [65] in the semifree case. Although the latter groups are only defined if the fixed point set has codimension ≥ 3, we shall need suitable versions for codimensions 1 and 2; a blanket assumption of codimension ≥ 3 for all embeddings of one fixed point set in another would eliminate a much larger class of examples than in the semifree case (e.g., numerous $\mathbf{Z}_p \times \mathbf{Z}_p$ actions to be considered in subsequent papers). For our purposes it will suffice to consider only those embeddings $A^n \subseteq B^{n+k}$ of one homotopy sphere in another for which the inclusion $S^{k-1} \subseteq B - A$ of a normal fiber to A is a homotopy equivalence; we shall say such an embedding is *homotopically unknotted* (an allusion to similar actions appears in [65]). It is well known that every embedding is homotopically unknotted if $k \neq 2$, while if $k = 2$ homotopy unknottedness is closely related to the stronger (topological, smooth, and PL) forms of unknottedness, mainly by results of Papakyriakopoulos [59], Stallings [85], Levine [43], and Shaneson [81]; in particular, if $n \geq 5$ and $k = 2$ all these notions are equivalent. Homotopy unknottedness for a proper embedding of one homotopy disk in another is defined similarly (both $B - A$ and $\partial B - \partial A$ are homotopic to S^{k-1}); as for spheres, homotopy unknottedness is automatic for $k \neq 2$ and equivalent to other forms if $k = 2$ and n is sufficiently large.

Given a linear representation U of a compact Lie group G, a *G-homotopy U-sphere* is an effective compact differentiable G-manifold Σ with basepoint $x \in \Sigma$ satisfying the following:

(i) (Σ, x) is U-oriented.

(ii) For every closed subgroup $H \subseteq G$, the fixed point set Σ^H of H is a homotopy (dim U^H)-sphere.[2]

[2]The term "semilinear" has been used for this.

(iii) If $K \subseteq H$ and K, H are closed subgroups of G, then Σ^H is homotopically unknotted in Σ^K.

It is an elementary exercise to verify that the connected sum of two G-homotopy U-spheres is again a homotopy U-sphere; the resulting binary operation makes the set of G-isomorphism classes of such objects into a monoid with unit S^U (commutative if $\dim U^G > 0$), and the isomorphism class of (Σ, x) is independent of x if $\dim U^G > 0$.

To make this monoid into a group, define an equivalence relation $\Sigma_1 \sim \Sigma_2$ if $\Sigma_1 \# -\Sigma_2$ bounds a homotopy $(U+1)$-disk, where $-\Sigma$ is the opposite U-orientation described in §1, and a homotopy $(U+1)$-disk is a pair (Δ, x) with $(\partial\Delta, x)$ a homotopy U-sphere and (i) – (iii) above true with Δ replacing Σ, $U+1$ replacing U, and with "disk" replacing "sphere". In analogy with [**65**], this defines a group Θ_U^G of h-cobordism classes of homotopy U-spheres, and this group is abelian if $\dim U^G > 0$.

Suppose that U is a simple ultrasemifree action as defined in §4 with subprincipal isotropy bound H, and write $U = W \oplus V$, where $W = U^H$. We wish to define a knot invariant homomorphism ω: $\Theta_U^G \to F/O_{G,\text{free},V}(S^W)$ along the lines of §4. By the results of that section we know that a knot invariant for a typical $\Sigma \in \Theta_U^G$ may be defined in $F/O_{G,\text{free},V}(\Sigma^H)$ and this invariant has reasonable additivity properties by (4.1). To replace Σ^H by S^W, it suffices to let κ: $\Sigma^H \to S^W$ be the map collapsing everything off a neighborhood of a fixed point and apply the following result:

PROPOSITION 6.1. *The map* κ^*: $F/O_{G,V,\text{free}}(S^W) \to F/O_{G,V,\text{free}}(\Sigma^H)$ *is an isomorphism.*

PROOF. This is an immediate consequence of the equivariant Whitehead theorem (e.g., see [**32**] or [**51**]). ■

REMARKS. 1. The above proposition is still true if all the fixed point sets are merely assumed to be $\mathbf{Z}$-homology spheres (homotopy unknottedness also being unnecessary). An equivariant obstruction-theoretic argument using techniques similar to [**68**, §4] replaced the Whitehead theorem in this case (compare (4.3) earlier).

2. Notice that the map κ actually comes from the map of triads κ: $(\Sigma; \Sigma - \operatorname{Int} D(W), D(W)) \to (S^W; D_+(W), D_-(W))$ where $\kappa|(D(W))$ is a canonical tubular neighborhood.

The knot invariant homomorphism given by the above discussion fits into the following exact sequence generalizing [**69**, (1.1)]:

$$(6.2) \qquad \cdots hS_{W+1}^G(S(V)) \to \Theta_U^G \xrightarrow{(\omega,\Sigma^H)} \bigoplus_{\Theta_W^{G/H}}^{F/O_{G,\text{free},V}(S^W)} \xrightarrow{\sigma} hS_W^G(S(V)).$$

The symbolism in (6.2) deserves some explanation. If Y is a free G-manifold, then $hS_W^G(Y)$ means $hS(D(W) \times_G Y, S(W) \times_G Y)$; if W is a trivial G-module this reduces to $hS_{\dim W}(Y/G)$, which appears in [**69**, (1.1)]. As in the case of trivial G-modules, the sets $hS_W^G(Y)$ have canonical group structures (induced by connected sums) if $\dim W \geqslant 1$, and the groups are abelian if $\dim W^G \geqslant 1$ (compare

[**75**, §6 and Footnote 2]). Passage to the fixed point set of H is denoted by Σ^H. The map γ is given by taking a homotopy smoothing h: $(M, \partial M) \to (D(W+1) \times_G S(V), S(W+1) \times_G S(V))$ that is a diffeomorphism on the boundary and gluing the total space $\tilde{M}$ of the principal G-bundle to $S^W \times D(V) = S(W+1) \times D(V)$ by the diffeomorphism ∂h: $\tilde{M} \to S(W+1) \times S(V)$. Finally, the map σ is given by taking (a) an equivariant fiber retraction t: $S(\xi) \to S(V)(\xi\downarrow S^W)$ that is linear near a fixed point, (b) the projection π: $S(\xi) \to S^W$, and (c) an equivariant collapsing map c: $\Sigma^H \to S^W$ onto a neighborhood of a fixed point, forming from them the homotopy smoothing $S(c^*\xi)/G \xrightarrow{(\pi,t)/G} S^W \times_G S(V)$. This is a map of triads into $(S^W; D_+(W), D_-(W)) \times_G S(V)$ and a diffeomorphism over $D_+ \times_G S(V)$.

Derivation of (6.2) (Sketch). With everything now at our disposal, this resembles the semifree case very closely. For the convenience of the reader we shall review the arguments briefly.

Consider first *exactness at* $F/O_G \cdots \oplus \Theta_W^{G/H}$. Given the retraction ρ: $S(\xi, S^W) \to S(V)$ from $F/O \cdots$ and the exotic G/H-sphere M, the map σ gives the homotopy smoothing

$$S(\xi\downarrow M)/G \to S(V) \times_G S^W.$$

This element is equivalent to zero if and only if it bounds a homotopy smoothing $\mathfrak{N} \to S(V) \times_G D(W+1)$ (compare [**75**, §§4 – 5] and [**92**, p. 33]). If (ρ, M) comes from a section on Σ, then $\sigma(\rho, M) = 0$ because $\Sigma - \operatorname{Int} D(\xi\downarrow M)/G$ is an explicit choice for $\mathfrak{N}$. On the other hand, if we already have $\mathfrak{N}$ then we may reconstruct $\Sigma = D(\xi\downarrow M) \cup \tilde{\mathfrak{N}}$. ($\tilde{\mathfrak{N}}$ denotes universal covering.)

Next consider exactness at Θ_U^G. To see that $(\omega, \Sigma)\gamma = 0$, notice that $\Sigma = \gamma(x)$ is the bottom of a cobordism $\mathfrak{W} = \Sigma \times I \cup D(W+1)$, where $\mathfrak{W}$ has the following further properties:

(i) The top boundary of $\mathfrak{W}$ has the equivariant homotopy type of $S(V) \times S^{W+1}$.

(ii) The top boundary is a deformation retract of $\mathfrak{W} - \mathfrak{W}^H$.

From these it follows that the knot invariant of Σ^H in Σ extends to a G-bundle with retraction $\mathfrak{W}^H$ in $\mathfrak{W}$. But $\mathfrak{W}^H$ is a disk $D(W+1)$, so $\Sigma^H = 0$ and the knot invariant must be trivial.

Conversely, if $(\omega, \Sigma^H)\Sigma = 0$, then one can reconstruct $\mathfrak{W}$ by a simple gluing operation. Using the fact that the knot invariant is null homotopic, one can construct the homotopy equivalence required for property (i).

