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INVOLUTIONS ON KLEIN SPACES $M(p, q)$

BY

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ABSTRACT. The Klein spaces $M(p, q)$ are defined (up to homeomorphisms) to be the class of closed, orientable, irreducible 3-manifolds with finite fundamental groups, in which a Klein bottle can be embedded. Their fundamental groups act freely on the 3-sphere S^3 in the natural way. We obtain a complete classification of the PL involutions on Klein spaces $M(p, q)$. It can be applied to the study of some transformation group actions on S^3 and double branched coverings of S^3.

0. Introduction. In an earlier study [**13**], we have defined the spaces $M(p, q)$ for coprime integers p, q. For convenience, we do not consider $S^1 \times S^2$ and $P^3 \# P^3$ as spaces $M(p, q)$. The spaces $M(p, q)$, which we shall call *Klein spaces*, are defined (up to homeomorphism) to be the class of closed, orientable, connected, irreducible 3-manifolds with finite fundamental groups, in which a Klein bottle can be embedded. In the present paper, we obtain a complete classification of the PL involutions on Klein spaces $M(p, q)$.

Recall [**13**] that $M(p, q)$ is homeomorphic to $M(p', q')$ if and only if $p = \pm p'$ and $q = \pm q'$, and $M(1, q)$ is homeomorphic to a lens space $L(4q, 2q - 1)$. We obtain the following classification theorems for PL involutions on Klein spaces $M(p, q)$.

THEOREM A. *A Klein space $M(p, q)$, $p > 1$, admits exactly five distinct nonconjugate PL involutions if $q > 1$, and three if $q = 1$. No such Klein spaces admit free involutions for q even.*

The case where $p = 1$ is stated in the following, which is of special interest. Theorem B classifies all PL involutions on the lens spaces $L(4q, 2q - 1)$.

THEOREM B. (1) *A lens space $L(4q, 2q - 1)$, $q > 1$, admits exactly eight distinct nonconjugate PL involutions if q is odd, and six if q is even.*

(2) *The lens space $L(4, 1)$ admits exactly five distinct nonconjugate PL involutions.*

The above two theorems are consequences of §5(I), (VII). The list of standard involutions is given in the same section. The involutions may be distinguished by the orbit spaces, which are again Klein spaces or lens spaces.

We shall review in §1 the definition of Klein spaces $M(p, q)$ [**13**]. Klein spaces can also be defined in terms of Seifert fibre spaces [**28**], as J. Rubinstein [**26**] has

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obtained independently. One may find that our direct definition **[13]** of Klein spaces $M(p, q)$ is quite workable for our purpose.

A main tool of the proofs of Theorems A and B is given in Theorem C. Theorem C for the case of free involutions has been obtained by Rubinstein **[26]**. The proof of Theorem C (§4), however, contains all the cases. It seems likely that the equivariant surgery on nonorientable surfaces developed in this paper will help to open up the study of involutions on 3-manifolds which contain incompressible, nonorientable surfaces, especially with finite fundamental groups.

THEOREM C. *Let h be a PL involution on $M = M(p, q)$. Then there exists an invariant torus S embedded in M such that S separates M into either a twisted I-bundle over a Klein bottle and a solid torus, or two solid tori. The former can be chosen in case the fixed-point set* $\text{Fix}(h)$ *is not empty.*

As applications of Theorems A and B, we obtain Corollaries 1 and 2. Let G be a group of order 8. The only such groups G which can act freely on the 3-sphere S^3 are Z_8 and Q (Quaternions). It has been shown **[25]**, **[26]**, **[6]** that a free action of Z_8 or Q on S^3 is conjugate to an orthogonal action. Indeed, every Q-action on S^3 acts freely (see §6), and the classification problem for the group actions on S^3 of Quaternions is essentially settled. We work in the PL category.

COROLLARY 1. *Suppose that G acts on S^3 and contains an element of order 4 acting freely. Then G is conjugate to an orthogonal action. The orbit space S^3/G is homeomorphic to S^3, P^3, $M(2, 1)$, or $L(8, q)$ $(q = 1, 3)$ according to $G = D_4$ (dihedral group), $Z_2 \times Z_4$, Q, or Z_8.*

To say that an element a of G acts freely means that $\langle a \rangle$ acts freely. Basic terminology on group actions in this paper (such as equivalence, weak-equivalence, equivariant map) may be found in **[1]**.

Let L_1 and L_2 be two links in S^3. Let $M(L_1)$ and $M(L_2)$ be the unique double branched coverings of S^3 branched over L_1 and L_2. It has been an open question of whether L_1 and L_2 are equivalent whenever $M(L_1)$ and $M(L_2)$ are homeomorphic as lens spaces (if one drops lens spaces in the hypothesis, it is well known that this is not true in general). The question, of interest in its own right, is directly related to the classification problem for PL involutions on lens spaces. As a direct consequence of Theorems A and B (see also Theorems 5.18, 5.19), we obtain a partial solution for the question. Note that $M(p, q)$ is a lens space if and only if $p = \pm 1$.

COROLLARY 2. *If $M(L_1)$ and $M(L_2)$ are homeomorphic as Klein spaces, then L_1 and L_2 are equivalent.*

We divide the paper into six sections. In §1, we give definitions, notations, and some known results. In §§2 and 3, we classify all orientation-preserving PL involutions on the orientable twisted I-bundle over the Klein bottle K. In §4, we prove Theorem C. In §5, we prove Theorems A and B. The details of Theorems A and B may be found in Parts (I) and (VII) of the same section. §6 contains the proof of Corollary 1 and the technique used here can be applied to a further study of group actions on S^3.

Throughout the paper all maps and spaces are assumed to be in the PL category exclusively. The author would like to thank the referee for his valuable suggestions.

1. Preliminaries. S^n, P^n, and I denote the n-sphere, the real projective n-space, and the unit interval. We refer to J. Hempel **[8]** for standard 3-manifold terms (such as irreducible, properly embedded, two-sided, one-sided, and incompressible). Let F be a surface properly embedded in a 3-manifold M. Observe that if F is not simply-connected and is π_1-injective in M (i.e., the homomorphism $i_*: \pi_1(F) \to \pi_1(M)$ induced by the inclusion map is injective), then F is incompressible. If F is two-sided and incompressible in M, then it follows from the loop theorem **[29]** that F is π_1-injective in M.

A neighborhood is said to be *small* provided it is contained in a second derived neighborhood with respect to a triangulation of M in which everything under discussion is simplicial. The manifold M' obtained by *separating* M along F is defined by the properties: (1) If F is two-sided in M, $\partial M'$ contains surfaces F' and F'' which are copies of F, and identification of F' and F'' gives a natural projection $(M', F' \cup F'') \to (M, F)$ that is a relative homeomorphism. (2) If F is one-sided in M, let U be a small neighborhood of F in M. ∂U may be viewed as a double cover of F in the usual way. Let $p: \partial U \to F$ be the covering map. Then $\partial M'$ contains a surface F' which is a copy of ∂U, and identification within F' by the nontrivial covering transformation of (F', p) gives a natural projection $(M', F') \to (M, F)$ that is a relative homeomorphism. In either case, observe that M' is homeomorphic to $\mathrm{cl}(M - U)$ where U is a regular neighborhood of F in M.

Let h be a simplicial involution of a triangulated 3-manifold M. Let F be a surface properly embedded in M as a subcomplex. In **[14]**, we describe an isotopy by which we move F into *h-general position*. Namely, first move F into general position with respect to $\mathrm{Fix}(h)$. Then, using only isotopies that keep $\mathrm{Fix}(h)$ constant, we move $F - \mathrm{Fix}(h)$ into general position with respect to $h(F) - \mathrm{Fix}(h)$. This can be done by the usual method of shifting subcomplexes into general position. Observe that there is no isolated point in $F \cap h(F)$ since F is in general position with respect to $\mathrm{Fix}(h)$.

A surface E contained in $h(F)$ is said to be *innermost* if $E \cap F \subset \partial E$ and $\partial E - (E \cap F) \subset \partial M$.

A simple closed curve J of a solid torus (or Möbius band, resp.) S is called *the center circle* if $\pi_1(S - J) = Z \oplus Z$ (or Z, resp.). It would be convenient to view the Klein bottle K both as an S^1-bundle over S^1 and a connected sum $P^2 \# P^2$. There are exactly five distinct isotopy classes of simple closed curves in K **[5]**: a contractible curve; the two obvious two-sided curves (one separating, the other nonseparating); the two obvious one-sided curves (center circles of the two Möbius bands after cutting (separating) K along the obvious separating curves in $P^2 \# P^2$).

We view S^1 as the set of complex numbers z with the norm $|z| = 1$, and D^2 as the set $\{\rho z | 0 \leqslant \rho \leqslant 1, |z| = 1\}$ where ρ is a real number. We let r always denote the free involution on $S^1 \times S^1$ defined by $r(z_1, z_2) = (-z_1, \bar{z}_2)$ for each $(z_1, z_2) \in S^1 \times S^1$. Then the orbit space $S^1 \times S^1/r$ is a Klein bottle K. We let $M(r)$ denote the (standard) twisted I-bundle over $S^1 \times S^1 \times I$ by identifying $(x, 0)$ with $(r(x), 0)$ for each $x \in S^1 \times S^1$. Each element of $M(r)$ may be represented by $[x, t]$ which is the image of $(x, t) \in S^1 \times S^1 \times I$ under the identification map, where $x \in S^1 \times S^1$ and $t \in I$. For simplicity, we view $\partial M(r)$ as $S^1 \times S^1$ in a natural way, induced by the map $[x, 1] \to x$.

(1.1) We review the definition of Klein spaces $M(p, q)$ **[13]**. Define a map f: $S^1 \times S^1 \to S^1 \times S^1$ by $f(z_1, z_2) = (z_1^p z_2^q, z_1^r z_2^s)$, where p, q, r, s are integers with $ps - qr = \pm 1$. We denote the adjunction space $M(r) \cup_f S^1 \times D^2$ by $M(p, q, r, s)$. In **[13]**, we assume $p \geqslant 0$ and $ps - qr = 1$, although this is only a matter of parametrization. Then it can be shown that $M(p', q', r', s')$ is homeomorphic to $M(p, q, r, s)$ if and only if $p' = \pm p$ and $q' = \pm q$. Therefore, the homeomorphic type of $M(p, q, r, s)$ is completely determined by its choice of p, q. From this point of view, we may define $M(p, q) = M(p, q, r, s)$. In this paper, we may assume $p, q \geqslant 0$. Then $M(p', q')$ is homeomorphic to $M(p, q)$ if and only if $p' = p$ and $q' = q$. Furthermore, $M(p, q)$ is homeomorphic to a lens space if and only if $p = 1$, and $M(1, q) \approx L(4q, 2q - 1)$.

It is also shown in **[13]** that the class of 3-manifolds $M = M(p, q)$ is precisely (up to homeomorphisms) that of closed, orientable, irreducible 3-manifolds with finite fundamental groups which contain Klein bottles, with an exception of the case where $pq = 0$ (if $p = 0$, then $M \approx P^3 \# P^3$, and if $q = 0$, then $M \approx S^1 \times S^2$).

(1.2) Let $(1, 0)$ and $(0, 1)$ be the elements of $\pi_1(S^1 \times S^1)$ represented by the (standard) paths $C_1 = (e^{2\pi ti}, 1)$ and $C_2 = (1, e^{2\pi si})$, $0 < s, t < 1$, respectively. Then the attaching map f in $M = M(p, q, r, s)$ induces an automorphism f_* on $\pi_1(S^1 \times S^1)$ such that $f_*(1, 0) = (p, r)$ and $f_*(0, 1) = (q, s)$. The matrix of f may be represented by $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$. Consider the orbit map g: $S^1 \times S^1 \to S^1 \times S^1/r$. We may assume that $\pi_1(S^1 \times S^1/r)$ is given by $\{a, b | aba^{-1}b = 1\}$ such that $g_*(1, 0) = a^2$ and $g_*(0, 1) = b$. Then each Klein space $M = M(p, q)$ may be identified by its fundamental group

$$\pi_1(M) = \{a, b | aba^{-1}b = 1, a^{2q} = b^p\}.$$

That is, two Klein spaces are homeomorphic if and only if their fundamental groups are isomorphic **[13]**.

The following theorem is shown in **[14]** and will be used in this paper.

THEOREM 1.3 **[14]**. *Let F be a compact surface and h a PL involution of $F \times I$ such that $h(F \times \partial I) = F \times \partial I$. Then there exists a product fibering of $F \times I$ and a map g of F such that relative to this fibering, $h(x, t) = (g(x), k(t))$, where $k(t) = t$ or $1 - t$ for $(x, t) \in F \times I$. The fibering can be chosen by an isotopy constant on $F \times \{0\}$.*

The following lemma is well known. It follows from the above theorem, using the results of **[19]** and **[34]** (for details, see Lemma 3 of **[18]** and also **[31]**).

LEMMA 1.4. *The solid torus $S^1 \times D^2$ admits exactly seven distinct nonconjugate PL involutions. The (standard) orientation-preserving involutions h on $S^1 \times D^2$ may be given by $h(z_1, \rho z_2) = (-z_1, \rho z_2)$, $(z_1, -\rho z_2)$, or $(\bar{z}_1, \rho \bar{z}_2)$.*

(1.5) Let S be a surface embedded in a compact 3-manifold M that is endowed with an involution h. Let P denote a property that is possessed by S. We assume that S is in h-general position. Suppose that there exists an innermost disk or annulus (possibly pinched at a point) E in $h(S)$ such that ∂E separates S into either two components E_1, E_2 (type A) or three components A_1, A_2, A_3 with $\partial E \subset \partial A_2$ (type B). Type B may occur when E is an annulus. The surface S is said to be P-permissible if one of the following holds; (1) If E is of type A, then at least one of $E \cup E_1$, $E \cup E_2$ possesses the property P, or (2) if E is of type B, then $A_1 \cup E \cup A_3$ possesses the property P. In (1.6) we give examples of P-permissible surfaces which will be used later in our paper.

Define the complexity $c(S)$ by $c(S) = a + b$ where a is the number of components of $S \cap \mathrm{Fix}(h)$ and b is the number of components of $S \cap h(S) - \mathrm{Fix}(h)$. Suppose that S is P-permissible. Let U be a small neighborhood of E such that $U \cap S$ is a regular neighborhood of ∂E in S. Assume that $S' = E_1 \cup E$, if E is of type A (or $S' = A_1 \cup E \cup A_2$, if E is of type B), possesses the property P and $h(S') \neq S'$. Then one can move S' equivariantly by a small isotopy (which can be chosen constant outside U) so as to shift it into h-general position and reduce the complexity at least by one. There are essentially three types of the construction of the isotopy above, depending on the types of the innermost surface E: (1) E is a disk and $J = S \cap E$ is a simple closed curve; (2) E is a disk and J is an arc; (3) E is a disk or an annulus but $J = S \cap E$ is neither a simple closed curve nor an arc (so J has two components which may be pinched at a point). If the construction involves Cases 1 or 2, then it is called α-operation or β-operation, respectively. Essentially α-, β-operations push S away from E in U so as to simplify the intersection curve J without increasing intersection curves elsewhere. The existence of α-, β-operations is similar to the proof of the existence of α-, β-operations in **[14]**. If the third occurs, the construction will be called γ-operation. Basically γ-operation pushes S' away from E in U so that $S' \cap E$ is either empty or has only one component (a simple closed curve or arc). The existence of γ-operation is similar to the discussions in Case (2) of the proof of Lemma 5 of **[14]** (also see **[17]**).

