ON $K_n(\mathbb{Z}/p^2\mathbb{Z})$ AND RELATED HOMOLOGY GROUPS

BY

LEONARD EVENS AND ERIC M. FRIEDLANDER

Abstract. It is shown that, for $p \geq 5$, 
$$R = \mathbb{Z}/p^2\mathbb{Z}, K_3(R) = \mathbb{Z}/p^2\mathbb{Z} \oplus \mathbb{Z}/(p^2 - 1)\mathbb{Z}$$
and $K_4(R) = 0$. Similar calculations are made for $R$ the ring of dual numbers over $\mathbb{Z}/p\mathbb{Z}$. The calculation reduces to finding homology groups of $\text{SL}(R)$. A key tool is the spectral sequence of the group extension of $\text{SL}(n, p)$ over $\text{SL}(n, p)$. The terms of this spectral sequence depend in turn on the homology of $\text{GL}(n, p)$ with coefficients various multilinear modules. Calculation of the differentials uses the Charlap-Vasquez description of $d^2$.

Introduction. In this paper we shall show that, for $p \geq 5$, 
$$K_3(\mathbb{Z}/p^2\mathbb{Z}) = \mathbb{Z}/p^2\mathbb{Z} \oplus \mathbb{Z}/(p^2 - 1)\mathbb{Z}$$
and 
$$K_4(\mathbb{Z}/p^2\mathbb{Z}) = 0.$$
These groups can be determined from the integral homology groups of $\text{SL}(\mathbb{Z}/p^2\mathbb{Z})$. The prime to $p$ homology has been calculated by Quillen. In this paper, we shall calculate the $p$-primary component of $H_r(\text{SL}(\mathbb{Z}/p^2\mathbb{Z}), \mathbb{Z})$ for $r = 1, 2, 3, 4$.

Our method is to consider $\text{SL}(n, p^2)$ as an extension of $\text{SL}(n, p)$ and to use the Lyndon-Hochschild-Serre spectral sequence. To determine the $E^2$ term of the spectral sequence requires rather extensive calculations of homology for $\text{GL}(n, p)$ with coefficients in various modules associated with its multilinear algebra. (For example, we show for $p \geq 5, n \geq 2$, that $H_2(\text{GL}(n, p), V_n \wedge V_n) = \mathbb{Z}/p\mathbb{Z}$ where $V_n$ is the module of $n \times n$ matrices of trace 0.) To calculate the differential $d^2$ in the spectral sequence, we restrict to the case $n = 2$ and use the theory of Charlap-Vasquez. That theory splits $d^2$ into two terms, one determined by the group extension and the other by the corresponding split extension. $d^2$ for this split extension is calculated by cocycle calculations in an explicit double complex.

With minor modifications, our calculations also apply to the split extension $\text{SL}(n, \mathbb{Z}/p\mathbb{Z}[\epsilon]) \to \text{SL}(n, \mathbb{Z}/p\mathbb{Z})$ where $\mathbb{Z}/p\mathbb{Z}[\epsilon]$ is the ring of dual numbers. We thus obtain the additional results 
$$K_3(\mathbb{Z}/p\mathbb{Z}[\epsilon]) = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/(p^2 - 1)\mathbb{Z},$$
$$K_4(\mathbb{Z}/p\mathbb{Z}[\epsilon]) = 0 \quad (p \geq 5).$$

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**Table of notations.**

- $p$: a prime $\geq 5$, unless otherwise noted.
- $F = \mathbb{Z}/p\mathbb{Z}$.
- $F[e] = F[X]/(X^2)$ = the ring of dual numbers over $F$.
- $GL(n, p) = \text{the group of } n \times n \text{ nonsingular matrices over } \mathbb{Z}/p\mathbb{Z}$. $GL(n - 1, p)$ is imbedded in $GL(n, p)$ in the upper left-hand square. Similarly, $GL(n, p^2), SL(n, p), SL(n, p^2)$ are defined as usual. $SL(\mathbb{Z}/p\mathbb{Z})$ and $SL(\mathbb{Z}/p^2\mathbb{Z})$ are the infinite limits.
- $SL$: a related group defined in §2.
- $M_n$: the module of $n \times n$ matrices over $F$. $GL(n, p)$ acts by conjugation.
- $V_n$: the submodule of $M_n$ of matrices of trace 0.
- $\wedge^k A$: the $k$-fold wedge product over $F$ of the vector space $A$.
- $S^k A$: the $k$-fold symmetric product over $F$ of $A$.
- $A \circ A$: the 2-fold symmetric product. $a \circ b$ denotes a typical generator.
- $\hat{A} = \text{Hom}_F(A, F)$.
- $B_n$: the subgroup of $GL(n, p)$ consisting of all upper triangular matrices with nonzero diagonal terms.
- $U_n$: the subgroup of $B_n$ consisting of all upper triangular matrices with 1's on the diagonal.
- $H_n$: the torus subgroup of $B_n$ consisting of all diagonal matrices with nonzero entries.
- $W_{n-1}$: the $n - 1$ dimensional vector space $\mathbb{F}^{n-1}$. $W_{n-1}$ is imbedded in $B_n$ as the matrices with 1's on the diagonal and all other entries 0 except those in the last column. Then $W_{n-1}$ is a normal subgroup of $U_n$ and conjugation yields the usual action. Elements $u$ of $W_{n-1}$ are considered column vectors, elements $f$ of $U_{n-1}$ are considered row vectors, $fu$ is a scalar, and $uf$ is a matrix.
- $C_n$: the normal subgroup of $B_n$ consisting of all matrices with 1's in the first $n - 1$ diagonal positions, a nonzero entry in the last diagonal position, and zeroes elsewhere except in the last column. $C_n$ is the semidirect $F^* \ltimes W_{n-1}$ where $F^*$ acts on $W_{n-1}$ by multiplication by the inverse.

- $e_{ij}$: the matrix with 1 in the $i,j$-position and zeroes elsewhere. The diagonal matrix $(t_1, t_2, \ldots)$ acts on $e_{ij}$ as $t_i t_j^{-1}$.
- $h_i = e_{ii} - e_{i+1,i+1}$.
- $x_{ij} = I + e_{ij}$.
- $[G, A]$ for a $G$-module $A$ ($G$ a group), the submodule generated by all $g(a) - a$.
- $X \otimes_{\mathbb{Z}[V]} A$: $X$ and $A$ are supposed left $V$-modules and $v x \otimes v a = x \otimes a$. If $V$ is written additively, $v x \otimes a = x \otimes (-v)a$.
- $[v_1, v_2, \ldots, v_r]$ : a typical basis element for the standard $\mathbb{Z}[V]$-free resolution $X$ of $\mathbb{Z}$. For $r = 0, [-]$ denotes the single basis element. $d[u] = u[-] - [-]$ (see §11). On occasion, the image in $X \otimes_{\mathbb{Z}[V]} F$ is denoted by the same symbol.
$x \cap \zeta$: the Pontryagin product $H_\bullet(V, F) \otimes H_\bullet(V, F) \to H_\bullet(V, F)$ induced by the multiplication homomorphism $V \times V \to V$ for $V$ an abelian group. The cup product in $H^\bullet(V, F)$ is denoted simply by multiplication.

$\delta$: $H^r(V, \mathbb{Z}/p \mathbb{Z}) \to H^{r+1}(V, \mathbb{Z}/p \mathbb{Z})$ is the Bockstein morphism, i.e. the connecting homomorphism arising from the exact sequence $0 \to \mathbb{Z}/p \mathbb{Z} \to \mathbb{Z}/p^2 \mathbb{Z} \to \mathbb{Z}/p \mathbb{Z} \to 0$.

$\rho(v)$: the element of $H_2(V, F)$ represented by $\sum_{j=0}^{p-1} [v, jv]$. $\rho: V \to H_2(V, F)$ is a monomorphism.

$H_{\cdot}(\cdot; p)$ the $p$-primary component of the relevant homology group.

**Chapter I. The basic argument**

1. **Reduction to $p$-primary homology.** From Quillen [Q], we know

\[
\begin{aligned}
H_r(\text{SL}(\mathbb{Z}/p \mathbb{Z}), \mathbb{Z}) &= \mathbb{Z}/(p^2 - 1) \mathbb{Z}, & r &= 3.
\end{aligned}
\]

Since for each $n$, $\text{Ker}{\text{SL}(n, p^2) \to \text{SL}(n, p)}$ is a $p$-group, it follows that

\[
H_r(\text{SL}(\mathbb{Z}/p^2 \mathbb{Z}), \mathbb{Z}; l) = H_r(\text{SL}(\mathbb{Z}/p \mathbb{Z}), \mathbb{Z}; l), \quad l \neq p.
\]

On the other hand, we shall show as a major result of this paper

**Theorem 3.4.** For $p \geq 5$,

\[
H_r(\text{SL}(\mathbb{Z}/p^2 \mathbb{Z}), \mathbb{Z}; p) = \begin{cases} 0, & r = 1, 2, 4, \\ \mathbb{Z}/p^2 \mathbb{Z}, & r = 3. \end{cases}
\]

Hence, we have

**Corollary 1.1.**

\[
\begin{aligned}
H_r(\text{SL}(\mathbb{Z}/p^2 \mathbb{Z}), \mathbb{Z}) &= 0, & r &= 1, 2, 4, \\
H_3(\text{SL}(\mathbb{Z}/p^2 \mathbb{Z}), \mathbb{Z}) &= \mathbb{Z}/p^2 \mathbb{Z} \oplus \mathbb{Z}/(p^2 - 1) \mathbb{Z}.
\end{aligned}
\]

From this we obtain our main result.

**Theorem 1.2.** Let $p \geq 5$. Then

\[
\begin{aligned}
K_1(\mathbb{Z}/p^2 \mathbb{Z}) &= \mathbb{Z}/p \mathbb{Z} \oplus \mathbb{Z}/(p - 1) \mathbb{Z}, & K_2(\mathbb{Z}/p^2 \mathbb{Z}) &= 0, \\
K_3(\mathbb{Z}/p^2 \mathbb{Z}) &= \mathbb{Z}/p^2 \mathbb{Z} \oplus \mathbb{Z}/(p^2 - 1) \mathbb{Z}, & K_4(\mathbb{Z}/p^2 \mathbb{Z}) &= 0.
\end{aligned}
\]

**Note.** $K_2$ has been calculated by Milnor [M], $K_1$ by Bass [B].

**Proof.** We recall that the $K$-groups of $\mathbb{Z}/p^2$ are defined to be the homotopy groups of $(BGL \mathbb{Z}/p^2)^+$, where $(\cdot)^+$ is the Quillen plus construction with respect to the (perfect) commutator subgroup of the fundamental group (see [W]). By Corollary 1.1, $H_i(\text{BSL} \mathbb{Z}/p^2) = 0$ so that $\text{SL} \mathbb{Z}/p^2$ is the commutator subgroup of $\text{GL} \mathbb{Z}/p^2$. This implies that $(\text{BSL} \mathbb{Z}/p^2)^+$ is the universal covering of $(\text{BGL} \mathbb{Z}/p^2)^+$ [W]. Consequently,

\[
\begin{aligned}
K_1(\mathbb{Z}/p^2) &= \pi_1((\text{BGL} \mathbb{Z}/p^2)^+) = (\text{GL} \mathbb{Z}/p^2)/ (\text{SL} \mathbb{Z}/p^2) \simeq \mathbb{Z}/p - 1 \oplus \mathbb{Z}/p, \\
K_i(\mathbb{Z}/p^2) &= \pi_i((\text{BSL} \mathbb{Z}/p^2)^+) \quad \text{for } i > 1.
\end{aligned}
\]

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Because $\text{BSL} \mathbb{Z}/p^2 \to (\text{BSL} \mathbb{Z}/p^2)^+$ is a homology equivalence, Corollary 1.1 and the Hurewicz theorem imply that
\[
\pi_2((\text{BSL} \mathbb{Z}/p^2)^+) \simeq H_2(\text{BSL} \mathbb{Z}/p^2) = 0,
\]
\[
\pi_3((\text{BSL} \mathbb{Z}/p^2)^+) \simeq H_3(\text{BSL} \mathbb{Z}/p^2) = \mathbb{Z}/p^2 - 1 \oplus \mathbb{Z}/p^2 = \Pi.
\]
Moreover, because $(\text{BSL} \mathbb{Z}/p^2)^+ \to (\text{BSL} \mathbb{Z}/p)^+$ is an $l$-equivalence of simply connected spaces for primes $l \neq p$, $\pi_d((\text{BSL} \mathbb{Z}/p^2)^+ ; l) \simeq \pi_d((\text{BSL} \mathbb{Z}/p)^+ ; l)$, and the latter is zero by [Q]. Hence, $\pi_d((\text{BSL} \mathbb{Z}/p^2)^+) \simeq \pi_d((\text{BSL} \mathbb{Z}/p^2)^+ ; p)$.

Let $(\text{BSL} \mathbb{Z}/p^2)^+ \to K(\Pi, 3)$ be the natural 3-equivalence obtained by attaching cells to $(\text{BSL} \mathbb{Z}/p^2)^+$ to kill the homotopy groups above dimension 3 and let $F$ denote the homotopy fibre of this map. The Hurewicz theorem and the long exact homotopy sequence imply that $F$ is 3-connected and $\pi_4((\text{BSL} \mathbb{Z}/p^2)^+) \to \pi_4(F)$, and the latter is zero by [Q]. Consequently, the Hurewicz theorem and the universal coefficient theorem imply that
\[
\pi_4((\text{BSL} \mathbb{Z}/p^2)^+) = \pi_4((\text{BSL} \mathbb{Z}/p^2)^+ ; p).
\]
Using the Serre spectral sequence
\[
E_2^{r,s} = H_r(K(\mathbb{Z}/p^2, 3), H_s(F)) \Rightarrow H_{r+s}((\text{BSL} \mathbb{Z}/p^2)^+)
\]
we conclude that $H_4(F, \mathbb{Z}; p) \simeq H_4((\text{BSL} \mathbb{Z}/p^2)^+, \mathbb{Z}; p) \simeq H_4(\text{SL} \mathbb{Z}/p^2, \mathbb{Z}; p)$. Consequently, Corollary 1.1 implies that
\[
\pi_4((\text{BSL} \mathbb{Z}/p^2)^+) = 0. \quad \Box
\]

Similar arguments will allow us to conclude for the ring of dual numbers $\mathbb{Z}/p\mathbb{Z}[\varepsilon | \varepsilon^2 = 0]$ the following. (Use Theorem 4.1.)

**Theorem 1.3.** Let $p \geq 5$. Then
\[
H_r(\text{SL}(F[\varepsilon]), \mathbb{Z}) = 0, \quad r = 1, 2, 4,
\]
\[
H_3(\text{SL}(F[\varepsilon]), \mathbb{Z}) = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/(p^2 - 1)\mathbb{Z}.
\]

Arguing as in the proof of Theorem 1.2, we have

**Corollary 1.4.** For $p \geq 5$
\[
K_1(F[\varepsilon]) = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/(p - 1)\mathbb{Z}, \quad K_2(F[\varepsilon]) = 0,
\]
\[
K_3(F[\varepsilon]) = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/(p^2 - 1)\mathbb{Z}, \quad K_4(F[\varepsilon]) = 0.
\]

$K_2$ was calculated by van der Kallen [V].

**2. Reduction to $\text{SL}$.** Define
\[
\frac{\text{SL}(n, p^2)}{\text{SL}(n, p^2) \oplus \text{SL}(n, p^2)} = \text{Ker} \left\{ \text{GL}(n, p^2)^{\text{det}^{-1}(p)} \to U(\mathbb{Z}/p^2\mathbb{Z}) \right\}.
\]
These groups form a direct limit system, and we define
\[
\frac{\text{SL}(n, p^2)}{\text{SL}(n, p^2) \oplus \text{SL}(n, p^2)} = \lim \frac{\text{SL}(n, p^2)}{\text{SL}(n, p^2) \oplus \text{SL}(n, p^2)}.
\]
Clearly, $\text{SL}(n, p^2) = \text{Ker}\bigl(\overline{\text{SL}}(n, p^2) \to U(\mathbb{Z}/p^2\mathbb{Z}) \to U(\mathbb{Z}/p\mathbb{Z})\bigr)$. Moreover, for $(n, p - 1) = 1$,

$$\overline{\text{SL}}(n, p^2) = \text{SL}(n, p^2) \times \mathbb{Z}/(p - 1)\mathbb{Z},$$

so $H_*(\text{SL}(n, p^2), \mathbb{Z}; p) \cong H_*(\overline{\text{SL}}(n, p^2), \mathbb{Z}; p)$, and by direct limit

$$H_*(\text{SL}(\mathbb{Z}/p^2\mathbb{Z}), \mathbb{Z}; p) \cong H_*(\overline{\text{SL}}(\mathbb{Z}/p^2\mathbb{Z}), \mathbb{Z}; p).$$

In what follows, we calculate the right-hand groups for $r = 1, 2, 3, 4$.

### 3. Calculation of $H_*(\overline{\text{SL}}(n, p^2), \mathbb{Z}; p)$, $r = 1, 2, 3, 4$, $p > 5$; the basic argument

The homomorphism $\pi: \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ induces an epimorphism $\pi: \text{SL}(n, p^2) \to \text{GL}(n, p)$. Its kernel consists of all $n \times n$ matrices of the form $I + pA$ where $A \in M_n(\mathbb{Z}/p^2\mathbb{Z})$ and $\det(I + pA)^{p-1} = 1$. However, $\det(I + pA)^{p-1} = 1 + (p - 1)\text{Tr}(pA) = 1$ if and only if $\text{Tr}(pA) = 0$. Since $I + pA_1 = I + pA_2$ if and only if $A_1 \equiv A_2 \mod p$, $I + pA \mapsto \pi(A)$ identifies the kernel with $V_n = \text{sl}(n, p) = \text{the additive group of } n \times n \text{ matrices over } \mathbb{Z}/p\mathbb{Z} \text{ with trace 0}$. $\text{GL}(n, p)$ acts on the kernel by conjugation $x: T \mapsto xTx^{-1}$.

To calculate the desired homology, we use the spectral sequence of the group extension described above, or rather its $p$-primary component

$$H_s(\pi: \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}) \Rightarrow H_{s+t}(\overline{\text{SL}}(n, p^2), \mathbb{Z}; p) \quad (s + t > 0).$$

Note that since all the relevant groups are finite, the spectral sequence does decompose into independent primary components. Also, for $t > 0$, we may omit the restriction to $p$-primary component since $H_*(V_n, \mathbb{Z})$ is a $p$-group.