Finally, consider *exactness at* $hS_W^G(V)$. To see that $\gamma\sigma = 0$, notice as before that we have a canonical equivariant cobordism $\mathfrak{W}$ from $\gamma\sigma(y)$ to $\Gamma\sigma(y)$, where Γ: $hS_W^G(F) \to hS(S^W \times_G S(V))$ is formed by gluing on $D_-(W) \times_G S(V)$. But now the top boundary of $\mathfrak{W}$ is just the sphere bundle $S(\xi\downarrow W)$, and it follows that $\mathfrak{W}$ embeds in $D(\xi\downarrow W)$ as a thickening of $S(\xi\downarrow W) \cup \text{Fiber } D(V)$. Hence $\gamma\sigma(\omega, M) = \overline{\partial(D(\xi\downarrow W) - \mathfrak{W})}$; but the resulting cobounding manifold is contractible. Conversely, if $\gamma(y) = 0$, then we can fill in $\mathfrak{W}$ with a cobounding disk for y. This gives a smooth G-manifold equivariantly homotopic to $D(V) \times S^W$, with fixed point set a homotopy W-sphere M and $M \to \mathfrak{W}^* = (\mathfrak{W} \cup \text{disk})$ a homotopy

equivalence; all relevant homotopy unknotting conditions also follow by routine calculations. It follows that $\partial\mathfrak{W}^*/_G$ is h-cobordant to a homotopy smoothing of $S^W \times_G S(V)$ given by the orbit bundle $S(\nu_{M,\mathfrak{W}})/G$. (This is just the standard embedding trick for constructing h-cobordisms.) A more careful analysis shows that one actually gets a homotopy smoothing of triads into $(S^W;\ D_+(W),\ D_-(W)) \times S(V)$ that is a diffeomorphism on $D_-(W) \times_G S(V)$, and the h-cobordism is an h-cobordism of triads. This concludes the verification of exactness at $hS_W^G(S(V))$. ■

REMARK. Modulo low-dimensional questions of a standard nature, the defining conditions for a homotopy U-sphere Σ are quite similar to an assumption that Σ is topologically equivalent to S^U. In fact, if Σ is a U-manifold and topologically equivalent to S^U, then it determines an element of Θ_U^G (unique modulo choice of U-orientation). Conversely, a theorem of Connell, Montgomery, and Yang [**19**] (strengthened in [**75**]) implies that all elements of Θ_U^G are topologically linear in a great many cases (a thorough discussion will appear in forthcoming work of S. Illman [**33**]). In such instances this almost proves that Θ_U^G are the groups of smooth G-manifolds topologically equivalent to S^U, generalizing what is known for $G = 1$ modulo low-dimensional problems. To conclude the proof (aside from low-dimensional considerations), it would be necessary to show that topologically equivalent representations are necessarily linearly equivalent. Although recent work of S. Cappell and J. Shaneson [**16**]–[**18**] shows the latter is not always true, this is the case under many reasonable conditions; e.g., finite p-groups with p an odd prime [**74**], all two-component groups and Weyl groups [**37**], and all semifree representations [unpublished work of W.-C. Hsiang and W. Pardon] (I am grateful to Wu-chung Hsiang for informing me of this work).[3]

The usefulness of (6.2) for studying Θ_U^G depends on the ability to describe the other two terms in the exact sequence. Surgery and homotopy smoothing theory provide effective means for studying the terms $hS_W^G(S(V))$ (as in the semifree case), and $F/O_{G,\text{free},V}(S^W)$ may be handled using the techniques of earlier sections. An adequate method for dealing with $\Theta_W^{G/H}$ is needed to complete the picture. Perhaps the simplest way to dispose of this is to say the group is known by "induction on the size of G/H"; however, it is not clear that such an assertion is justifiable unless the isotropy subgroups are linearly ordered and all normal (in this case Σ^H is an ultrasemifree G/H-manifold, so an exact sequence of type (6.2) exists for $\Theta_W^{G/H}$). Despite this, one can often get useful results from (6.2) without thinking about $\Theta_W^{G/H}$; for example, one obtains the actions in [**75**] by restricting to elements whose $\Theta_W^{G/H}$-coordinate vanishes (more examples will be given in paper IV).

In certain contexts (e.g., see Theorem 6.5 below) it is useful to consider instead a related group Es Θ_H^G of equivariant s-cobordism classes of *equivariantly simple homotopy U-spheres*. By the latter we mean that the equivariant collapsing map $\Sigma \to S^U$ has trivial equivariant Whitehead torsion in the sense of M. Rothenberg

[3]This is also true for representations of odd order groups; one proof is due to Hsiang and Pardon, and another to I. Madsen and M. Rothenberg.

[**64**] and H. Hauschild [**29**]. For these groups we may derive an exact sequence entirely parallel to (6.2):

(6.3) *There is an exact sequence analogous to* (6.2) *with* Es Θ *and shS* (= *simple homotopy smoothings*) *replacing* Θ *and hS.* ■

A vague allusion to the existence of this exact sequence appears in [**75**, Remark, p. 27]. Under suitable dimension $\geqslant 5$ and codimension $\geqslant 3$ hypothesis, the two exact sequences (6.2) and (6.3) are braided with a third exact sequence analogous to the Rothenberg sequence for homotopy and simple homotopy Wall groups ([**82**, §4]; compare the remark cited above):

$$\begin{aligned} \cdots \to H^{*+1}(Z_2; \mathrm{Wh}(G; \mathbf{Z})) \to \mathrm{Es}\,\Theta_U^G \xrightarrow{h} \Theta_U^G \\ \xrightarrow{\tau} H^*(Z_2; \mathrm{Wh}(G, W; \mathbf{Z})) \to \cdots. \end{aligned} \tag{6.4}$$

(We shall often write $\Theta_U^{G,s}$ in place of Es Θ_U^G.)

The equivariant Whitehead group $\mathrm{Wh}(G, W; \mathbf{Z})$ is the group $\tilde{K}_1(B(W); \mathbf{Z})$ considered in [**64**, p. 289], and the involution is essentially a conjugation; the family W is all orbit types occurring in U. In fact, if G and all subgroups act orientation preservingly on everything in U, then the involution is induced by the usual algebraic map $g \mapsto g^{-1}$; furthermore, under this assumption the involution is consistent with the splitting of $\mathrm{Wh}(G, W; \mathbf{Z})$ noted in [**64**, p. 289]. The maps h and τ in the sequence are defined by passing to the underlying h-cobordism class and taking Whitehead torsion modulo a natural indeterminacy (torsion is not well defined on h-cobordism classes). Of course, if G acts semifreely then the Whitehead group is just the usual $\mathrm{Wh}_{\mathbf{Z}}(G)$. In this case, with $G = \mathbf{Z}_p$ (p prime), groups resembling Es Θ were studied by Löffler [**45**].

Application. To indicate how $\Theta_U^{G,s}$ is sometimes more useful than Θ_U^G, we shall consider two results on the extendibility of group actions from [**63**] and [**69**]. Since both proofs are basically formal manipulations of algebraic exact sequences, the arguments generalize word for word if one replaces all "homotopy" objects with their "simple homotopy" analogs.

THEOREM 6.5. (*Compare* [**63**, *Corollary to Theorem* A, *p.* 463].) *Let* V^n *be a semifree G-module, with G finite,* $\dim V \geqslant 5$, $\dim V^G \geqslant 1$ *and* $\dim V - \dim V^G \geqslant 3$. *Let H be a subgroup of G that is either central or a direct summand, and suppose that the simple Wall group* $L_n^s(G)$ *has exponent m* ($n = \dim V$). *Suppose that* $\Sigma \in \Theta_V^{H,s}$ *has fixed point set F, and the equivariant normal bundle's structural group reduces to the centralizer of G* (*its structural group is given by the centralizer of H*). *Then for some* $t \geqslant 0$, *the element* $|G/H|^t m\Sigma$ *lies in the image of the forgetful map* $\Theta_V^{G,s} \to \Theta_{V|H}^{H,s}$. ■

Convention. If $H = 1$, *take the corresponding group of knotted homotopy sphere pairs.*

Since $L_{\mathrm{odd}}^s(G) = 0$ if G has odd order [**7**], [**88**], this theorem gives strong results when n is odd; for instance it applies for G cyclic of odd order and $H = 1$. If n is even, the relevant Wall groups have elements of infinite order in general, and thus we have a slightly different statement then:

THEOREM 6.6. *Let V, H, G, n be given as before with n even. Suppose that $\Sigma \in \Theta^{H,s}_{V|H}$ has order prime to 2^ε* order G, *where $\varepsilon = 0$ or 1 depending on the existence of 2-torsion in $L^s_n(G)$. Furthermore, assume the structural group of the normal bundle $\nu_{F,\Sigma}$ reduces as in Theorem 6.5, and the map*

$$B \otimes \mathbf{Q}\colon \pi_k(F_H(V_G)/C_H(V_G)) \to ShS_k(L(V_G)) \otimes \mathbf{Q}$$

from (6.2) *is monic. Then Σ lies in the image of $\Theta^{G,s}_V$.* ■

Notation. The normal bundle of A in B is called $\nu_{A,B}$, the module V_G is V/V^G, and $L(V_G)$ denotes $S(V_G)/G$.

By results of [7] and [88], these assumptions hold with $\varepsilon = 0$ if G is cyclic of odd order and $H = 1$. However, the homotopy analog in [58, Theorem 2.2] (contrary to the assertion made on p. 313 of that paper) needs $\varepsilon = 1$. It is now established that contrary to some previous announcements $L^h_0(\mathbf{Z}_p)$ has 2-torsion for at least some values of p, with 29 and 113 being specific examples [6]. Since $\Theta^{1,s}_V = \Theta^1_V$, there is clearly a substantial gain in using the simple Θ-groups instead of the ordinary Θ-groups.

7. Rational calculations. Historically speaking, the most basic question concerning the groups Θ^G_V of §6 has been the calculation of $\Theta^G_V \otimes \mathbf{Q}$. In fact this goes back to the original Kervaire-Milnor paper on the nonequivariant case [38], whose principal stated aim is to prove Θ_n is finite for $n = 3$.[4] More significantly, the work of Browder and Petrie on Θ^G_V for $G = S^1$ and V a semifree G-module [12] has the computation of $\Theta^G_V \otimes \mathbf{Q}$ as one of its major goals. The techniques of [12] also led to rational calculations for $G = \mathbf{Z}_2$, S^3, Pin_2 (= normalizer of S^1 in S^3), and a partial description for $G = \mathbf{Z}_n$ (n arbitrary) up to the sorts of G-signature calculations done in [25], [26] and [70] (it should be noted that Browder and Petrie knew the basic G-signature expression studied in these papers but did not pursue the matter; T. Petrie has kindly shown me an unpublished manuscript with the details of [12]). In [26] Ewing has given some explicit calculations. In this section we shall present methods for calculating $\Theta^G_V \otimes \mathbf{Q}$ in the spirit of [12], concentrating on the ultrasemifree case. Our objective is *not* to reproduce the precise computations of Browder and Petrie; they produced systematic tabulations in terms of dim V and dim V^G mod 4, omitting the case of $\mathbf{Z}_n$ where both dimensions are even. Instead we intend to supply enough information to make such precise information readily accessible. In passing we also notice some new applications to nonsemifree actions, most notably Corollaries 7.5 and 7.12.

