

WEAK P -POINTS IN COMPACT CCC F -SPACES

BY

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ABSTRACT. Using a technique due to van Mill we show that each compact ccc F -space of weight greater than 2^ω contains a weak P -point, i.e. a point $x \in X$ such that $x \notin \bar{F}$ for each countable $F \subset X - \{x\}$. We show that, assuming $BF(c)$, each nowhere separable compact F -space has a weak P -point. We show the existence of points which are not limit points of any countable nowhere dense set in compact F -spaces of weight \aleph_1 . We also discuss remote points and points not the limit point of any countable discrete set.

Introduction. All spaces considered are completely regular and X^* denotes $\beta X - X$. A space X is an F -space if each cozero set is C^* -embedded. A ccc F -space is easily seen to be extremally disconnected; the closure of each open set is open. A point $x \in X$ is a *weak P -point* if $x \notin \bar{F}$ for each countable $F \subset X - \{x\}$. Kunen [K] has shown that ω^* has a dense set of weak P -points. Jan van Mill then showed that each compact infinite F -space of weight 2^ω in which nonempty G_δ 's have nonempty interior has weak P -points [vM]. He also showed that if there is a ccc nonseparable growth of ω then he could remove the weight restriction. Subsequently Murray Bell [B] constructed such a growth of ω . Then in [DvM] the author and van Mill extended van Mill's result to "each compact nowhere ccc F -space has weak P -points". It is easy to see that a separable space cannot have weak P -points. In §5 we give an example of a nonseparable F -space in which there are no weak P -points. We see, therefore, that we need to assume nowhere separable rather than nonseparable. However in the case of ccc spaces we can consider all nonseparable spaces because such a space contains a nowhere separable open set. We address the open question "do all compact nowhere separable F -spaces have weak P -points?"

We are able to show that for compact ccc F -spaces of weight greater than c the answer is yes and that assuming $BF(c)$ it is also true for nonseparable compact ccc F -spaces of weight $\leq c$. We are also able to show that for spaces of weight \aleph_1 we do not need to assume $BF(c)$, that is, there are points not the limit point of any countable nowhere dense set.

The point $x \in X^*$ is called a *remote point* of X if $x \notin \text{cl}_{\beta x} A$ for each nowhere dense subset of A of X . It is known that if X is a nonpseudocompact space with countable Π -weight then X has a remote point [vD, 1.5], [CS]. In [vDvM] the authors show that not every nonpseudocompact space has remote points but ask if $\omega \times 2^{\omega^2}$ has remote points if CH fails. Our methods enable us to show that under $BF(c)$ each nonpseudocompact ccc space of weight $\leq c$ has a remote point.

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Finally we also show that under some further assumptions on the Π -character of a ccc F -space there are points which are not the limit point of any countable discrete set.

1. Nice filters. Let X be a normal space. A *filter base* on X is a collection of closed subsets of X which is closed under finite intersections and which does not contain the empty set. The *filter* generated by the filterbase \mathcal{F} is the collection of $\{A \subset X: A = \overline{A} \text{ and } \exists F \in \mathcal{F} \text{ with } F \subset A\}$. Let X be the topological sum of countably many nonempty compact spaces, say X_n ($n < \omega$). In [vM] van Mill defines a filter \mathcal{F} to be *nice* provided that for each $F \in \mathcal{F}$, the set $\{n < \omega: F \cap X_n = \emptyset\}$ is finite while, in addition, $\bigcap \mathcal{F} = \emptyset$. We will say that a nice filter \mathcal{F} on X *avoids* countable sets if for each countable $D \subset X$ there is an $F \in \mathcal{F}$ with $\overline{D} \cap F = \emptyset$. In [vM], Jan van Mill has done most of the work of constructing weak P -points.