We list here the relevant homology groups of $V_n$. The decompositions are as $\text{GL}(n, p)$-modules and hold for $p > 2$ (but not $p = 2$). Also, they are natural with respect to $n$. (These facts are easily established using the known structure of $H_*(V_n, \mathbb{Z}/p\mathbb{Z})$ for $V_n$ an elementary abelian $p$-group.)

- $H_0(V_n, \mathbb{Z}) = \mathbb{Z}, \quad H_1(V_n, \mathbb{Z}) = V_n, \quad H_2(V_n, \mathbb{Z}) = \wedge^2 V_n$. ($H_2$ is spanned by all Pontryagin products of elements of $H_1(V_n, \mathbb{Z})$.)
- $H_3(V_n, \mathbb{Z}) = \wedge^3 V_n \oplus S^2 V_n$,
- $H_4(V_n, \mathbb{Z}) = \wedge^4 V_n \oplus \text{Ker}\bigl\{V_n \otimes \wedge^2 V_n \to \wedge^3 V_n\bigr\}$.

**Proposition 3.0.** We have the following basic table of results for $n \geq 2$, $p \geq 5$.

(a) $H_s(\pi: \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}) = 0, \quad 1 \leq s \leq 2p - 4$.

(b)

$$H_s(\text{GL}(n, p), V_n) = \begin{cases} 0, & s = 0, 1, 3, \\ \mathbb{Z}/p\mathbb{Z}, & s = 2 \text{ (stably, i.e. } H(\text{GL}(2, p), V_2) \\ \rightarrow H_2(\text{GL}(n, p), V_n) \text{ is an isomorphism}) \end{cases}.$$  

(c)

$$H_s(\text{GL}(n, p), \wedge^2 V_n) = \begin{cases} 0, & s = 0, 1, \\ \mathbb{Z}/p\mathbb{Z}, & s = 2. \end{cases}$$
The proof of this proposition takes up the major part of this paper. See Chapter II, Propositions 5.1, 5.2, 7.1, 7.3, 7.5, 7.6, and 8.1–8.4.

Using this basic table, we may conclude that the $E^2$ term of the spectral sequence looks like:

In more detail, we have

**Proposition 3.1.** ($p > 5$, $n \geq 2$.)

(i) $H_s(\text{GL}(n, p), H_0(V_n, \mathbb{Z}); p) = 0$, $s = 1, 2, 3, 4$.

(ii) $H_s(\text{GL}(n, p), H_1(V_n, \mathbb{Z})) = 0$, $s = 0, 1, 3$.

(iii) $H_s(\text{GL}(n, p), H_2(V_n, \mathbb{Z})) = 0$, $s = 0, 1, 2$.

(iv) $H_s(\text{GL}(n, p), H_3(V_n, \mathbb{Z})) = 0$, $s = 0, 1$.

(v) $H_0(\text{GL}(n, p), H_4(V_n, \mathbb{Z})) = 0$. 

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Proof. (i) is a restatement of (a) in the basic table. (ii) is (b) and (iii) is (c). (iv) follows from (d) and (e). To prove (v), consider the exact sequence \( 0 \to \text{Ker} \to V_n \otimes \wedge^2 V_n \to \wedge^3 V_n \to 0 \). Since \( H_1(\text{GL}(n, p), \wedge^3 V_n) = 0 \) by (e), we obtain \( H_0(\text{GL}(n, p), \text{Ker}) = 0 \) from:

\[
0 \to H_0(\text{GL}, \text{Ker}) \to H_0(\text{GL}, V_n \otimes \wedge^2 V_n) \to H_0(\text{GL}, \wedge^3 V_n) \to 0
\]

\[
\begin{array}{c|c}
\mathbb{Z}/p\mathbb{Z} & \mathbb{Z}/p\mathbb{Z} \\
\end{array}
\]

Since \( H_0(\text{GL}, \wedge^4 V) = 0 \) by (g), the desired result follows. \( \square \)

To complete the calculation, we must investigate the differential:

\[
d^2_{2,2}: E^2_{2,2}(\mathbb{Z}) \to E^2_{0,3}(\mathbb{Z})
\]

\[
\begin{array}{c|c}
\mathbb{Z}/p\mathbb{Z} & \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p \\
\end{array}
\]

We first consider the case \( n = 2 \). From the basic table (c), for \( n = 2 \), \( E^2_{2,2}(\mathbb{Z}) = \mathbb{Z}/p \). Moreover, we have

**Proposition 12.4(b).** For \( n = 2 \), \( p > 3 \), and coefficient module \( \mathbb{Z} \), \( d^2_{2,2} \) is a monomorphism.

The proof of this fact is based on the Charlap-Vasquez theory and is discussed at length in Chapter III.

To deal with \( n \geq 2 \), consider the commutative diagram:

\[
\begin{array}{c|c}
H_0(\text{GL}(n, p), H_3(V_n, \mathbb{Z})) & \xleftarrow{d^1_{2,2}} H_2(\text{GL}(n, p), H_2(V_n, \mathbb{Z})) = \mathbb{Z}/p\mathbb{Z} \\
\uparrow & \uparrow \\
H_0(\text{GL}(2, p), H_3(V_2, \mathbb{Z})) & \xleftarrow{d^1_{2,2}} H_2(\text{GL}(n, 2), H_2(V_2, \mathbb{Z})) = \mathbb{Z}/p\mathbb{Z} \\
\end{array}
\]

We conclude that \( d^2_{2,2} \) is a monomorphism for \( n \geq 2 \).

To continue with the basic argument, we can now conclude that the \( E^3 \) term of the spectral sequence looks like:

\[
\begin{array}{c|c|c|c|c|c|c|c}
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
0 & & & & & & & \\
0 & & & & & & & \\
0 & & & & & & & \\
0 & & & & & & & \\
0 & & & & & & & \\
\mathbb{Z}/p & & & & & & & \\
0 & & & & & & & \\
\end{array}
\]

It follows that for total homology \( H_1 = 0, H_2 = 0, H_3 = \mathbb{Z}/p^2 \) or \( \mathbb{Z}/p \oplus \mathbb{Z}/p \), \( H_4 = 0 \).
To determine precisely what happens for $H_3$, we must repeat the above argument for coefficient group $\mathbb{Z}/p\mathbb{Z}$. We have

$$H_3(\text{GL}(n, p), H_1(V_n, \mathbb{Z}/p\mathbb{Z})) \to H_3(\text{SL}(n, p^2), \mathbb{Z}/p^2).$$

Also, for $p > 2$, we have decompositions of GL-modules

1. $H_0(V_n, \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$,
2. $H_1(V_n, \mathbb{Z}/p\mathbb{Z}) = V_n$,
3. $H_2(V_n, \mathbb{Z}/p\mathbb{Z}) = V_n \oplus \wedge^2 V_n$,
4. $H_3(V_n, \mathbb{Z}/p\mathbb{Z}) = \wedge^3 V_n \oplus V_n \otimes V_n = \wedge^3 V_n \oplus \wedge^2 V_n \oplus S^2 V_n$.

$V_n$ is imbedded in $H_2(V_n, \mathbb{Z}/p\mathbb{Z})$ as all $\rho(v)$, $v \in V_n$, where $\rho(v)$ is represented by the chain $\sum_{i=0}^{p-1} [v, iv]$.

$\wedge^2 V_n$ is imbedded in $H_2(V_n, \mathbb{Z}/p\mathbb{Z})$ as Pontryagin products $u \cap v$, $u, v \in V_n$, and similarly for $\wedge^3 V_n$ in $H_3(V_n, \mathbb{Z}/p\mathbb{Z})$.

$V_n \otimes V_n$ is imbedded in $H_3$ as all $u \cap \rho(v)$.

Hence, from the basic table, we have

**Proposition 3.2.** For $p > 5$, $n > 2$,

- $H_s(\text{GL}(n, p), H_0(V_n, \mathbb{Z}/p\mathbb{Z})) = 0$, $s = 1, 2, 3, 4$;
- $H_s(\text{GL}(n, p), H_1(V_n, \mathbb{Z}/p\mathbb{Z})) = 0$, $s = 0, 1, 3$;
- $H_s(\text{GL}(n, p), H_2(V_n, \mathbb{Z}/p\mathbb{Z})) = \mathbb{Z}/p\mathbb{Z}$, $s = 2$;
- $H_s(\text{GL}(n, p), H_3(V_n, \mathbb{Z}/p\mathbb{Z})) = \mathbb{Z}/p \oplus \mathbb{Z}/p$ (stably).

Again, we will show in Chapter III

**Proposition 12.3(a).** $d^2_{2, 2}: E^2_{2, 2}(\mathbb{Z}/p\mathbb{Z}) \to E^2_{0, 3}(\mathbb{Z}/p\mathbb{Z})$ is an isomorphism for $n = 2$.

It follows as before that $d^2_{2, 2}$ is an isomorphism for $\mathbb{Z}/p\mathbb{Z}$ for $n \geq 2$. Hence, $E^3(\mathbb{Z}/p\mathbb{Z})$ looks like:

![Diagram](image-url)

It follows that $H_3(\text{SL}(n, p^2), \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$. Hence, by the universal coefficient theorem we may conclude $H_3(\text{SL}(n, p^2), \mathbb{Z}/p) = \mathbb{Z}/p^2$. □
To summarize, we may now state one of our basic results.

**Theorem 3.3.** For $p \geq 5$, $n \geq 2$, we have stably

\[
\begin{align*}
H_1(\overline{\text{SL}}(n, p^2), \mathbb{Z}; p) &= 0, \\
H_2(\overline{\text{SL}}(n, p^2), \mathbb{Z}; p) &= 0, \\
H_3(\overline{\text{SL}}(n, p^2), \mathbb{Z}; p) &= \mathbb{Z}/p^2\mathbb{Z}, \\
H_4(\overline{\text{SL}}(n, p^2), \mathbb{Z}; p) &= 0.
\end{align*}
\]

**Proof.** The only thing to check is stability for degree 3. However, we know comparison with $n - 2$ yields isomorphisms for $E^2_{i,i}(\mathbb{Z})$ in all relevant positions. Thus, we conclude that comparison with $n = 2$ yields an isomorphism also for the relevant total homology. \(\square\)

We have now proved

**Theorem 3.4.** For $p \geq 5$,

\[
H_r(\text{SL}(\mathbb{Z}/p^2), \mathbb{Z}; p) = \begin{cases} 
0, & r = 1, 2, 4, \\
\mathbb{Z}/p^2\mathbb{Z}, & r = 3.
\end{cases}
\]

4. Calculations for the split extension. For the ring of dual numbers $\mathbb{F}[\varepsilon]$, we may argue very much as above. Define

\[
\overline{\text{SL}}(n, \mathbb{F}[\varepsilon]) = \text{Ker} \left\{ \text{GL}(n, \mathbb{F}[\varepsilon]) \to U(\mathbb{F}[\varepsilon]) \right\}
\]

and

\[
\overline{\text{SL}}(\mathbb{F}[\varepsilon]) = \lim_{\rightarrow} \overline{\text{SL}}(n, \mathbb{F}[\varepsilon]).
\]

The latter group is related to $K$-theory as before. Its prime to $p$ homology is given by Quillen's results. To calculate $H_r(\text{SL}(n, \mathbb{F}[\varepsilon]), \mathbb{Z}; p)$, we argue as above except that in this case the extension $0 \to V_n \to \text{SL}(n, \mathbb{F}[\varepsilon]) \to \text{GL}(n, p) \to 1$ splits. The $E^2$-terms of the spectral sequences are as before.

To calculate the relevant $d^2$'s, we rely on the following result from Chapter III.

**Propositions 12.4(a), 12.3(b).** For $n = 2$, $p > 3$, $\text{Im} d^2_{2,2}$ is 1 dimensional both for $\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z}$. ©
To determine $d^2_{2,2}$ for $n > 2$, consider the commutative diagrams

\[
\begin{array}{c}
H_0(\text{GL}(n, p), H_3(V_n, C)) &\xrightarrow{d^2_{2,2}}& H_2(\text{GL}(n, p), H_2(V_n, C))\\
\uparrow\cong&&\uparrow\cong\\
H_0(\text{GL}(2, p), H_3(V_2, C)) &\xrightarrow{d^2_{2,2}}& H_2(\text{GL}(2, p), H_2(V_2, C))
\end{array}
\]

for $C = \mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z}$. Note that as a result of our previous analysis, we know both vertical arrows are isomorphisms. Hence, $\dim \text{Im } d^2_{2,2} = 1$ for arbitrary $n > 2$.

Thus the relevant $E^3$ terms look (stably) like:

\[
\begin{array}{c}
\text{E}^3(\mathbb{Z}) \\
\mathbb{Z}/p
\end{array}
\quad
\begin{array}{c}
\text{E}^3(\mathbb{Z}/p\mathbb{Z}) \\
\mathbb{Z}/p
\end{array}
\]

Hence, we have proved

\begin{align*}
\text{THEOREM 4.1.} & \text{ For } n \geq 2, p \geq 5, \text{ we have stably} \\
& H_1(\text{SL}(n, F[e]), \mathbb{Z}; p) = 0, \\
& H_2(\text{SL}(n, F[e]), \mathbb{Z}; p) = 0, \\
& H_3(\text{SL}(n, F[e]), \mathbb{Z}; p) = \mathbb{Z}/p \oplus \mathbb{Z}/p, \\
& H_4(\text{SL}(n, F[e]), \mathbb{Z}; p) = 0.
\end{align*}

\section{Chapter II. Homology calculations to fill in the basic table}

\subsection{5. Initial calculations.}

\begin{proposition}
\text{For } p > 2, n \geq 1, 1 \leq r \leq 2p - 4, \\
H_r(\text{GL}(n, p), \mathbb{Z}; p) = 0, \quad H_r(\text{GL}(n, p), \mathbb{Z}/p\mathbb{Z}) = 0.
\end{proposition}

\text{\textbf{Proof.}} Using the universal coefficient theorem and $B_n \supset U_n$ (a $p$-Sylow subgroup), we need only prove $H_r(B_n, \mathbb{Z}/p\mathbb{Z}) = 0, 1 \leq r \leq 2p - 4$.

To establish this, note first that since $B_n = H_n \rtimes U_n$, we have

\[H_r(B_n, \mathbb{Z}/p\mathbb{Z}) = H_r(U_n, \mathbb{Z}/p\mathbb{Z})_{H_n}.\]
We look for the irreducible constituents (all one dimensional) of \( H_r(U, \mathbb{Z}/p) \) as a \( \mathbb{Z}/p(\mathbb{Z}/p) \)-module. We shall see that, for the relevant \( r \), these are all nontrivial—from which \( H_r(U, \mathbb{Z}/p)_{H_r} = 0 \) follows. \( U = U_n \) may be built up as successive extensions

\[
1 \to \frac{\Gamma_k}{\Gamma_{k+1}} \to \frac{U}{\Gamma_{k+1}} \to \frac{U}{\Gamma_k} \to 1, \quad k = 2, 3, \ldots, n - 1,
\]

where \( \Gamma_k \) is generated by \( 1 + e_{ij}, j - i \geq k \).

The \( E_2 \)-term of the spectral sequence of this central extension is \( H_*(U/\Gamma_k, \mathbb{Z}/p\mathbb{Z}) \otimes H_*(\Gamma_k/\Gamma_{k+1}, \mathbb{Z}/p) \) so we need only look at the irreducible \( H_\ast \)-constituents of this, or by induction the irreducible constituents of \( H_*(U/\Gamma_2) \otimes H_*(\Gamma_2/\Gamma_3) \otimes \cdots \otimes H_*(\Gamma_{n-1}) \). Recalling that this is a graded tensor product, we may identify this as the homology of the elementary abelian group

\[ L_n = \bigoplus_k \frac{\Gamma_k}{\Gamma_{k+1}} = \bigoplus_{j-i \geq 1} \mathbb{Z}/p\mathbb{Z} e_{ij} \]

(upper triangular matrices with 0’s on the diagonal).

As usual,

\[ H_*(L_n, \mathbb{Z}/p\mathbb{Z}) \cong \bigwedge L_n \otimes \mathcal{P}(pL_n) \leftarrow \text{Divided power algebra} \]

Given the action, \( h \cdot e_{ij} = t_i t_j^{-1} e_{ij} \) where

\[
h = \begin{bmatrix}
t_1 & 0 \\
t_2 & \\
\vdots & \\
0 & t_n
\end{bmatrix},
\]

it is straightforward to see that the first trivial irreducible constituent in a positive degree occurs in degree \( 2(p - 2) + 1 = 2p - 3 \). (This is easier to see in the dual cohomology module which is an exterior algebra tensored with a symmetric algebra.)

In what follows, we shall make frequent use of the spectral sequence of a filtered module. The general theory is as follows.

Let \( G \) be a group and \( A \) a \( G \)-module. Suppose in addition \( A \) is filtered by submodules \( A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A_f \supseteq A_{f+1} = 0 \). We obtain from this a filtration of the standard chain complex

\[ C_r(G, A) = F_0 C_r \supseteq F_{-1} C_r \supseteq F_{-2} C_r \supseteq \cdots \supseteq F_{-f} C_r \supseteq F_{-f-1} C_r = 0 \]
with $F_s C_r = C_r(G, A_{-s})$ viewed naturally as a submodule of $C_r(G, A)$. In general, a filtered complex yields a spectral sequence. (See [C-E, Chapter XV] for an extensive discussion.) The $E_1$-term is given by

$$E_{s,t}^1 = H_{s+t} \left( \frac{F_s C_*}{F_{s-1} C_*} \right) \cong H_{s+t} \left( C_* \left( G, \frac{A_{-s}}{A_{-s+1}} \right) \right)$$

$$= H_{s+t} \left( G, \frac{A_{-s}}{A_{-s+1}} \right).$$

Note that $E_{s,t}^1 = 0$ outside the set defined by $s + t \geq 0, -l \leq s \leq 0$. To calculate $d^k_s \epsilon^k$ for $c^k \in E_{s,t}^1$, choose $\tilde{c} \in F_s C_{s+t} = C_{s+t}(G, A_{-s})$ representing $c^k$ such that $d\tilde{c} \in F_{s-1} C_{s+t}$. (That is possible because $d\tilde{c}^1 = d^2 \tilde{c}^2 = \ldots = d^{k-1} \tilde{c}^{k-1} = 0$ where $\tilde{c}$ represents $c^l$ in $E_{s,t}^1$.) Then $d^k(\epsilon^k)$ is represented by $d\tilde{c}$.