Of course, the starting point is (6.2), which tells us the dimension of $\Theta^G_V \otimes \mathbf{Q}$ can be calculated from the numerical invariants of the mappings

$$\Theta^{G/H}_{W+\varepsilon} \otimes \mathbf{Q} \to hS^G_{W+\varepsilon}(S(N)) \otimes \mathbf{Q} \qquad (\varepsilon = 0, 1), \tag{7.1}$$

$$F/O_{G,\mathrm{free},N}(S^{W+\varepsilon}) \otimes \mathbf{Q} \to hS^G_{W+\varepsilon}(S(N)) \otimes \mathbf{Q} \qquad (= 0, 1), \tag{7.2}$$

[4] I am not asserting that the finiteness result is the most important feature of [38] for lat7er work in differential topology. On the other hand, a hypothetical reader with no knowledge of subsequent research could understandably view the finiteness theorem as the most important point.

where H is the subprincipal isotropy subgroup for V, $W \subset V$ is the fixed point set of H, and N is the free G-module V/W (so $V \cong W \oplus N$). In other words, we must know the dimensions of the vector spaces in (7.1) and (7.2) together with the ranks of the corresponding mappings. The dimension of the codomain is readily computable from the surgery exact sequence and the results is **[88]**.

Here is a summary of our principal results on these maps. We shall deal with the codomains in the next paragraph and assume that something is already known about the domain of (7.1) (see remarks in the last paragraph of §6). The domain of (7.2) is described in Lemma 7.6 for G finite and in Proposition 7.7 for G infinite. Finally, the kernel of (7.2) is given by Theorem 7.9 for G finite and by Theorem 7.11 for G infinite.

The dimensions of codomains are easily disposed of by tensoring the appropriate surgery exact sequence with $\mathbf{Q}$:

$$\cdots L_{\dim V - \dim G + 1}(G/G_0, \omega) \to hS_W^G(S(N)) \otimes \mathbf{Q} \to [S(N)^+ \wedge_G S^W, F/O] \otimes Q \cdots.$$

In this sequence the orientation homomorphism ω has the form $\omega_W \otimes \omega_N$, where $\omega_W(g)$ is the degree of $g \in G$ acting on S^W and ω_N is the orientation homomorphism for $S(N)/G$. Rationally, the normal invariant map is easy to calculate; its image is isomorphic to $H^{4*}((S(N)^+ \wedge_G S^W) - \{\mathrm{pt}\}; \mathbf{Q})$; deletion of a point is necessary to eliminate normal maps with nontrivial index obstructions. On the other hand, if G is *infinite or finite cyclic*, the rationalized Wall groups in the above sequence are known (see **[88]**, for example), and their images are isomorphic to

(7.2A) 0 if $\dim V - \dim G$ is odd,

(7.2B) $[RO(\mathrm{Ker}\ \omega)/RO(1)] \otimes \mathbf{Q}$ if $\dim V - \dim G \equiv 0 \bmod 4$,

(7.2C) $[R(\ker \omega)/RO(\ker \omega)] \otimes \mathbf{Q}$ if $\dim V - \dim G \equiv 2 \bmod 4$.

Thus the dimension of $hS_{W+\varepsilon}^G(S(N)) \otimes \mathbf{Q}$ can be easily read off in any particular case. Furthermore, all of the summands have natural geometric interpretations: Those corresponding to normal invariants measure changes in rational Pontrjagin classes (because the classifying map $F/O \to BO$ is a rational equivalence), and those corresponding to Wall group elements measure changes in the Atiyah-Singer σ-invariant (compare **[12]**, **[70]**).

Before disposing of (7.1), we state the following elementary formula for Atiyah-Singer invariants:

(7.3) *Suppose M and N are oriented G-manifolds of even and odd dimensions, respectively, and N is G-free. Then* $\sigma(M \times N) = \mathrm{sign}_G(M)\sigma(N)$.

The proof of (7.3) is a routine calculation, for if $kN = \partial W$, then $k(M \times N) = \partial(M \times W)$. ■

We can now determine (7.1) completely.

PROPOSITION 7.4. *Assume G is infinite or finite cyclic, and if $G = \mathrm{Pin}_2$ also assume the action on S^W preserves orientation. Then the map $\Theta_W^{G/H} \otimes \mathbf{Q} \to hS_W^G(S(N)) \otimes \mathbf{Q}$, which is determined by taking a W-sphere Σ to the relative homotopy smoothing*

$$\kappa^\# \times_G 1\colon (\Sigma\text{-int } D(N), S(N)) \times_G S(N) \to (D(W), S(W)) \times_G S(N)$$

($\kappa^\#$ is defined in Remark 2 following Proposition 6.1), is trivial.

PROOF. We must check that there is no change in rational Pontrjagin classes or Atiyah-Singer invariants. It is convenient to calculate these using the map $\kappa^{\#}$ defined using *all* of Σ.

Case 1. G is finite cyclic. To see that there is no change in rational Pontrjagin classes for a homotopy smoothing $\kappa^{\#} \times_G 1\colon\ \Sigma \times_G S(N) \to S^W \times_G S(N)$, it suffices to pass to the universal covering (since the latter operation is injective in rational cohomology). If this is done, we get back $\kappa^{\#} \times 1$; since Σ is (nonequivariantly) a π-manifold, there is no rational change in Pontrjagin classes.

Next, assume $hS_W^G(S(N)) \otimes \mathbf{Q}$ has a portion detected by Atiyah-Singer invariants; it follows that $\dim V = \dim W + \dim N$ must be even. Suppose that $\dim W$ and $\dim N$ are both even and G/H acts orientation-preservingly on S^W. Then (7.3) and the lack of middle-dimensional cohomology in a G/H-homotopy sphere Σ imply there is no change in Atiyah-Singer invariants. If we remove the orientation condition, then the same conclusion holds because the relevant Atiyah-Singer invariants are defined and Kernel ω = kernel ω_W (note that $\omega_N = 1$ here). Finally, if $\dim W$ and $\dim N$ are both odd, then we must have $G = \mathbf{Z}_2$; but then H must also be $\mathbf{Z}_2$ (since $1 \neq H \subseteq \mathbf{Z}_2$), so that ω_W is trivial but $\omega = \omega_N$ is nontrivial. Since $L_*(Z_2^-) \otimes \mathbf{Q}$ is zero [**88**], there are no Atiyah-Singer invariants present in this subcase.

Case 2. G is infinite (hence S^1, Pin_2, or S^3). The relevant surgery groups are $L_*(1)$ for S^1 or S^3, and $L_*(\mathbf{Z}_2^-)$ for Pin_2 (because Pin_2 preserves orientation on S^W). Hence there are no Atiyah-Singer invariants to be detected in this case.

To prove there is no change in rational Pontrjagin clases, first note that is suffices to consider the action restricted to S^1 (as in the finite case, the appropriate maps $M/S^1 \to M/G$ induce injections in rational cohomology). Next, note that the change in Pontrjagin classes is given by taking the class τ_Σ of the tangent bundle in $KO_G(\Sigma)$, mapping it to $KO_G(S^W)$ by κ^{*-1} (just as in Proposition 6.1, κ^* is bijective), and mapping $\kappa^{*-1}\tau_\Sigma$ into $KO_G(S^W \wedge_G S(N)^+)$ by taking a balanced product with $S(N)$. Therefore it is enough to know that τ_Σ goes to zero in $KO_G(\Sigma) \otimes \mathbf{Q}$ for $G = S^1$. But the induced S^1 action of Σ is pseudolinear in the sense of [**31**] (i.e., all fixed point sets are topological spheres), and accordingly τ_Σ has order $\leqslant 2$ in $KO_G(\Sigma)$ by a result of W. Iberkleid [**31**]. ■

COROLLARY 7.5. *Under the hypotheses of Proposition* 7.4, *we have* $\dim \Theta_V^G \otimes \mathbf{Q} > \dim \Theta_W^{G/H} \otimes \mathbf{Q}$. ■

Of course, this result is immediate from the proposition and our basic exact sequence (6.2). In particular, it follows that the infinite families of semifree G/H-actions constructed in [**18**], [**19**], [**57**] generate corresponding infinite families of *nonsemifree* G-actions.

We now focus attention on (7.2). In dealing with (7.1), we avoided an explicit tabulation of $\dim \Theta_W^{G/H} \otimes \mathbf{Q}$, the idea being (as in §6) that whatever we wanted to know had been previously obtained by some inductive process (with $\Theta_k \otimes \mathbf{Q} = 0$, $k \neq 3$, as a starting point). However, we shall require more specific information about $F/O_{G,\mathrm{free},N}(S^W) \otimes \mathbf{Q}$ to recover all the calculations implicit in [**12**] and [**75**]. To keep the discussion within bounds, we shall assume G is finite cyclic, S^1, Pin_2,

or S^3, and the isotropy subgroups are linearly ordered by inclusion; if $G = \mathrm{Pin}_2$ or S^3, we further assume that H acts trivially on W.