(1.6) *Examples of P-permissible surfaces.* (1) $S \subset M$ is an annulus whose boundary is invariant (one needs $\mathrm{Int}(S)$ in h-general position rather than S itself and the complexity is defined for $\mathrm{Int}(S)$).

(2) Let a, b be two disjoint simple closed curves $\subset \partial M$ such that $a \cup b$ is invariant and $(a \cup b) \cap \mathrm{Fix}(h) = \varnothing$. S is a disk properly embedded in M and meeting each of a, b in exactly one point.

(3) A Möbius band $B \subset M$ whose boundary is invariant (again one needs $\mathrm{Int}(S)$ in h-general position rather than S itself and the complexity is defined for $\mathrm{Int}(S)$).

(4) S is a Klein bottle in $M = M(p, q)$.

The P-permissibility of S in the first three examples is clear. The Klein bottle in (4) will be discussed in §4.

2. Involutions on Klein bottle K.

PROPOSITION 2.1. *There are exactly five distinct nonequivalent involutions on K. The involutions may be distinguished by the fixed-point sets*: (1) $\varnothing$, (2) 2 *points*, (3) *a simple closed curve*, (4) *a simple closed curve plus two points, and* (5) *two simple closed curves*.

PROOF. Let h be an involution on K. If $\mathrm{Fix}(h) = \varnothing$, the result is obvious. Now assume that $\mathrm{Fix}(h) \neq \varnothing$ in the following. Note $\Sigma \dim H_i(\mathrm{Fix}(h)) \leq \Sigma \dim H_i(K) = 4$ over Z_2 [7], and $\lambda(h) = 0$, or 2. Now, since $\lambda(h) = \chi(\mathrm{Fix}(h))$ (see [3]), the above list of fixed-point sets is complete.

In the following we will show that there exists a separating, invariant simple closed curve d in K which does not bound a 2-cell. Also d separates K into two Möbius bands B_1 and B_2 such that either B_1 and B_2 are interchanged or $h(B_i) = B_i$ $(i = 1, 2)$ (note that if h interchanges B_1 and B_2, then $\mathrm{Fix}(h)$ is either two points or a simple closed curve). Therefore, it is not difficult to complete the proof. Here, note that there exist two nonequivalent involutions on a Möbius band, which can be distinguished by the fixed-point sets: an arc plus a point, and a simple closed curve (or equivalently there exists a unique involution on P^2 (see [23])).

By a simple version of [15], there exists a simple closed curve s in K not bounding a 2-cell such that either $h(s) \cap s = \varnothing$ or $h(s) = s$ and s is in general position with respect to $\mathrm{Fix}(h)$. If s separates K, then one can find a simple closed curve d as desired (if $h(s) \cap s = \varnothing$, then s and $h(s)$ are parallel). Also note that an involution k on $S^1 \times I$ is essentially of the form $\alpha \times \beta$, where $\alpha^2 = 1$ and $\beta(t) = t$ or $1 - t$ for $t \in I$. Now assume that s is nonseparating.

Case 1. s is one-sided in K. Find a regular neighborhood B of s in K such that either $h(B) \cap B = \varnothing$ or $h(B) = B$ and B is in general position with respect to $\mathrm{Fix}(h)$. Then ∂B separates K, and we have finished this case.

Case 2. s is two-sided in K. Cut K along s, and we obtain an annulus A $(= S^1 \times I)$.

Subcase 1. $h(s) = s$. If $s \cap \mathrm{Fix}(h) = \varnothing$, we claim that there exists another simple closed curve s' in K parallel to s such that $h(s') \cap s' = \varnothing$. For, consider the involution h' on $S^1 \times I$ induced by h. Since $s \cap \mathrm{Fix}(h) = \varnothing$ and $\mathrm{Fix}(h) \neq \varnothing$ (and $\mathrm{Fix}(h') \neq \varnothing$), h' must interchange the two boundary components of A. Thus, there exists another invariant simple closed curve S_1 in A (under h') parallel to each boundary component of A such that h' interchanges the sides of S_1 in A. Thus the claim follows. This will be treated in the following Subcase 2. Now assume that $S \cap \mathrm{Fix}(h) \neq \varnothing$. If $\mathrm{Fix}(h)$ is a union of two simple closed curves, then each component of $\mathrm{Fix}(h)$ must be one-sided in order to recover K from A equivariantly. Now this is a similar situation to Case 1, and we obtain a simple closed curve as desired. If $\mathrm{Fix}(h)$ is a simple closed curve, then $\mathrm{Fix}(h)$ must separate in K in order to recover K from A equivariantly, and so take $d = \mathrm{Fix}(h)$.

Subcase 2. $h(s) \cap s = \varnothing$. $s \cup h(s)$ separates K into two annuli A_1 and A_2. Since $\mathrm{Fix}(h) \neq \varnothing$, $h(A_i) = A_i$ $(i = 1, 2)$. Let $h_i = h|A_i$. Since K is nonorientable, we may assume that either $\mathrm{Fix}(h_1) = S^0$ and $\mathrm{Fix}(h_2) = S^1$, or $\mathrm{Fix}(h_1) = S^0$ and $\mathrm{Fix}(h_2) = \varnothing$ (see an argument of [11]).

First, assume that $\mathrm{Fix}(h)$ is a simple closed curve plus two points. Let $x \in \partial A \cap \partial A_1$. Take an arc c_1 in A_1 joining x and a point y_1 of $\mathrm{Fix}(h_1)$ and another arc c_2 in A_2 joining $h(x)$ and a point y_2 of $\mathrm{Fix}(h_2)$ such that $h_i(c_i) \cap c_i = \{y_i\}$, $(i = 1, 2)$. Let $c = c_1 \cup h(c_1) \cup c_2 \cup h(c_2)$. Since h^2 is the identity, c must be a simple closed curve, and, moreover, c is one-sided in K (this is the only way to recover K from A). Now the result follows from Case 1.

We now assume that $\mathrm{Fix}(h)$ is two points. Let $x_1, x_1' \in \partial A \cap \partial A_1$ and $y, y' \in \mathrm{Fix}(h_1)$ be four distinct points. Take two disjoint arcs c_1 and c_1' in A_1 joining x_1 and y, and x_1' and y', respectively, such that $c_1 \cap h_1(c_1) = \{y\}$ and $c_1' \cap h_1(c_1') = \{y'\}$. Let $x_2 = h_2(h_1(x_1'))$ and $x_2' = h_2(h_1(x))$. Then $x_2 = x_1'$ and $x_2' = x_1$ in A (after identification). Let c_2 and c_2' be two arcs in A_2 joining $h_1(x_1)$ and x_2, and $h_1(x_1')$ and x_2', respectively, such that $c_2 \cap c_2' = \varnothing$ and $c_2' = h_2(c_2)$. Let c be the union of c_1, $h(c_1)$, c_1', $h(c_1')$, c_2, and c_2'. Then c is a simple closed curve in K. In order to recover K from A, the simple closed curve c must separate K into two Möbius bands B_1 and B_2, and h interchanges B_1 and B_2. This completes the proof.

3. Involutions on the twisted I-bundle $M(r)$. In this section we classify all orientation-preserving involutions on the orientable twisted I-bundle $M(r)$ over K. The standard involutions will be given after the statement of the classification theorem.

THEOREM 3.1. *The orientable twisted I-bundle over K admits exactly five distinct nonconjugate orientation-preserving involutions.*

In order to give the standard involutions on $M(r)$, we need to define some involutions g_i $(i = 1, \dots, 5)$ on $S^1 \times S^1$ by

$$g_1(z_1, z_2) = (z_1, -z_2), \quad g_2(z_1, z_2) = (\bar{z}_1, -\bar{z}_2), \quad g_3(z_1, z_2) = (-z_1, -z_2),$$
$$g_4(z_1, z_2) = (\bar{z}_1, \bar{z}_2) \quad \text{and} \quad g_5(z_1, z_2) = (-z_1, z_2).$$

In fact, the involutions on K $(= S^1 \times S^1/r)$ are the ones induced by the involutions defined above. Now an orientation-preserving involution h on $M(r)$ is conjugate to an involution h_i $(i = 1, \dots, 5)$ defined by

$$h_i[x, t] = [g_i(x), t] \quad \text{for } [x, t] \in M(r).$$

The fixed-point sets $F_i = \mathrm{Fix}(h_i)$ are given as: (1) $F_1 = \varnothing$; (2) $F_2 =$ two arcs; (3) a simple closed curve; (4) a simple closed curve plus two arcs; (5) two simple closed curves. In fact, it is possible to classify all involutions on $M(r)$, but we do not discuss it this time (it is relatively lengthy).

Let h be an involution on $M(r)$ with $h(K) = K$. Then it is elementary to check that K is already in general position with respect to $\mathrm{Fix}(h)$ off components of $\mathrm{Fix}(h)$ which are contained in K entirely.

LEMMA 3.2. *Let K be a Klein bottle in* $\mathrm{Int}(M)$ *where* $M = M(r)$. *Then M is a regular neighborhood of K.*

PROOF. Let $U(K)$ be a regular neighborhood of K in $\mathrm{Int}(M)$. Let $M' = \mathrm{cl}(M - U)$. Then the inclusion map $i\colon \partial U \to M'$ induces an injection i_* of fundamental groups. For, otherwise, it follows from the loop theorem **[29]** that there exists a properly embedded disk in M'. Therefore, one can find a 3-cell D in M (note that M' is irreducible) such that D contains either ∂M or a Klein bottle, which is a contradiction. Furthermore, the injection i_* is, in fact, an isomorphism (as can be easily seen), it follows from [2] that $M' = S^1 \times S^1 \times I$. This completes the proof.

LEMMA 3.3. *Let h_1 and h_2 be involutions on $M = M(r)$. Suppose that there exists a Klein bottle K_i in M which is invariant under h_i $(i = 1, 2)$. If h_1 and h_2 are orientation-preserving (or orientation-reversing), and $h_1|K_1$ and $h_2|K_2$ are equivalent, then h_1 and h_2 are also equivalent.*

PROOF. Let h be an involution on M, and $k = h|K$ where K is a Klein bottle in M invariant under h. Then it follows from Lemma 3.2 that M is a regular neighborhood of K. By cutting M along K, we may view M as a quotient space of $S^1 \times S^1 \times I$, in the usual way, by identifying $(x, 0)$ with $(q(x), 0)$ for each $x \in S^1 \times S^1$, where q is a free involution on $S^1 \times S^1$ (see §1). Let g be the orbit map induced by q, and k' an involution on $S^1 \times S^1$ which covers k (i.e., $gk' = kg$). Then there exists an involution h' of $S^1 \times S^1 \times I$ such that $h'(x, 0) = (\bar{k}(x), 0)$ and h' naturally induces h, where $\bar{k}$ denotes either k' or qk' ($\bar{k}$ is determined whether h' is orientation-preserving or orientation-reversing). By Theorem 1.3, h' may be assumed to be given by $h'(x, t) = (\bar{k}(x), t)$ for $x \in S^1 \times S^1$ and $t \in I$. Therefore, h is given by $h[x, t] = [\bar{k}(x), t]$ for $(x, t) \in S^1 \times S^1$ and $t \in I$, up to equivalence.

Let h_1 and h_2 be orientation-preserving involutions on M such that $f_1 = h_1|K_1$ and $f_2 = h_2|K_2$ are equivalent. Then there exists an equivariant map t: $K_1 \to K_2$ such that $t^{-1}f_2t = f_1$. For simplicity, in connection with h_1 and h_2 we use the same notations as above by just adding indices. Let t' be a homeomorphism of $S^1 \times S^1$ such that $g_2t' = tg_1$. Then we have $g_1t'^{-1}\bar{k}_2t' = g_1\bar{k}_1$. Therefore, we see that $t'^{-1}\bar{k}_2t' = \bar{k}_1$ (it cannot be $q_1\bar{k}_1$ since $\bar{k}_2$ is orientation-preserving). The map t' can be extended to an equivariant map t'' of $S^1 \times S^1 \times I$ such that $t''^{-1}h_2't'' = h_1'$. Certainly, t'' induces an equivariant map of M between h_1 and h_2. This argument goes through for orientation-reversing involutions h_1 and h_2 as well, and thus we have completed the proof.

(3.4) PROOF OF THEOREM 3.1. We divide the proof into several parts. Let h be an orientation-preserving involution on $M = M(r)$. First, consider the case where $\mathrm{Fix}(h) = \varnothing$. Let p: $M \to M''$ be the orbit map where $M'' = M/h$ is the orbit space. Then the subgroup $\pi_1(\partial M'')$ has index two in $\pi_1(M'')$, and so M'' is a twisted I-bundle (see [33]). Therefore, we see that there exists exactly one obvious orientation-preserving free involution.

In what follows we assume that $\mathrm{Fix}(h) \neq \varnothing$ and h is orientation-preserving. By Lemma 3.3, it is enough to show that there exists an invariant Klein bottle in M.

(3.5) Observe that there exists a double covering g: $S^1 \times S^1 \times I \to M$. Then there exists an (orientation-preserving) involution h' on M' with $\mathrm{Fix}(h') \neq \varnothing$ which covers h (i.e., $gh' = hg$). We may assume that the involution h' on $M' = S^1 \times S^1 \times I$ is given by $h'(x, t) = (\alpha(x), \beta(t))$ for $x \in S^1 \times S^1$ and $t \in I$, where $\alpha^2 = 1$ and $\beta(t) = t$ or $1 - t$ (see Theorem 1.3). Obviously, $T = S^1 \times S^1 \times \{\frac{1}{2}\}$ is invariant under h'. It is well known that the involutions on $S^1 \times S^1$ are obvious ones which may be distinguished by the fixed-point sets with the exception of the two obvious free ones: (1) four points; (2) one simple closed curve; (3) two simple closed curves.

Let $\bar{k}$ be the nontrivial covering transformation of (M', g). Then $\langle \bar{k} \rangle \times \langle h' \rangle$ ($\cong Z_2 \times Z_2$) acts on M'. Let N be the orbit space M'/h', and k the involution on N induced by $\bar{k}$. That is, $\bar{g}\bar{k} = k\bar{g}$, where $\bar{g}$: $M' \to N$ is the orbit map induced by h'. Note that $\mathrm{Fix}(k) = \bar{g}(\mathrm{Fix}(\bar{k}h'))$ and $\mathrm{Fix}(k) \cap \bar{g}(\mathrm{Fix}(h')) = \varnothing$, and $\bar{g}(\mathrm{Fix}(h'))$ is invariant under k. Also note that $\mathrm{Fix}(h) = g(\mathrm{Fix}(h')) \cup g(\mathrm{Fix}(\bar{k}h'))$ and $\bar{k}$ is orientation-preserving.