LEMMA 1.1. *Let K be a compact ccc F -space and let $\{Z_n: n \in \omega\}$ be disjoint nonempty clopen subsets of K . Then if \mathcal{F} is a nice filter on $X = \bigcup Z_n$, there is a point $x \in \bigcap \{\text{cl}_K F: F \in \mathcal{F}\}$ which is not in the closure of any countable subset of $K \setminus (X \cup \{x\})$. In particular, if \mathcal{F} avoids countable subsets of X then x is a weak P -point of K .*

Due to the length and complexity we will not include this proof. The reader is referred to the proof of Theorem 0.3 in [vM]. One need only observe that the restriction of \mathcal{F} to any infinite subcollection of the Z_n 's is again a nice filter on this union and that Bell [B] has shown the existence of a ccc nowhere separable growth of ω . We then follow the proof of Theorem 0.3 in [vM] verbatim. \square

Our investigation therefore turns to constructing nice filters on topological sums of compact ccc F -spaces which avoid countable sets.

2. Large F -spaces. We will first investigate compact ccc F -spaces with weight greater than c . We begin by stating some results which we will require.

THEOREM 2.1 [BF]. *Every infinite complete boolean algebra B contains a free subalgebra A with $|A| = |B|$.*

The clopen subsets of a compact ccc F -space form a complete boolean algebra so by the above result and Stone's duality theorem [W] we obtain

THEOREM 2.2. *Each compact ccc F -space of weight greater than c can be mapped onto 2^{c^+} .*

Using Theorem 2.2 we can now prove

THEOREM 2.3. *Each compact ccc F -space X of weight greater than c contains weak P -points.*

PROOF. It follows easily from 2.2 that we can choose countably many disjoint clopen subsets $\{Z_n: n \in \omega\}$ of X each of weight greater than c . From Lemma 1.1 we need only construct a nice filter \mathcal{F} on $\bigcup Z_n$ which avoids countable subsets of $\bigcup Z_n$. Our technique is very similar to that used in [vM]. By Theorem 2.2, we can let g_n be a continuous surjection from Z_n to 2^{c^+} . Note that 2^{c^+} is ccc nowhere separable. For

each countable $D \subset \bigcup Z_n$, let $D' = \bigcup_n g_n[D \cap Z_n]$. Choose a cellular family of clopen subsets $\{A_k^D: k \in \omega\}$ of 2^{c^+} whose union is dense and $A_k^D \cap D' = \emptyset$ for each k . This can be done because D' is nowhere dense in 2^{c^+} . We will let \mathfrak{F} be the filter generated by $\{\bigcup_{n \in \omega} \bigcup_{k \leq n} g_n^-[A_k^D]: D \in [\bigcup Z_n]^\omega\}$.

It is a simple matter to check that \mathfrak{F} is a nice filter on $\bigcup Z_n$ and obviously \mathfrak{F} avoids countable sets. Let $\{D_j: 1 \leq j \leq m\}$ be countable subsets of $\bigcup Z_n$. Then for each of the cellular families $\{A_k^{D_j}: k \in \omega\}$, $\bigcup_{k \in \omega} A_k^{D_j}$ is dense in 2^{c^+} . Hence we can recursively choose k_j , $1 \leq j \leq m$, so that $\bigcap_{j=1}^m A_{k_j}^{D_j} \neq \emptyset$. Let $N = \max\{k_j: 1 \leq j \leq m\}$ and let $n \geq N$. Hence

$$Z_n \cap \left(\bigcup_{k \leq n} g_n^-[A_k^{D_1}] \right) \cap \cdots \cap \left(\bigcup_{k \leq n} g_n^-[A_k^{D_m}] \right)$$

contains $g_n^-[\bigcap_{j=1}^m A_{k_j}^{D_j}]$ and is therefore not empty. This proves that \mathfrak{F} is a nice filter and completes the proof of the theorem. \square

REMARK. It is worth noting that any ccc nowhere separable space Y could have taken the place of 2^{c^+} in the above proof so long as one has surjections from each Z_n to Y . This fact can be used to conclude that many well-known F -spaces have weak P -points, for instance any ccc nowhere separable space which has countably many disjoint clopen sets which are pairwise homeomorphic.