If G is finite cyclic, we have the following analog of [**75**, Theorem 3.5]:

LEMMA 7.6. *Let G be a finite cyclic group acting linearly on W and N as above; label the isotropy subgroups of W as $G = H_1 \supsetneqq \cdots \supsetneqq H_r = H$, and let n_j be the dimension of the fixed point set of H_j. Define*

$$\rho_j\colon F/O_{G,\mathrm{free},N}(S^W) \otimes \mathbf{Q} \to \pi_{n_j}\big(F_{H_j}(N)/C_{H_j}(N)\big) \otimes \mathbf{Q}$$

via restriction to the induced H_j-vector bundle on the fixed point set of W_j, and define $\rho\colon F/O_{G,\mathrm{free},N}(S^W) \otimes \mathbf{Q} \to \pi_{n_j}$ [etc.] to be the product map. Then ρ is injective, and its image is the set of all r-tuples (x_j) such that x_j maps to zero in $\pi_{n_j}(F_{H_{j-1}}(N)) \otimes \mathbf{Q}$ for $j \geqslant 2$. ■

The proof is similar to [**75**, 3.5] and therefore omitted.

We shall not pursue the computations further here. Complete information about the groups $\pi_{n_j}(F_{H_j}(N)/C_{H_j}(N)) \otimes \mathbf{Q}$ and the forgetful maps

$$\pi_{n_j}\big(F_{H_j}(N)/C_{H_j}(N)\big) \otimes \mathbf{Q} \to \pi_{n_j}\big(F_{H_{j-1}}(N)/CH_{j-1}(N)\big) \otimes \mathbf{Q}$$

is recoverable from the results of [**80**], the rational equivalence of $F_H(N)$ with $F_1(N)$ (compare [**67**]), and the known structure of $\pi_*(G_k) \otimes \mathbf{Q}$ and $\pi_*(G_k/O_k) \otimes \mathbf{Q}$. The specific computations are often tedious, but always elementary.

Comment on the case $G = \mathbf{Z}_2$, $\dim N$ odd. Strictly speaking, this case is not covered by the results quoted in the previous paragraph, so we shall explain here how one calculates $\pi_*(F_G(N)/C_G(N))$ along the same lines. The spectral sequences of [**49**] yield spectral sequences converging to $\pi_*(F_G(N))$ with

$$E^1_{p,q} = \mathcal{C}_p\big(\mathbf{R}P^{2n};\, \pi_{2n+q}(S^{2n})\big) \cong \pi_{2n+q}(S^{2n}),$$
$$E^2_{p,q} = \mathcal{H}_p\big(\mathbf{R}P^{2n};\, \pi_{2n+q}(S^{2n})\big) \quad (\text{where } \dim N = 2n+1),$$

the $\mathbf{Z}_2$-twisted coefficients $\pi_*(S^{2n})$ being given by $-r_*$, where r is a self map of S^{2n} with degree -1. On the other hand, there is a centralizer spectral sequence as in [**67**, §5] obtained by filtering O_{2n+1} as $\bigcup_{k \leqslant 2n+1} O_k$ with

$$E^1_{p,q} = \mathcal{C}_p\big(\mathbf{R}P^{2n};\, \pi_{q+p-1}(S^{p-1})\big) \cong \pi_{q+p-1}(S^{p-1}).$$

Furthermore, the latter spectral sequence maps into the former one, the map on the E^1 level being an iterated suspension as [**67**]. ■

If $G = S^1$, Pin_2, or S^3, the corresponding calculation is entirely different:

PROPOSITION 7.7. *Let W be a noneffective S^1, Pin_2, or S^3 module with fixed point set of dimension $\geqslant 2$, isotropy subgroups linearly ordered if $G = S^1$, and trivial action if $G \neq S^1$. Denote the ineffective kernel of the action by H.*

(i) *If W is even dimensional or G acts trivially, then $F/O_{G,\mathrm{free},N}(S^W) \otimes \mathbf{Q} = 0$.*

(ii) *If W is odd dimensional and $G = S^1$, write the isotropy subgroups of the action in order $G = H_1 \supsetneqq \cdots \supsetneqq H_r = H$, and let $2m_i + 1 =$ dimension of $\mathrm{Fix}(S^W, H_j)$. Write $2n = \dim_R N$.*

Then

$$\text{(a)}\quad F/O_{G,\mathrm{free},N}(S^W)\otimes\mathbf{Q}\cong\bigotimes_{k=m_1+2}^{m_r}\pi_{2k}(BU_{m-1})\otimes\mathbf{Q}\quad \textit{if } H\neq\mathbf{Z}_2,$$

$$\text{(b)}\qquad =\left[\bigoplus_{k=m_1+2}^{m_r-1}\pi_{2k}(BU_{n-1})\right]\otimes\mathbf{Q}\oplus\left[\bigoplus_{k=m_{r-1}+1}^{m_r}\pi_{2k}(BSO_{2n-1})\right]\otimes\mathbf{Q}\quad \textit{if } H=\mathbf{Z}_2.$$

In either case the sum is zero by definition if the lower limits exceed the upper ones.

PROOF. Since $F/O_{G,\mathrm{free},N}$ is a representable functor (see §1), we may use the spectral sequence of [**68**, §4] to study $F/O_{G,\mathrm{free},N}(S^{d+W})\otimes\mathbf{Q}$. Notice that this spectral sequence disposes of the even- and odd-dimensional cases at the same time; therefore let us assume W is even dimensional and recover the odd case by looking at odd total degrees of the spectral sequence. Also, if G acts trivially the proposition is well known (compare [**7**]; the case $G=\mathrm{Pin}_2$ is not discussed there, but everything goes through), so let us assume $G=S^1$ and the action is nontrivial.

We shall adopt the notational scheme of (ii) for orbit types in the *even-dimensional case*, stipulating that $2m_j=$ dimension of $\mathrm{Fix}(S^W, H_j)$. Then the terms of the spectral sequence of [**68**] converging to $F/O\cdots\otimes\mathbf{Q}$ may be written as follows:

$$\begin{aligned}
&H^{d+j}\big(S^{2m_1+1};\pi_j(F_{S^1}(N)/U_n)\big)\otimes\mathbf{Q},\\
&H^{d+j}\big(S^{2m_1+1}\wedge\mathbf{C}P^{m_2-m_1-1};\pi_j(F_{H_2}(N)/C_{H_3}(N))\big)\otimes\mathbf{Q},\\
&H^{d+j}\big(S^{2m_1+1}\wedge\mathbf{C}P^{m_3-m_1-1}_{m_2-m_1};\pi_j(F_{H_3}(N)/C_{H_3}(N))\big)\otimes\mathbf{Q},\\
&\qquad\vdots\\
&\qquad\text{etc.}
\end{aligned}\tag{7.8}$$

(Notation: $\mathbf{C}P^s_t=\mathbf{C}P^s/\mathbf{C}P^{t-1}$ as in [**27**], [**46**].)

Actually, one deals with the rational cohomology of certain orbit space pieces derived from S^W, but the *rational* identification of these pieces with stunted projective spaces is a standard technique (e.g., see [**11**, Chapter III]). Since $F_{S^1}(N)/U_n$ is rationally acyclic (compare [**7**]), terms of the first type never appear. Furthermore, by the spectral sequences of [**67**] (or even more elementary considerations), $F_{H_j}(N)/C_{H_j}(N)$ has the same rational homotopy as BU_{n-1} if $H_j\neq\mathbf{Z}_2$ ($j\geq 2$) or BSO_{2n-1} if $H_j=\mathbf{Z}_2$ (the centralizer is either U_n or SO_{2n}). It follows that the rational spectral sequence vanishes in all even total degrees; thus (i) follows immediately. On the other hand, the spectral sequence obviously collapses because the domain or codomain of a differential is always zero, and (ii) follows by adding up all the nontrivial rational E_2 terms in particular odd degree. ■

We are now in a position to describe the map in (7.2). Since the statement in the finite case is rather long, we concentrate on that case first:

THEOREM 7.9. *Suppose G is finite cyclic and acts on W as in Lemma 6.6. Let V_j: $F_{H_j}(N)/C_{H_j}(N) \to BC_{H_j}(N)$ be the usual "fiber inclusion" (notation of 7.6 again) and let H_j have order k_j (hence $|G| = k_1 > \cdots > k_r = |H|$ and k_j/k_{j+1} is an integer).*

(Characteristic classes). *The kernel of the composite*

$$F/O_{G,\text{free},N}(S^W) \otimes \mathbf{Q} \xrightarrow{\text{map of (7.2)}} hS_W^G(S(N)) \otimes \mathbf{Q}$$
$$\xrightarrow{\eta} KO(S^W \wedge_G S(N)^+) \otimes \mathbf{Q}$$

(η = *normal invariant*) *equals the kernel of*

$$F/O_{G,\text{free},N}(S^W) \otimes \mathbf{Q} \underset{G\text{-action}}{\overset{\text{forget}}{\to}} \pi_{\dim W}(F/O) \otimes \mathbf{Q} \underset{\text{vector bundle}}{\overset{\text{underlying}}{\to}} KO(S^{\dim W}) \otimes \mathbf{Q}.$$

(Atiyah-Singer invariants). *The kernel of the Atiyah-Singer invariant map*

$$F/O_{G,\text{free},N}(S^W) \otimes \mathbf{Q} \to \begin{cases} [RO(\operatorname{Ker}\omega)/RO(1)] \otimes \mathbf{Q} \\ \text{or} \\ [R(\operatorname{Ker}\omega)/RO(\ker\omega)] \otimes \mathbf{Q} \end{cases} \tag{7.10}$$

(*see* (7.2B) *and* (7.2C) *for further description of the codomain*) *may be described as follows*:

For each j with $k_j > 2$ and each y in the domain of (7.10), *the vector bundle $V_{j*}(y) \otimes \mathbf{C}$ rationally admits an expression $\sum \xi_s \otimes t^s$ where ξ_s is a complex vector bundle over S^{n_j}, and s runs over all the primitive roots of unity* mod$|G|$, *with $\xi_{-s} = \bar{\xi}_s$. Then the kernel consists of all y for which*

$$(E_{qj})\sum \Phi_{n_j/2}(2\pi qs/k_j)c_{n_j/2}(\xi_s) \cap [S^{n_j}] = 0$$

is satisfied for each q and j, where j takes values as above and q runs over all integers mod k_j *for which* $2q \not\equiv 0 \bmod k_j/k_{j+1}$.