(3.6) *In what follows we always assume that* $\mathrm{Fix}(h')$ *contains a simple closed curve, whenever* $\mathrm{Fix}(h)$ *does* (this is a matter of choosing base points). Recall that h, h' are orientation-preserving. We divide the proof of Theorem 3.1 into three cases according to the fixed-point sets: (1) $\mathrm{Fix}(h')$ is two simple closed curves; (2) $\mathrm{Fix}(h')$ is a simple closed curve; (3) $\mathrm{Fix}(h')$ is four arcs (see (3.5)). It is easy to observe that $\mathrm{Fix}(\bar{k}h')$ would be $\varnothing$ or four arcs for the first two cases, and it is empty for the third case (note that each boundary component of M' is invariant under $\bar{k}h'$ and see (3.5)). In the following we shall use the same notations as in (3.5).

(3.7) Now we consider the case where $\mathrm{Fix}(h')$ is two simple closed curves. It follows from (3.5) that T is invariant under h', and $\mathrm{Fix}(h'|T)$ is two simple closed curves, say a, b. Furthermore, the orbit space N is a solid torus such that $a' = \bar{g}(a)$ and $b' = \bar{g}(b)$ are isotopic to the center circle of N. Observe that $\bar{g}(T)$ is an annulus with $\partial\bar{g}(T) = a' \cup b'$. It follows from (3.5) and (3.6) that $\mathrm{Fix}(k)$ would be empty or two arcs. Obviously, we see that $\bar{g}(\mathrm{Fix}(h')) = a' \cup b'$ and it is invariant under k.

(3.8) In (3.9) we will show that there exists an invariant annulus A (under k) in N such tht $\partial A = a' \cup b'$. If this is the case, then one sees that $T' = \bar{g}^{-1}(A)$ is a torus, which is invariant under both $\bar{k}$ and h', such that $T' \cap \mathrm{Fix}(h') = a \cup b$. Moreover, T is incompressible in M'. For, otherwise, it follows from the loop theorem **[29]** that T' bounds a solid torus S, and hence $h'(S) = S$ (since M' has nonempty boundary components). However, this cannot occur since, otherwise, $\dim(\mathrm{Fix}(h'))$ $= 2$ (recall that $\mathrm{Fix}(h') = a \cup b$ and see Lemma 1.4). Therefore, it follows from **[2]** that T' is isotopic to each component of $\partial M'$. Since $\bar{k}$ is orientation-preserving and interchanges the sides of T', we see that $g(T')$ is a Klein bottle in M which is invariant under h. Thus, we find an invariant Klein bottle in M as desired.

(3.9) Now we apply (1.5) to show that there exists an invariant annulus in N as desired in (3.8). It follows from (3.7) that there exists an annulus A in N such that $\partial A = a' \cup b'$. If A is invariant, then we are done. Suppose that A is not invariant. Then we move $\mathrm{Int}(A)$ into k-general position by an isotopy keeping ∂A constant. Observe that there exists an innermost disk or annulus (possibly pinched at a point of $\mathrm{Fix}(k) \cup \partial A$) (note that $\mathrm{cl}(k(A) \cap A - \partial A)$ cannot be a simple arc since N is a solid torus and $\mathrm{Int}(A)$ is in h-general position). We may assume that $\mathrm{Int}(A) \subset \mathrm{Int}(N)$.

Case (1). There exists an innermost disk E in $k(A)$. Let $J = \mathrm{cl}(E \cap A - \partial A)$ (J may be a simple closed curve, arc, or disjoint union of two arcs). Then J separates A into two components E_1 and E_2. We may assume that E_2 is a disk. Then $A' = E \cup E_1$ is an annulus. If $k(A') = A'$, then we are done. So assume that $k(A') \neq A'$. We apply α-operation (see (1.5) and (1.6)(1)) to move $\mathrm{Int}(A')$ by a small isotopy (constant on $\partial A'$) so as to shift it into k-general position and remove the intersection curve(s) J without increasing intersection curves elsewhere. By a finite number of these operations, we can remove all contractible intersection curves in $k(A)$. So in the following we assume that there is no such intersection curve in $k(A)$.

Case (2). There exists an innermost annulus E in $k(A)$ (possibly pinched at a point). Especially let E be the one such that one boundary component of E is a'. Let J be the other boundary component of E. We see that J separates A into two annuli E_1, E_2. Let E_1 be the one with $E_1 \supset b'$. Let $A' = E_1 \cup E$. If $k(A') = A'$, we are done. So in the following we assume that $k(A') \neq A'$. Now use an γ-operation (see (1.5) and (1.6)(1)) to move $\mathrm{Int}(A')$ by a small isotopy (constant on $\partial A'$) so as to shift it into k-general position and simplify the intersection curve along J. By repeating these operations, we can remove all the intersection curves of $\mathrm{Int}(A)$ and $\mathrm{Int}(k(A))$. Consequently we have the three possibilities as in the following.

Subcase (1). $k(A) \cup A$ is a torus and $\mathrm{Fix}(k) = \varnothing$. Let $T' = k(A) \cup A$. Obviously T' bounds a solid torus S, and a', b' are isotopic to the center circle of S (recall that a', b' are isotopic to the center circle of N). Thus, there exists a meridional disk D, properly embedded in S, which meets $a' \cup b'$ at exactly two points, say x, y. Note that $k(S) = S$. Let $k' = k|S$. One may apply α-, β-operations (see (1.5) and (1.6)(2)) in order to obtain a new disk D properly embedded in S such that either $k'(D) = D$ or $k'(D) \cap D = \varnothing$, and $D \cap (a' \cup b') = \{x, y\}$. However, $k'(D) = D$ cannot occur since $\mathrm{Fix}(k') = \varnothing$, and we have $k'(D) \cap D = \varnothing$. Since $k'(D) \cup D$ separates S into two 3-cells, and k' interchanges the two 3-cells, it is not difficult to find an invariant annulus B in S with $\partial B = a' \cup b'$, as desired.

Subcase (2). $k(A) \cup A$ is a torus and $\mathrm{Fix}(k) \neq \varnothing$. Since k is orientation-preserving and $\mathrm{Fix}(k) \cap (k(A) \cup A) = \varnothing$, it follows from Theorem 1.3 and Lemma 1.4 that $\mathrm{Fix}(k)$ must be a simple closed curve (note $\mathrm{cl}(N - S) \approx S^1 \times S^1 \times I$ and $k(S) = S$). This is a contradiction to the fact that $\mathrm{Fix}(k)$ is two arcs (see (3.6)). Thus, this case cannot occur.

Subcase (3). $k(A) = A$. Nothing to show. This completes the case where $\mathrm{Fix}(h')$ is two simple closed curves.

(3.10) Now we consider the case where $\mathrm{Fix}(h')$ is a simple closed curve. It follows from (3.5) that the orbit space N is a solid torus and $\bar{g}(T)$ is a Möbius band with $\partial\bar{g}(T) = \bar{g}(\mathrm{Fix}(h'))$. By performing α-, γ-operations (see (1.6)(3) and also (3.9)), we obtain a new Möbius band B with $\partial B = \bar{g}(\mathrm{Fix}(h'))$ such that $k(B) = B$, $k(B) \cap B = \partial B$, or $k(B) \cap B = \partial B \cup c$ where c is isotopic to the center circle of B (recall that $\bar{g}(\mathrm{Fix}(h'))$ is invariant under k). But the case where $k(B) \cap B = \partial B$ cannot occur since a Klein bottle cannot be embedded in a solid torus. We consider the following two subcases.

Subcase (1). $k(B) = B$. Then $T' = \bar{g}^{-1}(B)$ is an invariant torus in M' (under both h' and $\bar{k}$). By a similar reason to that in (3.8), we see that $g(T')$ is an invariant Klein bottle in M (if T' bounds a solid torus S in M', then $h'|T'$ must be extended to an involution on S. However, since $\mathrm{Fix}(h'|T')$ is a simple closed curve, this is impossible (see Lemma 1.4)).

Subcase (2). $h(B) \cap B = \partial B \cup c$. Observe that $T_1 = \bar{g}^{-1}(B)$ and $T_2 = \bar{g}^{-1}(k(B))$ are tori such that $T_1 \cap T_2$ consists of two disjoint invariant simple closed curves (under h') (one of them is $\mathrm{Fix}(h')$). It is clear that $T_1 \cap T_2$ separates T_2 into two components $A_1, A_2 \approx S^1 \times I$. First, suppose that T_1 is incompressible. Since $T_2 = \bar{k}(T_1)$, one sees that T_2 is also incompressible. It follows from [2] that T_1 separates M' into two components $M_1', M_2' \approx S^1 \times S^1 \times I$. Since T_1 separates M', we may assume that $A_1 \subset M_1'$ and $A_2 \subset M_2'$. Then an annulus ($\subset \partial M_i'$) and A_i ($i = 1, 2$) cobound a solid torus in M_i (see Lemma 4.4). One sees that T_2 is compressible, which is a contradiction. Now suppose that each T_i is compressible. Then T_i bounds a solid torus G_i in M'. Notice that T_i is invariant under h', and therefore so is G_i. Also $h'|G_i$ is orientation-preserving but $\mathrm{Fix}(h'|G_i) \subset T_i$, which is a contradiction (see Lemma 1.4). Thus this case cannot occur.

(3.11) We now consider the remaining case. That is, $\mathrm{Fix}(h')$ is four arcs. Since $\mathrm{Fix}(k) = \bar{g}(\mathrm{Fix}(\bar{k}h'))$, it follows from (3.6) that $\mathrm{Fix}(k) = \emptyset$. Let U be a small h'-invariant regular neighborhood of $\mathrm{Fix}(h')$. We may assume that $\bar{k}(U) = U$. Let $X = \mathrm{cl}(M' - U)$. Then it follows from Theorem 1.3 that $X' = X/h'$ has a product structure $Y \times I$, naturally induced by X, where Y is a 2-sphere minus four 2-cells. It follows, again, from Theorem 1.3 that there exists an invariant Y' ($\approx Y$) properly embedded in $Y \times (0, 1)$. Then one sees that $\bar{g}^{-1}(Y')$ is a torus minus four 2-cells which is invariant under h'. Of course, U consists of four 3-cells U_i, each of which is invariant under h'. Since $h'|U_i$ is a rotation (see [34]), it is not difficult to extend $\bar{g}^{-1}(Y')$ to an invariant torus T' (under h'). Furthermore, the extension can be chosen so that T' is also invariant under $\bar{k}$. Namely, if U_1, U_2, U_3, U_4 are the components of U such that $\bar{k}(U_i) = U_{i+2}$ ($i = 1, 2$), let c_1, c_2 be the components of $\bar{g}^{-1}(Y') \cap (U_1 \cup U_2)$. Let D_i be an invariant disk in U_i (under h') with $\partial D_i = c_i$. Then $T' = \bar{g}^{-1}(Y') \cup D_1 \cup D_2 \cup \bar{k}(D_1 \cup D_2)$ is the one desired (recall that $\bar{k}h' = h'\bar{k}$, and $g^{-1}(Y')$ is invariant under $\bar{k}$). By a similar reason to that in (3.5), we see that $g(T')$ is an invariant Klein bottle in M. This completes the proof of Theorem 3.1.

4. Proof of Theorem C. Let k be an orientation-preserving involution on the solid torus $D^2 \times S^1$. Suppose that there exists an annulus or a Möbius band S properly embedded in $D^2 \times S^1$ such that S meets $k(S)$ only in a simple closed curve J transversally. Observe that J is isotopic to the center circle of $D^2 \times S^1$. We show the following two lemmas.

LEMMA 4.1. *If S is an annulus, then* $\mathrm{Fix}(k) \neq \emptyset$, *and either J is the fixed-point set or $J \cap$* $\mathrm{Fix}(k)$ *is two points.*

PROOF. Suppose that $\mathrm{Fix}(k) = \emptyset$. The solid torus is separated by $S \cup k(S)$ into two pairs of nonadjacent solid tori $\{B_1, C_1\}$, $\{B_2, C_2\}$. Observe that $k(B_i \cup C_i) = B_i \cup C_i$ $(i = 1, 2)$, and for some i, B_i and C_i must be interchanged. We may assume that $k(B_1) = C_1$, and it follows that each of B_2 and C_2 is invariant. Let A_1 be an annulus in B_1 such that $A_1 \cap \partial(D^2 \times S^1)$ is a simple closed curve and $A_1 \cap k(A_1) = J$. Then since $k(B_1) = C_1$, $A = A_1 \cup k(A_1)$ is an invariant annulus. Note that $k|A$ is orientation-reversing (since k interchanges the sides of J in A and $\mathrm{Fix}(k) = \emptyset$). Therefore, since each side of A is invariant, k must be orientation-reversing. This completes the proof.

If S is a Möbius band, then $S \cup k(S)$ separates $\partial(D^2 \times S^1)$ into two annuli A_1 and A_2.

LEMMA 4.2. *If S is a Möbius band and* $\mathrm{Fix}(k) = \emptyset$, *then k interchanges the two annuli A_1 and A_2.*

PROOF. Observe that there exists a double covering $g: D^2 \times S^1 \to D^2 \times S^1$ such that $A = g^{-1}(S)$ is an annulus. Let $A' = g^{-1}(k(S))$. Then $A \cap A'$ is the simple closed curve $g^{-1}(J)$. Let k' be a homeomorphism of $D^2 \times S^1$ which covers k (i.e., $gk' = kg$). Then k'^2 is the identity or the covering transformation of g. Since $k'(A) = A'$, and A meets A' in the simple closed curve $g^{-1}(J)$ transversally, it follows from Lemma 4.1 that k' cannot be an involution. Therefore, k' has the period 4. We see that $A \cup A'$ separates $\partial(D^2 \times S^1)$ into four components which are permuted by k'. Therefore, since the covering map g is induced by k'^2, the involution k must interchange A_1 and A_2.

Let F be a nonorientable surface in an orientable 3-manifold endowed with an involution T. Suppose that F is in T-general position. In general, a one-sided (or two-sided) simple closed curve c in F may not be one-sided (or two-sided) in $T(F)$ where the curve c is an intersection curve of F and $T(F)$. The following lemma may give the idea of how they are associated. Since the proof is not difficult, we omit the proof.

LEMMA 4.3. *Let S_α $(\alpha = 1, 2)$ be an annulus or a Möbius band properly embedded in $S^1 \times D^2$ (S_1 may not be homeomorphic to S_2). Let $\{x_i\}$ be a set of finite points of the center circle c of $S^1 \times D^2$. Suppose that $S_1 \cap S_2$ is a union of the simple closed curve c plus a finite system of arcs $\{a_{ij}, b_{ik}\}_{i,j,k}$ $(j = 1, \ldots, n_i)$ $(k = 1, \ldots, m_i)$ which satisfies the following conditions*: (1) *$e_{ij} \cap e_{ik} = \{x_i\}$ whenever $e_{ij} \neq e_{ik}$, and $e_{ij} \cap e_{tk} = \emptyset$ for $i \neq t$, where $e_{ij} = a_{ij}$ or b_{ij}*; (2) *$a_{ij} \subset S^+$ and $b_{ik} \subset S^-$, where S^+ and S^- are the closures of the components of $S_1 - c$; and* (3) *S_1 and S_2 are in general positions off $\{x_i\}$. Then S_1 and S_2 are homeomorphic if and only if $n_i \equiv m_i$* (mod 2) *for each i and $r \equiv 0$* (mod 2) *where r is the number of x_i's with $n_i \equiv 1$* (mod 2).