3. Small F -spaces. In the case of small F -spaces, that is, spaces of weight less than or equal to c , we cannot use the above method because 2^κ for $\kappa \leq c$ is separable. The method we use is an attempt to capture within the space the essential idea behind the above method. We only managed to succeed with the aid of the set-theoretic principle $BF(c)$. Let F be the set of all functions from ω into ω . If f and g belong to F , define $g \leq f$ provided $\{n \in \omega: g(n) > f(n)\}$ is finite. A subset G of F is bounded if there is an $f \in F$ such that for each $g \in G$, $g \leq f$. $BF(c)$ is equivalent to the statement: each subset of F of cardinality less than c is bounded. $BF(c)$ is known to be consistent with the usual axioms of set theory and follows from MA or even $P(c)$ [R, pp. 82, 88].

To prove our result for small nowhere separable compact ccc F -spaces we will first prove a lemma in greater generality than is needed for weak P -points.

LEMMA 3.1. *Assume $BF(c)$. Let $\{Z_n: n \in \omega\}$ be compact ccc spaces of weight less than or equal to c and let X be the topological sum of $\{Z_n: n \in \omega\}$. There is a nice filter \mathfrak{F} on X which avoids all nowhere dense subsets of X . (Such a filter has been called a remote filter [vM].)*

PROOF. Since X is ccc and of weight less than or equal to c there are only c maximal cellular families of regular closed sets. Let $\{\{A_{n,m}^\alpha: n, m \in \omega\}: \alpha < c\}$ list all maximal cellular families of regular closed sets such that for each $\alpha < c$ and $n \in \omega$, $A_{n,m}^\alpha \subset Z_n$ for all $m \in \omega$. Let \mathfrak{D} be the set of nowhere dense subsets of X . Notice that for each $D \in \mathfrak{D}$ there is an $\alpha < c$ such that $\bar{D} \cap (\bigcup_{n,m} A_{n,m}^\alpha) = \emptyset$.

Our plan is to select, for each $\alpha < c$, a function h_α from ω into ω . We will define our filter \mathfrak{F} to be generated by the set of closed sets $\{\bigcup_{n \in \omega} \bigcup_{j \leq h_\alpha(n)} A_{n,j}^\alpha: \alpha < c\}$.

So the idea is to select the h_α 's to ensure that this filter is nice. This procedure is actually a simple recursion using $BF(c)$.

Let $h_0(n) = n$ for each $n \in \omega$. Suppose we have defined h_γ for $\gamma < \alpha < c$ such that for any finite sequence $\gamma_1 < \gamma_2 < \dots < \gamma_k < \alpha$ there is an $N \in \omega$ such that for $n \geq N$,

$$Z_n \cap \left[\bigcup_{j \leq h_{\gamma_1}(n)} A_{n,j}^{\gamma_1} \right] \cap \dots \cap \left[\bigcup_{j \leq h_{\gamma_k}(n)} A_{n,j}^{\gamma_k} \right] \neq \emptyset.$$

(The Z_n is here only for emphasis.) This is the condition we require to ensure we get a nice filter.

Let us select h_α . For each $E \in [\alpha]^{<\omega}$ we define a function g_E as follows. Let E be the sequence $\gamma_1 < \gamma_2 < \dots < \gamma_k$. Let $g_E(n) = 0$ if $\bigcap_{i=1}^k [\bigcup_{j \leq h_{\gamma_i}(n)} A_{n,j}^{\gamma_i}] = \emptyset$. Otherwise let $g_E(n)$ be the smallest integer p such that

$$A_{n,p}^\alpha \cap \bigcap_{i=1}^k \left[\bigcup_{j \leq h_{\gamma_i}(n)} A_{n,j}^{\gamma_i} \right] \neq \emptyset.$$

Since $|\{g_E: E \in [\alpha]^{<\omega}\}| \leq |[\alpha]^{<\omega}| < c$, the set $\{g_E: E \in [\alpha]^{<\omega}\}$ is a bounded family if we assume $BF(c)$. Hence we can choose a function h_α so that for each $E \in [\alpha]^{<\omega}$ the set $\{n: g_E(n) > h_\alpha(n)\}$ is finite.