Notation. If n_j is ood all the terms in E_{qj} are taken to be zero. If n_j is even, then $c_{n_j/2}$ refers to the appropriate Chern class and $\Phi_{n_j/2}(\theta)$ is given by

$$\sum(-1)^m\Phi_m(\theta)z^m = 1 + iz\csc(iz - \theta)$$

(compare [**70**]).

PROOFS. The statement regarding the map into $KO(S^W \wedge_G S(N)^+) \otimes \mathbf{Q}$ follows because the inclusion

$$S^W = S^W \wedge_G G^+ \xrightarrow{1\wedge j} S^W \wedge_G S(N)^+$$

is monic in $KO \otimes \mathbf{Q}$ (an elementary verification). The statement about Atiyah-Singer invariants is essentially a direct generalization of [**70**, (2.2)]. In particular, the condition that $2q \equiv 0 \bmod k_j/k_{j+1}$ corresponds to the vanishing of the Atiyah-Singer invariant if $g \in G$ has an eigenspace with eigenvalue -1 (see [**75**, (2.2c)]). Since similar calculations have appeared in several other places [**25**], [**26**], [**70**], [**75**], [**89**] we shall not include a verification that the (E_{rj}) are the nontrivial restrictions forced upon y by the vanishing of the Atiyah-Singer invariant; using the given references, the reader should be able to do this himself. ■

Of course, the equations (E_{rj}) are not very informative as they stand; in general, to complete the calculation, one needs further information about the rational linear

independence of certain elements in algebraic number fields. Some fairly strong and effective results in this direction have been obtained by J. Ewing [25], [26]. On the other hand, there are some cases where one gets nontrivial elements in the kernel of (7.2) rather easily; the examples pursued in [75, Theorem A] illustrate this phenomenon.

In contrast to the finite case, the infinite case is remarkably simple.

THEOREM 7.11. *If $G = S^1$ or S^3 and the G-action satisfies the hypothesis of Proposition* 7.7, *then the map of* (7.2) *is zero.*

PROOF. There are no Atiyah-Singer invariants once again, so it suffices to check that the map

$$\begin{array}{c} F/O_{G,\mathrm{free},N}(S^W) \otimes \mathbf{Q} \to hS^G_W(S(N)) \otimes \mathbf{Q} \to [S^W \wedge_H S(N)^+, F/O] \otimes \mathbf{Q} \\ \downarrow \cong \\ \widetilde{KO}(S^W \wedge_G S(N)^+) \otimes \mathbf{Q} \end{array} \tag{7.11a}$$

is trivial. As in 7.4, it suffices to consider the case $G = S^1$. But $F/O_{G,\mathrm{free},N}(S^W) \otimes \mathbf{Q} = 0$ if W is even dimensional by Proposition 7.7, and thus we are left with the odd-dimensional case. To dispose of this case, note that the composite in (7.11a) can also be expressed as follows:

$$\begin{array}{c} F/O_{G,\mathrm{free},N}(S^W) \otimes \mathbf{Q} \to \widetilde{KO}_{S^1,\mathrm{free}}(S^W) \otimes \mathbf{Q} \\ \downarrow \text{forgetful map} \\ \widetilde{KO}_{S^1}(S^W) \otimes \mathbf{Q} \\ \downarrow \text{balanced product with } S(N) \\ \widetilde{KO}(S^W \wedge_G S(N)^+) \otimes \mathbf{Q} \end{array} \tag{7.11b}$$

Since all nontrivial irreducible real S^1-modules comes from *complex* S^1-modules, the odd-dimensional W may be rewritten as $1 \oplus W^*$ where W^* comes from a complex S^1-module. Therefore the Thom isomorphism theorem for K_G implies $K_G(S^W) = 0$; on the other hand, the rational complexification map $KO_G \otimes \mathbf{Q} \to K \otimes \mathbf{Q}$ is split injective, and thus $KO_G(S^W) \otimes \mathbf{Q} = 0$ in our case. It follows that the composite given by either (7.11a) or (7.11b) is zero.

If we combine this result with Proposition 7.7, we get some curious examples of S^1 actions that have no counterparts in the semifree case.

COROLLARY 7.12. *Let W be an odd-dimensional, noneffective S^1-module with linearly ordered isotropy subgroups. Assume the fixed point set has dimension $\geqslant 3$ and codimension $\geqslant 4$. If the ineffective kernel of the action is $\mathbf{Z}_2$ and C is the smallest isotropy subgroup properly containing $\mathbf{Z}_2$, further assume that* $\dim W - \dim W^C \geqslant 6$. *Then for every integer $n \geqslant (\dim W + 3)/2$, there exists an infinite family $F(n, W)$ of ultrasemifree actions on homotopy $(2n + \dim W)$-spheres with the following properties*:

(i) *If K is the ineffective kernel of the S^1 action on W, then K is a maximal subprincipal isotropy bound of the actions.*

(ii) *The fixed point set of each action in $F(n, W)$ is S^W.*

(iii) *The actions are distinguished by the equivalence class of the equivariant normal bundle of* S^W *in the ambient sphere, taken to lie in the group* $KO_{S^1,\mathrm{free}}(S^W) \otimes \mathbf{Q}$.

PROOF. The hypotheses were devised so that 7.7 would guarantee that $F/O_{S^1,\mathrm{free},n\mathbf{C}}(S^W) \otimes \mathbf{Q}$ is nonzero ($n\mathbf{C}$ denotes the usual $2n$-dimensional free S^1-module). Consequently, the result follows from the calculation of 7.11 and the exactness of (6.2). ■

COMPLEMENT TO 7.12. (iv) *For each* (n, W) *as above, an infinite subfamily of the actions in* $F(n, W)$ *are on the standard sphere* (*take connected sums along the fixed point set*). ■

FINAL REMARK. Since $KO_{S^1}(S^W) \otimes \mathbf{Q} = 0$, it follows that some nonzero multiple of the equivariant normal bundle of S^W in any such Σ becomes trivial when a suitable product bundle is added. However, the nontriviality of this class in $KO_{S^1,\mathrm{free}} \otimes \mathbf{Q}$ implies that the product bundle to be added is *not* a *free* S^1-module (compare the discussion in [**75**, §1], especially the example following (1.6)).

8. Actions on the exotic 8-sphere. The original motivation for the results in this paper (and its successors) was the degree of symmetry problem for homotopy spheres. In particular, one would like to know whether every homotopy sphere admits smooth effective actions of rank one compact Lie groups. Empirical evidence strongly suggests that every exotic sphere of dimension $\geqslant 7$ admits a smooth effective circle action. For dimension $\leqslant 13$ this has been done in print except for the generator of $\Theta_{10} = \mathbf{Z}_6$, and we shall do this case in Part III. The results of Part VI raise the bound on n still further, and it seems clear that one could proceed indefinitely with enough perseverance and explicit knowledge of the way elements in π_* are built from lower-dimensional ones. Although a more global approach is surely preferable, at this time it seems just out of reach.

In contrast to the case of circle actions, analytic methods imply that some exotic spheres admit no effective smooth actions of S^3 or SO_3. In particular, the existence of such an action implies that the exotic sphere bounds a spin manifold [**93**]. It is known that all exotic spheres bounding π-manifolds admit smooth SO_3-actions and, in dimensions $\geqslant 9$, effective S^3 actions [**30**]. Furthermore, it is known that many exotic spheres not bounding π-manifolds admit effective semifree S^3 actions [**14**]. At this time it is not known whether every spin boundary admits a smooth effective action of a nonabelian rank one group. However, the following result suggests that some do not:

THEOREM 8.1. *Let* Σ^8 *be the exotic* 8-*sphere* (*unique up to orientation-preserving diffeomorphism* [**38**]). *Then* Σ^8 *admits no effective smooth* S^3 *action.*

We shall prove this by an extensive study of the symmetry properties of an exotic 8-sphere. With very little work we can prove somewhat more:

THEOREM 8.2. *The only compact connected Lie groups that can act effectively and smoothly on* Σ^8 *are* S^1, T^2, *and* SO_3.

In the S^1 case we know an action exists [**66**]; both the remaining cases are still undecided.

Theorem 8.1 raises a curious point that one might have guessed was already settled. Namely, *does an exotic-seven-sphere admit an effective smooth S^3 action*? If such actions exist, they must resemble the pseudofree circle actions considered by Montgomery and Yang and others (e.g., [**20**]).

The first step in proving Theorem 8.1 is technically quite simple.