For the proof of the following lemma, see **[27]**.

LEMMA 4.4. *Let B be a properly embedded annulus in $X = D^2 \times S^1$. If B is not contractible in $D^2 \times S^1$, then there exists an isotopy which sends B into ∂X keeping ∂B fixed.*

(4.5) Proof of Theorem C. We divide the proof of Theorem C into several steps. Recall that there is no orientation-reversing involution on $M = M(p, q)$ [23]. Let K be a Klein bottle in M. If $h(K) = K$, then one sees that K is already in h-general position off components of $\mathrm{Fix}(h)$ which are contained in K entirely. Let U be a small invariant regular neighborhood of K. Then ∂U is the torus as desired. So assume that $h(K) \neq K$. We put K into h-general position. In the following the proof will be divided into four steps: In Step 1, we remove all the innermost disks and annuli in $h(K)$. In Step 2, we prove the theorem for the case where $h(K) \cap K$ is a single simple closed curve J. In Step 3, we define a system Q (which will be called a good system) of simple closed curves of $K \cap h(K)$ such that $K \cap h(K) = \bigcup Q$ and show that the system Q can be reduced to another good system which consists of only two nonisotopic one-sided curves in K. In Step 4, we prove the theorem for the case where the system Q (and $K \cap h(K)$) consists of only two nonisotopic one-sided curves in K, which concludes the proof.

Step 1. Suppose that there exists an innermost disk or annulus in $h(K)$. By performing α-operations (see (1.5)), we may assume that there is no innermost disk in $h(K)$. Suppose there exists an innermost annulus E in $h(K)$ (possibly pinched at a point). Let J_1 and J_2 be the boundary components of E (possibly meeting at a point). Since J_1 and J_2 are isotopic in $h(K)$, both of them are either one-sided or two-sided in $h(K)$.

Lemma. (1) *Both J_1, J_2 are two-sided in $h(K)$.*

(2) *J_1 is either tangent to J_2 in $h(K)$ at a point or disjoint from J_2.*

Proof. (1) Suppose that J_1, J_2 are one-sided in $h(K)$. Then J_1 and J_2 must meet transversally in $h(K)$ at a point. For, otherwise, one has two isotopic disjoint one-sided simple closed curves in $h(K)$, which is impossible in a Klein bottle. Therefore, since K is in h-general position, it follows from Lemma 4.3 that J_1, J_2 are two-sided in K (this can be seen by taking regular neighborhood U_i of J_i ($i = 1, 2$) such that $U_i \cap K$, $U_i \cap h(K)$ are regular neighborhoods of J_i in K and $h(K)$, respectively). Furthermore, J_1 and J_2 also meet transversally in K at a single point. But there are no two two-sided simple closed curves in a Klein bottle which meet transversally at exactly one point (see Lemma 4.6) which is a contradiction.

(2) Since J_1 and J_2 are isotopic in $h(K)$, the result follows directly from (1) and the last sentence of the proof of (1).

Now it follows from Lemma 4.3 and the above lemma that each of J_1, J_2 is two-sided in K. Then $J = \partial E$ separates K into two components E_1 and E_2. Observe that one of $E \cup E_1$ and $E \cup E_2$ is a Klein botle. Assume that $K' = E \cup E_1$ is a Klein bottle. Let U be a small neighborhood of E such that $U \cap K$ is a regular neighborhood of J in K. If $h(K') = K'$, then we are done. Now we assume that $h(K') \neq K'$. Use γ-operation (see (1.5)) to find a Klein bottle K'' with $c(K'') < c(K)$. By performing a finite number of γ-operations, we may assume that there is no such innermost annulus in $h(K)$.

Step 2. Suppose that $K \cap h(K)$ is a single simple closed curve J. We claim the following.

LEMMA. *J is a two-sided, nonseparating simple closed curve in K (and $h(K)$).*

PROOF. Suppose that J is one-sided in K. Let $g: L \to M$ be a double covering such that $g^{-1}(K) \approx S^1 \times S^1$ where L is a lens space (this can be done by the technique of cutting M along K). Since J is one-sided in K, $J' = g^{-1}(J)$ is a simple closed curve. Let $H = g^{-1}(K)$ and $H' = g^{-1}(h(K))$. Since H separates L, obviously H separates H' (note that H meets H' along J' transversally). However, since $H \cap H' = J'$, the simple closed curve J' must separate the torus H'. This implies that J' is a contractible curve in H', which is a contradiction. Thus J must be two-sided in K. Now suppose that J separates K into two Möbius bands B_1 and B_2. Then $B_1 \cup h(B_1)$ and $B_2 \cup h(B_2)$ are tangent along J. So, by pulling $B_1 \cup h(B_1)$ away from $B_2 \cup h(B_2)$ slightly, we obtain two disjoint Klein bottles in M, which is a contradiction. Thus the claim follows.

Now we will show that there exists an invariant torus S which separates M into a twisted I-bundle over K and a solid torus. In fact, this case may occur when $\mathrm{Fix}(h) \neq \varnothing$ (see Lemma 4.1).

Let N be a small neighborhood of K such that $N \cap h(K)$ is an annulus A, and $h(A) \cap A = J$. N may be viewed as an orientable annulus bundle over S^1 with connected boundary having A as a fibre. (It would be convenient to view N as the following: parametrize $A = S^1 \times I$, and $J = S^1 \times \{\frac{1}{2}\}$. Define a map $f: A \to A$ by $f(z, t) = (\bar{z}, 1 - t)$ for each $z \in S^1$ and $t \in I$. Then N may be viewed as a quotient space obtained from $A \times I$ by identifying $(x, 0)$ to $(f(x), 1)$ for each $x \in A$.) Let $A^+ = S^1 \times [0, \frac{1}{2}] \subset A$ and $A^- = S^1 \times [\frac{1}{2}, 1]$. We see that ∂A separates ∂N into two annuli B_1 and B_2. It is obvious that each $\mathrm{Int}(B_i)$ $(i = 1, 2)$ is isotopic to $K - J$ in N in a natural way. Let $N' = \mathrm{cl}(M - N)$. Then N' is a solid torus (see [13]). Since J is two-sided in $h(K)$ and A is a regular neighborhood of J in $h(K)$, we see that ∂A bounds an annulus B $(\subset h(K))$ in N' (note that $N \cap h(K) = A$). Since ∂A bounds an annulus $(= B_i)$ in ∂N $(= \partial N')$, it follows from Lemma 4.4 that $B \cup B_i$ bounds a solid torus in N' and there exists an isotopy of M keeping ∂A constant so that it carries B onto B_i (note that each boundary component of A does not bound a disk in N'; otherwise, $M \approx S^1 \times S^2$ (see Proposition 2.1 of [13])). Thus, we may assume that $B = B_i$ for some i; we let $B = B_1$. Obviously $N - (A \cup K)$ consists of two components Q_1 and Q_2, exactly one of which meets B; let $B \cap Q_1 \neq \varnothing$ (in fact, $\mathrm{Int}(B) = \partial Q_1$). On the other hand, we let U be a small invariant regular neighborhood of J such that $U \cap (K \cup h(K))$ consists of two annuli which cross along the curve J. We may assume that one of the annuli is A. Let $A' = h(A)$. Then A' is the other annulus. $A \cup A'$ separates ∂U into two pairs of nonadjacent annuli $\{G_1, H_1\}$, $\{G_2, H_2\}$. One sees that each $G_i \cup H_i$ $(i = 1, 2)$ is invariant, and the two adjacent annuli G_1, G_2 cannot meet the same Q_i (recall that the two copies $A^+ \times \{0\}$ and $A^- \times \{1\}$ are the same in N); let $G_2 \cap Q_1 = \varnothing$. Then we see that $H_2 \cap Q_1 = \varnothing$ (note that G_2 and H_2 are nonadjacent annuli). Let $S = [(K \cup h(K)) - (A \cup A')] \cup (G_2 \cup H_2)$. Then it is not difficult to see that S is an invariant torus isotopic to N. Then S is the torus, as desired, which separates M into a twisted I-bundle over K and a solid torus.

Step 3. Consider a system $\{J_i\}$ of simple closed curves of $K \cap h(K)$ such that (1) $J_i \cap \mathrm{Fix}(h)$ for each i is at most two points, (2) $J_i \cap J_j$ $(i \neq j)$ is at most two points, and (3) $\bigcup J_i = K \cap h(K)$. Such a system will be called a good system of intersection curves of K and $h(K)$. Observe that one can always choose such a system. One way to do this is: First, collect all arcs $J \subset K \cap h(K)$ joining two fixed points. Then one sees that $h(J) \cap J = J \cap \mathrm{Fix}(h)$, and $J \cup h(J)$ is a simple closed curve. So, among the arcs, each proper pair of two arcs will give a simple closed curve J which meets $\mathrm{Fix}(h)$ at exactly two points. Second, collect all simple closed curves, each of which meets $\mathrm{Fix}(h)$ in less than two points. Then the whole collection will give a good system of intersection curves as described above (note that a branched point of any connected graph of $h(K) \cap K$ is a fixed point).

We observe the following lemma which will simplify the proof of Theorem C. Let Q be a good system of intersection curves of K and $h(K)$. Recall that any two different elements of Q have at most two points in common.

LEMMA 4.6. (1) *Let J_1 and J_2 be two elements of Q which are two-sided* (*or one-sided*) *in K. If J_1 and J_2 meet transversally in K at one point x, then there exists another point* (*other than x*) *where they meet transversally in K.*

(2) *Let J_1 be an element of Q which is two-sided in K. Suppose that J_1 meets another element J_2 of Q at two points. If J_2 is either isotopic to J_1 or a one-sided simple closed curve in K, then there exists an innermost disk or annulus in $h(K)$.*

(3) *Let J_1 and J_2 be two distinct elements of Q which are one-sided in K. Then J_1 and J_2 are not isotopic in K. Furthermore, if J_1 and J_2 meet transversally in K at two points, then there exists an innermost disk or annulus in $h(K)$.*

PROOF. In order to prove the above, it would be helpful if one views h as a simplicial involution of some triangulation of M, so that K, $h(K)$ and $\mathrm{Fix}(h)$ are subcomplexes, and among them all cuts are locally piercing (see §1). So each element of $Q = \{J_i\}$ is a subcomplex of one dimension and the common points of $J_i \cap J_j$ $(i \neq j)$ are fixed points. Suppose that there exists a disk or an annulus A (maybe pinched) in K bounded by subpolyhedra of the complexes J_1 and J_2. Then one may easily find an innermost disk or annulus (maybe pinched) in K (and therefore in $h(K)$) which is a subcomplex of the complex A (see h-general position in §1).

(1) First of all, observe that there are no two two-sided simple closed curves in K which meet transversally at only one point. Suppose that such curves J_i $(i = 1, 2)$ do exist. Then neither can separate K. Cut K along J_1, and we obtain $S^1 \times I$. Then the arc, resulted from J_2, must be isotopic to a vertical line in $S^1 \times I$, and therefore J_2 cannot be isotopic to J_1, which is a contradiction (there is only one isotopy class of nonseparating two-sided curves). Now suppose that J_1 and J_2 are one-sided curves in K which meet transversally at only one point (in K). It follows from Lemma 4.3 that J_1 and J_2 are both two-sided curves in $h(K)$ which meet transversally (in $h(K)$) at only one point, which is again impossible.

(2) Suppose that J_1 is a nonseparating two-sided curve in K. Cut K along J_1, and we have $S^1 \times I = T$. Let C_1, C_2 be the components of $\mathrm{cl}(J_2 - J_1)$. If C_i, for some

i, is separating in T, observe that C_i can be deformed to an arc in J_1 keeping the endpoints constant. Therefore, in this case, J_2 is isotopic to J_1, or it is a one-sided curve in K (obviously there is a disk in K bounded by subpolyhedra of J_1 and J_2). If C_1 and C_2 are both nonseparating in T, they must be isotopic to vertical lines of $S^1 \times I$. Obviously we see that in this case J_2 is a separating two-sided curve in K. Consequently, if J_2 is isotopic to J_1 or a one-sided curve in K, C_i must be separating in T for some i, and therefore the claim follows.

Suppose J_1 is a separating two-sided curve in K (noncontractible). Cutting K along J_1, we have two Möbius bands, B_1, B_2. Let C_1, C_2 be the components of $\mathrm{cl}(J_2 - J_1)$. If C_1 and C_2 do not separate B_1 or B_2, then one sees that J_2 is a nonseparating two-sided curve in K. If one of C_1 and C_2 separate B_1 or B_2, then it can be deformed to an arc J_1 keeping the endpoints constant (of course, in this case we have a disk in K bounded by arcs in J_1 and J_2). Therefore, J_2 is either isotopic to J_1, or it is a one-sided curve. Thus, the result follows.

(3) If J_1 and J_2 have just one point in common, it follows from (1) that they are locally tangent in K at that point. Therefore J_1 is not isotopic to J_2 in K (since there are no two isotopic disjoint one-sided curves in a Klein bottle). So we assume that $J_1 \cap J_2$ is two points. By (1), J_1 must meet J_2 transversally in K at both points. Cut K along J_1, and we have a Möbius band B. J_1 separates J_2 into two components C_1, C_2. Observe that both C_1, C_2 cannot be nonseparating in B. Then we see that one of C_1, C_2 separates B into two components, one of which is a 2-cell D. One can easily argue that the 2-cell is a nonsingular disk in K. Hence, the result follows.

(4.7) Let Q be a good system of intersection curves of K and $h(K)$. We have eliminated all innermost disks and annuli from the intersection curves of K and $h(K)$. Therefore, one sees from Lemma 4.6 that any two distinct elements of Q are not isotopic in K. Thus, we may assume that Q consists of at most four elements (nonisotopic). If Q has a nonseparating curve in K and has no other element, then the result of Theorem C follows from Step 2.

LEMMA 4.8. *If Q contains two nonisotopic two-sided curves J_1 and J_2 in K, then Q can be replaced by another good system Q' which consists of only one-sided curves in K.*

PROOF. Observe that J_1 and J_2 meet transversally in K at two points. Then $\mathrm{Fix}(h)$ separates J_1 and J_2 into two pairs of arcs $\{J_{11}, J_{12}\}$ and $\{J_{21}, J_{22}\}$ such that $J_1 = J_{11} \cup J_{12}$ and $J_2 = J_{21} \cup J_{22}$. It is not difficult to see that the simple closed curve $C_i = J_{1i} \cup J_{2i}$ $(i = 1, 2)$ is one-sided in K. Now replace J_1, J_2 by C_1, C_2, so that we have a new good system. In this case we may assume (by Lemma 4.6) that $Q = \{C_1, C_2\}$ (since C_1 and C_2 are not isotopic in K and there is no innermost disk or annulus in $h(K)$).

(4.9) Note that there are no three disjoint nonisotopic one-sided simple closed curves in K. Now it follows from Lemmas 4.6 and 4.8 that the only remaining possible good systems Q would be (i) Q consists of a nonseparating two-sided curve and a one-sided curve, (ii) Q consists of two nonisotopic one-sided curves and one nonseparating two-sided curve in K, (iii) Q consists of two nonisotopic one-sided curves and one separating (two-sided) curve in K, and (iv) Q consists of two nonisotopic one-sided curves in K. In the following we rule out the first two cases and reduce the third case to the fourth which will be taken care of in Step 4.