**Proposition 5.2.** For $p \geq 5, n \geq 2$,

$$H_0(\text{GL}(n, p), V_n) = 0, \quad H_0(\text{GL}(n, p), M_n) = \mathbb{Z}/p\mathbb{Z}.$$

$$H_1(\text{GL}(n, p), V_n) = H_1(\text{GL}(n, p), M_n) = 0.$$

$$H_2(\text{GL}(n, p), V_n) = H_2(\text{GL}(n, p), M_n) = \mathbb{Z}/p\mathbb{Z}.$$

Moreover, this isomorphism is stable; it commutes with the induced map from $n = 2$.

$$H_3(\text{GL}(n, p), V_n) = H_3(\text{GL}(n, p), M_n) = 0.$$

Similar formulas hold for $B_n$.

**Proof.** The sequence $0 \to V_n \to M_n \xrightarrow{\text{Tr}} F = \mathbb{Z}/p\mathbb{Z} \to 0$ together with Proposition 5.1 allows us to relate the homology of $V_n$ to that of $M_n$ or vice versa.

For $V_n$, we first calculate $H_r(B_n, V_n)$ ($r = 0, 1, 2, 3$). We use $B_n = B_{n-1} \times C_n$, $C_n = F^* \times W_{n-1}$. Recall $[W, V]$ is the submodule of $V$ spanned by all $u(s) - s, u \in W, s \in V$. Consider the tower

$$A_0 = V_n$$

$$A_1 = X_n = \left[ W_{n-1}, V_n \right]$$

$$A_2 = W_{n-1} = \left[ W_{n-1}, X_n \right]$$

where $W_{n-1}$ acts trivially on the factors, $F^*$ acts trivially on $M_{n-1}$, by multiplication on $W_{n-1}$, and by reciprocal multiplication on $F^*$. Note $H_0(C_n, V_n) = H_0(W_{n-1}, V_n)_{F^*} = (W_{n-1})_{F^*} = 0$.

The given filtration yields a spectral sequence

$$E_{s,t}^1 = H_{s+t}(C_n, A_{-s}/A_{-s+1}) \Rightarrow H_{s+t}(C_n, V_n).$$

The $E_1$-term is indicated below. (Subscripts $n, n-1$ are deleted for convenience.)
\[
H_3(C, W) = 0 \quad (\wedge^3 W \otimes W \text{ for } p = 5)
\]

\[
H_2(C, W) = 0 \quad H_3(C, M) = 0
\]

\[
H_1(C, W) = 0 \quad H_2(C, M) = 0 
\quad H_3(C, \hat{W}) = 0
\]

\[
H_0(C, W) = 0 \quad H_1(C, M) = 0 \quad H_2(C, \hat{W}) = M
\]

\[
H_0(C, M) = M \quad d_{0,1} \quad H_1(C, \hat{W}) = M
\]

\[
H_0(C, \hat{W}) = 0 \quad s + t = 3
\]

\[
s + t = 2
\]

\[
s + t = 1
\]

\[
s + t = 0
\]

(For example, \(H_2(C, \hat{W}) = H_2(W, \hat{W})_{F^*} = (\wedge^2 W \otimes \hat{W} \otimes W \otimes \hat{W})_{F^*} = M\). Since \(H_0(C, V) = 0\), \(d_{0,1}\) must be onto and hence it is an isomorphism. Hence, \(E^2\) looks like:

\[
\begin{array}{c}
0 \quad (\wedge^3 W \otimes W, \text{ p = 5}) \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{array}
\]

\[
\begin{array}{c}
0 \quad 0 \\
0 \quad 0 \\
0 \quad 0 \\
0 \quad M \quad n-1 \\
0 \quad 0 \\
0 \\
0 \\
0 \\
0
\end{array}
\]

Thus we have

\[\text{Lemma 5.3.}\]

\[
H_3(C_n, V_n) = 0 \quad (p > 5), \quad H_2(C_n, V_n) \cong M_{n-1},
\]

\[
H_1(C_n, V_n) = 0, \quad H_0(C_n, V_n) = 0.
\]

For \(p = 5\), \(H_3(C_n, V_n)\) is a quotient of \(\wedge^3 W_{n-1} \otimes W_{n-1}\).
We next consider the spectral sequence associated with the decomposition $B_n = B_{n-1} \times C_n$. From $H_0(C_n, V_n) = 0$, we conclude $H_0(B_n, V_n) = 0$ for all $n \geq 1$. Replacing $n$ by $n - 1$, we see $H_0(B_{n-1}, M_{n-1}) = \mathbb{Z}/p\mathbb{Z}$. $H_1(B_n, M_n) = H_1(F^*, F) = 0$, so we may assume inductively that $H_1(B_{n-1}, M_{n-1}) = 0$. It follows easily that $H_r(B_n, V_n) = 0$, $r = 1, 3$, and $H_2(B_n, V_n) = \mathbb{Z}/p\mathbb{Z}$. See the $E^2$ diagram below. (For $p = 5$, it suffices to note that $H_0(H_{n-1}, \wedge^3 W_{n-1} \otimes W_{n-1}) = 0$ for $H_{n-1}$ the subgroup of diagonal matrices.)

![E2 diagram](image)

To complete the proof, we need only show the existence of a stable nonzero class in $H_2(GL(n, p), M_n)$. (For, we know by the above that $H_2(GL(n, p), M_n) \cong H_2(GL(n, p), V_n) \otimes \mathbb{Z}/p\mathbb{Z}.$) To see this, we show that

$$H_2(GL(2, p), M_2) \rightarrow H_2(GL(n, p), M_n)$$

is a monomorphism with the left-hand side $\neq 0$. Dually, we can show instead $H^2(GL(n, p), M_n) \rightarrow H^2(GL(2, p), M_2) \neq 0$ is an epimorphism. However, $A \otimes B \rightarrow \text{Tr}(AB)$ provides a $GL(n, p)$-pairing of $M_n \otimes M_n \rightarrow F$ which gives an isomorphism $M_n \cong M_2$. Under this isomorphism, the dual of the inclusion $i: M_2 \rightarrow M_n$ is the map $j: M_n \rightarrow M_2$ which given $A \in M_n$ picks out the $2 \times 2$ block of $A$ in the upper left corner. Note $j \circ i = \text{id}$.

From the comparison,

$$1 \rightarrow M_2 \rightarrow GL(2, p^2) \rightarrow GL(2, p) \rightarrow 1: \iota_2 \in H^2(GL_2, M_2)$$

$$1 \rightarrow M_n \rightarrow GL(n, p^2) \rightarrow GL(n, p) \rightarrow 1: \iota_n \in H^2(GL_n, M_n)$$

we conclude $i_*(\iota_2) = k^* (\iota_n) \in H^2(GL(2, p), M_n).$ However, the map under consideration is $j_* \circ k^*: H^2(GL(n, p), M_n) \rightarrow H^2(GL(2, p), M_2)$, so

$$j_*(k^*(\iota_n)) = j_*(i_*(\iota_2)) = \iota_2.$$  

To complete the proof, we show that $\iota_2$ is nontrivial. If $\iota_2$ were trivial, we could find an element $z$ in $GL(2, p^2)$ in the coset of $[1 1]$ such that $z^p = 1$. But $z$ can be written $z = 1 + E_{12} + pA$ where

$$E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A \in M_n(\mathbb{Z}/p\mathbb{Z}).$$

Hence,

$$(1 + E_{12} + pA)^p = 1 + p(E_{12} + pA) + \binom{p}{2}(E_{12} + pA)^2 + \cdots + (E_{12} + pA)^p.$$ 

It is easy to see that $(\binom{p}{j})(E_{12} + pA)^j = 0$, $2 \leq j < p$. Also, $(E_{12} + pA)^p = 0$ for $p \geq 5$ since every term has at least one factor $E_{12}^2$ or two factors $pA$. Hence,
6. Calculation of $H_*(C_n, V_n \wedge V_n)$. In what follows, we shall calculate $H_r(B_n, V_n \circ V_n)$ for $r = 0, 1$, $H_r(B_n, V_n \wedge V_n)$ for $r = 0, 1, 2$, $H_0(B_n, V_n \otimes \wedge^2 V_n)$, $H_r(B_n, \wedge^3 V_n)$ for $r = 0, 1$, and $H_0(B_n, \wedge^4 V_n)$. Our basic method is to couple the spectral sequence of a filtered module with appropriate coinvariant (dually invariant) calculations. We proceed generally in the indicated order except where necessary to pursue inductive arguments. In particular, we calculate $H_r(C_n, V_n \wedge V_n)$ (to be used primarily for $H_r(B_n, V_n \wedge V_n)$) in this section.

**Proposition 6.1.**

(i) For $n \geq 3$, $p > 5$, there is an exact sequence

$$0 \to H_1(C_n, V_n \wedge V_n) \to \ker\left(M_{n-1} \otimes M_{n-1} \to M_{n-1}\right) \to M_{n-1} \wedge M_{n-1} \to 0$$

which splits if $n \not\equiv -1 \mod p$.

(ii) For $n \equiv 3$, $p > 5$ or $n \equiv 4$, $p = 5$,

$$H_2(C_n, V_n \wedge V_n) \cong \frac{M_{n-1} \otimes M_{n-1}}{d^1(\otimes^2 W_{n-1} \otimes \wedge^2 \hat{W}_{n-1})}.$$

For $n = 3$, $p = 5$, there is an exact sequence $0 \to E \to H_2(C_3, V_3 \wedge V_3) \to M_2 \otimes M_2/\text{Im} \ d^1 \to 0$, where $E$ is a quotient of $\wedge^2 W_2 \otimes \wedge^2 W_2 \cong \mathbb{Z}/5\mathbb{Z}$.

(A fuller explanation of the notation appears below.)

**Proof.** (i) for $n = 2$ follows since $A \wedge B \mapsto [A, B]$ defines an isomorphism $V_2 \wedge V_2 \cong V_2$.

We consider the following filtration of $V_n \wedge V_n$.

\[
\begin{array}{c}
A_0 = V_n \wedge V_n \\
\downarrow \\
A_1 = V_n \wedge X_n \\
\downarrow \\
A_2 \\
\downarrow \\
A_3 = X_n \wedge X_n \\
\downarrow \\
A_4 = X_n \wedge W_{n-1} \\
\downarrow \\
A_5 = W_{n-1} \wedge W_{n-1} \\
\end{array}
\]

\[
\begin{array}{c}
\hat{W}_{n-1} \wedge \hat{W}_{n-1} \\
\downarrow \\
\hat{W}_{n-1} \wedge X_n \\
\downarrow \\
\hat{W}_{n-1} \wedge W_{n-1} \\
\downarrow \\
M_{n-1} \wedge M_{n-1} \\
\downarrow \\
M_{n-1} \otimes W_{n-1} \\
\end{array}
\]

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This induces the spectral sequence

\[ E_{s,t}^1 = H_{s+t}(C_n, \frac{A_{-s}}{A_{-s+1}}) \Rightarrow H_{s+t}(C_n, V_n \wedge V_n). \]

We exhibit the \( E^1 \)-term for \( p > 5 \). (Subscripts and \( C_n \) are omitted for brevity.) (See diagram.) The necessary calculations and the case \( p = 5 \) are outlined in the appendix. The differentials are given as follows: we have the complementary inclusion onto a direct summand (\( p > 2 \))

\[ i: \bigwedge^2 W \to W \otimes W, \quad i(u \wedge v) = u \otimes v - v \otimes u. \]

Use \( i' \) for the corresponding morphism for \( \hat{W} \). Then \( d_{0,2}^1 \) is the composition

\[ \bigwedge^2 W \otimes \bigwedge^2 \hat{W} \to W \otimes W \otimes \hat{W} \otimes \hat{W} \to W \otimes \hat{W} \otimes W \otimes \hat{W} = M \otimes M. \]

\( d_{0,3}^1 \) is the composition

\[ \bigotimes^2 W \otimes \bigwedge^2 \hat{W} \to \bigotimes^2 W \otimes \bigotimes^2 \hat{W} = M \otimes M. \]

Note that \( d_{0,2}^1 \) and \( d_{0,3}^1 \) are monomorphisms onto direct summands.

\( d_{-1,2}^2: M \otimes M \to M \) is given by

\[ d_{-1,2}^1(S \otimes T) = (T + \text{Tr}(T))S, \]

which is easily seen to be an epimorphism. It follows that \( E^2 \) looks like

![Diagram](https://via.placeholder.com/150)

where \( E_{-1,3}^2 = \text{Coker}(d_{0,3}^1: \bigotimes^2 W \otimes \bigwedge^2 \hat{W} \to W \otimes M) \) and

\[ E_{-1,2}^2 = \frac{\text{Ker}(d_{-1,2}^1: M \otimes M \to M)}{\text{Im}(d_{0,2}^1: \bigwedge^2 W \otimes \bigwedge^2 \hat{W} \to M \otimes M)}. \]

Then \( d_{-1,2}^2 \) is the negative of the morphism obtained by restricting the standard projection \( \pi: M \otimes M \to M \wedge M \) to the kernel indicated above. (One sees easily that \( \pi: M \otimes M \to M \wedge M \) kills \( \text{Im} d_{0,2}^1 \).)

**Lemma 6.2.** \( d_{-1,2}^2 \) is an epimorphism; it is a split epimorphism if \( n \not\equiv -1 \mod p \).
Proof. It suffices to show \( \ker(M \otimes M \to M) \to M \otimes M \) is an epimorphism. 

\( e_{ij} \wedge e_{kl} (i \neq k \text{ or } j \neq l) \) is the image of the following element in the kernel:

\[
\begin{align*}
& e_{ij} \otimes e_{kl} & \text{if } i \neq l \text{ and } k \neq l, \\
& -e_{kl} \otimes e_{ij} & \text{if } i = l \text{, but } i \neq j \text{ and } k \neq j, \\
& \frac{1}{2}(e_{ij} \otimes e_{kl} - e_{kl} \otimes e_{ij}) & \text{if } i = j \text{ and } (k \neq i = j), \\
& e_{ij} \otimes e_{ij} - \frac{1}{2}e_{jj} \otimes e_{jj} & \text{if } i = l, k = j \text{ (then } i \neq k = j), \\
& -e_{kk} \otimes e_{ij} & \text{if } k = l \text{ but } j \neq k, i \neq j, \\
& -\frac{1}{2}(e_{jj} \otimes e_{ij} - e_{ij} \otimes e_{jj}) & \text{if } k = l \text{ and } (i \neq k = j), \\
& e_{ii} \otimes e_{kk} - \frac{1}{2}(e_{ii} \otimes e_{ii}) & \text{if } k = l \text{ and } i = j \text{ (then } j = i \neq k). 
\end{align*}
\]

For the splitting, define the \( \text{GL}(n-1, p) \)-map \( \psi : M \otimes M \to M \otimes M \) by

\[
\psi(S \otimes T) = S \otimes T - T \otimes S + I \otimes [S, T] + I \otimes ((\text{Tr}S)T - (\text{Tr}T)S).
\]

Note that \( d_{1,2}^1(\psi(S \otimes T)) = TS + (\text{Tr}T)S - ST - (\text{Tr}S)T + [S, T] + 0 \cdot I + ((\text{Tr}S)T - (\text{Tr}T)S)I + 0 \cdot I = 0 \). Read modulo \( \text{Im } d_{0,2} \), \( \psi \) splits \( \pi \) since

\[
\pi(\psi(S \otimes T)) = 2S \otimes T + I \otimes [S, T] + I \otimes ((\text{Tr}S)T - (\text{Tr}T)S).
\]

[To see \( \psi \) is an isomorphism, examine its effect on the sequence \( 0 \to V \to V \otimes M \to V \otimes V \to 0 \) \((\text{rep } \tilde{S}) \) where \( \lambda(\tilde{S}) = I \otimes S \) \( (S \text{ representing } \tilde{S}) \). On \( \lambda(\tilde{V}) = I \otimes V \), \( \pi \psi(\lambda(\tilde{T})) = 2I \otimes T + I \otimes (n - 1)T = (n + 1)I \otimes T \). \( \pi \psi \) clearly induces multiplication by 2 modulo \( I \otimes V \). Hence, for \( p > 2 \), \( \pi \psi \) is an isomorphism \( \leftrightarrow n + 1 \equiv 0 \mod p \).]

Looking at \( E^3 \), we are done.

Appendix. We indicate briefly how some of the terms in \( E^1 \) are calculated.

\[
\begin{align*}
H_0(C_n, \wedge^2 \hat{W}) &= H_0(W, \wedge^2 \hat{W})_{F^*} = (\wedge^2 \hat{W})_{F^*} = 0, \\
H_1(C_n, \wedge^2 \hat{W}) &= (W \otimes \wedge^2 \hat{W})_{F^*} = 0, \\
H_2(C_n, \wedge^2 \hat{W}) &= [(\wedge^2 W \otimes W) \otimes \wedge^2 \hat{W}]_{F^*} = \wedge^2 W \otimes \wedge^2 \hat{W}, \\
H_3(C_n, \wedge^2 \hat{W}) &= [(\wedge^3 W \otimes \wedge^2 W) \otimes \wedge^2 \hat{W}]_{F^*} = \wedge^2 W \otimes \wedge^2 \hat{W}, \\
H_1(C_n, \hat{W} \otimes M) &= (W \otimes \hat{W} \otimes M)_{F^*} = M \otimes M, \\
H_2(C_n, \hat{W} \otimes M) &= [(\wedge^2 W \otimes W) \otimes \hat{W} \otimes M]_{F^*} = W \otimes \hat{W} \otimes M = M \otimes M. 
\end{align*}
\]

The other calculations are similar. For \( p = 5 \) we have exceptionally

\[
H_2(C_n, W \wedge W) = [\wedge^2 W \wedge W]_{F^*} = \wedge^2 W \wedge \wedge^2 W
\]

and \( H_3(C_n, M \otimes W) = \wedge^3 W \otimes M \otimes W (t^4 = 1 \text{ in } F^*_5) \). Note for \( n = 3 \) \((n - 1 = 2)\), the latter term is zero, but \( H_2(C_3, W \wedge W) \approx \wedge^2 W \otimes \wedge^2 W \approx \mathbb{Z}/5\mathbb{Z} \).