PROPOSITION 8.3. *Let $\mathbf{Z}_4$ act effectively, orientation-preservingly and smoothly on Σ^8 as above. Then the fixed point set of $\mathbf{Z}_2$ is 4-dimensional.*

PROOF. In view of [**72**, 4.16] it suffices to exclude the case where $\mathbf{Z}_2$ has a zero-dimensional fixed point set. But if the fixed point set of $\mathbf{Z}_2$ is zero dimensional, it follows that $\mathbf{Z}_4$ must act semifreely with zero-dimensional fixed point set. As in the work of R. Lee [**40**], we know that such actions are essentially given by diffeomorphisms of a 7-dimensional $\mathbf{Z}_4$-lens space L^7 homotopic to the identity (in our setup, this corresponds to the vanishing of the knot invariant). By surgery (compare [**69**]) this means the Pontrjagin-Thom invariant of an exotic sphere M^8 admitting such an action must lie in the image of

$$[\Sigma L^7, F/O] \xrightarrow{(\Sigma p)^*} [\Sigma S^7, F/O] = \pi_8(F/O),$$

where $p\colon S^7 \to L^7$ is projection. Inspection of the first k-invariant of the space Coker J localized at 2 and the action of the mod 2 Steenrod algebra on $S^9/\mathbf{Z}_4$ shows that the Pontrjagin-Thom invariant of Σ^8 does not lie in this image. (One can replace Coker $J_{(2)}$ with the two stage system

$$E_2 = \text{fiber } k_1\colon K(\mathbf{Z}_2, 6) \to K(\mathbf{Z}_2, 9).)$$

Suppose now that F^4 is the fixed point set of $\mathbf{Z}_2$ for some effective smooth S^3 action on a homotopy 8-sphere. Since $\mathbf{Z}_2$ is a central subgroup, it follows that F^4 is S^3 invariant and thereby inherits a smooth action of SO_3. The following is now a consequence of considerations from [**58**, p. 277]:

PROPOSITION 8.4. *If SO_3 acts smoothly and nontrivially on a $\mathbf{Z}_2$ homology sphere F^4, then the action is smoothly equivalent to a linear action on S^4.* ■

There are then three possible cases, which we label for future use:

(8.4A) *SO_3 acts trivially on F^4.* We treat this case–which has been long understood–in the first paragraph of §9. It turns out that F^4 must be a $\mathbf{Z}$-homology sphere bounding a contractible 5-manifold K, and in fact Σ is diffeomorphic to S^8. Therefore *we shall disregard this case for the rest of this section.*

(8.4B) *F^4 is $S^{\rho+1}$, where ρ is the standard representation of SO_3 on $\mathbf{R}^3$.*

(8.4C) *F^4 is the unit sphere in $\Sigma^2(\rho) - 1$, the representation obtained by factoring out a trivial 1-dimensional representation from the symmetric 2-tensors on ρ.*

In order to make the proof less unpleasant, we shall postpone the proof of the next result to §9; the ideas are easy, but the details are tedious.

THEOREM 8.5. *Let S^3 act effectively on the homotopy sphere Σ with fixed point set of $\mathbf{Z}_2$ given by F^4. Let $\nu \downarrow F^4$ be the S^3-equivariant normal bundle of F^4. Then ν is S^3-trivial in case* (B), *and ν is* Pin_2*-trivial in case* (C).

PROOF OF THEOREM 8.1. Consider the action restricted to $G = Q_{2^n}$ for some large quaternionic group $Q_{2^n} \subseteq \mathrm{Pin}_2 \subseteq S^3$. Let $F^4 = S^{(W)}$ and let V be the normal representation of G at the fixed point set. By 8.5 the knot invariant lies in the image of

$$\left[S^{(W)}, F(V)\right]_G, \tag{8.6}$$

where we adopt the notation of §§2–4. If we stabilize this knot invariant (or more accurately, some chosen preimage in (8.6)) by 3.5, we obtain a class ω in the group

$$\pi_4^S(BG^{4-W}). \tag{8.7}$$

We next locate this class in the Atiyah-Hirzebruch spectral sequence, at least for some suitably large quaternionic subgroup. Since $H_4(G; \mathbf{Z}) = 0$, we cannot find ω in $E_{4,0}^\infty = 0$. Consider next $H_3(G; \mathbf{Z}_2) = \mathbf{Z}_2$. If we knew that

$$\mathrm{Sq}^2\colon H^3(BG^{4-W}) \to H^5(BG^{4-W}) \tag{8.8}$$

was nonzero ($\mathbf{Z}_2$-coefficients), then as in **[14]**, **[72]** it would follow that the Atiyah-Hirzebruch spectral sequence for (8.7) has $d_{5,0}^2\colon E_{5,0}^2 \to E_{3,1}^2$ nontrivial, being multiplication by η on one summand and zero on the other. Therefore we digress:

SUBLEMMA 8.9. *The map in* (8.8) *is nonzero.*

PROOF OF SUBLEMMA. It is a straightforward exercise to prove that the first two Stiefel-Whitney classes of the vector bundle determined by $-W$ are zero. If one recalls that the representation $1 + W$ actually extends to a representation of S^3 with $\mathbf{Z}_2$ acting ineffectively, the calculation is in fact trivial.

Next, notice that the transfer map $H_3(Q_N; \mathbf{Z}_2) \to H_3(\mathbf{Z}_2; \mathbf{Z}_2)$ is onto, where Q_N denotes a large quaternionic group (look over the integers first, where the map is just a surjection of cyclic groups). It follows then that the restriction map in $H^3(\ ; \mathbf{Z}_2)$ is also onto, and we may proceed using the functoriality of Sq^2. ■

Returning to our main line of reasoning, we are now interested in seeing if ω is nontrivial in $E_{2,2}^\infty$. If $Q' \subseteq Q$ is an inclusion of quaternion groups with index 2, then the transfer map

$$\begin{array}{c} H_2(Q; \mathbf{Z}_2) \to H_2(Q'; \mathbf{Z}_2) \\ = \\ \mathbf{Z}_2 \oplus \mathbf{Z}_2 \to \mathbf{Z}_2 \oplus \mathbf{Z}_2 \end{array}$$

is trivial on one $\mathbf{Z}_2$ factor and nontrivial on the other. In $E_{2,2}^2$ the $\mathbf{Z}_2$ corresponds to π_2 and, hence, there is a nontrivial composition operation from $E_{2,1}^2$ to $E_{2,2}^2$ in the spectral sequence corresponding to composition with η (also compare **[67**, §6**]**). We then get the following information:

SUBLEMMA 8.10. *The image of* $\pi_4(BQ^{4-W}) \to \pi_4(BQ'^{4-W})$ *is generated by* $\eta\pi_3(BQ'^{4-W})$ *and elements of filtration* $\leqslant 1$. *In fact, the* η*-composites may be assumed to have filtration* $\leqslant 2$.

REMARK ON PROOF. The only serious problem could be that $d_{2,1}^2\colon E_{2,1}^2 \to E_{0,2}^2$ would be nonzero; i.e., multiplication by η. This cannot happen because $H^0 \to H^2$ is trivial by the methods used in 8.9. ■

This brings us to the calculations we really need.

PROPOSITION 8.11. *If the $C \subseteq Q$ is the index two cyclic group and likewise for $C' \subseteq Q' \subseteq Q$, then the umkehr map $\pi_4(BQ^{4-W}) \to \pi_4(BC'^{4-W})$ has exponent two and lies in filtration $\leqslant 1$. Furthermore, for sufficiently large Q, the image of $\pi_4(BQ^{4-W})$ in $\pi_4(B\mathbf{Z}_4^{4-W})$ is trivial.*

PROOF. Since $\pi_4 = 0$ we know $E^2_{0,4} = E^\infty_{0,4}$ everywhere. Since $H_1(Q'; A)$ has exponent two for all coefficients A, we see that the image of $\pi_4(BQ^{4-W})$ in $\pi_4(BQ'^{4-W})$ has exponent two by 8.10 and naturality of the spectral sequence. On the other hand, the transfer map $H_2(C; \mathbf{Z}_2) \to H_2(C'; \mathbf{Z}_2)$ is trivial, and therefore by naturality the image of $\pi_4(BQ^{4-W})$ in $\pi_4(BC'^{4-W})$ also must have filtration $\leqslant 1$. To prove the claim about $\mathbf{Z}_4$, let C' have order $\leqslant 16$. It then follows that $E^2_{3,1}(Q') \to E^2_{3,1}(\mathbf{Z}_4)$ is trivial, and therefore the image of $\pi_4(BQ^{4-W})$ in $\pi_4(B\mathbf{Z}_4^{4-W})$ is generated by composition with η on elements in $\pi_3(BQ'^{4-W})$ having filtration $\leqslant 2$. Let C'' be a cyclic group strictly between $\mathbf{Z}_4$ and C', and suppose that $\alpha \in \pi_3(BQ'^{4-W})$ has filtration $\leqslant 2$. Then by naturality of the spectral sequence it follows that the image of α in $\pi_3(BC''^{4-W})$ has filtration $\leqslant 1$. Therefore in $E^\infty(B\mathbf{Z}_4^{4-W})$ the class $\alpha\eta$ is given by a composition operation $\eta^*\colon E^\infty_{1,2} \to E^\infty_{1,3}$. But in E^2 this merely corresponds to the map $\eta^*\colon \pi_2 \otimes \mathbf{Z}_4 \to \pi_3 \otimes \mathbf{Z}_4$, and by $\eta^3 = 4\nu$ this is trivial. ■

Having made these rather tedious calculations, the rest of the proof of Theorem 8.1 is relatively easy. By 4.8 we know that the Pontrjagin-Thom invariant of the exotic sphere Σ^8 can be recovered from

(i) the fixed point set F of $\mathbf{Z}_2$ as a $\mathbf{Z}_4$-manifold,

(ii) the equivariant normal bundle of F in Σ,

(iii) the knot invariant of Σ as a $\mathbf{Z}_4$-manifold.

By 8.4 the action on F is linear, and therefore in the formula ($G = \mathbf{Z}_4$)

$$(1 \times_G \kappa)^* c^*(-q(\Sigma) + \gamma') = q\theta[\omega_0 \oplus M] + q(1 \times_G \kappa)$$

the term $q(1 \times_G \kappa)$ vanishes. Likewise, the term $q\theta[\omega_0 \oplus M]$ vanishes because (i) the Pin_2-equivariant normal bundle is trivial, (ii) the calculations in 8.11 say that ω_0 must then be at least *stably* trivial, (iii) $q\theta[\omega_0 \oplus M]$ only depends on the stabilization of ω_0 by the remarks in §4.