To see that we have preserved our induction assumption, let $E = \{\gamma_i: 1 \leq i \leq k, \gamma_i < \alpha\}$. By assumption, there is an integer N such that for $n \geq N$,

$$Z_n \cap \bigcap_{i=1}^k \left[\bigcup_{j \leq h_{\gamma_i}(n)} A_{n,j}^{\gamma_i} \right] \neq \emptyset.$$

Therefore, for $n \geq N$,

$$A_{n,g_E(n)}^\alpha \cap \bigcap_{i=1}^k \left[\bigcup_{j \leq h_{\gamma_i}(n)} A_{n,j}^{\gamma_i} \right] \neq \emptyset.$$

By the definition of h_α , there is an integer N_1 , so that for $n \geq N_1$, $h_\alpha(n) \geq g_E(n)$. Therefore, for $n \geq \max(N, N_1)$,

$$Z_n \cap \bigcap_{i=1}^k \left[\bigcup_{j \leq h_{\gamma_i}(n)} A_{n,j}^{\gamma_i} \right] \cap \left[\bigcup_{j \leq h_\alpha(n)} A_{n,j}^\alpha \right] \neq \emptyset.$$

This completes the induction.

Let \mathfrak{F} be the filter generated by the set $\{\bigcup_{n < \omega} \bigcup_{j \leq h_\alpha(n)} A_{n,j}^\alpha: \alpha < c\}$. Since each $A_{n,j}^\alpha$ is closed in Z_n and we are only taking a finite union in each Z_n , \mathfrak{F} is indeed a filter of closed sets. It follows easily from the induction that \mathfrak{F} is nice. Finally, for each $D \in \mathfrak{D}$, as pointed out above, there is an $\alpha < c$ such that $\overline{D} \cap (\bigcup_{n,m} A_{n,m}^\alpha) = \emptyset$. This completes the proof. \square

We can now state the main result of this section.

THEOREM 3.2. *Assume $BF(c)$. Each compact ccc nowhere separable F -space of weight less than or equal to c contains weak P -points.*

PROOF. Let K be such an F -space and let $\{Z_n: n \in \omega\}$ be nonempty disjoint clopen subsets of K (recall that K is extremally disconnected). By Lemma 1.1, we need only construct a nice filter \mathfrak{F} on $X = \bigcup_n Z_n$ which avoids countable sets. From Lemma 3.1, we have a nice filter \mathfrak{F} on X which avoids all nowhere dense sets. Since X is nowhere separable we are done. \square

Our theorem concerning remote points also follows from Lemma 3.1.

THEOREM 3.3. *Assume $BF(c)$. Each nonpseudocompact ccc space of weight less than or equal to c has remote points.*

PROOF. Let X be such a space. Since X is not pseudocompact there is a nonempty zero set Z of βX contained in X^* . Let $Y = \beta X \setminus Z$. Since X is ccc and of weight less than or equal to c , so is Y . Let $\{Z_n: n \in \omega\}$ be a locally finite collection of disjoint nonempty regular closed subsets of Y . Let \mathfrak{F} be a remote filter on $\bigcup Z_n$ as constructed in Lemma 3.1. Then any $p \in \bigcap_{F \in \mathfrak{F}} \text{cl}_{\beta X} F$ is a remote point of Y and hence of X . To see this, first observe that since each Z_n is a regular closed subset of Y , there is an $F \in \mathfrak{F}$ such that $F \subset \text{int}_Y(\bigcup Z_n)$. Since Y is normal, $\text{cl}_{\beta X} F \cap \text{cl}_{\beta X}(Y \setminus \bigcup Z_n) = \emptyset$. Therefore the only nowhere dense subsets of X we have to worry about are those contained in $\bigcup Z_n$, but \mathfrak{F} avoids all such sets. \square

4. Another technique. In this section we develop another technique of constructing nice filters, which is an adaptation of Eric van Douwen's construction of remote points [vD]; the construction in [CS] is also similar. The Π -weight, ΠwX , of a space X is the least cardinality of a Π -base. A Π -base is a collection of nonempty open sets such that every nonempty open set of the space contains one from the collection. We construct nice filters in compact ccc F -spaces of Π -weight \aleph_1 which avoid all countable nowhere dense sets. We cannot hope to avoid all countable sets because it is consistent that all such spaces are separable [T]. We use the same method to construct nice filters which avoid countable discrete sets in compact ccc F -spaces with further Π -weight assumptions.

We first need a result of Efimov.