To calculate the differentials we follow the procedure indicated in §5. See the table of notations for terminology.
$d_{0,2}^1: \wedge^2 W \subset H^2(W, \mathbb{Z}/p\mathbb{Z})$ is spanned by products $u \cap v$, so we need only calculate $d_{0,2}^1$ on elements of the form $u \cap v \otimes f \wedge g, f, g \in \hat{W}$. Choose
\[
\hat{f} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in V
\]
to represent $f \in \hat{W}$ and similarly for $g$. We use the fact that $d$ is a derivation relative to the Pontryagin product on the chain level and related facts outlined in §10.3. We have
\[
d([u] \cap [v] \otimes \hat{f} \wedge \hat{g}) = (d[u] \cap [v] - [u] \cap d[v]) \otimes \hat{f} \wedge \hat{g}
\]
\[= ((u[\cdot] - [\cdot]) \cap [v] - [u] \cap (v[\cdot] - [\cdot])) \otimes \hat{f} \wedge \hat{g}
\]
\[= [v] \otimes ((-u)(\hat{f} \wedge \hat{g}) - \hat{f} \wedge \hat{g})
\]
\[= [u] \otimes ((-v)(\hat{f} \wedge \hat{g}) - \hat{f} \wedge \hat{g}).
\]
However, calculation shows
\[
(-u)f = \begin{bmatrix} uf & - (fu)u \\ f & fu \end{bmatrix}
\]
(note $fu = \text{Tr}(uf)$) and similarly for $g$. It is sufficient to calculate $(-u)(\hat{f} \wedge \hat{g}) - \hat{f} \wedge \hat{g}$ modulo $A_2$. That means we may ignore terms of the form
\[
\begin{bmatrix} 0 & 0 \\ * & 0 \end{bmatrix} \wedge \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \wedge \begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix}.
\]
We are left with $(-u)(\hat{f} \wedge \hat{g}) - \hat{f} \wedge \hat{g} \equiv \hat{f} \wedge (\hat{u}g) + (-uf) \wedge \hat{g}$ where
\[
\hat{T} = \begin{bmatrix} T & 0 \\ 0 & -\text{Tr}T \end{bmatrix}.
\]
Hence,
\[
d([u] \cap [v] \otimes \hat{f} \wedge \hat{g}) = [v] \otimes [\hat{g} \wedge (\hat{u}f) - \hat{f} \wedge (\hat{u}g)]
\]
\[- [u] \otimes [\hat{g} \wedge (\hat{v}f) - \hat{f} \wedge (\hat{v}g)].
\]
In the identification $M \cong W \otimes \hat{W}$, $uf \cong u \otimes f$. Unraveling what the above expression represents in $H_1(C_n, W \otimes M) = W \otimes \hat{W} \otimes M = W \otimes \hat{W} \otimes W \otimes \hat{W}$, we have
\[
d_{0,2}^1(u \cap v \otimes f \wedge g) = (v \otimes g) \otimes (u \otimes f) - (v \otimes f) \otimes (u \otimes g)
\]
\[= -(u \otimes g) \otimes (v \otimes f) + (u \otimes f) \otimes (v \otimes g)
\]
which after twisting yields
\[
(v \otimes u - u \otimes v) \otimes (g \otimes f - f \otimes g) = (u \otimes v - v \otimes u) \otimes (f \otimes g - g \otimes f)
\]
as claimed.

$d_{0,3}^1: \otimes^2 W \subset H_2(W, \mathbb{Z}/p\mathbb{Z})$ is spanned by elements of the form $u \cap \rho(v)$ where $\rho(v) \in H_2(W, \mathbb{Z}/p\mathbb{Z})$ is represented by $\rho[v] = \Sigma_{i=0}^1 [v, iv]$. We have
\[
d([u] \cap \rho[v] \otimes \hat{f} \wedge \hat{g}) = (u[\cdot] - [\cdot]) \cap \rho[v] \otimes \hat{f} \wedge \hat{g} - [u] \cap d\rho[v] \otimes \hat{f} \wedge \hat{g}.
\]
However, 
\[ dp[v] = d \sum [v, iv] = \sum (v[iv] - [(i + 1)v] + [v]) \]
\[ = v(\sum [iv]) - \sum [iv]. \]

Hence, 
\[ d([u] \cap \rho[v] \otimes \hat{f} \wedge \hat{g}) = \rho[v] \otimes \left( (-u)(\hat{f} \wedge \hat{g}) - \hat{f} \wedge \hat{g} \right) \]
\[ - \sum [u] \cap [iv] \otimes \left( (-v)(\hat{f} \wedge \hat{g}) - \hat{f} \wedge \hat{g} \right). \]

Calculating modulo \( A_2 \) as above, we have 
\[ d([u] \cap \rho[v] \otimes \hat{f} \wedge \hat{g}) = \rho[v] \otimes \left[ \hat{g} \wedge (\overline{uf}) - \hat{f} \wedge (\overline{vg}) \right] \]
\[ - \sum [u] \cap [iv] \otimes \left[ \hat{g} \wedge (\overline{vf}) - \hat{f} \wedge (\overline{vg}) \right]. \]

Given \( H_2(C_n, \hat{W} \otimes M) = ((\rho(W) \otimes \wedge^2 W) \otimes \hat{W} \otimes M) \cong \rho(W) \otimes \hat{W} \otimes M \cong W \otimes \hat{W} \otimes M \cong M \otimes M \), and noting that the second group of terms is in \( \wedge^2 W \otimes \hat{W} \otimes M \), we have 
\[ d^1_{0, 3}(u \cap \rho(v) \otimes f \wedge g) = \rho(v) \otimes (g \otimes u \otimes f - f \otimes u \otimes g). \]

Identifying \( \rho(V) \cong V \), we have 
\[ d^1_{0, 3}(u \otimes v \otimes f \wedge g) = v \otimes g \otimes u \otimes f - v \otimes f \otimes u \otimes g \]
\[ = - (v \otimes u) \otimes (f \otimes g - g \otimes f) \] (after twisting).

\( d^1_{1, 2} \) and \( d^2_{1, 2} \): \( H_1(C_n, \hat{W} \otimes M) = W \otimes \hat{W} \otimes M = M \otimes M \). Represent \( u \otimes f \otimes T \) by \([u] \otimes \hat{f} \wedge \hat{T} \). Then 
\[ d([u] \otimes \hat{f} \wedge \hat{T}) = (u[\cdot] - [\cdot]) \otimes \hat{f} \wedge \hat{T} \]
\[ = [\cdot] \otimes \left( (u)(\hat{f} \wedge \hat{T}) - \hat{f} \wedge \hat{T} \right). \]

However, 
\[ (u)(\hat{f}) = \hat{f} - (uf) - (fu)u \quad \text{where } u = \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix}. \]

Also \( (u)\hat{T} = \hat{T} + \hat{T}u \) where \( T'u = (T + \text{Tr} T)u \). Thus, modulo \( A_4 = X \wedge W \), 
\[ d([u] \otimes \hat{f} \wedge \hat{T}) \equiv [\cdot] \otimes \left[ \hat{f} \wedge \hat{T}u - uf \wedge \hat{T} \right]. \]

Hence, under the identification \( u \otimes f \approx uf \in M \), \( \hat{W} \otimes W \approx W \otimes \hat{W} \approx M \), we have 
\[ d^1_{1, 2}(uf \otimes T) = (T'u)f = T'(uf). \]
Since the matrices \( uf \) span \( M \), we may write 
\[ d^1_{1, 2}(S \otimes T) = (T + \text{Tr}(T))S \]
as claimed.

Examining formula (\( \ast \)) a bit more carefully, we see that \( d^2_{1, 2} \) is represented by the restriction to \( \text{Ker} d^1_{1, 2} \) of the morphism defined by \( uf \otimes T \mapsto -uf \wedge T \) or \( S \otimes T \mapsto -S \wedge T \) as claimed.

It remains to deal with the case \( p = 5 \). We show 
\[ d^1_{4, 7}: H_3(C_n, M \otimes W) = \wedge^3 W \otimes M \otimes W \to H_3(C_n, W \wedge W) \]
\[ = \wedge^2 W \otimes \wedge^2 W \]
is an epimorphism. Then $E^{2, 5, 6} = 0$ and the argument proceeds as before. To calculate $d_{-4, 7}^{1}$, we calculate:

$$d((u) \cap ((v) \cap [w])) \otimes \overline{T} \wedge \overline{x}), \quad u, v, w, x \in W, T \in M.$$

Use $(-u)\overline{T} = \overline{T} + \overline{T}u$, $(-u)\overline{x} = \overline{x}$ and the fact that $d$ is a derivation with $d[u] = u[\cdot] - [\cdot]$. The above expression is

$$[v] \cap [w] \otimes \overline{T}u \wedge \overline{x} - [u] \cap [w] \otimes \overline{T}v \wedge \overline{x} + [u] \cap [v] \otimes \overline{T}w \otimes \overline{x}.$$

Hence,

$$d_{-4, 7}^{1}(u \cap v \wedge u \wedge x)$$

$$= v \wedge w \otimes T'u \wedge x - u \wedge w \otimes T'v \wedge x + u \wedge v \otimes T'w \wedge x$$

where $T' = T + \text{Tr}(T)$.

To see this is an epimorphism, note that, for $l \neq i, j, k$, $e_i \cap e_j \cap e_k \wedge x \mapsto e_i \cap e_j \otimes e_k \wedge x$. For $n = 4 (n - 1 = 3)$

$$e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge x \mapsto e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge x.$$

Similarly, $e_1 \wedge e_2 \otimes e_2 \wedge x$, $e_2 \wedge e_3 \otimes e_2 \wedge x$, $e_2 \wedge e_3 \otimes e_3 \wedge x$, $e_1 \wedge e_3 \otimes e_1 \wedge x$, and $e_1 \wedge e_3 \otimes e_3 \wedge x$ are in the image. Since $x$ is arbitrary, we are done with the case $n \geq 4$. For $n = 3$, there is an additional nonzero term $\wedge^2 W_2 \otimes \wedge^2 W_2 = \mathbf{Z}/5\mathbf{Z}$ (at the upper left corner of the diagram) in $E^2$. It may persist in $E^\infty$ leading to the exact sequence given in the statement of the proposition.

7. Homology of $V_n \circ V_n$ and $V_n \wedge V_n$.

**Proposition 7.1.** For $n \geq 2$,

$$H_0(B_n, V_n \circ V_n) = H_0(\text{GL}(n, \mathbb{F}), V_n \circ V_n) = \mathbf{Z}/p\mathbf{Z}e_1 \circ e_2,$$

and

$$H_0(B_n, M_n \circ M_n) = H_0(\text{GL}(n, \mathbb{F}), M_n \circ M_n) = \mathbf{Z}/p\mathbf{Z}e_1 \circ e_1 + \mathbf{Z}/p\mathbf{Z}e_2 \circ e_2.$$

**Proof.** We start with $M_n \circ M_n$. Write $[x, a] = x(a) - a$ for $x \in B_n, a \in M_n \circ M_n$. Using the action of diagonal matrices, we see that $e_{ij} \circ e_{kl} = 0$ mod $[B_n, M_n \circ M_n]$ unless $i \neq j, k = j, l = i$ or $i = j, k = l$. Hence, $H_0(B_n, M_n \circ M_n)$ is spanned by the cosets of $e_{ij} \circ e_{ji}, i < j$, and $e_{ii} \circ e_{jj}, i \leq j$. Consideration of $[x_{si}, e_{ij} \circ e_{js}]$ shows $e_{sj} \circ e_{js} \equiv e_{ij} \circ e_{ji}$ for $s < i < j$. Similarly, $e_{ii} \circ e_{ii} \equiv e_{ij} \circ e_{ji}$ for $i < j < t$. Consideration of $[x_{st}, e_{is} \circ e_{jj}]$ shows $e_{ss} \circ e_{jj} \equiv e_{ii} \circ e_{jj}$ for $s < i < j$, and similarly $e_{ii} \circ e_{jj} \equiv e_{ii} \circ e_{ii}$ for $i < j < t$. Finally, consideration of $[x_{ij}, e_{ii} \circ e_{ji}]$ shows $e_{ii} \circ e_{jj} \equiv e_{ii} \circ e_{ii} - e_{ij} \circ e_{ij}$ for $i < j$. It follows that $e_{11} \circ e_{11}$ and $e_{12} \circ e_{21}$ span. That these are independent follows by considering the values assumed by the $B_n$-invariant forms defined by $\phi(A \circ B) = \text{Tr}(A)\text{Tr}(B)$ and $\psi(A \circ B) = \text{Tr}(AB)$. In fact, since those functionals are GL$(n, \mathbb{F})$-invariant, the result also follows for the larger group.

To obtain the result for $V_n \circ V_n$, consider the exact sequence $0 \to V_n \circ V_n \to M_n \circ M_n \to M_n \to 0$ where $\lambda^+(A \circ B) = (\text{Tr} A)B + (\text{Tr} B)A$. Note $\lambda^+(e_{12} \circ e_{21}) = 0$ and use Proposition 5.2.
COROLLARY 7.2. In $H_0(B_n, M_n \otimes M_n)$,
\[ e_{11} \circ e_{22} \equiv e_{11} \circ e_{11} - e_{12} \circ e_{21}. \]

PROPOSITION 7.3. For $n \geq 2$,
\[ H_0(B_n, V_n \wedge V_n) = 0, \quad H_0(B_n, M_n \wedge M_n) = 0, \]
\[ H_0(B_n, V_n \otimes V_n) = \mathbb{Z}/p\mathbb{Z} e_{12} \otimes e_{21}, \]
\[ H_0(B_n, M_n \otimes M_n) = \mathbb{Z}/p\mathbb{Z} e_{11} \otimes e_{11} + \mathbb{Z}/p\mathbb{Z} e_{12} \otimes e_{21}. \]
Similarly for $\text{GL}(n, p)$.

PROOF. Use Proposition 6.1(0) to derive the first statement. For the second, use the exact sequence
\[ 0 \to V_n \wedge V_n \to M_n \wedge M_n \to V_n \to 0, \]
where $\lambda^-(A \wedge B) = (\text{Tr} A)B - (\text{Tr} B)A$, and then use Proposition 5.2.

The third result follows from Proposition 7.1 and the decomposition $V_n \otimes V_n = \wedge^2 V_n \otimes S^2 V_n (p > 2)$. The last result is analogous. \qed

We shall need some congruences in $H_0(B_n, M_n \otimes M_n)$ for later reference.

PROPOSITION 7.4. For $n \geq 2$, we have
\[ e \otimes f \equiv f \otimes e, \quad e_{11} \otimes e_{22} \equiv e_{11} \otimes e_{11} - e_{12} \otimes e_{21} \]
modulo $[B_n, M_n \otimes M_n]$.

PROOF. Use $H_0(B_n, \wedge^2 V_n) = 0$ and Corollary 7.2.

PROPOSITION 7.5. For $n \geq 2$, $H_1(B_n, V_n \circ V_n) = H_1(\text{GL}(n, p), V_n \circ V_n) = 0$. Also, $H_1(B_n, M_n \circ M_n) = H_1(\text{GL}(n, p), M_n \circ M_n) = 0$.

PROOF. Note first that for each $n$ the first statement implies the second. This follows from Proposition 5.2 and the sequence $0 \to V_n \to M_n \to M_n \to 0$.

We argue inductively. For $n = 2$, $H_1(B_2, V_2 \circ V_2) = H_1(U_2, V_2 \circ V_2)_{H_2}$. Let $x$ generate the cyclic group $U_2$. As an $H_2$-module
\[ H_1(U_2, V_2 \circ V_2) = W_1 \otimes \frac{(V_2 \circ V_2)^{U_2}}{T_x(V_2 \circ V_2)}, \]
where $T_x = 1 + x + x^2 + \cdots + x^{p-1}$. Direct calculation shows $(V_2 \circ V_2)^{U_2} = \mathbb{Z}/p\mathbb{Z}(h_1 \circ h_1 + 4e_{21} \circ e_{12}) + \mathbb{Z}/p\mathbb{Z} e_{12} \circ e_{12}.$

\[ t = \begin{bmatrix} t_1 & 0 \\ 0 & t_2 \end{bmatrix} \]
acts trivially on the first factor and as $(t_1 t_2^{-1})^2$ on the second. Since it acts as $t_1$ on $W_1, W_1 \otimes (V_2 \circ V_2)^{U_2}/T_x(V_2 \circ V_2)$ does not have a trivial irreducible $H_2$-constituent. Hence, $H_1(B_2, V_2 \circ V_2) = 0$ as required. For $n > 2$, we consider the following filtration.
This filtration yields a spectral sequence

\[ E_{s,t}^1 = H_{s+t} \left( B_n, \frac{A_s}{A_{s+1}} \right) \Rightarrow H_{s+t}(B_n, A). \]

To calculate the $E^1$-term, we proceed as follows: Each $H_s(B_n, A_s/A_{s+1})$ is calculated using the spectral sequence of the split extension $B_n = B_{n-1} \ltimes C_n (C_n = F^\wedge \ltimes W_{n-1})$.

$s = 0$. $H_0(C_n, S^2\hat{W}_{n-1}) = (S^2\hat{W}_{n-1})_{F^\wedge} = 0$, so

\[ H_0(B_n, S^2\hat{W}_{n-1}) = H_0(B_{n-1}, H_0(C_n, S^2\hat{W}_{n-1})) = 0. \]

$H_1(C_n, S^2\hat{W}_{n-1}) = (\hat{W}_{n-1} \otimes S^2\hat{W}_{n-1})_{F^\wedge} = 0$, so

\[ H_1(B_{n-1}, S^2\hat{W}_{n-1}) = H_0(B_{n-1}, H_1(C_n, S^2\hat{W}_{n-1})) = 0. \]

$H_2(C_n, S^2\hat{W}_{n-1}) = \wedge^2 W_{n-1} \otimes S^2\hat{W}_{n-1}$, so

\[ H_2(B_n, S^2\hat{W}_{n-1}) = H_0(B_{n-1}, H_2(C_n, S^2\hat{W}_{n-1})) = H_0(B_{n-1}, \wedge^2 W_{n-1} \otimes S^2\hat{W}_{n-1}). \]

By Proposition 7.3, $H_0(B_{n-1}, M_{n-1} \otimes M_{n-1}) = \mathbb{Z}/p\mathbb{Z}e_{11} \otimes e_{11} + \mathbb{Z}/p\mathbb{Z}e_{12} \otimes e_{21}$. Also, $\wedge^2 W \otimes S^2\hat{W}$ is a direct summand of $(W \otimes W) \otimes (\hat{W} \otimes \hat{W}) \cong M \otimes M$. Hence, $H_0(B_{n-1}, \wedge^2 W_{n-1} \otimes S^2\hat{W}_{n-1})$ is generated by the projections of $e_{11} \otimes e_{11}$ and $e_{12} \otimes e_{21}$. However,

\[ e_{11} \otimes e_{11} = e_1 \otimes e_1 \otimes e^1 \otimes e^1 \mapsto e_1 \wedge e_1 \otimes e^1 \otimes e^1 = 0, \]

\[ e_{12} \otimes e_{21} = e_1 \otimes e_2 \otimes e^2 \otimes e^1 \mapsto e_1 \wedge e_2 \otimes e^2 \otimes e^1. \]

Going the other way yields

\[ e_1 \wedge e_2 \otimes e^2 \otimes e^1 \mapsto \frac{1}{2}(e_1 \otimes e_2 - e_2 \otimes e_1) \otimes \frac{1}{2}(e^2 \otimes e^1 + e^1 \otimes e^2) \]

\[ \cong \frac{1}{4}[e_{12} \otimes e_{21} - e_{22} \otimes e_{11} + e_{11} \otimes e_{22} - e_{21} \otimes e_{12}]. \]
However, we know from Proposition 7.4 that $e_{12} \otimes e_{21} \equiv e_{21} \otimes e_{12}$, $e_{11} \otimes e_{22} \equiv e_{22} \otimes e_{11}$, so $e_1 \wedge e_2 \otimes e^2 \circ e^1 \mapsto 0$. Hence, $H_2(B_n, S^2W_{n-1}) = 0$.