It follows that $c^*Q(\Sigma^8) = 0$; by the Puppe sequence for the cofibering

$$(S^3)^+ \wedge_{\mathbf{Z}_4} S^W \to (X^4)^+ \wedge_{\mathbf{Z}_4} S^W \to S^8$$

(X^4 = inverse image of 4-skeleton $S^5/\mathbf{Z}_4$ in S^5), this implies that $q(\Sigma)$ lies in the image of the map

$$\left[\Sigma\{(S^{3+}) \wedge_{\mathbf{Z}_4} S^W\}, F/O\right]_{(2)} \to \pi_8(F/O)_{(2)}.$$

But now an argument similar to the one excluding semifree $\mathbf{Z}_4$ actions with two fixed points can be used to prove that $q(\Sigma)$ must vanish. ■

In §10 we shall use the same types of methods to prove the following result:

PROPOSITION 8.12. *Let S^1 act smoothly and effectively on the exotic 8-sphere Σ^8. Then the knot invariant of the induced $\mathbf{Z}_4$ action is nontrivial.* ■

This result will be used in paper III of this series.

PROOF OF THEOREM 8.2. Aside from S^1, T^2, and SO_3 every compact connected Lie group contains a copy of S^3, T^3, or $SO_3 \times SO_3$. In the latter two cases $\mathbf{Z}_2^3$ and $\mathbf{Z}_2^4$ would have smooth effective actions on Σ^8; furthermore, in the first case the fixed point set is nonempty and even dimensional. Since each element of T^3 or $SO_3 \times SO_3$ (indeed any compact connected Lie group) lies on a circle subgroup, by 8.3 we know that each $\mathbf{Z}_2$-subgroup has a 4-dimensional fixed point set. It is now a simple exercise in the use of the Borel formulas [**9**, Chapters XII–XIII] to prove that no such actions of $\mathbf{Z}_2^4$ are possible. ■

9. Proofs of (8.4A) and (8.5). We first consider case (8.4A). If SO_3 acts trivially on F^4, then by Smith theory it follows that S^3 must act semifreely. In this case it is well known that Σ^8 is equivariantly diffeomorphic to $\partial(K^5 \times D^4)$ (with rounded corners), where K^5 is a suitable contractible smooth 5-manifold with boundary F^4 [**30**], [**92**]. Triviality of the equivariant normal bundle is immediate as is $\Sigma^8 \cong S^8$.

Consider now Case B (8.4B). In this case S^3 acts on Σ^8 with fixed points. If $x_0 \in F^4$ is a fixed point, write the local representation at x_0 as $W \oplus V$, where W corresponds to the fixed set of $\mathbf{Z}_2$. The only admissible choice for V is then the standard free 4-dimensional representation of K.

It suffices to consider now the knot invariant of Σ, which is a class in

$$F/O_{S^3,\mathbf{K},\mathrm{free}}(S^{\rho+1}). \tag{9.1}$$

Suppose we restrict to the fixed point set. Then by the discussion in §2 we get an element of $\pi_1(F_{S^3}(\mathbf{K})/Sp_1)$, which by [**7**] or [**67**] is zero. Thus the knot invariant lifts back to

$$F/O_{S^3,\mathbf{K},\mathrm{free}}(S(\rho) \times D^2/S(\rho) \times S^1). \tag{9.2}$$

Since $S(\rho) = S^3/S^1$, we may apply a variant of 2.1 to show that (9.2) is isomorphic to

$$F/O_{S^1,K,\mathrm{free}}(D^2/S^1). \tag{9.3}$$

But the latter is just $\pi_2(F_{S^1}(\mathbf{K})/U_2)$, which is not zero but nevertheless maps trivially into $\pi_1(U_2) = \mathrm{Vect}_{S^1,\mathbf{K},\mathrm{free}}(D^2/S^1)$. From this we see the map

$$F/O_{S^3,\mathbf{K},\mathrm{free}}(S^{\rho+1}) \to \mathrm{Vect}_{S^3,\mathbf{K},\mathrm{free}}(S^{\rho+1}) \tag{9.4}$$

is trivial, and therefore the equivariant normal bundle of F and Σ must be trivial as claimed.

In the final case $F = S(\Sigma^2\rho - 1)$ *we must restrict to the subgroup*, Pin_2 in order to get an action on F with a fixed point. The restriction of $S^2(\rho) - 1$ to O_2 is readily determined to be $1 + W_2 + W_1$, where W_1 is the standard two-dimensional representation of O_2 and W_2 is given by the two-dimensional representation associated to the homomorphism $\psi\colon O_2 \to O_2$ with $\psi(z) = z^2$ for $z \in SO_2$ (note that $O_2/\{\pm 1\} \cong O_2$). The representation of Pin_2 on a fixed point of F now takes the form $1 + W_2 + W_1 + V$, where *every 2-primary element of* Pin_2 *acts freely on* V. Notice that an application of the usual Atiyah-Bott method [**5**] plus the classification of $\mathbf{Z}_4$-representations show that the local representations at both fixed points coincide (Pin_2 consists of a circle plus elements of order 4).

Let $F_2 \subseteq F$ denote the fixed point set of $\{\pm 1\} \subseteq O_2$. Then the restriction of ν to F_2 as a Pin_2-bundle is given by a Pin_2-equivariant automorphism of V; i.e., an element of the centralizer of V. Another tedious, but direct, calculation shows that V is irreducible of symplectic type, and hence the automorphism may be deformed to the identity. Hence the bundle restricted to F_2 is trivial.

Finally, ν must come from a bundle over

$$\begin{aligned} F/F_2 &= S(W_1) \times D(W_2 \oplus 1)/S(W_1) \times S^{W_2} \\ &= O_2/O_1 \times D(W_2 \oplus 1)/O_2/O_1 \times S^{W_2} \\ &= O_2 \times_{O_1} D(W_2 \oplus 1)/O_2 \times_{O_1} S^{W_2}. \end{aligned} \tag{9.5}$$

Thus the Pin_2-vector bundles over F/F_2 are (by another variant of (2.1)) just the free $\mathbf{Z}_4$-vector bundles over $D(2+T)/S(2+T) = S^{2+T}$, where T denotes the nontrivial one-dimensional representation of $\mathbf{Z}_4$.

Consider the restriction of this bundle to S^2. By retracting the definitions and the proof of 2.1, it is fairly straightforward to prove that this restriction merely corresponds to the normal bundle of the fixed point set of a $\mathbf{Z}_4 \subseteq \mathrm{Pin}_2$ not in the identity component. Since we do know that this $\mathbf{Z}_4$ action extends to an S^1 action (since $\mathrm{Pin}_2 \subseteq S^3$) this equivariant normal bundle is trivial by the result of [**75**, §6]. Thus we are left to consider a free $\mathbf{Z}_4$-vector bundle over

$$S^{2+T}/S^2 = (\mathbf{Z}_4/\mathbf{Z}_2) \times D^3/(\mathbf{Z}_4/\mathbf{Z}_2) \times S^2.$$

Again this amounts to considering a free $\mathbf{Z}_2$ vector bundle over $D^3/S^2 = S^3$, and such a bundle is trivial because $\pi_2(G) = 0$ for every compact Lie group G. Therefore in the third case the equivalent normal bundle for the induced Pin_2 action is trivial. ■

REMARK. It is by no means clear that V must be a semifree representation of Pin_2, and we have not used such an assumption anywhere above. In fact, it would not be surprising to learn that counterexamples exist.

10. Proof of (8.12). First of all, let us indicate that the proof reduces to verifying the following assertions.

(10.1) *F^4 is stably framable as a $\mathbf{Z}_4/\mathbf{Z}_2$-manifold.*

(10.2) *F^4 is a $\mathbf{Z}_2$ framed boundary in the appropriate sense.*

Suppose these are true. A routine check of the definitions shows that the normal invariant of the map

$$1 \times_G \kappa\colon S(V \oplus M) \times_G F^4 \to S(V \oplus M) \times_G S^W$$

(here $G = \mathbf{Z}_4$) is merely $j_*(\mathrm{id}_{S(V\oplus M)} + \wedge_G \mathfrak{B}(F^4))$, where $\mathfrak{B}(F^4) \in \pi_W^{\mathbf{Z}_2}$ is the equivariant Pontrjagin-Thom construction on F^4 and $j\colon QS^0 \to F \to F/O \to F/O_{(2)}$ is the usual map. Since $\mathfrak{B}(F) = 0$, the normal invariant of $1 \times_G \kappa$ is trivial too. If the knot invariant of the action were also trivial, we would be exactly in the situation at the end of §8 and could prove that Σ^8 is the standard sphere.

There are three cases, depending upon whether $\mathbf{Z}_2$ has F^4, S^2, or S^0 as its fixed point set.

Case I. The fixed point set is F^4; i.e., $\mathbf{Z}_4/\mathbf{Z}_2 = \mathbf{Z}_2$ acts trivially on F. By obstruction theory and the Hirzebruch signature theorem, F^4 is stably parallelizable. But $\pi_4 = 0$ and hence F^4 is also a framed boundary.

Case II. The fixed point set is S^2. In this case we need two lemmas:

(10.3) *Let F^4 be a smooth $\mathbf{Z}_2$-homology sphere with a smooth involution having S^2 as fixed point set. Then F^4 is stably $(2 + 2T)$ framable in the sense of Segal* **[80]**.

Here $2 + 2T$ denotes two trivial one-dimensional $\mathbf{Z}_2$-representations plus two nontrivial ones.