THEOREM 4.1 [E, p. 260]. *A compact ccc F -space X with no isolated points can be represented uniquely as $X = \bigcup_{n \in \mathbb{N}} \overline{U_{\kappa_n}}$, where the U_{κ_n} are disjoint open and closed sets of homogeneous Π -weight: that is, $\Pi wV = \Pi wU_{\kappa_n}$ for any open $V \subset U_{\kappa_n}$, and if $n \neq m$ then $\kappa_n \neq \kappa_m$.*

From the above theorem it follows that when we are considering compact ccc F -spaces we may assume they have homogeneous Π -weight. For the remainder of this section all hypothesized spaces will be compact ccc F -spaces with homogeneous Π -weight. The Π -character of a point x of X , denoted $\Pi\chi(x, X)$, is the least cardinal of a local Π -base at x .

LEMMA 4.2. *Let X have Π -weight \aleph_1 . Then*

- (i) $\Pi\chi(x, X) = \aleph_1$ for all $x \in X$, and
- (ii) $\Pi\chi(D, X) = \aleph_1$ for any countable nowhere dense set D , i.e. given countably many open subsets of X there is a neighborhood of D containing none of them.

PROOF. Recall that we are assuming that the Π -weight of every open set is \aleph_1 .

(i) Let $x \in X$ and let $\{A_n : n \in \omega\}$ be nonempty clopen subsets of X . It suffices to find a neighborhood of x which does not contain any of the A_n 's. Since $\Pi w(A_n) = \aleph_1$ for each $n \in \omega$, we can begin by finding a clopen set B_0 such that $B_0 \subset A_0$ and $A_n \setminus B_0 \neq \emptyset$ for all $n \in \omega$. Similarly the set $\{A_n \setminus B_0 : n \in \omega\}$ is not a Π -base for any open set, in particular, not for $A_0 \setminus B_0$. We can therefore choose a clopen set B_1 such that $B_1 \subset A_0 \setminus B_0$ and $(A_n \setminus B_0) \setminus B_1 \neq \emptyset$ for all $n \in \omega$. Similarly, recursively select clopen sets B_{2n} and B_{2n+1} such that

- (1) $B_i \cap B_j = \emptyset$ for $i < j \leq 2n + 1$,
- (2) $B_{2n} \cup B_{2n+1} \subset A_n$, and
- (3) $A_k \setminus [\bigcup_{j=1}^{2n+1} B_j] \neq \emptyset$ for all $k \in \omega$.

Therefore both $\bigcup_{j \in \omega} B_{2j}$ and $V = \overline{\bigcup_{j \in \omega} B_{2j+1}}$ intersect each A_n by (2) and $U \cap V = \emptyset$ because X is extremally disconnected. Hence either U or $X \setminus U$ is the required neighborhood of x because neither set contains any of the A_n 's.

(ii) Let $D = \{d_n : n \in \omega\}$ be nowhere dense in X and let $\{A_n : n \in \omega\}$ be clopen subsets of X . We again wish to find a neighborhood of D which does not contain any A_n . One recursively selects nonempty clopen sets B_k and C_k for $k \in \omega$ such that:

- (1) $d_k \in B_k$, $B_k \cap [\bigcup_{j < k} C_j] = \emptyset$ and $A_n \setminus [B_k \cup \bigcup_{j < k} (B_j \cup C_j)] \neq \emptyset$ for $n \in \omega$; and

- (2) $C_k \cap \overline{D} = \emptyset$, $C_k \subset A_k \setminus [\bigcup_{j \leq k} B_j]$ and $A_n \setminus [\bigcup_{j \leq k} (B_j \cup C_j)] \neq \emptyset$ for $n \in \omega$.