$s = -1$. $H_0(C_n, \hat{W}_{n-1} \otimes M_{n-1}) = 0$, so $H_0(B_n, \hat{W}_{n-1} \otimes M_{n-1}) = 0$.

$H_1(C_n, \hat{W}_{n-1} \otimes M_{n-1}) = M_{n-1} \otimes M_{n-1}$, so $H_1(B_n, \hat{W}_{n-1} \otimes M_{n-1}) = H_0(B_n, M_{n-1} \otimes M_{n-1}) = \mathbb{Z}/pe_{11} \otimes e_{11} \oplus \mathbb{Z}/pe_{12} \otimes e_{21}$.

$s = -2$. $H_0(C_n, \hat{W}_{n-1} \otimes W_{n-1}) = M_{n-1}$, so $H_0(B_n, \hat{W}_{n-1} \otimes W_{n-1}) = H_0(B_n, M_{n-1}) = \mathbb{Z}/p\mathbb{Z}$.

$H_1(C_n, \hat{W}_{n-1} \otimes W_{n-1}) = 0$, so $H_1(B_n, \hat{W}_{n-1} \otimes W_{n-1}) = H_1(B_n, M_{n-1}) = 0$.

$s = -3$. $H_0(C_n, S^2M_{n-1}) = S^2M_{n-1}$, so $H_0(B_n, S^2M_{n-1}) = H_0(B_n, S^2M_{n-1}) = \mathbb{Z}/p\mathbb{Z}e_{11} + \mathbb{Z}/p\mathbb{Z}e_{12} \otimes e_{21}$.

$H_1(C_n, S^2M_{n-1}) = 0$, so $H_1(B_n, S^2M_{n-1}) = H_1(B_n, S^2M_{n-1}) = 0$ (assumed inductively).

$s = -4$. $H_r(C_n, A_s/A_{s+1}) = 0$, $r = 0, 1$, in each case. Hence, $H_r(B_n, A_s/A_{s+1}) = 0$, $r = 0, 1$.

The $E_{s,t}$-term of the spectral sequence looks like

![Diagram]

Since $H_0(B_n, V_n \circ V_n) = \mathbb{Z}/p\mathbb{Z}e_{12} \circ e_{21}$, it follows easily that $E_{-1,2}^3 = 0$, so $H_1(B_n, V_n \circ V_n) = 0$. (In fact, since $e_{12} \circ e_{21}$ is in position $(-3, 3)$, we can say further that $d_{-1,2}^1$ is an epimorphism and $d_{-1,2}^2$ a monomorphism.)

**Proposition 7.6.** For $n \geq 2$, $p \geq 5$, we have

(a) $H_1(B_n, V_n \wedge V_n) = H_1(GL(n, p), V_n \wedge V_n) = 0$;

(b) $H_1(B_n, M_n \wedge M_n) = H_1(GL(n, p), M_n \wedge M_n) = 0$;
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(b) \( H_2(B_n, V_n \land V_n) = H_2(GL(n, p), V_n \land V_n) = \mathbb{Z}/p \mathbb{Z}. \)

**Proof.** First note that using Proposition 5.2 and the sequence \( 0 \to V_n \land V_n \to M_n \land M_n \to 0 \) we derive \( H_i(B_n, M_n \land M_n) = 0 \) from \( H_i(B_n, V_n \land V_n) = 0. \)

To prove (a) and (b) we argue inductively.

For \( n = 2, \) as in the proof of Proposition 6.1(0), \( H_i(B_2, V_2 \land V_2) = H_i(B_2, V_2) = 0, \)
and \( H_2(B_2, V_2 \land V_2) \) is \( \mathbb{Z}/p \mathbb{Z} \) by Proposition 5.2.

For \( n > 2, \) use the spectral sequence of the group extension \( B_n = B_{n-1} \times C_n. \) To prove (a), since \( H_1(B_{n-1}, H_0(C_n, V_n \land V_n)) = 0 \) by Proposition 6.1, we need only show \( H_0(B_{n-1}, H_1(C_n, V_n \land V_n)) = 0. \) From Proposition 7.3,
\[
H_0(B_{n-1}, M_{n-1} \land M_{n-1}) = 0,
\]
and we may assume inductively that \( H_1(B_{n-1}, M_{n-1} \land M_{n-1}) = 0. \) Hence, by Proposition 6.1,
\[
H_0(B_{n-1}, H_1(C_n, V_n \land V_n)) = H_0(B_{n-1}, \text{Ker}/\text{Im})
\]
which in turn is a direct summand of \( H_0(B_{n-1}, \text{Ker}). \) Since \( H_1(B_{n-1}, M_{n-1}) = 0, \)
\[
0 \to H_0(B_{n-1}, \text{Ker}) \to H_0(B_{n-1}, M_{n-1} \land M_{n-1}) \to H_0(B_{n-1}, M_{n-1}) \to 0
\]
is exact. Since \( e_{12} \otimes e_{21} \mapsto e_{11} \) and \( e_{11} \otimes e_{11} \mapsto 2e_{11} \),
\[
H_0(B_{n-1}, \text{Ker}) = \mathbb{Z}/p \mathbb{Z}(2e_{12} \otimes e_{12} - e_{11} \otimes e_{11}).
\]
Consider next the morphism \( d^1: \land^2 W \otimes \land^2 \hat{W} \to \land^2 W \otimes \land^2 W = M \otimes M. \)
\[
e_1 \land e_2 \otimes e^1 \land e^2 \mapsto (e_1 \otimes e_2 - e_2 \otimes e_1) \otimes (e^1 \otimes e^2 - e^2 \otimes e^1)
\]
\[
\equiv e_{11} \otimes e_{22} - e_{12} \otimes e_{21} - e_{21} \otimes e_{12} + e_{22} \otimes e_{11}
\]
(Proposition 7.4)
\[
\equiv 2(e_{11} \otimes e_{22} - e_{12} \otimes e_{21})
\]
\[
\equiv 2(e_{11} \otimes e_{11} - e_{12} \otimes e_{21} - e_{12} \otimes e_{21})
\]
\[
\equiv 2(e_{11} \otimes e_{11} - 2e_{12} \otimes e_{21}).
\]
Hence, \( H_0(B_{n-1}, \text{Ker}/\text{Im}) = 0 \) as required.

We next consider (b). By Proposition 6.1, \( H_0(B_{n-1}, H_2(V_n \land V_n)) = H_0(B_{n-1}, (M_{n-1} \otimes M_{n-1})/\text{Im} d^1) \) which (as shown in the proof of 6.1) is a direct summand of \( H_0(B_{n-1}, M_{n} \otimes M_{n-1}) = \mathbb{Z}/p \mathbb{Z}e_{12} \otimes e_{21} + \mathbb{Z}/p \mathbb{Z}e_{11} \otimes e_{11}). \) (For \( n = 3, p = 5, \) use the fact that \( H_0(B_2, \land^2 W_2 \otimes \land^2 \hat{W}_2) = 0. \) Referring again to the proof of 6.1, we recall that \( \text{Im} d^1 \simeq \land^2 W_{n-1} \otimes \land^2 \hat{W}_{n-1}. \) Note that \( e_{11} \otimes e_{11} \equiv (e_1 \otimes e_1) \otimes (e^1 \otimes e^1) \mapsto e_1 \otimes e_1 \otimes e^1 \land e^1 = 0, \) so \( H_0(B_{n-1}, \text{Im} d^1) \equiv \mathbb{Z}/p \mathbb{Z}. \) On the other hand, going the other way,
\[
e_1 \otimes e_2 \otimes e^1 \land e^2 \mapsto e_1 \otimes e_2 \otimes (e^1 \otimes e^2 - e^2 \otimes e^1)
\]
\[
\equiv e_{11} \otimes e_{22} - e_{12} \otimes e_{21} \not\equiv 0.
\]
Hence, $H_0(B_{n-1}, \text{Im } d^1) = \mathbb{Z}/p\mathbb{Z}$, and thus

$$H_0(B_{n-1}, (M_{n-1} \otimes M_{n-1})/\text{Im } d^1) = \mathbb{Z}/p\mathbb{Z}.$$ 

Continuing with (b), we shall show that $H_1(B_{n-1}, H_1(C_n, V_n \wedge V_n)) = 0$. We use the short exact sequence

$$0 \rightarrow H_1(C_n, V_n \wedge V_n) \rightarrow \frac{\text{Ker}(M_{n-1} \otimes M_{n-1} \rightarrow M_{n-1})}{\text{Im } d^1} \rightarrow M_{n-1} \wedge M_{n-1} \rightarrow 0$$

of Proposition 6.1(i). If $n \equiv -1 \mod p$, the sequence splits, so we are reduced to showing $H_1(B_{n-1}, \text{Ker/Im}) = 0$. If $n \equiv -1 \mod p$, we obtain the same reduction by means of the following argument (suggested by the referee). It suffices to show that $H_2(B_{n-1}, \text{Ker}) \rightarrow H_2(B_{n-1}, M_{n-1} \wedge M_{n-1})$ is an epimorphism. In the proof of 6.1, we derived the splitting (for $n \equiv -1$) by showing that $\pi\psi$ is an isomorphism of $M_{n-1} \wedge M_{n-1}$. (Refer to Lemma 6.2 for notation.) That in turn was deduced by examining the effect of $\pi\psi$ on the short exact sequence $0 \rightarrow V_{n-1} \rightarrow M_{n-1} \otimes M_{n-1} \rightarrow \lambda \rightarrow \lambda \rightarrow 0$ (where $\lambda(T) = I_{n-1} \wedge T$). If $n \equiv -1$, $\pi\psi$ is unfortunately trivial on $\text{Im } \lambda$. However, we still have $\chi \pi\psi = 2\chi$ so $\text{Im}(2 \text{id} - \pi\psi) \subset \text{Im } \lambda$. Thus, it suffices to show $\lambda_*(H_2(B_{n-1}, V_{n-1})) \subset \pi_*(H_2(B_{n-1}, \text{Ker}))$. For this, consider $\sigma$: $M_{n-2} \rightarrow M_{n-1} \otimes M_{n-1}$ defined by

$$\sigma(T) = I_{n-2} \otimes T - T \otimes \iota_{n-1, n-1} - (\text{Tr}(T)/(n-1))I_{n-2} \otimes I_{n-2}.$$ 

Note that $\text{Im } \sigma \subset \text{Ker}(M_{n-1} \otimes M_{n-1} \rightarrow M_{n-1})$. Also, if we set $e(T) = \overline{T}$, then

$$M_{n-2} \rightarrow \text{Ker}(M_{n-1} \otimes M_{n-1} \rightarrow M_{n-1})$$

$$\downarrow \iota \quad \downarrow \pi$$

$$\overline{V}_{n-1} \rightarrow M_{n-1} \wedge M_{n-1}$$

commutes. Since $\iota_*$: $H_2(B_{n-2}, M_{n-2}) \rightarrow H_2(B_{n-1}, \overline{V}_{n-1})$ is an isomorphism by Propositions 5.1 and 5.2, it follows readily that $\lambda_*(H_2(B_{n-1}, \overline{V}_{n-1})) \subset \text{Im } \pi_*$ as required.

To show that $H_1(B_{n-1}, \text{Ker/Im}) = 0$, we use the fact established in the proof of Proposition 6.1 that $\text{Im } \lambda$ is the direct summand of $\text{Ker}$; thus, we need only show that $H_1(B_{n-1}, \text{Ker}) = 0$. Since $H_1(B_{n-1}, M_{n-1} \otimes M_{n-1}) = 0$ by part (a) and Proposition 7.5, it suffices to show $H_2(B_{n-1}, M_{n-1} \otimes M_{n-1}) \rightarrow H_2(B_{n-1}, M_{n-1})$ is an epimorphism. This follows from Proposition 5.2 and the commutative diagram

$$M_{n-1} \otimes M_{n-1} \rightarrow M_{n-1}$$

$$\uparrow \rho \rightarrow \rho$$

$$V_{n-1}$$

where we define $\rho(T) = I_{n-1} \otimes T$.

Since $H_2(B_{n-1}, H_0(C_n, V_n \wedge V_n)) = 0$, we may now conclude

$$H_2(\text{Gl}(n, p), V_n \wedge V_n) \cong H_2(B_{n-1}, V_n \wedge V_n) \cong \mathbb{Z}/p\mathbb{Z}.$$ 

To conclude that $H_2(\text{Gl}(n, p), V_n \wedge V_n) = \mathbb{Z}/p\mathbb{Z}$, we need only show it is nontrivial.
For this, consider
\[ H_2(\text{Gl}(2, p), V_2 \wedge V_2) \to H_2(\text{Gl}(2, p), V_2) \]
\[ \downarrow \]
\[ H_2(\text{Gl}(n, p), V_n \wedge V_n) \to H_2(\text{Gl}(n, p), V_n). \]

The upper horizontal arrow is an isomorphism since \( V_2 \wedge V_2 \cong V_2 \), and the right-hand vertical arrow is an isomorphism by Proposition 5.2. It follows that the lower horizontal arrow is onto something nontrivial, and we are done. (This argument was also suggested by the referee.)

8. Higher multilinear modules.

**Proposition 8.1.** For \( n \geq 2 \), \( H_0(B_n, M_n \wedge M_n \otimes M_n) = \mathbb{Z}/p \oplus \mathbb{Z}/p \). Also, \( H_0(\text{GL}(n, p), V_n \wedge V_n \otimes V_n) = H_0(B_n, V_n \wedge V_n \otimes V_n) = \mathbb{Z}/p \mathbb{Z} \) (stably).

**Proof.** First note that the exact sequence \( 0 \to \wedge^2 M_n \otimes V_n \to \wedge^2 M_n \otimes M_n \to \wedge^2 M_n \to 0 \) yields
\[ H_0(B_n, \wedge^2 M_n \otimes V_n) = H_0(B_n, \wedge^2 M_n \otimes M_n). \]
In addition, \( 0 \to \wedge^2 V_n \otimes V_n \to \wedge^2 M_n \otimes V_n \to V_n \otimes V_n \to 0 \) yields
\[ 0 \to H_0(B_n, \wedge^2 V_n \otimes V_n) \to H_0(B_n, \wedge^2 M_n \otimes V_n)
\[ \to H_0(B_n, V_n \otimes V_n) = \mathbb{Z}/p \to 0 \]
(similarly, for \( \text{GL}(n, p) \)).

For \( n = 2 \), \( \wedge^2 V_2 \otimes V_2 \cong V_2 \otimes V_2 \), and \( H_0(B_2, V_2 \otimes V_2) = \mathbb{Z}/p \mathbb{Z} \). The exact sequence above gives the desired result for \( \wedge^2 M_2 \otimes M_2 \). On the other hand, for \( n \geq 3 \), using those sequences, we need only show \( H_0(B_n, \wedge^2 M_n \otimes M_n) = \mathbb{Z}/p \oplus \mathbb{Z}/p \) (stably) (similarly for \( \text{GL}(n, p) \)).

Assume \( n \geq 3 \). By duality, we have
\[ H_0(B_n, M_n \wedge M_n \otimes M_n) \cong H^0(B_n, M_n \wedge M_n \otimes M_n) \cong H^0(B_n, M_n \wedge M_n \otimes M_n). \]
(Recall that \( M_n \) is self-dual as noted in the proof of Proposition 5.2.) Since it is technically easier to calculate invariants, we calculate the last group.

Let \( z \in M_n \wedge M_n \otimes M_n \) be invariant under \( B_n \). Using the action of the torus, we conclude that \( z \) must be of the form
\[ \sum_{i \neq j \neq k \neq i} \alpha_{ijk} e_{ij} \wedge e_{jk} \otimes e_{ki} + \sum_{i \neq j \neq k \neq i} \beta_{ijk} e_{ij} \wedge e_{ji} \otimes e_{kk} \]
\[ + \sum_{i \neq j \neq k \neq i} \gamma_{ijk} e_{ij} \wedge e_{kk} \otimes e_{ji} + \sum_{i \neq j} \delta_{ij} e_{ij} \wedge e_{ji} \otimes e_{ii} \]
\[ + \sum_{i \neq j} e_{ij} e_{ij} \wedge e_{ii} \otimes e_{ji} + \sum_{i \neq j} \xi_{ij} e_{ij} \wedge e_{jj} \otimes e_{ji} \]
\[ + \sum_{i \neq j} \eta_{ijk} e_{ii} \wedge e_{jj} \otimes e_{kk} + \sum_{i \neq j} \theta_{ij} e_{ii} \wedge e_{jj} \otimes e_{ii}. \]
We now make use of the following argument suggested by the referee. Any \( z \) of the above form which is invariant under \( B_n \) (defined over the prime field \( \mathbb{F} \)) is also invariant under \( B_n(\overline{\mathbb{F}}) \) (defined over the algebraic closure). (Note that \( M_n \otimes M_n \) is contained naturally in the corresponding object defined over the algebraic closure.) Namely, such a \( z \) is a sum of terms each of which has weight 0 and hence is invariant under the torus \( H_n(\overline{\mathbb{F}}) \) over the algebraic closure. If such a \( z \) is also \( B_n \)-invariant, then it must also be \( B_n(\overline{\mathbb{F}}) \)-invariant because \( B_n(\overline{\mathbb{F}}) \) is generated by \( B_n \) and \( H_n(\overline{\mathbb{F}}) \). On the other hand, it is well known that the \( B_n(\overline{\mathbb{F}}) \)-invariants are identical with the \( \text{GL}_n(\overline{\mathbb{F}}) \)-invariants. (To see this, note that the orbit of a \( B_n(\overline{\mathbb{F}}) \)-invariant element is the image of the projective variety \( \text{GL}_n(\overline{\mathbb{F}})/B_n(\overline{\mathbb{F}}) \) in the affine variety \( (M_n \otimes M_n) \otimes_{\mathbb{F}} \overline{\mathbb{F}} \) and is thus a single point.)