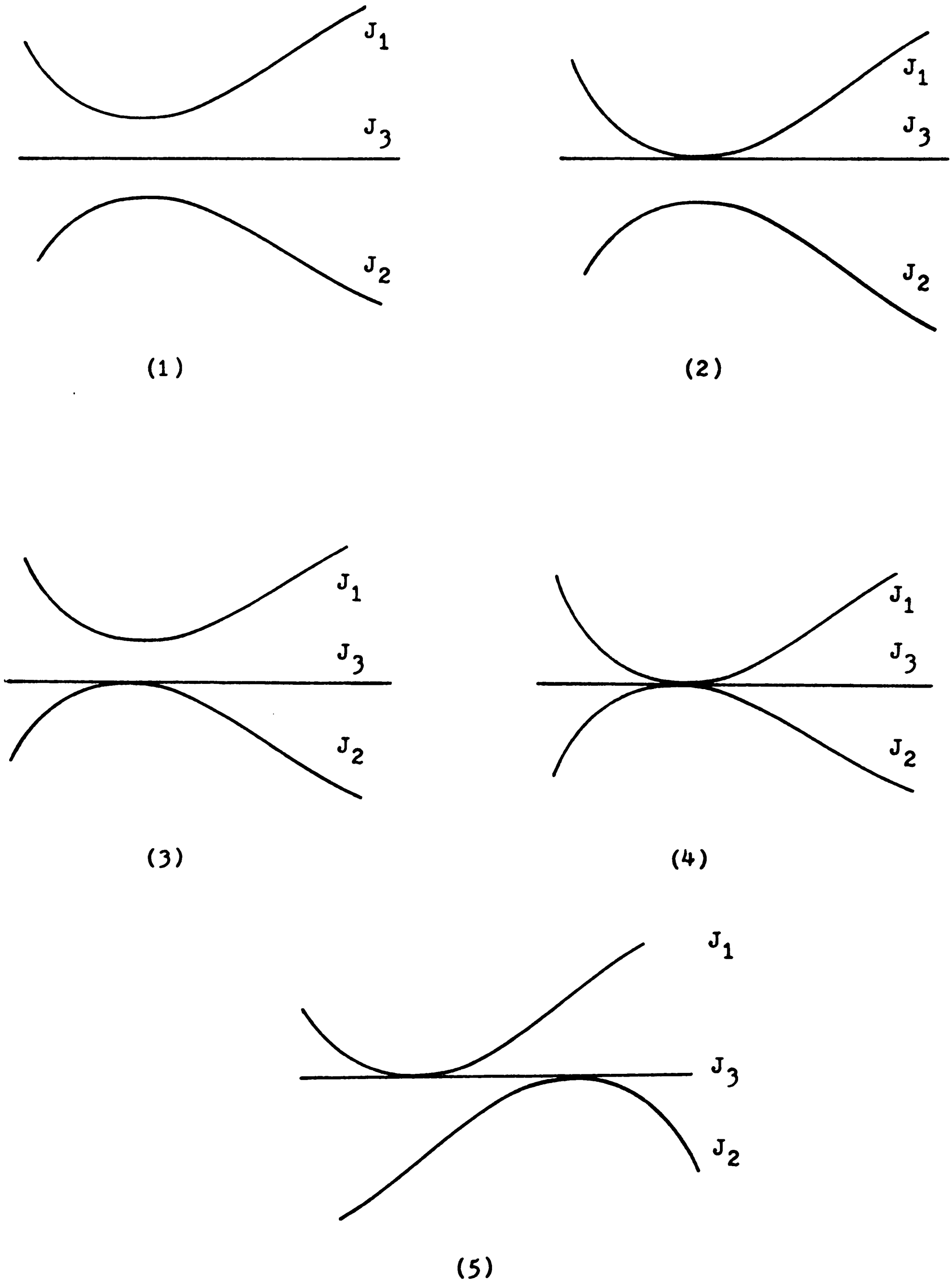

FIGURE A

Case (1). Q consists of a nonseparating two-sided curve J_1 and a one-sided curve J_2. We rule out this case. Observe that if J_1 and J_2 meet at two points, then it has an innermost disk in $h(K)$. Therefore, we may assume that J_1 and J_2 meet at exactly one point x. Let B be a small neighborhood of x in M such that $D = B \cap K$ is a regular neighborhood of x in K. Then D separates B into two balls B^+ and B^-. Furthermore, since K is in h-general position, $h(D)$ is separated by D into two pairs of nonadjacent disks $\{B_1, C_1\}$, $\{B_2, C_2\}$ such that $B_1 \cup C_1 \subset B^+$ and $B_2 \cup C_2 \subset B^-$. Note that $K - (J_1 \cup J_2)$ is connected. Let G be a small neighborhood of J_1 in K. Then J_1 separates G into two components G^+, G^-. We may assume that $D \subset G$. Suppose that B_1, B_2 are the two adjacent disks such that $B_1 \cap D$ and $B_2 \cap D$ are contained in the same side G^+ of J_1 in G. Then one may easily see that both B_1 and B_2 must be in B^+ (or B^-, otherwise) by following along J_1 in G^+ (make a complete circle trip along J_1 in G^+ without crossing the arc $B_1 \cap B_2$). This is a contradiction to the earlier observation.

Case (2). Q consists of two nonisotopic one-sided curves J_1, J_2 in K and one nonseparating two-sided curve J_3 in K. We rule out this case. It follows from Lemma 4.6 that J_3 meets each of J_1 and J_2 at a single point (transversally in K), and J_1 is either locally tangent to or disjoint from J_2. Thus, one can easily see that either $h(J_1) = J_2$ or one of J_1 and J_2 is invariant. Therefore, one of J_1 and J_2 must be one-sided in $h(K)$ (say J_1). Since J_1 and J_3 meet at exactly one point x locally piercingly in K, it follows from Lemma 4.3 that we need another (third) curve of Q which meets J_1 transversally at a point (other than x) in order to have J_1 to be one-sided in both K and $h(K)$ (treat J_1 as c in Lemma 4.3 and take a regular neighborhood of J_1). But, there is no such third curve in Q (J_2 is disjoint from or locally tangent to J_1 at one or two points).

Case (3). Q consists of two nonisotopic one-sided curves J_1, J_2 and one separating (two-sided) curve J_3 in K. It follows from Lemma 4.6 that all cuts among J_1, J_2 and J_3 cannot be locally piercing in K. Thus, J_1 or J_2 may be tangent to J_3 in K at a point or disjoint from J_3. Therefore, one sees that J_1, J_2 are one-sided in $h(K)$ and J_3 is two-sided in $h(K)$. The three curves J_1, J_2 and J_3 may intersect in five similar ways described in Figure A.

Our plan is to simplify the curve J_3, so that we reduce this case to the case where Q consists of only two nonisotopic one-sided curves in K. We demonstrate this only for the case where J_1, J_2, J_3 are pairwise disjoint. One can easily see that our demonstration also goes through for the other cases with minor changes (also see Step 1).

Since J_3 is two-sided in K and $h(K)$, it separates $K \cup h(K)$ into two components T_1 and T_2, each of which consists of two Möbius bands crossing along J_i for some i ($i = 1, 2$) transversally. We may assume that $J_i \subset T_i$ ($i = 1, 2$). Let U be a small neighborhood of J_3 such that $U \cap (K \cup h(K))$ consists of two annuli A_1, A_2 with center circles in common. Then $A_1 \cup A_2$ separates ∂U into two pairs of nonadjacent annuli $\{G_1, H_1\}$, $\{G_2, H_2\}$. We see that $h(G_i \cup H_i) = G_i \cup H_i$ ($i = 1, 2$). Observe that exactly one of the two pairs of nonadjacent annuli meet both T_1 and T_2, say $\{G_1, H_1\}$. We may assume that H_1 meets K in T_1 and $h(K)$ in T_2 (if this is not the case, then G_1 does the job). Then we see that G_1 meets $h(K)$ in T_1 and K in

T_2 (see Figure B). Let $K' = H_1 \cup [T_1 \cap (K - U)] \cup [T_2 \cap (h(K) - U)]$. Then K' is a Klein bottle such that either $h(K') = K'$ or $h(k') \cap K'$ consists of two nonisotopic one-sided curves in K'. Thus, we have reduced this case to the case where G consists of two nonisotopic one-sided curves in K.

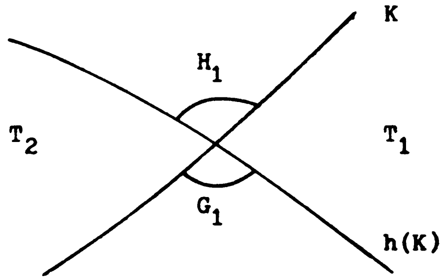

FIGURE B

Step 4. Finally we consider the case where Q consists of two nonisotopic one-sided curves J_1, J_2 in K.

Case (1). J_1 and J_2 are disjoint. Let U_i ($i = 1, 2$) be a small regular neighborhood of J_i in M such that $U_i \cap (K \cup h(K))$ is two Möbius bands B_i, B_i' with center circles in common. Let $U = U_1 \cup U_2$. We may assume that $B_i' \subset K$ and $B_i \subset h(K)$, and $h(U_i) = U_i$ or $h(U_i) \cap U_i = \varnothing$. Let N_0 be a regular neighborhood of $\mathrm{cl}(K - U)$ ($\approx S^1 \times I$) in $\mathrm{cl}(M - U - h(K))$. Let $N = N_0 \cup U$ and $N_1 = \mathrm{cl}(M - N)$. Then N is a regular neighborhood of K in M, and therefore N_1 is a solid torus (see **[13]**).

Let $E_i = U_i \cap N_0$ and c_i be a boundary component of E_i. Then E_i is a regular neighborhood of $\partial B_i'$ in ∂U_i, and $c_i \cup \partial B_i$ bounds an annulus H_i in ∂U_i. Therefore, $\partial B_1 \cup \partial B_2$ bounds an annulus B in ∂N ($= \partial N_1$) (here, one can choose H_1, H_2 so that $B = H_1 \cup H_2 \cup G$ where G is a component of $\mathrm{cl}(\partial N_0 - (E_1 \cup E_2))$ which meets both H_1 and H_2). Clearly, $\partial B_1 \cup \partial B_2$ bounds an annulus B' ($\subset h(K)$) in N_1. Therefore $B \cup B'$ bounds a solid torus and there exists an isotopy which sends B to B', keeping $\partial B_1 \cup \partial B_2$ constant (note that ∂B_i does not bound a 2-cell in N_1; otherwise, $M \approx P^3 \# P^3$ (see Proposition 2.1 of **[13]**)) (see Lemma 4.4). Furthermore, each component of $\mathrm{cl}(\partial N_0 - (E_1 \cup E_2))$ is isotopic to $\mathrm{cl}(K - U)$ in N_0, and each component of ∂E_i is also isotopic to $\partial B_i'$. Consequently, letting R_1 and R_2 be the closures of $M - (U \cup K \cup h(K))$, we see that $R_1 \cup U$ or $R_2 \cup U$ is a twisted I-bundle over a Klein bottle. We assume that $X = R_1 \cup U$ is the twisted I-bundle. Observe that each R_i ($i = 1, 2$) is invariant or h interchanges R_1 and R_2. One also sees that each ∂U_i is separated by $K \cup h(K)$ into two annuli A_i, A_i', one of which is contained in R_1. We may assume that $A_i' \subset R_1$. Then $\partial X = A_1 \cup A_2 \cup [(K \cup h(K)) - U]$.

Subcase (1). $h(U_i) = U_i$. Suppose that $J_i \cap \mathrm{Fix}(h) = \varnothing$ for some i. It follows from Lemma 4.2 that $h(A_i) = A_i'$, and therefore we see that $h(R_1) = R_2$ (those points in R_2 near A_i and those in R_1 near A_i' are interchanged). So this is the case where $\mathrm{Fix}(h) = \varnothing$. Observe that R_1 is deformable into A_i' ($i = 1, 2$) (see how X is shown to be a twisted I-bundle in the above). Furthermore, since $h(R_1) = R_2$ and

$h(A_2') = A_2$, we see that R_2 is deformable into A_2. Therefore, $\pi_1(M - J_1 - J_2) = Z \oplus Z$. Since U_1 and U_2 are disjoint regular neighborhoods of J_1 and J_2, it follows from [30] that $\text{cl}(M - U_1 = U_2) \approx S^1 \times S^1 \times I$, and $R_1 \cup R_2 \approx S^1 \times S^1 \times I$. Since U_1 is invariant, it follows that ∂U_1 is the desired torus which splits M into two invariant solid tori.

Now suppose that $J_i \cap \text{Fix}(h) \neq \varnothing$ for all i. Then obviously $h(A_i) = A_i$. Therefore ∂X is invariant, and we are done.

Subcase (2). $h(U_1) = U_2$. Observe that $h(A_1) = A_2$ or A_2'. Suppose that $h(A_1) = A_2$. Then we see that ∂X is an invariant torus as desired. Now suppose that $h(A_1) = A_2'$. Then we see that $h(R_1) = R_2$ (those points in R_2 near A_1 and those in R_1 near A_2' are interchanged). So this is the case where $\text{Fix}(h) = \varnothing$. Let $S = A_2' \cup (\partial X - A_2)$. Then we see that S is an invariant torus such that h interchanges the sides of S (since $h(R_1) = R_2$). Now the result ensues from the following lemma whose proof is clear.

LEMMA. *Let L be a closed, orientable, irreducible* 3-*manifold with $|\pi_1(L)| < \infty$. Given a homeomorphism f of L, if there exists an invariant torus S such that f interchanges the sides of S, then L is a lens space* (*in fact, S splits L into two solid tori*).

Case (2). $J_1 \cap J_2$ is one or two points. It follows from Lemma 4.6 that J_1 is locally tangent to J_2 in K at each intersection point of J_1 and J_2. The following is based on the case where $J_1 \cap J_2$ is a single point. We will state the differences by inserting parentheses for the case where $J_1 \cap J_2$ is two points. The proof is almost the same as Case (1) above. Let T be a small invariant regular neighborhood of $J_1 \cap J_2$ in M such that $T \cap K$ is a regular neighborhood of $J_1 \cap J_2$ in K. Let U_i ($i = 1, 2$) be a small regular neighborhood of $\text{cl}(J_i - T)$ in $\text{cl}(M - T)$ such that $U_i \cap (K \cup h(K))$ is a pair of disks B_i, B_i' (or two disjoint pairs of disks, resp.) crossing each other along an arc transversally. Using the fact that $J_1 \cap J_2 \subset \text{Fix}(h)$, one can easily argue that $h(J_1) = J_1$ or J_2. Thus we may assume that $h(U_i) = U_i$ or $h(U_1) = U_2$. Let $U = U_1 \cup U_2 \cup T$ and N_0 be a regular neighborhood of $\text{cl}(K - U)$ ($\approx$ disk) in $\text{cl}(M - U - h(K))$. Then $N = N_0 \cup U$ is a regular neighborhood of K in M. We see that $M - (U \cup K \cup h(K))$ has two components (three components, resp.) whose closures are denoted by R_1, R_2 (R_1, R_2, R_3, resp.). Observe that one of R_1, R_2 (two of R_1, R_2, R_3, resp.) has the boundary homeomorphic to S^2, say R_1 (R_1, R_3, resp.). Since M is irreducible and a 3-cell does not contain a Klein bottle, we see that R_1 (R_1, R_3, resp.) is a 3-cell. Since each boundary component of N_0 is isotopic to $\text{cl}(K - U)$ in $\text{cl}(M - U - h(K))$, it is clear that $X = U \cup R_1$ ($X = U \cup R_1 \cup R_3$, resp.) is homeomorphic to N. Since R_2 is homeomorphic to a solid torus, one sees that R_1 ($R_1 \cup R_3$, resp.) is invariant under h. Therefore ∂X is invariant, and ∂X is a torus as desired. This completes the proof of Theorem C.