This recursion can be carried out because $\Pi \chi(d_k, X) = \aleph_1$ for each $k \in \omega$, $\Pi w(A_n \setminus [\bigcup_{j \leq k} (B_j \cup C_j)]) = \aleph_1$ and because \overline{D} is nowhere dense in X . This completes the proof. \square

THEOREM 4.3. *Let X have Π -weight \aleph_1 . There are points in X which are not in the closure of any countable nowhere dense sets.*

PROOF. Let $\{Z_n : n \in \omega\}$ be disjoint nonempty clopen subsets of X . By Lemma 1.1 we must construct a nice filter on $\bigcup Z_n$ which avoids all countable nowhere dense subsets of $\bigcup Z_n$. For each $n \in \omega$, let $\{B_{n,\alpha} : \alpha < \omega_1\}$ be nonempty clopen subsets of Z_n which form a Π -base for Z_n . Let $\mathfrak{D} = \{D \subset \bigcup Z_n : D \text{ is countable and nowhere dense}\}$. We will construct a family $\{F_{D,n} : D \in \mathfrak{D}, n \in \omega\}$ of clopen sets satisfying:

- (1) $F_{D,n} \cap \overline{D} = \emptyset$;
- (2) for any $\mathfrak{E} \subset \mathfrak{D}$ with $1 \leq |\mathfrak{E}| \leq n$, $\bigcap_{D \in \mathfrak{E}} F_{D,n} \neq \emptyset$; and
- (3) $F_{D,n} \subseteq Z_n$; and then define $F_D = \bigcup_n F_{D,n}$.

Our nice filter will be \mathfrak{F} which has $\{F_D : D \in \mathfrak{D}\}$ as a filter base.

For each $D \in \mathfrak{D}$ and $n \in \omega$ we define $\alpha(D, n, 0) = \min\{\alpha : B_{n,\alpha} \cap \overline{D} = \emptyset\}$. Note that $\alpha(D, n, 0)$ exists because \overline{D} is nowhere dense. By Lemma 4.2(ii) there is a clopen neighborhood $U(D, n, 1)$ of D such that $B_{n,\gamma} \setminus U(D, n, 1) \neq \emptyset$ for each $\gamma < \alpha(D, n, 0)$. Let $\alpha(D, n, 1) = \min\{\alpha : \text{for each } \gamma \leq \alpha(D, n, 0) \text{ there is a } \xi < \alpha \text{ with } B_{n,\xi} \subset B_{n,\gamma} \setminus U(D, n, 1)\}$.

Then let $K(D, n, 1) = \{\xi < \alpha(D, n, 1) : B_{n,\xi} \cap U(D, n, 1) = \emptyset\}$. Note that $\bigcup \{B_{n,\xi} : \xi \in K(D, n, 1)\} \cap \overline{D} = \emptyset$ and that, for each $\gamma < \alpha(D, n, 0)$, there is a $\xi \in K(D, n, 1)$ with $B_{n,\xi} \subset B_{n,\gamma}$. Recursively construct for $m < n$, $U(D, n, m + 1)$, clopen neighborhoods of D such that for each $\gamma < \alpha(D, n, m)$,

$$B_{n,\gamma} \setminus U(D, n, m + 1) \neq \emptyset,$$

ordinals $\alpha(D, n, m + 1) = \min\{\alpha: \text{for each } \gamma \leq \alpha(D, n, m) \text{ there is a } \xi < \alpha \text{ with } B_{n,\xi} \subset B_{n,\gamma} \setminus U(D, n, m + 1)\}$, and sets $K(D, n, m) = \{\xi < \alpha(D, n, m + 1): B_{n,\xi} \cap U(D, n, m + 1) = \emptyset\}$. Define

$$F_{D,n} = B_{n,\alpha(D,n,0)} \cup \bigcup_{m=1}^n \left[\overline{\bigcup \{B_{n,\xi}; \xi \in K(D, n, m)\}} \right].$$

Let us check that this satisfies (2). Let \mathcal{E} be a subfamily of \mathfrak{D} with $1 \leq |\mathcal{E}| \leq n$. Let $|\mathcal{E}| = e$. With recursion on j pick $E_j \in \mathcal{E} - \{E_i: 0 \leq i \text{ and } i < j\}$, for $0 \leq j < e$ in such a way that

$$(4) \alpha(E_j, n, j) \leq \alpha(E, n, j) \text{ for all } E \in \mathcal{E} - \{E_i: 0 \leq i \text{ and } i < j\}.$$

Next define $s(j) \in \omega_1$, for $0 \leq j < e$ by $s(0) = \alpha(E_0, n, 0)$;

$$s(j + 1) = \min\{\xi \in K(E_{j+1}, n, j + 1): B_{n,\xi} \subseteq B_{n,s(j)}\}.$$