In particular, any \( B_n \)-invariant element \( z \) of the above form is invariant under the symmetric group \( \Sigma_n \) (contained in \( \text{GL}_n(\mathbb{F}) \)). Using the transpositions \( (ij) \), we conclude that \( \beta_{ijk} = \eta_{ijk} = 0 \) for all \( i \neq j \neq k \neq i \). Using permutations sending \( i \) to \( i' \), \( j \) to \( j' \), and \( k \) to \( k' \), for various triples of indices, we see that \( \alpha_{ijk} = \alpha \) is independent of \( i, j, \) and \( k \). Similarly, we may write \( \gamma_{ijk} = \gamma, \delta_{ij} = \delta, \epsilon_{ij} = \epsilon, \zeta_{ij} = \zeta, \) and \( \theta_{ij} = \theta \) since these coefficients are also independent of the relevant indices.

Consider the coefficient of \( e_{ij} \wedge e_{jk} \otimes e_{ki} \) in \( \chi_{ij}(z) - z \) where \( i \neq j \neq k \neq i \). Because \( \beta_{kji} = 0 \), this coefficient is \( \alpha_{ijk} - \delta_{ijk} \). Thus, \( \alpha = \delta \).

Consider the coefficient of \( e_{ij} \wedge e_{ji} \otimes e_{ii} \) in \( \chi_{ji}(z) - z \) where \( i \neq j \). It is \( \delta_{ij} + \delta_{ji} + \epsilon_{ij} - \zeta_{ij} \), so that \( \zeta = 2\delta + \epsilon \).

Consider the coefficient of \( e_{ij} \wedge e_{jk} \otimes e_{ki} \) in \( \chi_{jk}(z) - z \) where \( i \neq j \neq k \neq i \). It is \( \alpha_{ijk} - \gamma_{ijk} \), so that \( \gamma = \zeta - \alpha \).

Consider the coefficient of \( e_{ii} \wedge e_{ij} \otimes e_{ii} \) in \( \chi_{ji}(z) - z \) where \( i \neq j \). It is \( \theta_{ij} - \epsilon_{ij} - \delta_{ij} \), so that \( \epsilon = \theta - \delta \).

Solving these equations yields \( \alpha = \delta, \zeta = \delta + \theta, \gamma = \theta, \) and \( \epsilon = -\delta + \theta \), so that we may conclude that a \( B_n \)-invariant \( z \) (if such exists) must be of the form

\[
\delta \sum_{i \neq j \neq k \neq i} e_{ij} \wedge e_{jk} \otimes e_{ki} + \theta \sum_{i \neq j \neq k \neq i} e_{ij} \wedge e_{kk} \otimes e_{ji} + \\
(\theta - \delta) \sum_{i \neq j} e_{ij} \wedge e_{ji} \otimes e_{ii} + \\
(\theta - \delta) \sum_{i \neq j} e_{ij} \wedge e_{jj} \otimes e_{ji} + \theta \sum_{i \neq j} e_{ii} \wedge e_{jj} \otimes e_{ii} \\
= \delta \sum_{i,j,k} e_{ij} \wedge e_{jk} \otimes e_{ki} + \theta \sum_{i,j,k} e_{ij} \wedge e_{kk} \otimes e_{ji}.
\]

Thus, by duality, \( H_0(B_n, \wedge^2 M_n \otimes M_n) \cong \mathbb{Z}/p \otimes \mathbb{Z}/p \). To see that we have equality (even if \( B_n \) is replaced by \( \text{GL}(n, p) \), and even stably), consider the (stable) \( \text{GL}(n, p) \)-invariant forms defined by

\[
\phi(A \wedge B \otimes C) = \text{Tr}(A)\text{Tr}(BC) - \text{Tr}(B)\text{Tr}(AC),
\]

and

\[
\psi(A \wedge B \otimes C) = \text{Tr}(ABC - BAC).
\]
These are independent since \( \phi(e_{12} \wedge e_{21} \otimes e_{11}) = 0 \), \( \phi(e_{11} \wedge e_{22} \otimes e_{11}) = -1 \), \\
\( \psi(e_{12} \wedge e_{21} \otimes e_{11}) = 1 \), \( \psi(e_{11} \wedge e_{22} \otimes e_{11}) = 0 \).

**Corollary 8.2.** For \( n \geq 2 \), \( H_0(B_n, \wedge^3 M_n) = \mathbb{Z}/p\mathbb{Z} \). Also, \( H_0(B_n, \wedge^3 V_n) = H_0(\text{GL}(n, p), \wedge^3 V_n) = \mathbb{Z}/p\mathbb{Z} \) (stably).

**Proof.** Consider the exact sequence \( 0 \to \wedge^3 V_n \to \wedge^3 M_n \xrightarrow{\lambda} \wedge^2 V_n \to 0 \), where \( \lambda(A \wedge B \wedge C) = \text{Tr}(A)B \wedge C - \text{Tr}(B)A \wedge C + \text{Tr}(C)A \wedge B \). (Note that the codomain of \( \lambda \) is \( \wedge^2 M \), but in fact \( \text{Im} \lambda = \wedge^2 V \).

\[
H_0(B_n, \wedge^3 V_n) \cong H_0(B_n, \wedge^3 M_n),
\]
(and similarly for \( \text{GL}(n, p) \)), so we need only consider \( \wedge^3 V_n \). From Proposition 8.1, it suffices to show the epimorphism

\[
\mathbb{Z}/p\mathbb{Z} = H_0(B_n, \wedge^3 V_n) \to H_0(B_n, \wedge^3 V_n)
\]
is an isomorphism (similarly for \( \text{GL}(n, p) \)). However, this is clear since the right-hand side is nontrivial. Indeed, the \( \text{GL}(n, p) \)-invariant form defined by \( \phi(A \wedge B \wedge C) = \text{Tr}(ABC - ACB) \) is nontrivial on \( \wedge^3 V_n \) (\( \phi(h_1 \wedge e_{12} \wedge e_{21}) = 2 \)). \( \square \)

**Proposition 8.3.** For \( p \geq 5 \), \( n \geq 2 \),

\[
H_1(B_n, \wedge^3 V_n) = H_1(\text{GL}(n, p), \wedge^3 V_n) = 0.
\]

**Proof.** For \( n = 2 \), \( \wedge^3 V_2 = \mathbb{Z}/p\mathbb{Z}e_{12} \wedge h_1 \wedge e_{21} \).

Since \( e_{12} \wedge h_1 \wedge e_{21} \) is fixed by \( x_{12} \) and \( H_2, B_2 \) acts trivially, and the result follows by Proposition 5.1.

For \( n > 2 \), we proceed by induction. We form the tower shown as Table 1 (subscripts are omitted). The calculations in the table are done as previously.

One calculation requires further comment.

\[
H_1(B_n, \wedge^2 \hat{W}_{n-1} \otimes W_{n-1}) = H_0(B_{n-1}, H_1(C_n, \wedge^2 \hat{W}_{n-1} \otimes W_{n-1}))
\]

\[
= H_0(B_{n-1}, \otimes^2 W \otimes \wedge^2 \hat{W}).
\]

However, \( \otimes^2 W \otimes \wedge^2 \hat{W} \) is a direct summand of \( \otimes^2 W \otimes \otimes^2 \hat{W} \cong M \otimes M \). Thus \( H_0(B_{n-1}, \otimes^2 W \otimes \wedge^2 \hat{W}) \) is generated by the images of \( e_{11} \otimes e_{11} \cong e_1 \otimes e_1 \otimes e_1 \otimes e_1 \otimes e_1 \otimes e_1 \otimes e_1 \otimes e_1 \otimes e_1 \otimes e_1 \).

On the other hand, going the other way, we get

\[
e_1 \otimes e_2 \otimes e_2 \otimes e_1 \otimes e_1 \otimes e_1 \otimes e_1 \otimes e_1 \otimes e_1 \otimes e_1
\]

\[
\cong \frac{1}{2}[e_{12} \otimes e_{21} - e_{11} \otimes e_{22}]
\]

\[
= \frac{1}{2}[e_{12} \otimes e_{21} - e_{11} \otimes e_{11} + e_{12} \otimes e_{21}]
\]

\[
= \frac{1}{2}[e_{12} \otimes e_{21} - e_{11} \otimes e_{11}] \cong 0.
\]

Hence, \( H_1(B_n, \wedge^2 \hat{W}_{n-1} \otimes W_{n-1}) = \mathbb{Z}/p\mathbb{Z} \).
Table 1

<table>
<thead>
<tr>
<th>Level</th>
<th>Factor</th>
<th>( H_r(C_n, -) )</th>
<th>( H_r(B_n, -) )</th>
</tr>
</thead>
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<tr>
<td>0</td>
<td>( \lambda^3 )</td>
<td>( H_0=H_1=H_2=0 )</td>
<td>( H_0=H_1=H_2=0 )</td>
</tr>
<tr>
<td>1</td>
<td>( \lambda^2 \wedge x )</td>
<td>( H_0=H_1=0, H_2=\lambda^2 \wedge \lambda^2 \wedge \wedge \wedge M )</td>
<td>( H_0=H_1=0 )</td>
</tr>
<tr>
<td>2</td>
<td>( \lambda^2 \wedge \wedge M )</td>
<td>( H_0=H_1=0, H_2=\lambda^2 \wedge \wedge M )</td>
<td>( H_0=H_1=0, H_2=0 )</td>
</tr>
<tr>
<td>3</td>
<td>( \wedge \wedge \wedge \wedge \wedge M )</td>
<td>( H_0=H_1=0, H_2=\lambda^2 \wedge \wedge M )</td>
<td>( H_0=H_1=0 )</td>
</tr>
<tr>
<td>4</td>
<td>( \wedge \wedge \wedge \wedge \wedge M )</td>
<td>( H_0=H_1=0, H_2=\lambda^2 \wedge \wedge M )</td>
<td>( H_0=H_1=0 )</td>
</tr>
<tr>
<td>5</td>
<td>( \wedge \wedge \wedge \wedge \wedge M )</td>
<td>( H_0=H_1=0, H_2=\lambda^2 \wedge \wedge M )</td>
<td>( H_0=H_1=0 )</td>
</tr>
<tr>
<td>6</td>
<td>( \lambda^3 \wedge \wedge M )</td>
<td>( H_0=H_1=0, H_2=\lambda^2 \wedge \wedge M )</td>
<td>( H_0=H_1=0, H_2=0 )</td>
</tr>
<tr>
<td>7</td>
<td>( \lambda^2 \wedge \wedge \wedge \wedge M )</td>
<td>( H_0=H_1=0, H_2=\lambda^2 \wedge \wedge M )</td>
<td>( H_0=H_1=0, H_2=0 )</td>
</tr>
<tr>
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<tr>
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<td>( H_0=H_1=0, H_2=0 )</td>
</tr>
<tr>
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<td>0</td>
<td>( H_0=H_1=0 ) (including ( p=5 ))</td>
<td>( H_0=H_1=0 )</td>
</tr>
</tbody>
</table>

We examine the spectral sequence

\[ E_{s+t}^1 = H_{s+r}(B_n, A_{-s}/A_{-s+1}) \Rightarrow H_{s+r}(B_n, A) \]

as previously. If we exhibit only the range containing nonzero terms, the \( E^1 \)-term looks like the following:

\[ \mathbb{Z}/p \mathbb{Z} \varepsilon_{12} \wedge h_1 \wedge e_{21} \]

\[ \mathbb{Z}/p \mathbb{Z} \varepsilon_{12} \wedge h_1 \wedge e_{21} \]

\[ \mathbb{Z}/p \mathbb{Z} \varepsilon_{12} \wedge h_1 \wedge e_{21} \]

\[ \mathbb{Z}/p \mathbb{Z} \varepsilon_{12} \wedge h_1 \wedge e_{21} \]

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To complete the proof it suffices to show \( d_{-1,3}^{1} \) is an epimorphism. (Then, \( d_{-3,4}^{1} \) is an isomorphism since \( H_{0}(B_{n}, \wedge^{2} V_{n}) = \mathbb{Z}/p^{2} \wedge h_{1} \wedge e_{21} \).

To show \( d_{-1,3}^{1} \) is an epimorphism, we argue as follows. The corresponding morphism for \( C_{n} \),

\[
d_{-1,3}^{1} : H_{2}(C_{n}, \wedge^{2} \hat{W}_{n-1} \otimes M_{n-1}) \rightarrow H_{1}(C_{n}, \wedge^{2} \hat{W}_{n-1} \otimes W_{n-1})
\]

is given by

\[
d_{-1,3}^{1}(u \wedge v \otimes f \wedge g \otimes S) = -[u \otimes (S + \Tr S)v - v \otimes (S + \Tr S)u] \otimes f \wedge g
\]

(see below). Since we showed above that

\[
H_{0}(B_{n-1}, \otimes^{2} W_{n-1} \otimes \wedge^{2} \hat{W}_{n-1}) = \mathbb{Z}/p^{2} \otimes e_{1} \otimes e_{2} \otimes e_{2} \otimes e_{1},
\]

it suffices to note that

\[
d_{-1,3}^{1}(e_{1} \wedge e_{2} \otimes e_{2} \otimes e_{1} \otimes e_{1}) = -[e_{1} \otimes e_{2} - 2e_{2} \otimes e_{1}] \otimes e_{2} \otimes e_{1},
\]

\[
d_{-1,3}^{1}(e_{1} \wedge e_{2} \otimes e_{2} \otimes e_{1} \otimes e_{22}) = -[2e_{1} \otimes e_{2} - e_{2} \otimes e_{1}] \otimes e_{2} \otimes e_{1}.
\]

(Subtracting twice the second from the first yields \( +3e_{1} \otimes e_{2} \otimes e_{2} \otimes e_{1} \) as required.)

**Appendix.** Calculation of \( d_{-1,3}^{1} \) for \( C_{n} \):

\[
d([u] \cap [v] \otimes \tilde{f} \wedge \tilde{g} \wedge \tilde{S}) = (d[u] \cap [v] - [u] \cap d[v]) \otimes \tilde{f} \wedge \tilde{g} \wedge \tilde{S}
\]

\[
= [v] \otimes [(-u)(\tilde{f} \wedge \tilde{g} \wedge \tilde{S}) - \tilde{f} \wedge \tilde{g} \wedge \tilde{S}]
- [u] \otimes [(-v)(\tilde{f} \wedge \tilde{g} \wedge \tilde{s}) - \tilde{f} \wedge \tilde{g} \wedge \tilde{s}].
\]

However, \((-u)\tilde{f} = \tilde{f} - (uf) - (fu)\tilde{u} \), similarly for \( \tilde{g} \), and

\[
(-u)\tilde{S} = \tilde{S} + \tilde{S'}u, \quad S' = S + \Tr(S).
\]

Hence, modulo

\[
A_{3} = V \wedge X \wedge X, \quad \tilde{f} \wedge \tilde{g} \wedge \tilde{S} \equiv \tilde{S} \wedge \tilde{S'}u.
\]

Hence, \( d([u] \cap [v] \otimes \tilde{f} \wedge \tilde{g} \wedge \tilde{S}) \equiv [v] \otimes (\tilde{f} \wedge \tilde{g} \wedge \tilde{S'}u) - [u] \otimes (\tilde{f} \wedge \tilde{g} \wedge \tilde{S'}v). \)

Hence, with some twisting, we get

\[
d_{-1,3}^{1}(u \wedge v \otimes f \wedge g \otimes S) = (v \otimes S' u - u \otimes S' v) \otimes f \wedge g
\]

as claimed. \( \square \)

**Proposition 8.4.** For \( n \geq 2 \), \( H_{0}(B_{n}, \wedge^{4} M_{n}) = \mathbb{Z}/p^{2} \), \( H_{0}(GL(n, p), \wedge^{4} V_{n}) = H_{0}(B_{n}, \wedge^{4} V_{n}) = 0 \).

**Proof.** We calculate \( H^{0}(B_{n}, \wedge^{4} M_{n}) \) instead and use duality. Let \( z \in M_{n} \wedge M_{n} \wedge M_{n} \wedge M_{n} \) be invariant under \( B_{n} \). Using the action of the torus, we conclude that \( z \) must be of the form...
\[
\sum_{i<j, k \neq i, j} \alpha_{ijk} e_{ij} \wedge e_{ji} \wedge e_{kl} \wedge e_{lk} + \sum_{i<j, k, l \neq i, j} \beta_{ijkl} e_{ij} \wedge e_{jk} \wedge e_{kl} \wedge e_{li} + \\
\sum_{j<k} \gamma_{ijk} e_{ij} \wedge e_{ji} \wedge e_{ik} \wedge e_{ki} + \\
\sum_{i<j, k, l, \neq i, j} \delta_{ijkl} e_{ij} \wedge e_{jk} \wedge e_{kl} \wedge e_{li} + \sum_{i<j, k, l, \neq i, j} \varepsilon_{ijkl} e_{ij} \wedge e_{jk} \wedge e_{kl} \wedge e_{li} + \\
\sum_{i<j, k, l, \neq i, j} \lambda_{ijkl} e_{ij} \wedge e_{ji} \wedge e_{kl} \wedge e_{lk} + \sum_{i<j, k, l, \neq i, j} \mu_{ijkl} e_{ij} \wedge e_{ji} \wedge e_{kl} \wedge e_{lk}.
\]

For \( n = 2 \) only terms of "type \( \theta \)" occur. For \( n = 3 \) only terms of type \( \alpha, \gamma, \epsilon, \eta, \) and \( \theta \) occur. However, with minor modifications, the arguments we shall introduce below are also valid in those cases.