(10.4) *The equivariant stable homotopy group $\pi_{2+2}^{\mathbf{Z}_2}$ (notation of* **[80]**) *is isomorphic to π_2 under restriction to the fixed point set.*

Case II is immediate from these because the nonzero class in π_2 contains no spherical representatives (as framed bordism group).

Proof of (10.3). We want to show that the class of τ_F in $KO_{\mathbf{Z}_2}(F^4)$ is zero. Consider the first restriction to S^2, which is $\tau_{S^2} \oplus \nu(S^2, F^4)$. First the summand is, of course, stably trivial, while the second summand is detected by its second Stiefel-Whitney class. By a standard characteristic class argument **[55]** the latter is automatically zero in our case. Thus τ_M lies in the image of $KO_{\mathbf{Z}_2}(M^4/S^2)$.

The functors KO_G are representable (compare **[50]**) and in analogy with **[24]** and **[68]** one has spectral sequences for this half exact functor. In particular, we have a spectral sequence with

$$E_2^{i,i} = \mathcal{H}^2(F^4/\mathbf{Z}_2;\, S^2;\, \pi_i(BO)) \Rightarrow KO_{\mathbf{Z}_2}(M, S^2).$$

Some routine homological calculations show that only $E_2^{4,4} \cong \mathbf{Z}$ is nonzero, and the naturality of everything with respect to restriction of the acting group (from $\mathbf{Z}_2$ to (1) here) shows that the nonzero elements are detected by their Pontrjagin classes through the nonequivariant group $KO(F^4) \otimes \mathbf{Q} = \mathbf{Q}$. But F^4 is a rational homology sphere, and therefore the top Pontrjagin class vanishes by the Hirzebruch signature theorem. ■

Proof of (10.4). The restriction map to the fixed point set fits into a long exact sequence of Conner-Floyd type:

$$\rightarrow \pi_2^S(P_{-2}^\infty) \rightarrow \pi_{2+2T}^{\mathbf{Z}_2} \xrightarrow{\rho} \pi_2^S \rightarrow \pi_1^S(P_{-2}^\infty). \tag{10.5}$$

This is described (for example) in **[45]**; actually the sequence presented has a metastable homotopy group for a Stiefel manifold in place of the stable homotopy of the stunted projective complex $\mathbf{R}P_{-2}^\infty$ (notation of **[33]**), but these groups agree by S-duality and the metastable equivalence between Stiefel manifolds and stunted projective spaces. Löffler has observed that ρ is onto **[45]**, and therefore it suffices to prove that the stable group $\pi_2^S(\mathbf{R}P_{-2}^\infty) = 0$.

Consider the Atiyah-Hirzebruch spectral sequence for this stable group with $E_{i,j}^2 = H_i(\mathbf{R}P_{-2}^\infty;\, \pi_j)$. By the same types of manipulations considered in **[72**, §4] and §5 of this paper, we have that $d_{i,j}^2$ is multiplication by η if $i \equiv 0, 1 \bmod 4$ and $d_{i,j}^4$ is multiplication by ν if $i \equiv 0, 1, 2, 3 \bmod 8$. From these relations it follows that $E_{i,j}^5 = 0$ if $i = j = 2$. ■

Remark. In **[96]**, **[97]** Bredon has already calculated the groups $\pi_W^{\mathbf{Z}_2}$ in many low-dimensional cases, and several of our calculations here can be extracted from

his work. The proof included here are meant to illustrate how things may be done from the viewpoint of this paper.

Case III. The fixed point set is S^0. In this case we need two lemmas similar to those needed for Case II.

(10.6) *If F^4 (as before) has S^0 as fixed point set, then F^4 is stably $4T$-framable.*

(10.7) *The map $\pi_{4T}^{\mathbf{Z}_2} \to \pi_0$ induced by restriction to the fixed point set is injective.*

Once again we see that for F^4 the fixed point set image is trivial *provided the involution extends to a smooth S^1 action.* For the Atiyah-Bott formula [5] implies that the induced orientation on S^0 yields an oriented boundary (i.e., the points must have opposite orientations).

PROOF OF (10.6). Consider first the restriction of τ_F to the fixed point S^0. Since the local representations at both points are $4T$, there is a lifting of τ_F to $KO_{\mathbf{Z}_2}(F/S^0)$; in fact, the result of Atiyah and Bott provides a canonical (up to isotopy) orientation preserving $\mathbf{Z}_2$-isomorphism that we shall call A.

Another useful observation is that the usual collapse map $f\colon F \to S^W$ is an isomorphism in equivariant KO-theory; to see this, notice that $F_0 = F - \operatorname{Int} D(W)$ is $\mathbf{Z}_2$-acyclic, and thus with a little effort one can prove that $\widetilde{KO}_{\mathbf{Z}_2}(F_0)$ is zero (recall that F_0 is a bounded 4-manifold and thus no homology above dimension 4). From this and a short diagram chase the assertion on F follows. Notice that the same thing is true for $F/S^0 \to S^W/S^0$.

If $\xi \in \widetilde{KO}_{\mathbf{Z}_2}(F/S^0)$ represents a canonical preimage of the tangent bundle, we can recover ξ as follows: Glue the top and bottom of $F^4 - 2 \operatorname{Int} D(4T)$ together by A, and take the mod 2 quotient X of the resulting free $\mathbf{Z}_2$-manifold. Then X maps onto $\Sigma(\mathbf{R}P^{3+})$ by collapsing the complement of a bicollar about $\mathbf{R}P^3$, and τ_S is the image of ξ under the composite

$$KO_{\mathbf{Z}_2}(F/S^0) \cong KO(F/\mathbf{Z}_2, S^0) \cong KO(S/\mathbf{Z}_2, S^0) \cong \widetilde{KO}(\Sigma(\mathbf{R}P^{3+})) \to \widetilde{KO}(X).$$

Now it is fairly routine to construct a map

$$g\colon X_0 = (F^4 - 2 \operatorname{Int} D(4T))/\mathbf{Z}_2 \to \mathbf{R}P^3$$

by obstruction theory, and it follows that g induces a $\mathbf{Z}_2$-homology equivalence of manifold triads from $(X_0; \mathbf{R}P^3; \mathbf{R}P^3)$ to $\mathbf{R}P^3 \times (I; 0, 1)$ that is homotopic to the identity on both ends. It follows that X has the 2-local homology of $S^1 \times \mathbf{R}P^3$, and this is induced by a map $\bar{g}\colon X \to S^1 \times \mathbf{R}P^3$.

By the Wu formulas and the parallelizability of $S^1 \times \mathbf{R}P^3$ we know that $w_1(X) = w_2(X) = 0$. Thus the only obstruction to stable parallelizability of X is the top Pontrjagin class, and by the Hirzebruch theorem this must vanish too. So ξ goes to zero under the composite below:

$$\begin{tikzcd}
 & KO(S^1 \times \mathbf{R}P^3) \arrow[d, "\bar{g}^*"] \\
KO(\Sigma(\mathbf{R}P^{3+})) \arrow[ur, "c^*"] \arrow[r] & KO(X)
\end{tikzcd}$$

But $\bar{g}^*$ is an isomorphism by spectral sequence considerations and c^* is split injective, and therefore ξ itself must also be zero. ■

PROOF OF (10.7). Consider the Conner-Floyd type sequence analogous to (10.5) occurring here:

$$\pi_1 \xrightarrow{\gamma} \pi_0(\mathbf{R}P^\infty_{-4}) \to \pi^{\mathbf{Z}_2}_{4T} \to \pi_0. \tag{10.8}$$

We first consider $\pi_0(\mathbf{R}P^\infty_{-4})$ and the Atiyah-Hirzebruch spectral sequence for computing it. By the same types of arguments used for (10.4) we find that the E^∞ term is concentrated in filtration -3. It is known (compare [**94**]) that this class corresponds to the map $S^0 \xrightarrow{\eta} S^{-1} \xrightarrow{\rho} \mathbf{R}P^\infty_{-4}$ where ρ is defined as in [**94**]. Using bordism theory it is a standard sort of exercise to verify the next result.

PROPOSITION 10.9. *In the Conner-Floyd type sequences*

$$\pi_{k+1} \xrightarrow{\gamma} \pi_k(\mathbf{R}P^\infty_{-n}) \to \pi^{\mathbf{Z}_2}_{k+nT} \to \pi_k \tag{10.10}$$

the map γ corresponds to the composition

$$\rho_*\colon \pi_{k+1} = \pi_k(S^{-1}) \to \pi_k(\mathbf{R}P^\infty_{-n}). \quad ■$$

It is now clear that γ in (10.8) is onto, and this yields (10.7) immediately. ■

Postscript. The first complete draft of this manuscript was written in the Summer of 1977 (although portions were first written in mid-1974), and since then there have been two major revisions (including this). In the course of these revisions a few items were deleted. To keep the record straight we shall describe them here.

The first draft did not contain the present §§8–10 but instead concluded with a proof of a theorem announced in [**98**, Problem 8, p. 261]. This and the supporting machinery will appear in a more detailed study of fixed point sets.

The second draft presented a few calculations in the spirit of §§8–10 that do not appear here (unfortunately–as noted before–a key one was incorrect). One general principle relating the methods of [**7**] and [**67**] more closely is somewhat interesting in its own right. In [**67**] a spectral sequence was given for $\pi_*(F_G)$, and in [**7**] it was shown that $\pi_*(F_G)$ was isomorphic to $\pi^S_*(BG^\zeta)$. There is a simple reason for guessing that the latter is true on the basis of [**67**]; namely, the spectral sequence looks suspiciously like the Atiyah-Hirzebruch spectral sequence for $\pi^s_*(BG^\zeta)$. In the second draft of this paper we outlined a proof that the two spectral sequences were in fact isomorphic using methods from [**7**] and [**67**]. A more complete version of this proof should appear some time in the future.

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DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907