5. Involutions on Klein spaces. In this section we obtain a complete classification of the involutions h on Klein spaces $M = M(p, q)$. It follows from Theorem C that

M may be given as

$$M = M(r) \cup_f S^1 \times D^2 \quad \text{or} \quad M = D^2 \times S^1 \cup_{f'} S^1 \times D^2$$

such that the twisted I-bundle $M(r)$ and the solid torus $S^1 \times D^2$ in the first case are invariant under h, and the two solid tori in the second case are either invariant or interchanged under h, where f and f' are equivariant attaching maps of $S^1 \times S^1$. Therefore, we have the following general characterization theorem. Theorem D for the case of free involutions has been obtained by Rubinstein **[26]** (also see **[22]** for free involutions on lens spaces).

THEOREM D. *The orbit space of an involution on $M(p, q)$ is either a Klein space or a lens space.*

According to the proof of Theorem C, if $\mathrm{Fix}(h) \neq \varnothing$, we may always assume the splitting of the first kind above. In the case where $\mathrm{Fix}(h) \neq \varnothing$, we let $T_1 = h|M(r)$ and $T_2 = h|S^1 \times D^2$.

Our plan is that we will first investigate the free involutions, and then consider the involutions h with $\mathrm{Fix}(h) \neq \varnothing$ according to the fixed-point sets of involutions T_1. It follows from Theorem 3.1 that the fixed-point sets $\mathrm{Fix}(T_1)$ would be: (1) $\varnothing$; (2) a simple closed curve; (3) two simple closed curves; (4) two arcs; (5) a simple closed curve plus two arcs. It follows from **[26]** that there exist no orientation-reversing involutions on Klein spaces.

It is known **[21]** that the isotopy classes of homeomorphisms of $S^1 \times S^1$ are completely determined by the subgroup S ($\subset \mathrm{Gl}(2, Z)$) of 2×2 matrices over Z with determinant ± 1. We always take the standard generators of $\pi_1(S^1 \times S^1)$ as defined in §1 (therefore, we understand that S is obtained from these generators).

(I) *Free involutions*. We prove the following classification theorem for free involutions on Klein spaces $M = M(p, q)$.

THEOREM 5.1. *$M(p, q)$, $p > 1$, admits free involutions if and only if q is odd, and it is unique* (*up to conjugation*). *The orbit space is homeomorphic to $M(2p, q)$.*

The remaining case where $p = 1$ is stated in the following. Recall that $M(1, q)$ is homeomorphic to a lens space $L(4q, 2q - 1)$ (see §1).

THEOREM 5.2. (1) *Let $q > 1$. $L(4q, 2q - 1)$ admits exactly four distinct nonconjugate free involutions for q odd and two for q even.*

(2) *$L(4, 1)$ admits exactly three distinct nonconjugate free involutions.*

REMARK. The orbit spaces (Theorem 5.2) are homeomorphic to $M(2, q)$, $M(q, 2)$, $L(8q, 2q - 1)$, or $L(8q, 2q + 1)$ (the first two cannot occur for q even). Note that, if $q = 1$, then $M(q, 2)$ is homeomorphic to $L(8q, 2q + 1)$ (see §1).

PROOF OF THEOREM 5.1. Since $p > 1$, we see that $\pi_1(M)$ is not abelian. Let M' be a Klein space which can be double-covered by $M = M(p, q)$. Then M' is homeomorphic to $M(p', q')$ for some p', q'. It follows from Proposition 2.3 of **[13]** that $p' = 2p$ and $q' = \pm q$. Therefore, it follows from Theorem D that $M(2p, q)$ is the only possible orbit space (up to homeomorphisms) induced by free involutions on $M(p, q)$. On the other hand, $\pi_1(M')$ has two distinct nonabelian subgroups of index

2, but there exists a homeomorphism H of M' such that H_* sends one nonabelian subgroup of index 2 to another (see Theorem 3 of **[13]**). Therefore, the result follows easily.

PROOF OF THEOREM 5.2. It follows from **[13**, Theorem 4 and Proposition 4.6] that there exist exactly two distinct nonconjugate free involutions on $L(4q, 2q-1)$, for q odd, whose orbit spaces contain Klein bottles (their orbit spaces are $M(2, q)$, $M(q, 2)$), and no such free involutions for q even. Now let $L(p', q')$ be a lens space which can be double-covered by $L(4q, 2q-1)$. Then we have either $q'q'' \equiv \pm 1$ or $q' \equiv \pm q''$ (mod $4q$) where $q'' = 2q - 1$ (see the proof of Theorem 3.6 of **[10]**. Theorem 3.6 there is not settled as stated; Ritter's result, used there, still remains unsolved, and so we need to insert the additional hypothesis $q = 2k - 1$, which is then a special case of our Theorem 5.2). On the other hand, since $q''^2 \equiv 1$ (mod $4q$), we see that $q' \equiv \pm q''$ (mod $4q$). Note that two lens spaces $L(a, b)$ and $L(a, b')$ are homeomorphic if and only if $bb' \equiv \pm 1$ or $b \equiv \pm b'$ (mod a). Now the result follows easily (see also Remark in the above).

(II) $\mathrm{Fix}(T_1)$ *is empty*. This is the case where $\mathrm{Fix}(T_2)$ is a simple closed curve. Thus, by Theorem 3.1 and Lemma 1.4, we may assume that

$$M = M(r) \cup_f S^1 \times D^2 \tag{$*$}$$

and h is given by

$$h[x, t] = [g_1(x), t] \quad \text{for } [x, t] \in M(r),$$
$$h(z_1, \rho z_2) = (z_1, -\rho z_2) \quad \text{for } (z_1, \rho z_2) \in S^1 \times D^2$$

such that $h(M(r)) = M(r)$, where f is an appropriate equivariant attaching map of $S^1 \times S^1$ (recall that g_1 is given by $g_1(z_1, z_2) = (z_1, -z_2)$ for $(z_1, z_2) \in S^1 \times S^1$). Here, the matrix of f is given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and we may assume that $a = p$, $b = \pm q$, and $\left|\begin{smallmatrix} p & b \\ c & d \end{smallmatrix}\right| = 1$ (by a proper choice of orientations of the standard paths C_1, C_2 in §1) (see §1). The following figure illustrates our situation.

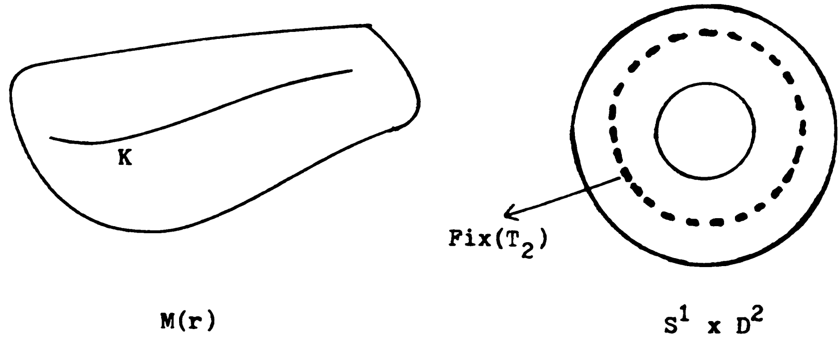

FIGURE C

LEMMA 5.3. *q is even.*

PROOF. The orbit space $M' = M/h$ may be given as $M' = M(r) \cup_{f'} S^1 \times D^2$ such that $k_1(M(r)) = M(r)$, where f' is the attaching map of $S^1 \times S^1$ induced by f, and k_i ($i = 1, 2$) is the orbit map induced by T_i. Let k_i' be the map of $S^1 \times S^1$ such that $k_i' = k_i|S^1 \times S^1$. Then one observes that there exist parametrizations of $M(r)$ and $S^1 \times D^2$ in M' such that $k'_{i*}(1, 0) = (1, 0)$ and $k'_{i*}(0, 1) = (0, 2)$ for $(1, 0)$, $(0, 1) \in \pi_1(S^1 \times S^1)$. Now it follows from simple computation that the matrix of f' is given by $\left(\begin{smallmatrix} p & b/2 \\ 2c & d \end{smallmatrix}\right)$, and the result follows immediately.

LEMMA 5.4. *Let h_i ($i = 1, 2$) be an involution of $M_i = M(r) \cup_{f_i} S^1 \times D^2$ such that $h_i(M(r)) = M(r)$, and $h_i|M(r) = T_1$ and $h_i|S^1 \times D^2 = T_2$. If f_1 and f_2 are isotopic, then h_1 and h_2 are conjugate.*

PROOF. The orbit space $M_i' = M_i/h_i$ can be given as $M_i' = M(r) \cup_{f_i'} S^1 \times D^2$ where f_i' is the attaching map induced by f_i. It follows from the proof of Lemma 5.3 that f_1' and f_2' are isotopic. Therefore, it is not difficult to see that h_1 and h_2 are conjugate.

By the above lemma, we may now assume that the attaching map f in $(*)$ may be given by

$$f(z_1, z_2) = \left(z_1^p z_2^b, z_1^c z_2^d\right) \quad \text{for } (z_1, z_2) \in S^1 \times S^1$$

where $b = \pm q$, and $\left|\begin{smallmatrix} p & b \\ c & d \end{smallmatrix}\right| = 1$. We denote this space M by $M = M(b, c, d)$. One may check that this is a well-defined equivariant map such that $fT_1 = T_2 f$ (note that b is even, and therefore d is odd). Our involutions h on $M = M(p, q)$ now depend on various possible integers b, c, d with $b = \pm q$ and $\left|\begin{smallmatrix} p & b \\ c & d \end{smallmatrix}\right| = 1$. We denote these involutions h by $h(b, c, d)$. In the following, $h_1 \sim h_2$ means that h_1 and h_2 are conjugate.

LEMMA 5.5. *Let $\left|\begin{smallmatrix} p & b \\ c & d \end{smallmatrix}\right| = 1$. Then,*
(1) $h(b, c, d) \sim h(b, c', d')$ for any c', d' with $\left|\begin{smallmatrix} p & b \\ c' & d' \end{smallmatrix}\right| = 1$.
(2) $h(b, c, d) \sim h(-b, -c, d)$.

PROOF. (1) Let $h_1 = h(b, c, d)$ and $h_2 = h(b, c', d')$. Since $pd' - c'b = 1 = pd - cb$, we see that $d' = d + mb$ and $c' = c + mp$ for some m. Define t_1: $M(b, c, d) \to M(b, c', d')$ such that $t_1(M(r)) = M(r)$ by $t_1[x, t] = [x, t]$ for $[x, t] \in M(r)$ and $t_1(z_1, \rho z_2) = (z_1, \rho z_2 z_1^m)$ for $(z_1, \rho z_2) \in S^1 \times D^2$. It is checked that t_1 is a well-defined equivalence such that $h_2 t_1 = t_1 h_1$.

(2) Let $h_3 = h(-b, -c, d)$. Define a map g_6 of $S^1 \times S^1$ by $g_6(z_1, z_2) = (z_1, \bar{z}_2)$. Then g_6 and r commute. Define t_2: $M(b, c, d) \to M(b, c', d')$ such that $t_2(M(r)) = M(r)$ by $t_2[x, t] = [g_6(x), t]$ for $[x, t] \in M(r)$ and $t_2(z_1, \rho z_2) = (z_1, \rho \bar{z}_2)$ for $(z_1, \rho z_2) \in S^1 \times D^2$. One may check that this is a well-defined equivalence such that $h_2 t_2 = t_2 h_1$.

THEOREM 5.6. *There exist an involution h on $M(p, q)$ with* $\mathrm{Fix}(T_1) = \varnothing$ *if and only if q is even, and it is unique (up to conjugation). The orbit space is homeomorphic to $M(p, q/2)$.*

PROOF. The first part of the thoerem is obvious from the previous work. Let $h_1 = h(q, r, s)$ and $h_2 = h(b, c, d)$. If $b = q$, then it follows from Lemma 5.5 that $h_1 \sim h_2$. If $b = -q$, again it follows from Lemma 5.5 that $h_1 \sim h(p, -q, -r, s) \sim h(p, b, c', d') = h_2$. Thus, we see that every involution of this type is conjugate to h_1. By the proof of Lemma 5.3, we see that the orbit space M/h is homeomorphic to $M(p, q/2)$.

(III) Fix(T_1) *is a simple closed curve.* The fixed-point set Fix(T_2) is either empty or a simple closed curve. By Theorem 3.1 and Lemma 1.4, we may assume that (*) holds (see (II)) and h is given by either

$$h[x, t] = [g_3(x), t] \quad \text{for } [x, t] \in M(r),$$
$$h(z_1, \rho z_2) = (-z_1, \rho z_2) \quad \text{for } (z_1, \rho z_2) \in S^1 \times D^2,$$

or

$$h[x, t] = [g_3(x), t] \quad \text{for } [x, t] \in M(r),$$
$$h(z_1, \rho z_2) = (z_1, -\rho z_2) \quad \text{for } (z_1, \rho z_2) \in S^1 \times D^2$$

such that $h(M(r)) = M(r)$, where f in (*) is an appropriate equivariant attaching map of $S^1 \times S^1$ (g_3 is defined by $g_3(z_1, z_2) = (-z_1, -z_2)$ for $(z_1, z_2) \in S^1 \times S^1$). We may assume that the matrix of f is given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ so that $a = p$, $b = \pm q$, and $\left|\begin{smallmatrix} p & b \\ c & d \end{smallmatrix}\right| = 1$. We illustrate our situations in the following figure (Fix(T_1) is contained in K).