As in the proof of Proposition 8.1, we use the referee’s argument to conclude that \( z \) is in fact \( \text{GL}(n, p) \)-invariant and hence invariant under the symmetric group \( S_n \). Using the transposition \((ij)\), we conclude that \( \alpha_{ijk} = \delta_{ijk} = \lambda_{ijk} = 0 \) for \( i, j, k \) distinct. Using the cycle \((ijkl)\), we conclude that \( \beta_{ijkl} = 0 \) for \( i, j, k, l \) distinct. Using permutations sending \( i \) to \( i' \), \( j \) to \( j' \), and \( k \) to \( k' \) for various sets of indices, we conclude that \( \gamma_{ijk} = \gamma, \delta_{ijkl} = \delta, \epsilon_{ijkl} = \epsilon, \eta_{ijkl} = \eta, \) and \( \theta_{ijkl} = \theta \) are independent of the indices.

Consider the coefficient of \( e_{ij} \wedge e_{ji} \wedge e_{kl} \wedge e_{lk} \) in \( x_{ik}(z) - z, j < k \). It is \( -\gamma_{ijk} - \eta_{ijk} + \epsilon_{kji}, \) so \( \gamma = \epsilon - \eta. \)

Consider the coefficient of \( e_{ij} \wedge e_{ji} \wedge e_{kii} \wedge e_{lii} \) in \( x_{ki}(z) - z, i < j \). It is \( \eta_{ijl} - \delta_{ijkl}, \) so \( \eta = \delta. \)

Consider the coefficient of \( e_{ij} \wedge e_{ji} \wedge e_{kij} \wedge e_{lji} \) in \( x_{ki}(z) - z, i < j \). It is \( \theta_{ij} - \eta_{ijk} - \epsilon_{kji}, \) so \( \theta = \eta + \epsilon. \)

Consider the coefficient of \( e_{ij} \wedge e_{jk} \wedge e_{ik} \wedge e_{kl} \) in \( x_{ik}(z) - z, j < i \) and \( i < j \). It is \( \epsilon_{ij} - \epsilon_{kji} - \epsilon_{kjj}, \) so \( -2\gamma = 0. \)

From these equations we conclude that \( \epsilon = \eta = \delta \) and \( \theta = 2\epsilon \) (with appropriate deletions for \( n = 2, 3 \)). It follows that an invariant \( z \), if such exists, must be of the form given below. (Make the appropriate deletions and modifications for \( n = 2, 3 \).)
Thus, by duality, \( H_0(B_n, \wedge^4 M_n) \cong \mathbb{Z}/p\mathbb{Z} \). To complete the proof, consider the exact sequence

\[
0 \to \wedge^4 V_n \to \wedge^4 M_n \to \wedge^3 V_n \to 0
\]

where

\[
\lambda(A_1 \wedge A_2 \wedge A_3 \wedge A_4) = \sum_{i=1}^{4} (-1)^{i+1} \text{Tr}(A_i) A_1 \wedge \cdots \wedge \hat{A}_i \wedge \cdots \wedge A_4.
\]

(Note as in the proof of Corollary 8.2 that \( \lambda \) initially takes its values in \( \wedge^3 M_n \), but it is possible to show that \( \text{Im} \lambda = \wedge^3 V_n \) since \( H_1(B_n, \wedge^3 V_n) = 0, H_0(B_n, \wedge^3 V_n) = \mathbb{Z}/p\mathbb{Z} \), it follows that \( H_0(B_n, \wedge^4 M_n) = \mathbb{Z}/p\mathbb{Z} \) and \( H_0(B_n, \wedge^4 V_n) = 0 \).)

**Chapter III. Calculations of \( d^2 \) for \( n = 2 \)**

**9. The relevant \( p \)-groups.** We now turn to the needed calculations in the spectral sequence for \( n = 2 \). We start by considering the spectral sequence of the group extension \( 1 = V \to U \to P \to 1 \) where (a) \( U \) is the \( p \)-Sylow subgroup of \( \text{SL}(2, p^2) \) consisting of all matrices over \( \mathbb{Z}/p^2\mathbb{Z} \) of the form

\[
\begin{bmatrix}
1 + pa & b \\
p & 1 + pd
\end{bmatrix}, \quad a + d - bc \equiv 0 \mod p,
\]

(b) \( P = U_2 \) consists of all matrices over \( \mathbb{Z}/p\mathbb{Z} \) of the form \([1, \ast] \), and (c) \( V = V_2 \) as before.

Using multiplicative notation, we choose the following generators:

For \( P \),

\[
y = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.
\]

(Let \( \bar{y} = [1, -1] \) in \( \bar{U} \).)

For \( V \),

\[
z_1 = \begin{bmatrix} 1 & 0 \\ p & 1 \end{bmatrix} = 1 + pE_{21} \quad \text{where} \quad \bar{E}_{21} = e_{21},
\]

\[
z_2 = \begin{bmatrix} 1 + p & 0 \\ 0 & 1 - p \end{bmatrix} = 1 + pH_1 \quad \text{where} \quad \bar{H}_1 = h_1,
\]

\[
z_3 = \begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix} = 1 + pE_{12} \quad \text{where} \quad \bar{E}_{12} = e_{12}.
\]

Then,

\[
y^{-1}z_1\bar{y} = z_1z_2z_3^{-1}, \quad \bar{y}^{-1}z_2\bar{y} = z_2z_3^{-2}, \quad \bar{y}^{-1}z_3\bar{y} = z_3.
\]

The calculations will be simpler if we change the basis for \( V \) to \( x_1 = z_1, x_2 = z_2z_3^{-1}, x_3 = z_3^{-2} \). Then

\[
y^{-1}x_1\bar{y} = x_1x_2, \quad \bar{y}^{-1}x_2\bar{y} = x_2x_3, \quad \bar{y}^{-1}x_3\bar{y} = x_3.
\]

(Additively, \( y^{-1} \) has matrix

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
on $V$.) Since the Charlap-Vasquez theory is for cohomology, we shall make our spectral sequence calculations for cohomology with coefficients in $F = \mathbb{Z}/p\mathbb{Z}$ and dualize later. Thus we shall need the dual bases. Define $\alpha_i(x_j) = \delta_{ij}$. Then,

\begin{equation}
\gamma \alpha_1 = \alpha_1, \quad \gamma \alpha_2 = \alpha_1 + \alpha_2, \quad \gamma \alpha_3 = \alpha_2 + \alpha_3.
\end{equation}

**10. The Charlap-Vasquez description of $d_2$.** Let $1 \to V \to U \to P \to 1$ be a group extension with $V$ a finitely generated abelian group. Charlap and Vasquez [C-V] have described

\[ d^+_2: E^+_2(A) \to E^+_{2+t-1}(A) \]

for $A$ any $\mathbb{F}(U)$-module on which $V$ acts trivially. ($\mathbb{F}$ should be a PID and each $H_t(V, F)$ should be $\mathbb{F}$-free.) We shall use their description in the case $F$ is an elementary abelian $p$-group, $F = \mathbb{Z}/p\mathbb{Z}$, and $A = F$.

In order to state the Charlap-Vasquez result, we need to introduce some terminology involving certain standard identifications. We have as usual $H^t(V, F) = H_t(V, F)$.

(In particular, for $\alpha \in H^t(V, F)$, $\mu \in H_t(V, F)$, $\alpha(\mu)$ is calculated by evaluating a cochain representing $\alpha$ on a chain representing $\mu$.) Moreover, $H^{-1}(V, H_t(V, F)) = H^{-1}(V, F) \otimes H_t(V, F)$, so we have a pairing $q: H^t(V, F) \otimes H^{-1}(V, H_t(V, F)) \to H^{-1}(V, F)$ defined by

\begin{equation}
q(\alpha \otimes (\beta \otimes \mu)) = \alpha(\mu)\beta.
\end{equation}

$q$ in turn induces a pairing

\[ q: H^t(P, H^t(V, F)) \otimes H^2(P, H^{-1}(V, H_t(V, F))) \to H^{t+2}(P, H^{-1}(V, F)), \]

that is, $q: E^+_2(A) \otimes E^+_{2+t-1}(H_t(V, F)) \to E^+_{2+t-1}(F)$. Then Charlap and Vasquez assert that for $x \in F^+(V, V)$, $\nu \in H^{t-1}(V, H_t(V, F))$, and $F' \in H^{t+2}(P, H^{-1}(V, H_t(V, F)))$ is a certain universal characteristic class.

Charlap and Vasquez (and later Sah [S]) have investigated $V^i$ for $F = \mathbb{Z}$ and $V$ free abelian, but little is known about it in our case. We shall rely simply on the fact that for the corresponding split extension ($x = 0$)

\[ d^+_2(\xi) = (-1)^x \xi \cup q(V^i - Q_*(x)) \]

where $\chi \in H^2(P, V)$ is the class of the extension, $Q: V \to H^{-1}(V, H_t(V, F))$ is derived from the Pontryagin product $\mu \cap \nu$ in $H_*(V, F)$, and $V^i \in H^2(P, H^{-1}(V, H_t(V, F)))$ is a certain universal characteristic class.

Charlap and Vasquez (and later Sah [S]) have investigated $V^i$ for $F = \mathbb{Z}$ and $V$ free abelian, but little is known about it in our case. We shall rely simply on the fact that for the corresponding split extension ($x = 0$)

\[ d^+_2(\xi) = (-1)^x \xi \cup q(V^i) \]

Thus we obtain the first term in (10.1) if we can calculate $d^+_2(\xi)$ by some other method. This is done in §11.

We concentrate now on calculating the second term $(-1)^x \xi \cup q Q_*(x)$. Let $\mu_1, \mu_2, \ldots, \mu_n$ be a basis for $H_t^{-1}(V, F)$, $\gamma_1, \gamma_2, \ldots, \gamma_n$ the dual basis for $H^{-1}(V, F)$, and $\xi_1, \xi_2, \ldots, \xi_m$ a basis for $H_t(V, F)$. For $z \in H_t(V, F) = V$, let $\mu_i \cap z = \sum_{i=1}^n a_{iz} \xi_i$. License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Then one may translate the definition of \( Q \) in Charlap-Vasquez (called \( P \) there) into the formula:

\[
Q(z) = \sum_{i, l} a_{i,l} \gamma_i \otimes \zeta_l \in H^{i-1}(V, F) \otimes H_i(V, F).
\]

We shall be especially interested in the case \( V = V_2 \) (as in \( \S 9 \)), \( t = 1, 2, 3 \). In order to calculate \( Q_\ast \) explicitly, we need bases and dual bases in the relevant dimensions. We start with a description of the Pontryagin product in terms of standard chains. (See the table of notations for definitions of these terms.)

**Proposition 10.3.** Let \( X \) be the standard chain complex over \( \mathbb{Z} \) for the abelian group \( V \). The Pontryagin product is induced by a map (of complexes over \( \mathbb{Z} \)) \( \partial : X \otimes X \to X \) which is consistent with the group homomorphism \( V \times V \to V \) defined by multiplication. (\( X \otimes X \) is viewed as a \( V \times V \)-complex by componentwise action.) That map necessarily satisfies the rule \( d(x \cap y) = dx \cap y + (-1)^{\deg x} x \cap dy \). One may choose this map of complexes so that in degrees 0, 1, 2, and 3 it is given by

\[
\begin{align*}
\{\cdot\} \cap \{\cdot\} &= \{\cdot\}, \\
\{u\} \cap \{\cdot\} &= \{\cdot\} \cap \{u\} = \{u\}, \\
\{u, v\} \cap \{\cdot\} &= \{\cdot\} \cap \{u, v\} = \{u, v\}, \\
\{u\} \cap \{v\} &= \{u, v\} - \{v, u\}, \\
\{u, v, w\} \cap \{\cdot\} &= \{\cdot\} \cap \{u, v, w\} = \{u, v, w\}, \\
\{u\} \cap \{v, w\} &= \{u, v, w\} - \{v, u, w\} + \{v, w, u\}, \\
\{u, v\} \cap \{w\} &= \{u, v, w\} - \{u, w, v\} + \{w, u, v\}.
\end{align*}
\]

**Proof.** The Pontryagin product may be defined as the composite map \( H_\ast(V, F) \otimes H_\ast(V, F) \to H_\ast(V \times V, F) \to H_\ast(V, F) \) where the first component is the Künneth isomorphism and the second is induced by the group homomorphism \( V \times V \to V \). To compute it on the chain level, we note that \( X \otimes X \) is a \( V \times V \)-resolution of \( \mathbb{Z} \). Hence, we need only define a map of complexes \( X \otimes X \to X \) (consistent with augmentations and differentials) which is consistent with the group homomorphism \( V \times V \to V \). (The map is unique up to chain homotopy.) Since the differential in \( X \otimes X \) is given by \( d(x \otimes y) = dx \otimes y + (-1)^{\deg x} x \otimes dy \), the map is necessarily a derivation as stated. The main content of the proposition is that the chain map defined above in low degrees has the right properties. One checks this by direct calculations. For example, \( d_i([\cdot] \cap [u]) = d_i([u]) = u[\cdot] - [\cdot] \), and

\[
\begin{align*}
\bigcap (d_0 \otimes \text{id} + \text{id} \otimes d_1)([\cdot] \otimes [u]) &= \bigcap ([\cdot] \otimes d_1[u]) \\
&= \bigcap ([\cdot] \otimes (u[\cdot] - [\cdot])) = [\cdot] \cap u[\cdot] - [\cdot] \cap [\cdot] \\
&= u([\cdot] \cap [\cdot]) - [\cdot] \cap [\cdot] = u[\cdot] - [\cdot].
\end{align*}
\]

The other calculations are similarly tedious and we omit them. Note that the map of complexes has been defined on a \( \mathbb{Z}[V \times V] \)-basis for \( X \otimes X \), and semilinearity with respect to \( V \times V \to V \) has been used extensively. \( \square \)

Recall the notation \( \rho(x) \in H_2(V, F) \) for \( x \in V \), \( \delta \alpha \in H^2(V, F) \) for \( \alpha \in \hat{V} \).
Proposition 10.4. Let \( V = V_2 \), and choose a basis \( x_1, x_2, x_3 \) for \( V = H_1(V, F) \) and dual basis \( \alpha_1, \alpha_2, \alpha_3 \) for \( \tilde{V} = H^1(V, F) \). Then the following are bases and dual bases for \( H_1(V, F) \) and \( H^1(V, F) \) respectively.

For \( t = 2 \), \( x_m \cap x_n, m < n, \rho(x_m), m = 1, 2, 3 \), with dual bases \( \alpha_m \alpha_n, m < n, \delta \alpha_m, m = 1, 2, 3 \).

For \( t = 3 \), \( x_l \cap x_m \cap x_n, l < m < n, \chi_i \rho(x_m), l, m = 1, 2, 3 \) with dual bases \( \alpha_l \alpha_m \alpha_n, l < m < n, \delta \alpha_m, l, m = 1, 2, 3 \).

Proof. Use the formulas in Proposition 10.3 to evaluate the cohomology classes on the homology classes. For example, \((\alpha_l \delta \alpha_m)(x_l \cap \rho(x_m))\) is calculated as follows. Let \( \alpha_l \) denote \( \alpha_l \) viewed as a 1-cocycle on \( V \), and let \( \delta \alpha_m \) be the standard cocycle representing \( \delta \alpha_m \). Then \( x_l \cap \rho(x_m) \) is represented by

\[
\sum_{i=0}^{p-1} [x_l] \cap [x_m, ix_m] = \sum_{i=0}^{p-1} ([x_l, x_m, ix_m] + [x_m, ix_m, x_l])
\]

and

\[
(\alpha_l \delta \alpha_m)(x_l \cap \rho(x_m)) = \sum_{i=0}^{p-1} (\alpha_l(x_l) \delta a_m(x_m, ix_m) - \alpha_l(x_m) \delta a_m(x_l, ix_m) + \alpha_l(x_m) \delta a_m(ix_m, x_l)).
\]

If \( l \neq m \), there is only one nonzero term, the first term with \( i = p - 1 \). If \( l = m \), all terms for \( i = p - 1 \) are nonzero, but two cancel. In any event, the net result is 1.

The other cases are similar. \( \square \)

Proposition 10.5. Let \( 1 \to V \to \tilde{U} \to P \to 1 \) be the extension in \( \S 9 \), and let \( p > 3 \). Then choosing \( y \) as generator for \( P \), we have \( H^2(P, V) = V^p = Z/pZ \times 3 \) and the class of the extension \( \chi \) is given by \( \frac{1}{3} x_3 \).

Proof. Generally, \( H^2(P, V) = V^p / TV \) where \( T = 1 + y + y^2 + \cdots + y^{p-1} \). In our case, it is easy to see that \( TV = 0 \). To determine the class of the extension, let

\[
\bar{y} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad \text{in} \quad SL(2, p^2).
\]

Then

\[
\bar{y}^p = \begin{bmatrix} 1 & -p \\ 0 & 1 \end{bmatrix} = z_3^{-1}.
\]

Consider now the commutative diagram:

\[
\begin{array}{ccccccc}
1 & \to & GP\langle z_3^{-1} \rangle = Z/pZ & \to & GP\langle \bar{y} \rangle = Z/p^2Z & \to & GP\langle y \rangle = Z/pZ & \to & 1 \\
\downarrow \phi & & \downarrow & & \downarrow & & \parallel \\
1 & \to & V & \to & \tilde{U} & \to & P & \to & 1
\end{array}
\]

The class of the upper extension is represented by \( z_3^{-1} \); hence so is that of the lower extension.

Changing to additive notation, we may identify \( \chi \) with \(-z_3 = \frac{1}{3} x_3 \). \( \square \)
Proposition 10.6. Let $P$ and $V$ be as in §9, and let $p > 3$. Then

(i) $H^1(V, F)^p = F\alpha_1$,

(ii) $H^2(V, F)^p = F\alpha_1\alpha_2 \oplus F\delta\alpha_1$,

(iii) $H^3(V, F)^p = F\alpha_1\alpha_2\alpha_3 \oplus F\alpha_1\delta\alpha_1 \oplus F(\alpha_1\delta\alpha_2 - \alpha_2\delta\alpha_1) \oplus \delta(\alpha_1\delta\alpha_3 + \alpha_3\delta\alpha_1 - \alpha_2\delta\alpha_2 + \alpha_2\delta\alpha_1)$.

Proof. (i) is clear, (ii) follows easily since $H^2(V, F) = \wedge^2 \hat{V} + \delta\hat{V}$ (as an $F(P)$-module), and $\wedge^2 \hat{V} \cong \hat{V}$.