This is possible by the definition of $K(E_j, n, j + 1)$ and the fact that $s(j) < \alpha(E_j, n, j + 1)$ for each $0 \leq j < e$ by (4). Since each $B_{n,s(j)} \subset F_{E_j,n}$ for $0 \leq j < e$, it follows that $\bigcap_{j < e} F_{E_j,n} \supseteq B_{n,s(e)} \neq \emptyset$. This completes the proof. \square

COROLLARY 4.4. *Let X have Π -weight \aleph_1 and be nowhere separable. Then X contains weak P -points.*

REMARK. An easy adaptation of the proof of Theorem 4.3 can be used to show the following. Let X be a nonpseudocompact space of Π -weight at most \aleph_1 . Then there is a point $x \in X^*$ such that $x \notin \bar{F}$ for any countable nowhere dense $F \subset X$ of Π -character \aleph_1 .

THEOREM 4.5. *Let $\Pi w(X) = \kappa$ be a regular cardinal and suppose that $\Pi\chi(x, X) = \kappa$ for each $x \in X$. Then X contains points which are not the limit point of any countable discrete set.*

PROOF. If $\Pi w(X) = \omega$, this is shown in [vM]. For larger cardinals κ , it is simple to show that $\Pi\chi(D, X) = \kappa$ for each countable discrete set D in X . Then simply replace \aleph_1 by κ in the proof of 4.3.

We do not know if it is necessarily true that $\Pi\chi(x, X) = \Pi w(X)$ for each $x \in X$ in a compact ccc F -space of homogeneous Π -weight. We showed that it was true for Π -weight \aleph_1 in 4.2 and in the following theorem we show it for separable compact F -spaces assuming Martin's axiom for σ -centred posets, denoted MAS.

THEOREM 4.6 (MAS). *Let X be a compact separable F -space. Then $\Pi\chi(x, X) = \Pi w(X)$ for each $x \in X$.*

PROOF. Let $\Pi w(X) = \kappa$ and note that $\kappa \leq c$ since X is separable. Let S be a countable dense subset of X and suppose there is an $x \in X$ with $\Pi\chi(x, X) < \kappa$. Let us first prove the following fact.

FACT 1. Suppose that x is the limit point of a discrete set D and that $\Pi\chi(d, X) > \lambda$ for each $d \in D$. Then $\Pi\chi(x, X) > \lambda$. Indeed, suppose $\{A_\alpha: \alpha < \lambda\}$ is a collection of clopen subsets of X . To show that it is not a Π -base at x we will find a clopen neighborhood of x not containing any A_α . Since D is discrete, let $D = \{d_n: n \in \omega\}$ and $\{V_n: n \in \omega\}$ be disjoint neighborhoods of the d_n 's. Since $\Pi\chi(d_n, X) > \lambda$ for

each n , we can choose a neighborhood U_n of d_n such that $U_n \subset V_n$ and, if $A_\alpha \cap V_n \neq \emptyset$, then $(A_\alpha \cap V_n) \setminus U_n \neq \emptyset$ for $\alpha < \lambda$. Since X is extremally disconnected and $x \in U = \overline{\bigcup U_n}$, U is a neighborhood of x . Let $\alpha < \lambda$ and suppose that $A_\alpha \subset U$. Then there is an $n < \omega$ such that $A_\alpha \cap U_n \neq \emptyset$ and therefore $A_\alpha \cap V_n \neq \emptyset$. This is a contradiction because $(A_\alpha \cap V_n) \setminus U_n \neq \emptyset$ by assumption and $(U \setminus U_n) \cap (A_\alpha \cap V_n) = \emptyset$. Therefore Fact 1 is true.

Now suppose that $x \in X$ and $\Pi\chi(x, X) = \lambda < \kappa$. Note that $\{s \in S: \Pi\chi(s, X) \leq \lambda\}$ is nowhere dense in X because the union of local Π -bases of a dense set is a Π -base for X . Therefore we assume $\Pi\chi(s, X) > \lambda$ for each $s \in S$. Our plan now is to find a discrete subset of S which has x as a limit point, which will complete the proof.