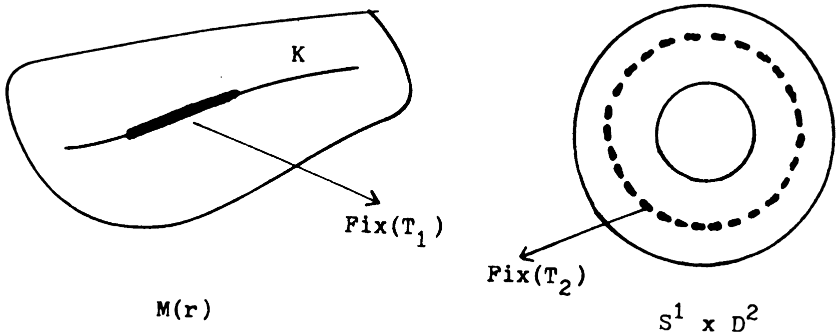

FIGURE D

Let M_1 be the orbit space $M(r)/T_1$. Then we see from the above that M_1 is a solid torus. Let k: $M(r) \to M_1$ be the orbit map induced by T_1. Let k': $\partial M(r) \to \partial M_1$ be the map so that $k' = k|\partial M(r)$. One observes (from the above) that there exists an orientable annulus bundle structure on $M(r)$ over S^1 with connected boundary such that each fibre A is an invariant annulus and A separates K into $S^1 \times I$. Notice that Fix(T_1) is two-sided in K. Now it is obvious that there exists a parametrization of M_1 in terms of $S^1 \times D^2$ such that $k'_*(1, 0) = (2, 1)$ and $k'_*(0, 1) = (0, \varepsilon)$, where $\varepsilon = 1$ or -1 (see also Theorem 1.3).

LEMMA 5.7. *If* Fix(T_2) $= \varnothing$, *then* $p + b$ *is odd and* $c + d$ *is even. If* Fix(T_2) $\neq \varnothing$, *then* $p + b$ *is even and* $c + d$ *is odd.*

PROOF. The orbit space $M' = M/h$ may be given as $M' = D^2 \times S^1 \cup_{f'} S^1 \times D^2$ such that $k_1(M(r)) = D^2 \times S^1$, where f' is the attaching map of $S^1 \times S^1$ induced by f, and k_i ($i = 1, 2$) is the orbit map induced by T_i. Let k'_i be the map of $S^1 \times S^1$ such that $k'_i = k_i|S^1 \times S^1$. By an earlier observation, we see that there exists a parametrization of $D^2 \times S^1$ such that $k'_{1*}(1, 0) = (1, 2)$ and $k'_{1*}(0, 1) = (\varepsilon, 0)$, where $\varepsilon = 1$ or -1 (note that we switch our parametrization of $S^1 \times D^2$ to

$D^2 \times S^1$ for a technical purpose). On the other hand, there exists an obvious parametrization of $S^1 \times D^2$ such that if $\mathrm{Fix}(T_2) = \varnothing$, we have $k'_{2*}(1, 0) = (2, 0)$ and $k'_{2*}(0, 1) = (0, 1)$, and if $\mathrm{Fix}(T_2) \neq \varnothing$, we have $k'_{2*}(1, 0) = (1, 0)$ and $k'_{2*}(0, 1) = (0, 2)$. By a simple computation, we see that the matrix of f' is given by

$$\begin{pmatrix} 2\varepsilon b & p - \varepsilon b \\ \varepsilon d & (c - \varepsilon d)/2 \end{pmatrix} \quad \text{if } \mathrm{Fix}(T_2) = \varnothing,$$

or

$$\begin{pmatrix} \varepsilon b & (p - \varepsilon b)/2 \\ 2\varepsilon d & c - \varepsilon d \end{pmatrix} \quad \text{if } \mathrm{Fix}(T_2) \neq \varnothing.$$

Now the result follows from a simple observation (note that the determinants are 1 or -1).

COROLLARY 5.8. *$\mathrm{Fix}(h)$ is a simple closed curve if $p + q$ is odd, and $\mathrm{Fix}(h)$ is two simple closed curves if $p + q$ is even.*

PROOF. Immediate from Lemma 5.7.

LEMMA 5.9. *Let h_i $(i = 1, 2)$ be an involution of $M_i = M(r) \cup_{f_i} S^1 \times D^2$ such that $h_i(M(r)) = M(r)$, and $h_i|M(r) = T_1$ and $h_i|S^1 \times D^2 = T_2$. If f_1 and f_2 are isotopic, then h_1 and h_2 are conjugate.*

PROOF. A similar proof to that of Lemma 5.4 may be used (use the proof of Lemma 5.7).

By the above lemma, we may assume that the attaching map f in $(*)$ is given by $f(z_1, z_2) = (z_1^p z_2^b, z_1^c z_2^d)$ for $(z_1, z_2) \in S^1 \times S^1$, where $b = \pm q$ and $\left|\begin{smallmatrix} p & b \\ c & d \end{smallmatrix}\right| = 1$, and $c + d$ is even or odd according to $p + b$ odd or even, respectively (note that such c, d always exist for each pair of coprime p, b).

LEMMA 5.10. *Let $\left|\begin{smallmatrix} p & b \\ c & d \end{smallmatrix}\right| = 1$. Then*
(1) $h(b, c, d) \sim h(b, c', d')$ for any c', d' with $\left|\begin{smallmatrix} p & b \\ c' & d' \end{smallmatrix}\right| = 1$,
(2) $h(b, c, d) \sim h(-b, -c, d)$.

PROOF. Use the same proof as that of Lemma 5.5. One will need the fact that m is even in this case if $\mathrm{Fix}(T_2) = \varnothing$ (note that $p + b$ is odd and $c + d \equiv c' + d' \equiv 0 \pmod 2$).

Now the following theorem follows easily from our earlier work (see also the proof of Theorem 5.6).

THEOREM 5.11. *There exists exactly one involution h on $M(p, q)$ with $\mathrm{Fix}(T_1) \approx S^1$, up to conjugation. The orbit space is homeomorphic to either a lens space $L(2q, p + q)$ (if $p + q$ is odd) or a lens space $L(q, (p + q)/2)$ (if $p + q$ is even).*

REMARK 1. Let h_1 and h_2 be two standard involutions on $M = M(p, q)$ with $\mathrm{Fix}(T_1) = \varnothing$ and $\mathrm{Fix}(T_1) \neq \varnothing$, respectively. Since q must be even (Theorem 5.6), we see that p is odd. Therefore, $\mathrm{Fix}(h_i)$ for each $i = 1, 2$ is a simple closed curve, and $M/h_1 \approx M(p, q/2)$ and $M/h_2 \approx L(2q, p + q)$ (see Theorems 5.6 and 5.11). Thus we see that h_1 is not conjugate to h_2 if $p \neq 1$. Now we claim that h_1 is conjugate to h_2 if $p = 1$. Let g: $S^1 \times S^1 \times I \to M(r)$ be the standard orbit map (see §1). Let J be the simple closed curve $(e^{2\pi ti}, i)$ $(0 \leq t \leq 1)$ in $S^1 \times S^1$. Then one observes that $g(J \times \{0\}) = \mathrm{Fix}(h_2)$. Furthermore, we see that $g(J \times I)$ is a nonsingular annulus in $M(r)$. However, $J \times \{1\}$ is isotopic to the simple closed curve C_1 (in (1.2)). Since $f_*(1, 0) = (1, r)$ for some r, we see that $\mathrm{Fix}(h_2)$ is isotopic to the center circle of $S^1 \times D^2$ in $M = M(r) \cup_f S^1 \times D^2$. Therefore, letting U be an invariant regular neighborhood of $\mathrm{Fix}(h_2)$, $\mathrm{cl}(M - U)$ is a twisted I-bundle over a Klein bottle. Thus, h_2 is of essentially the same type as h_1, and therefore it follows from Theorem 5.6 that h_1 and h_2 are conjugate. For further reference, the involutions h_1 and h_2 will be said to be of types (II) and (III), respectively.

(IV) $\mathrm{Fix}(T_1)$ *is two simple closed curves*. The fixed-point set $\mathrm{Fix}(T_2)$ is either empty or a simple closed curve. Therefore, by Theorem 3.1 and Lemma 1.4, we may assume that the Klein space $M = M(p, q)$ is as in (*) and h as in (III) (replace g_3 by g_5; g_5 is defined by $g_5(z_1, z_2) = (-z_1, z_2)$ for $(z_1, z_2) \in S^1 \times S^1$). Our situation is as described in the following figure. ($\mathrm{Fix}(T_1)$ is contained in K.)

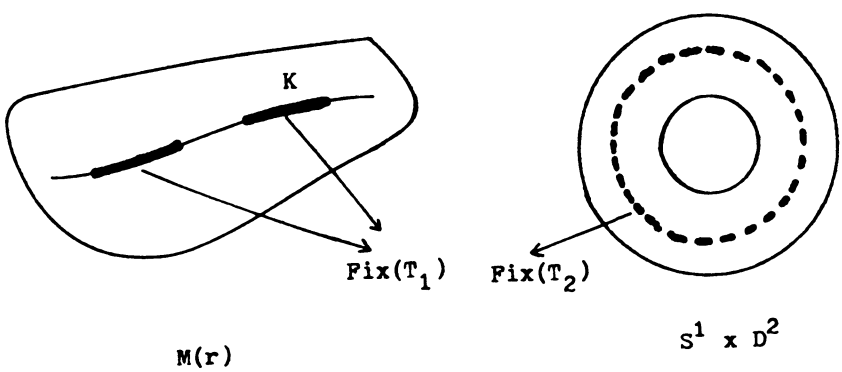

FIGURE E

Let M_1 be the orbit space $M(r)/T_1$. Then M_1 is a solid torus. Let k: $M(r) \to M_1$ be the orbit map induced by T_1. Let k': $\partial M(r) \to \partial M_1$ be the map so that $k' = k|\partial M(r)$. Observe that the simple closed curves of $\mathrm{Fix}(T_1)$ are one-sided in K, and each component of $k(\mathrm{Fix}(T_1))$ is isotopic to the center circle of the solid torus M_1. Furthermore, we see that $k(C_2 \times \{1\})$ bounds a disk in M_1 where C_2 is the standard path in (1.2). Therefore, there exists a parametrization of M_1 in terms of $D^2 \times S^1$ such that $k'_*(1, 0) = (0, 2)$ and $k'_*(0, 1) = (1, 0)$.

Now we may easily fill in the proofs of the following results by following the same steps as in (III) (see also (II)).

PROPOSITION 5.12. *If p is odd, then* $\mathrm{Fix}(h)$ *is two simple closed curves. If p is even, then* $\mathrm{Fix}(h)$ *is three simple closed curves.*

REMARK. The equivariant attaching map f in $M = M(r) \cup_f S^1 \times D^2$ may be given as $f(z_1, z_2) = (z_1^p z_2^q, z_1^r z_2^s)$ for $(z_1, z_2) \in S^1 \times S^1$ (here, r is even if $\mathrm{Fix}(T_2) = \varnothing$, and p is even if $\mathrm{Fix}(T_2) \neq \varnothing$ where $\left|\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right| = 1$).

THEOREM 5.13. *There exists exactly one involution h on $M(p, q)$ with* $\mathrm{Fix}(T_1) \approx S^1 \dot{\cup} S^1$, *up to conjugation. The orbit space is homeomorphic to either a lens space* $L(2q, p)$ (*if p is odd*) *or a lens space* $L(q, p/2)$ (*if p is even*).

REMARK 2. The standard involution h on $M(p, q)$ with $\mathrm{Fix}(T_1) \approx S^1 \dot{\cup} S^1$ will be said to be of type (IV). One may easily check that any involution of type (IV) is not conjugate to an involution of type (II) or (III) (compare the number of components of the fixed-point sets and their orbit spaces).

(V) $\mathrm{Fix}(T_1)$ *is two arcs*. The fixed-point set $\mathrm{Fix}(T_2)$ is also two arcs. By Theorem 3.1 and Lemma 1.4, we may assume that $(*)$ holds and h is given by

$$h[x, t] = [g_2(x), t] \quad \text{for } [x, t] \in M(r),$$

$$h(z_1, \rho z_2) = (\bar{z}_1, -\rho\bar{z}) \quad \text{for } (z_1, \rho z_2) \in S^1 \times D^2$$

such that $h(M(r)) = M(r)$, where f in $(*)$ is an appropriate equivariant attaching map of $S^1 \times S^1$ (recall that g_2 is given by $g_2(z_1, z_2) = (\bar{z}_1, -\bar{z}_2)$ for $(z_1, z_2) \in S^1 \times S^1$). Again, the matrix of f is given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and we may assume that $a = p$, $b = \pm q$, and $\left|\begin{smallmatrix} p & b \\ c & d \end{smallmatrix}\right| = 1$. Our situation may be described in the following figure.

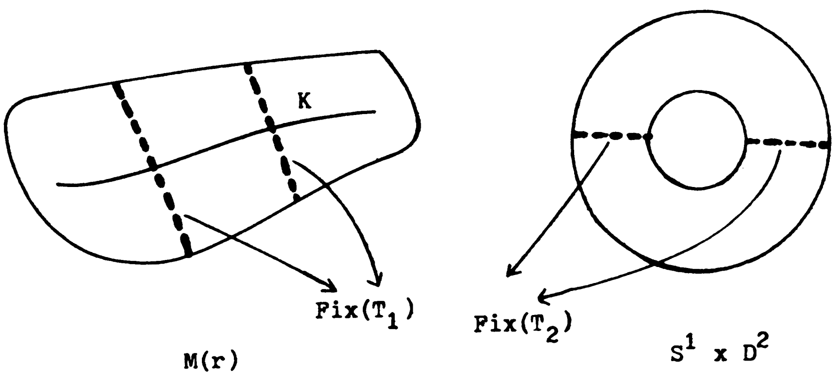

FIGURE F

The following is a result of Tollefson [32].

LEMMA 5.14 [32]. *Let v be an involution on $S^1 \times S^1$ with four fixed points. Let f_1, f_2 be two isotopic, equivariant homeomorphisms of T^2 (i.e., $f_i v = v f_i$ for each $i = 1, 2$). If f_1 and f_2 agree at a point of* $\mathrm{Fix}(v)$, *then there exists an equivariant ambient isotopy H_t: $S^1 \times S^1 \to S^1 \times S^1$ $(0 \leqslant t \leqslant 1)$ such that $H_1 f_1 = f_2$ (i.e., $H_t v = v H_t$).*

It follows from Lemma 5.14 that we may assume in $(*)$ that

$$f(z_1, z_2) = \left((-1)^k i^b z_1^p z_2^b,\ (-1)^l i^{d+1} z_1^c z_2^d\right) \quad \text{for some } k, l = 0, 1$$

where i is a nonreal fourth root of 1.

We denote the space M by $M(b, c, d)$ if $k, l = 0$, and the involution h on $M(b, c, d)$ by $h(b, c, d)$. Define a homeomorphism G: $M \to M(b, c, d)$ by

$$G[x, t] = [x, t] \qquad \text{for } [x, t] \in M(r),$$

$$G(z_1, \rho z_2) = \left((-1)^k z_1, (-1)^l \rho z_2\right) \quad \text{for } (z_1, \rho z_2) \in S^1 \times D^2.$$

Then G serves as an equivalence between h and $h(b, c, d)$. Now our classification depends on the possible integers b, c, d. Here, $b = \pm q$ and $\left|\begin{smallmatrix} p & b \\ c & d \end{smallmatrix}\right| = 1$.

LEMMA 5.15. *let* $\left|\begin{smallmatrix} p & b \\ c & d \end{smallmatrix}\right| = 1$. *Then*
(1) $h(b, c, d) \sim h(b, c', d')$ *for any* c', d' *with* $\left|\begin{smallmatrix} p & b \\ c' & d' \end{smallmatrix}\right| = 1$,
(2) $h(b, c, d) \sim h(-b, -c, d)$.

PROOF. Use the same proof as that of Lemma 5.5.

Now the rest of the steps are almost identical to those in (II) (see also (III)), and one may obtain the following result.

THEOREM 5.16. *There exists exactly one involution h on $M(p, q)$ with* $\mathrm{Fix}(T_1) \approx$ *two arcs, up to conjugation. The orbit space is homeomorphic to the projective* 3-*space* P^3.

REMARK 3. The involution h with $\mathrm{Fix}(T_1) \approx$ two arcs will be said to be of type (V). We observe the following: (1) Let h_1 be an involution on $M = M(p, q)$ of type (III). If $\mathrm{Fix}(h_1)$ is a simple closed curve and $q = 1$, then h_1 is conjugate to an involution of type (V). (2) Let h_2 be an involution on $M = M(p, q)$ of type (IV). If $\mathrm{Fix}(h_2)$ is two simple closed curves and $q = 1$, then h_2 is conjugate to an involution of type (V). The former can occur if p is even, and the latter can occur if p is odd (see Corollary 5.8 and Proposition 5.12). Recall that there exists an invariant Klein bottle K in M containing $\mathrm{Fix}(h_i)$ for each i ($i = 1, 2$). Then $\mathrm{Fix}(h_1)$ is a separating two-sided simple closed curve in K, and each component of $\mathrm{Fix}(h_2)$ is one-sided in K. In either case, there exists an invariant nonseparating two-sided simple closed curve J in K which meets $\mathrm{Fix}(h_i)$ at two points transversally in K. One sees that J is isotopic to the standard path C_2 (in §1). Furthermore, one observes that $f_*(0, 1) = (1, s)$ for some s (since $q = 1$) where f is the attaching map in $M = M(r) \cup_f S^1 \times D^2$. Therefore, J is isotopic to the center circle of $S^1 \times D^2$. Now, we see that $\mathrm{cl}(M - U)$ is a twisted I-bundle over S^1 where U is an invariant regular neighborhood of J in M. Thus, each h_i is essentially of type (V).

One can easily check that an involution on $M(p, q)$ of type (V) cannot be conjugate to any involution of other type unless $q = 1$ (compare the orbit spaces, and conditions on p, q (odd or even); recall that $M(p, q)$ and $M(p', q')$ are homeomorphic if and only if $p' = p$ and $q' = \pm q$).

(VI) $\mathrm{Fix}(T_1)$ *is a simple closed curve plus two arcs*. The fixed-point set $\mathrm{Fix}(T_2)$ is then two arcs. By Theorem 3.1 and Lemma 1.4, we may assume that M ($= M(p, q)$) and h are given by $M = M(r) \cup_f S^1 \times D^2$, and

$$h[x, t] = [g_4(x), t] \quad \text{on } M(r)$$
$$h(z_1, \rho z_2) = (\bar{z}_1, \rho \bar{z}_2) \quad \text{on } S^1 \times D^2$$

such that $h(M(r)) = M(r)$. (Recall that g_4 is defined by $g_4(z_1, z_2) = (\bar{z}_1, \bar{z}_2)$ for each $(z_1, z_2) \in S^1 \times S^1$.) An example of these involutions can be given by defining $f(z_1, z_2) = (z_1^p z_2^q, z_1^r z_2^s)$ where $\left|\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right| = 1$. The following figure describes our situation (the simple closed curve of $\mathrm{Fix}(T_1)$ is contained in K).

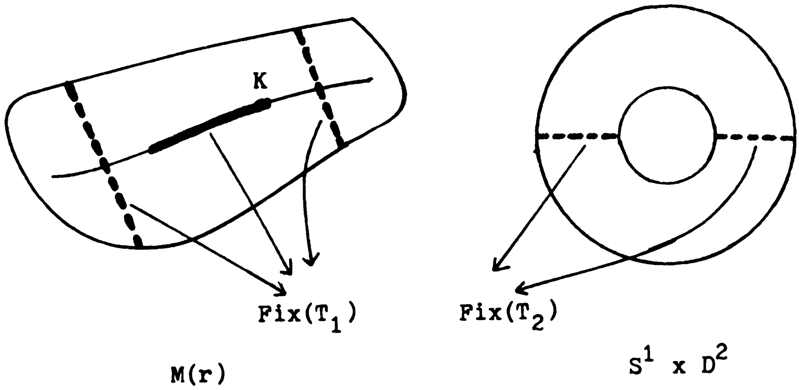

FIGURE G

One may easily see the proof of Theorem 5.17 by following similar steps to those in (V).

THEOREM 5.17. *There exists exactly one involution h on $M(p, q)$ such that* $\mathrm{Fix}(T_1)$ *is a simple closed curve plus two arcs up to conjugation. The orbit space is homeomorphic to S^3.*

REMARK 4. The involution in Theorem 5.17 will be said to be of type (VI). Let h be an involution of $M = M(p, q)$ of type (VI). Let K be an invariant Klein bottle in $M(r)$ such that $K \cap \mathrm{Fix}(T_1)$ is a simple closed curve J plus two points. Suppose that $q = 1$. Observe that J is a nonseparating two-sided simple closed curve in K. Thus, by a similar reason to that of Remark 3, $\mathrm{cl}(M - U)$ is a twisted I-bundle over S^1 where U is an invariant regular neighborhood of J in M such that $U \cap \mathrm{Fix}(h) = J$. Therefore, the involution h is essentially of type (III) if $\mathrm{Fix}(h)$ is two simple closed curves, and of type (IV) if $\mathrm{Fix}(h)$ is three simple closed curves. The former can occur when p is odd, and the latter can occur when p is even (see Corollary 5.8 and Proposition 5.12).

One may easily observe that h cannot be conjugate to an involution of another type unless $q = 1$.

(VII) *Involutions with nonempty fixed-point set.* We summarize our work in (II) through (VI). In the following chart, we indicate the occurence of types of involutions h on $M(p, q)$, according to the coprime integers p, q. (Equality between two types means that the involutions arising from the two different types are conjugate.)

CHART A

$p > 1$		$p = 1$		$q = 1$	
q even > 1	q odd > 1	q even > 1	q odd > 1	p even	p odd
(II) (III) (IV) (V) (VI)	(III) (IV) (V) (VI)	(II) (= (III)) (IV) (V) (VI)	(III) (IV) (V) (VI)	(III) (= (V)) (IV) (= (VI))	(III) (= (VI)) (IV) (= (V))

The above chart follows from our earlier work in this section (also see Remarks 1–4). We are now in a position to state the classification theorem for involutions on $M(p, q)$ with nonempty fixed-point set. The proof follows from theorems in this section (see Chart A and Remarks 1–4).

THEOREM 5.18. *There exist exactly five distinct nonconjugate involutions on* $M(p, q)$, $p > 1$, *with nonempty fixed-point set if* q *is even, four if* q *is odd* > 1, *and two if* $q = 1$.

For $p = 1$, we have the following. Recall that $M(1, q)$ is homeomorphic to the lens space $L(4q, 2q - 1)$.

THEOREM 5.19. *There exists exactly four distinct nonconjugate involutions on* $L(4q, 2q - 1)$ *with nonempty fixed-point set if* $q > 1$ *and two if* $q = 1$.

REMARK. The involutions in the above two theorems may be distinguished by the nonhomeomorphic orbit spaces: S^3, P^3, $M(p, q/2)$ (q even), $L(2q, p + q)$ (or $L(q, (p + q)/2)$), and $L(2q, p)$ (or $L(q, p/2)$). The choice of the last two depends on whether $p + q$ in the former (or p in the latter, resp.) is odd or even. Note that some of the orbit spaces are homeomorphic according to p and q (for example, $M(p, q/2) \approx L(2q, p + q)$ in case $p = 1$ and q is even).

6. Some group actions on S^3. In this section we study some group actions on S^3 and prove Corollary 1. Let G be a group of order 8. The only such groups G which can act freely on the 3-sphere are Z_8 and Q (Quaternions) (see **[4]**, **[20]**). It has been shown **[25]**, **[26]**, **[6]** that a free action of Z_8 or Q on S^3 is conjugate to an orthogonal action. Indeed, since any Q-action on S^3 acts freely (as shown in the following), the classification problem for group actions on S^3 of Quaternions is essentially settled.

PROPOSITION 6.1. *Every Q-action on S^3 acts freely.*

PROOF. First, observe that Q contains an orientation-preserving element a of order 4 (if c, d generate G and they are orientation-reversing, then cd is an orientation-preserving element of order 4). Suppose that Q does not act freely. Then, since there is only one element of order 2 in Q, we see that $\mathrm{Fix}(b) \subset \mathrm{Fix}(a^2)$ for any element b ($\neq 1$) $\in Q$, and $\mathrm{Fix}(a^2) \neq \varnothing$. Furthermore, we see that $F = \mathrm{Fix}(a^2)$ is invariant under any element of b of Q (since $a^2 = b$ or b^2 if $b \neq 1$). Here, F is a simple closed curve.

Let $p\colon S^3 \to S^3/\langle a \rangle$ be the orbit map and h the involution on $S^3/\langle a \rangle$ induced by Q. Since F is invariant under each element of Q, $F' = p(F)$ is invariant under h. Therefore, there exists an invariant regular neighborhood U' of F' in $S^3/\langle a \rangle$. Let $U = p^{-1}(U')$. Then U is a regular neighborhood of F which is invariant under Q. Since F is unknotted **[34]**, we see that $X = \mathrm{cl}(S^3 - U)$ is a solid torus. Since X is invariant under Q, for simplicity we assume that Q acts on X. Then we see (from the first paragraph) that Q acts freely on X. Let $X' = X/\langle a \rangle$. Then, since a is orientation-preserving, it is well known that X' is a solid torus (see **[12]** for example). Let h' be the involution on X' induced by Q. Since h' is free, the orbit space $X'' = X'/\langle h' \rangle$ is a solid torus or a solid Klein bottle. Note that $X'' = X/Q$. Let $g: X \to X''$ be the orbit map induced by Q. Then $\pi_1(X'')/g_*\pi_1(X) \cong Q$, which is impossible since $\pi_1(X'') \cong Z$. This completes the proof.

Now in what follows we suppose that G acts on S^3 and contains an element a of order 4 acting freely. Observe that, if $G = Z_8$, then G must act freely in this case (since $\mathrm{Fix}(b) \subset \mathrm{Fix}(a^2)$ for any b $(\neq 1) \in G$). Let h be the involution on $L = S^3/\langle a \rangle$ induced by G. Then we see that $\mathrm{Fix}(h) = \varnothing$ if and only if $G = Q$ or Z_8 (by the above argument). Recall that L is homeomorphic to $L(4, 1)$ (see **[24]**). Thus, if $G = Q$ or Z_8, then it follows from Theorem 5.2 that S^3/G is homeomorphic to either $M(2, 1)$ or $L(8, k)$ $(k = 1, 3)$.

Now assume that $F = \mathrm{Fix}(h) \neq \varnothing$. It follows from the lifting theorem that there exists an involution b on S^3 with $\mathrm{Fix}(b) \neq \varnothing$ such that $hg = gb$ where $g: S^3 \to L$ is the orbit map. Then we see that $g^{-1}(F)$ has one or two components, and exactly one of components of $g^{-1}(F)$ is the fixed-point set $\mathrm{Fix}(b)$. Observe that the elements a, b generate G. A homeomorphism t on $Y = L(p, q)$ is called sense-preserving if t induces the identity on $H_1(Y)$.

Lemma 6.2. *If h is sense-preserving, then $G = Z_2 \times Z_4$. Otherwise, $G = D_4$ (dihedral group).*

Proof. It follows from **[9]** and **[16]** that h is sense-preserving if and only if $g^{-1}(F)$ is a simple closed curve. Suppose that $F' = g^{-1}(F)$ is a simple closed curve and $G = D_4$. Then we see that $\mathrm{Fix}(b) = F'$ and $a(F') = F'$. Since $a^3 = bab$, we see that $a^3(x) = bab(x) = ba(x) = a(x)$, and therefore $a^2(x) = x$ where x is a point of F'. Since $\langle a \rangle$ acts freely on S^3, this cannot occur. Thus, if h is sense-preserving, then $G = Z_2 \times Z_4$.

Now suppose that $g^{-1}(F)$ is two simple closed curves F_1, F_2 and $G = Z_2 \times Z_4$. We may assume that $\mathrm{Fix}(b) = F_1$. Observe that $a(F_1) = F_2$ and $b(F_2) = F_2$. Let $x \in F_1$. Then, since $a = bab$, we see that $a(x) = bab(x) = ba(x)$. Therefore, $a(x) \in F_1$ $(= \mathrm{Fix}(b))$. However, since $a(F_1) = F_2$ and $x \in F_1$, we see that $a(x) \in F_2$, which is impossible. Thus, if h is not sense-preserving, then $G = D_4$. This completes the proof.

Proof of Corollary 1. We use the above notations. It follows from Theorem 5.21 that $L/h \approx S^3$ or P^3 where $L = L(4, 1)$. If h is sense-preserving, then $L/h \approx P^3$ (see **[9]**). Since there exist exactly two distinct nonconjugate involutions on L with nonempty fixed-point (Theorem 5.19), L/h must be S^3 if h is not sense-preserving. Observe that $L/h = S^3/G$.

Now we show that G is conjugate to an orthogonal action. We only need to consider the case where G is either D_4 or $Z_2 \times Z_4$. Let G_1 and G_2 be two groups $\approx G$ which act on S^3. For simplicity, in connection with G_i $(i = 1, 2)$ we use the same notations as above by just adding indices i. It follows from Theorem 5.19 that there exists an equivalence t of L_1 to L_2 such that $th_1 = h_2t$. We also have a lifting homeomorphism t' of S^3 such that $g_2t' = tg_1$. One can easily check that $t'a_1t'^{-1} = a_2^j$ $(j = 1$ or $3)$ and $t'b_1t'^{-1} = a_2^kb_2$ (note that $g_ib_i = h_ig_i$) where $k = 0$ or 2 if $G_i = Z_2 \times Z_4$, and $k = 0, 1, 2$, or 3 if $G_i = D_4$ (in D_4, each a^kb has order 2). Since

a_2^j and $a_2^k b_2$ generate the group G_2, we see that G is conjugate to an orthogonal action. We remark that the standard action on S^3 of D_4 may be given by $a(z_1, z_2) = (iz_1, iz_2)$ and $b(z_1, z_2) = (\bar{z}_1, \bar{z}_2)$, where S^3 is viewed as $\{(z_1, z_2) | z_1\bar{z}_1 + z_2\bar{z}_2 = 1\}$ and i is a nonreal fourth root of 1. This completes the proof.

ADDED DURING REVISION. The author would like to take this opportunity to add to the paper (by the author) *Some 3-manifolds which admit Klein bottles*, Trans. Amer. Math. Soc. **244** (1978), 299–312, the following diagram (omitted by the printer). It should be inserted in the third line from the top, p. 304, right after the sentence "The sequence can be factored as:"

$$\begin{array}{ccccc} \pi_1(M) & \xrightarrow{k} & Z_2 & \to & 0 \\ \downarrow & \nearrow & & & \\ H_1(M) & & & & \end{array}$$

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