Let $\{A_\alpha: \alpha < \lambda\}$ be a Π -base at x consisting of clopen subsets of X . Also let $\{B_\alpha: \alpha < \kappa\}$ be clopen subsets of X which form a Π -base for X . Recall that we are assuming that X has homogeneous Π -weight. We can inductively define B'_α so that $B'_\alpha \subset B_\alpha$ and for each $\xi < \lambda$ and $\gamma_1 < \dots < \gamma_n < \alpha$, $B'_\alpha \not\supset A_\xi \setminus \bigcup_{i=1}^n B'_{\gamma_i}$. Therefore $\{B'_\alpha: \alpha < \kappa\}$ is a Π -base such that no finite union of its members contains any A_ξ . Let (P, \leq) be the poset whose members are $\{(F, V): F \in [S]^{<\omega}, V \text{ is a finite union of sets from the } \Pi\text{-base; and } F \subset V\}$. We will define $(G, W) \leq (F, V)$ if $F \subset G$, $V \subset W$ and $G \setminus F \cap V = \emptyset$. To see that (P, \leq) is σ -centred, we simply let $P_F = \{(F, V): (F, V) \in P\}$ for each $F \in [S]^{<\omega}$. Let $E_\alpha = \{(F, V): F \cap A_\alpha \neq \emptyset\}$ for each $\alpha < \lambda$. It is easy to check that E_α is dense in (P, \leq) . Therefore MAS allows us to choose a generic filter \mathcal{G} such that for each $\alpha < \lambda$ there is an $(F_\alpha, V_\alpha) \in \mathcal{G}$ with (F_α, V_α) also in E_α . Define $D = \bigcup \{F: \exists V \text{ with } (F, V) \in \mathcal{G}\}$.

First note that $D \cap A_\alpha \neq \emptyset$ for each $\alpha < \lambda$ and so $x \in \overline{D}$. All we have to show is that D is discrete. Let $d \in D$ and find $(F_D, V_D) \in \mathcal{G}$ with $d \in F_D$. By the definition of the partial ordering we see that $D \cap V_D = F_D$. It follows that D is discrete. \square

We remark that MA implies that if X is compact, ccc and $\Pi w(X) < c$ then X is separable [T].

COROLLARY 4.7 (MAS). *Let X be a compact separable F -space with $\Pi w(X)$ regular; then X contains points not the limit point of any countable discrete set.*

PROOF. Theorems 4.5 and 4.6.

5. Example and remarks.

EXAMPLE. We now give the promised example of a compact nonseparable F -space which does not have any weak P -points. The author is grateful to the referee and also to Jan van Mill for suggesting this example. Let $\omega_1 + 1$ be the ordinals less than or equal to ω_1 endowed with the order topology. Let E be the projective cover of $\omega_1 + 1$ and let $k: E \rightarrow \omega_1 + 1$ be the canonical map. Let $Z = k^{-1}[\{\omega_1\}]$ and let $Y = \beta(\omega \times E)$. Observe that $Y - \text{cl}_Y(\omega \times Z)$ is locally separable. Let $\Pi: \omega \times Z \rightarrow \omega$ be the projection map and let $\beta\Pi: \beta(\omega \times Z) \rightarrow \beta\omega$ be its Stone extension. Note that $\beta(\omega \times Z) = \text{cl}_Y(\omega \times Z)$, as Y is an F -space and $\omega \times Z$ is σ -compact. We can form the adjunction space $M = Y \cup_{\beta\Pi} \beta\omega$ (see [W, Chapter 10]). Since $\text{cl}_Y(\omega \times Z)$ is a P -set of Y it is easy to show that M is an F -space. Clearly M is not separable and has no weak P -points.

REMARKS. One would naturally conjecture that all compact nowhere separable F -spaces contain weak P -points. The remaining problem is to remove special set-theoretic assumptions for the case of compact ccc F -spaces of weight $\leq c$. Murray Bell has observed that if such a space has a σ - n -linked base for each n then it contains weak P -points. One would also like to show that all compact F -spaces contain points which are not the limit point of any countable discrete set. Is it true that all compact ccc F -spaces of homogeneous Π -weight also have homogeneous Π -character?

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