To prove (iii), note first that $H^3(V, F) = \wedge^3 \hat{V} \oplus \delta\hat{V}$ (as an $F(P)$-module). Moreover, $\wedge^3 \hat{V} \cong F$, and

$$\delta\hat{V} \cong \hat{V} \otimes \hat{V} = \wedge^2 \hat{V} \oplus \hat{V} \circ \hat{V}.$$ 

Again, $\wedge^2 \hat{V} = \hat{V}$ and a basic invariant is $\alpha_1 \wedge \alpha_2$. Tracing back through the identifications yields first $\alpha_1 \otimes \alpha_2 = \alpha_2 \otimes \alpha_1$ and then $\alpha_1\delta\alpha_2 = \alpha_2\delta\alpha_1$.

To calculate $(\hat{V} \circ \hat{V})^p$, let $\gamma = \sum_{i=1}^3 b_i \alpha_i \circ \alpha_i + \sum_{i<j} b_{ij} \alpha_i \circ \alpha_j$ be invariant. Then

$$y(\gamma) = b_1\alpha_1 \circ \alpha_1 + b_2\alpha_2 \circ \alpha_2 + 2b_2\alpha_1 \circ \alpha_2 + b_2\alpha_1 \circ \alpha_1 + b_3\alpha_3 \circ \alpha_3 + 2b_3\alpha_2 \circ \alpha_2 + b_2\alpha_2 \circ \alpha_2 + b_{12}\alpha_1 \circ \alpha_1 + b_{12}\alpha_1 \circ \alpha_2 + b_{23}\alpha_1 \circ \alpha_2 + b_{23}\alpha_2 \circ \alpha_2 + b_{23}\alpha_1 \circ \alpha_3 + b_{13}\alpha_1 \circ \alpha_2 + b_{13}\alpha_1 \circ \alpha_3.$$ 

Hence, $b_2 + b_{12} = b_3 + b_{23} = 2b_2 + b_{23} + b_{13} = 2b_3 = b_{23} = 0$. Thus, $b_3 = b_{23} = 0$, $b_{12} = -b_2$, $b_{13} = -2b_2$, and

$$\gamma = b_1\alpha_1 \circ \alpha_1 + b_2(\alpha_2 \circ \alpha_2 - \alpha_1 \circ \alpha_2 - 2\alpha_1 \circ \alpha_3).$$

Tracing through the identifications yields $\alpha_1 \circ \alpha_1 \mapsto 2\alpha_1\delta\alpha_1$ and $\alpha_2 \circ \alpha_2 - \alpha_1 \circ \alpha_2 - 2\alpha_1 \circ \alpha_3 \mapsto 2\alpha_2\delta\alpha_2 - \alpha_1\delta\alpha_2 - \alpha_2\delta\alpha_1 - 2\alpha_1\delta\alpha_3 - 2\alpha_3\delta\alpha_1$. Adding $\alpha_1\delta\alpha_2 - \alpha_2\delta\alpha_1$ yields

$$2(\alpha_2\delta\alpha_2 - \alpha_2\delta\alpha_1 - \alpha_1\delta\alpha_3 - \alpha_3\delta\alpha_1)$$

as required. \square

Proposition 10.7. Let $1 \to V \to \overline{U} \to P \to 1$ be the extension in 10.5, and assume $p > 3$. The second term $\Delta = (-1)^{s+1} \xi \cup_q (Q_s(\chi))$ in the Charlap-Vasquez decomposition (10.1) is given as follows ($s = 0$ in all cases):

(i) $\xi = \alpha_1$: $\Delta = 0$ in $H^2(P, F) = F$,

(ii) $\xi = \alpha_1\alpha_2$: $\Delta = 0$ in $H^2(P, \hat{V}) = \hat{V}^p$.
(iii) 
\[ \begin{align*}
\xi &= a_1a_2a_3 ; \\
\xi &= a_1 \delta a_1 ; \\
\xi &= a_1 \delta a_2 - a_2 \delta a_1 ; \\
\xi &= a_1 \delta a_3 + a_3 \delta a_1 - a_2 \delta a_2 + a_2 \delta a_1 ; \\
\Delta &= -\frac{1}{2} \delta a_1
\end{align*} \]

in \( H^2(P, H^2(V, F)) = (\wedge^2 \hat{\nu})^P \oplus (\delta \hat{\nu})^P \). \square

Proof. Recall first that given a \( P \)-pairing \( q : A \otimes B \to C \), if \( \alpha \in H^3(P, A) \) is represented by \( a \in A \) and \( \beta \in H^3(P, B) \) is represented by \( b \in B \), then \( \alpha \cup_q \beta \in H^{3+i}(P, C) \) is represented by \( q(a \otimes b) \) provided \( s \) or \( t \) is even. (See [C-E, Chapter XII, §7].)

Abbreviate \( z = \frac{1}{2}x_3 \), and use the bases and dual bases given in Proposition 10.4

(i) 
\[ 1 \cap z = \frac{1}{2}x_3, \text{ so } Q(z) = \frac{1}{2} 1 \otimes x_3 \left( 1 \in H^0(V, F) \right) \text{ (use (10.2))}. \]

Hence (from (10.0)),
\[ \alpha_1 \cup_q Q_* (x) = \frac{1}{2} q(\alpha_1 \otimes (1 \otimes x_3)) = \frac{1}{2} \alpha_1 (x_3) = 0. \]

(ii) \( x_1 \cap z = \frac{1}{2}x_1 \cap x_3, x_2 \cap z = \frac{1}{2}x_2 \cap x_3, x_3 \cap z = 0 \), so
\[ Q(z) = \frac{1}{2} \left( \alpha_1 \otimes x_1 \cap x_3 + \alpha_2 \otimes x_2 \cap x_3 \right). \]

Hence, \( \alpha_1 \alpha_2 \cup_q Q_* (x) = \frac{1}{2} \left[ q(\alpha_1 \alpha_2 \otimes (\alpha_1 \otimes x_1 \cap x_3)) + q(\alpha_1 \alpha_2 \otimes (\alpha_2 \otimes x_2 \cap x_3)) \right] = 0 \), and \( \delta \alpha_1 \cup_q Q_* (x) = 0 \).

(iii) \( x_1 \cap x_2 \cap z = \frac{1}{2}x_1 \cap x_2 \cap x_3, x_1 \cap x_3 \cap z = x_2 \cap x_3 \cap z = 0 \),
\( \rho(x_i) \cap z = \frac{1}{2}x_3 \cap \rho(x_i), i = 1, 2, 3, \) so
\[ Q(z) = \frac{1}{2} \left[ \alpha_1 \alpha_2 \otimes x_1 \cap x_2 \cap x_3 + \sum_{i=1}^{3} \delta \alpha_i \otimes x_3 \cap \rho(x_i) \right]. \]

Hence,
\[ \alpha_1 \alpha_2 a_3 \cup_q Q_* (x) = \frac{1}{2} q(\alpha_1 a_2 a_3) \otimes (\alpha_1 a_2 \otimes x_1 \cap x_2 \cap x_3) \]
\[ = \frac{1}{2} \alpha_1 \alpha_2 a_3 (x_1 \cap x_2 \cap x_3) \alpha_1 a_2 = \frac{1}{2} \alpha_1 a_2, \]
\[ \alpha_1 \delta a_1 \cup_q Q_* (x) = 0, \quad (\alpha_1 \delta a_2 - a_2 \delta a_1) \cup_q Q_* (x) = 0, \]
\[ (\alpha_1 \delta a_3 + a_3 \delta a_1 - a_2 \delta a_2 + a_2 \delta a_1) \cup_q Q_* (x) \]
\[ = \frac{1}{2} \left[ (\alpha_3 \delta a_1) (x_3 \cap \rho(x_1)) \delta a_1 \right] = \frac{1}{2} \delta a_1. \quad \square \]

11. \( d_2 \) for the split extension. Let \( \hat{U} = P \times V \). Write \( V \) additively, and denote the action of \( P \) on \( V \) by \( y(v) \). We shall calculate \( d_2^{3p}(\xi) \) by making use of explicit resolutions. Let \( X = \bigoplus \mathbb{Z}[V][v_1, v_2, \ldots, v_n] \) be the standard \( \mathbb{Z}[V] \)-free resolution of \( \mathbb{Z} \). Recall that
\[ d_V [v_1, v_2, \ldots, v_n] = v_1 [v_2, \ldots, v_n] + \sum_{i=1}^{n-1} (-1)^i [v_1, \ldots, v_i + v_{i+1}, \ldots, v_n] \]
\[ = \bigoplus \mathbb{Z}[V][v_1, v_2, \ldots, v_n] \text{ for } \mathbb{Z}[V] \text{ free over } \mathbb{Z}. \]

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and \( \epsilon_v(\cdot) = 1 \). Note also that \( P \) acts on \( X \) by \( y(v[v_1, v_2, \ldots, v_n]) = y(v)[y(v_1), y(v_2), \ldots, y(v_n)] \). Also, \( d_v y(x) = y(d_v x), \epsilon(y(x)) = \epsilon(x) \), and \( y(v)x = y(v)y(x) \). Let \( Y \) be the usual minimal \( \mathbb{Z}[P] \)-free resolution of \( \mathbb{Z} \). In particular, \( y_n = \mathbb{Z}[P]y_n, d_py_2n = Ty_{2n-1}, d_py_{2n-1} = (y-1)y_{2n-2} \) \((n > 0)\), and \( \epsilon_py_0 = 1 \). \((T = 1 + y + y^2 + \cdots + y^{p-1})\)

Make \( Y \otimes X \) into a \( \mathbb{Z}[P \times V] \)-resolution of \( \mathbb{Z} \) as usual. In particular, define \( y(y_s \otimes x_i) = y(y_s) \otimes y(x_i), \nu(y_s \otimes x_i) = y_s \otimes \nu x_i \).

**Proposition 11.1.** \( Y \otimes X \) is a \( \mathbb{Z}[\hat{U}] \)-free resolution of \( \mathbb{Z} \).

**Proof.** We have an isomorphism of \( \mathbb{Z}[\hat{U}] \)-modules \( \mathbb{Z}[\hat{U}] \otimes V X_i \cong \mathbb{Z}[P] \otimes X_i \cong Y_s \otimes X_i \) defined by \( vy \otimes x \mapsto y \otimes (v(y(x))) \). The left-hand side is \( \mathbb{Z}[\hat{U}] \)-free since \( X_i \) is \( \mathbb{Z}[V] \)-free. \( \Box \)

It follows that \( H^*(\hat{U}, \mathbb{F}) \) is the homology of the complex \( C = \bigoplus_{s} \mathbb{Z}[\hat{U}](Y \otimes X, \mathbb{F}) = \mathbb{Hom}_{\mathbb{Z}[\hat{U}]}(Y, \mathbb{Hom}_{\mathbb{Z}[\hat{U}]}(X, \mathbb{F})) \).

\( C \) is in fact a double complex with differentials defined as follows: Since \( Y_s = \mathbb{Z}[P]y_s \), we may identify \( C^{s,t} = \mathbb{Hom}_{\mathbb{F}}(\mathbb{Z}[P], \mathbb{Hom}_{\mathbb{F}}(X_t, \mathbb{F})) = \mathbb{Hom}_{\mathbb{F}}(X_t, \mathbb{F}) \). Then \( d_1^{s,t} = (y-1)d^t \), and

\[
d_{11} f = \begin{cases} (y-1)f, & s \text{ even}, s > 0, \\ f, & s \text{ odd}. \end{cases}
\]

As usual, \( d_1d_{11} + d_{11}d_1 = 0 \).

The double complex yields a spectral sequence with \( E_2^{s,t} = H^s(P, H^t(V, \mathbb{F})) = H_{11}^s(H_1^t(C)) \) and limit \( H_*^{\text{total}}(C) = H^*(\hat{U}, \mathbb{F}) \).

**Proposition 11.2.** The spectral sequence is the Lyndon-Hochschild-Serre spectral sequence of the split group extension \( 1 \to V \to \hat{U} \to P \to 1 \).

**Proof.** The Lyndon-Hochschild-Serre spectral sequence may be realized as the spectral sequence of the double complex (as above) \( \mathbb{Hom}_{\mathbb{Z}[\hat{U}]}(Y, \mathbb{Hom}_{\mathbb{Z}[V]}(W, \mathbb{F})) \) where \( W \) is any \( \mathbb{Z}(\hat{U}) \)-projective resolution of \( \mathbb{Z} \) (see Evens [E]). Since \( X \) is an acyclic \( \mathbb{Z}(\hat{U}) \)-complex over \( \mathbb{Z} \), there is a map of \( \mathbb{Z}(\hat{U}) \)-complexes over \( \mathbb{Z} \), \( G: W \to X \) (and \( G \) is unique up to \( \mathbb{Z}(\hat{U}) \)-chain homotopy). Since such a \( G \) is also a \( \mathbb{Z}(V) \)-map, and since \( X \) and \( W \) are projective \( \mathbb{Z}(V) \)-resolutions of \( \mathbb{Z} \), \( G \) is an isomorphism up to \( \mathbb{Z}(V) \)-chain homotopy. \( G \) induces a morphism of double complexes

\[
\mathbb{Hom}_{\mathbb{Z}[P]}(Y, \mathbb{Hom}_{\mathbb{Z}[V]}(X, \mathbb{F})) \to \mathbb{Hom}_{\mathbb{Z}[P]}(Y, \mathbb{Hom}_{\mathbb{Z}[V]}(W, \mathbb{F}))
\]

which by the above remark yields an isomorphism (the identity)

\[
\mathbb{Hom}_{\mathbb{Z}[P]}(Y, H^*(V, \mathbb{F})) \to \mathbb{Hom}_{\mathbb{Z}[P]}(Y, H^*(V, \mathbb{F}))
\]

on the \( E_1 \)-level (and hence for \( E_r, r \geq 1 \)).

Arguing for \( W \) exactly as for \( X \), we see \( Y \otimes W \) is also a \( \mathbb{Z}(\hat{U}) \)-projective resolution of \( \mathbb{Z} \) and \( id \otimes G: Y \otimes W \to Y \otimes X \) induces the identity \( H^*(\hat{U}, \mathbb{F}) \to H^*(\hat{U}, \mathbb{F}) \) for cohomology of the two corresponding total complexes \( \mathbb{Hom}_{\mathbb{Z}[P]}(Y, \mathbb{Hom}_{\mathbb{Z}[V]}(X, \mathbb{F})) \) and \( \mathbb{Hom}_{\mathbb{Z}[P]}(Y, \mathbb{Hom}_{\mathbb{Z}[V]}(W, \mathbb{F})) \).
To calculate \( d_2^{0.1} \alpha \) for \( \alpha \in H^t(V, \mathbb{F}) \), proceed as follows. Choose a cocycle \( a \in \text{Hom}_\nu(X, \mathbb{F}) \) representing \( \alpha \). Since \( y a = \alpha \), we have \( (y - 1)a = d \nu b \) where \( b \in \text{Hom}_\nu(X_{t-1}, \mathbb{F}) \). Then

\[
d(a + b) = (d + d_\nu)(a + b) = d \nu a + (y - 1)a + (-1)d \nu b + Tb = 0 + (y - 1)a - d \nu b + Tb = Tb.
\]

Hence,

\[\text{(11.3)} \quad a + b \text{ represents } \alpha \in E_2^{0.1} \text{ and } d(a + b) = Tb \text{ represents } d_2 \alpha \in E_2^{2.1-1}.\]

We are now ready to calculate \( d_2^{0.1} \) for \( t = 1, 2, 3 \). First, \( d_2^{2.1} = 0 \) since the splitting implies \( E_2^{2.0} = H^* (P, \mathbb{F}) \) is a direct summand of \( H^*(\bar{U}, \mathbb{F}) \).

**Proposition 11.4.** For \( p > 3 \),

\[
d_2^{0.2} (\alpha_1 \alpha_2) = 0, \quad d_2^{0.2} \delta \alpha_1 = 0.
\]

**Proof.** Let \( a_i \) be identical with \( \alpha \), but viewed as a 1-cocycle on \( V \). Then \( a_1 a_2 \) represents \( \alpha_1 \alpha_2 \), and \( y a_1 = a_1, y a_2 = a_1 + a_2, y a_3 = a_2 + a_3 \), so \( y(a_1 a_2) = a_1^2 + a_1 a_2 \). Hence, \( (y - 1)(a_1 a_2) = a_1^2 \). Moreover, \( a_1^2 = d \nu u_1 \) where \( u_1(v) = -\frac{1}{2} i_1^2 i_1 \) for \( v = i_1 x_1 + i_2 x_2 + i_3 x_3 \) in \( V \). Hence, \( d_2 (\alpha_1 \alpha_2) \) is represented by \( Tu_1 \). However, by 9.2, \( y^{-1}(v) = i_1 x_1 + (i_1 + i_2) x_2 + (i_2 + i_3) x_3 \), so \( yu_1(v) = -\frac{1}{2} i_1^2 \) and \( yu_1 = u_1 \). Hence, \( Tu_1 = pu_1 = 0 \).

To show \( d_2 \delta \alpha_1 = 0 \), we use \( d_2 \alpha_1 = 0 \) and the following result.

**Lemma 11.5.** The diagram below anticommutes.

\[
\begin{array}{ccc}
H^t(P, H^t(V, \mathbb{F})) & \xrightarrow{H^t(P, \delta)} & H^2(P, H^t+1(V, \mathbb{F})) \\
\downarrow d_2^{2.1-t} & & \downarrow d_2^{t+1-1} \\
H^{t+2}(P, H^t-1(V, \mathbb{F})) & \xrightarrow{H^{t+2}(P, \delta)} & H^{t+2}(P, H^t(V, \mathbb{F}))
\end{array}
\]

(By abuse of notation \( d_2^2 + \delta d_2 = 0 \).)

**Proof.** We prove a somewhat more general result. Let \( A \) be a \( \mathbb{Z} \)-free double complex and put \( \bar{A} = A/pA \). (In particular, consider \( A = \text{Hom}_p(Y, \text{Hom}_p(X, \mathbb{Z})) \).) Let \( \alpha \in H^t_{\bar{A}}(H_t^1(\bar{A})) \) be represented by \( \tilde{a} + \tilde{b} \) where \( \tilde{a} \in \bar{A}^{t-1}, \tilde{b} \in \bar{A}^{t+1-1}, d_\bar{t} \tilde{a} = 0, d_\bar{t} \tilde{a} + d \nu \tilde{b} = 0 \). Then \( d_2 \alpha \) is represented by \( d(\tilde{a} + \tilde{b}) = d_\bar{t} \tilde{b} \). Let \( a \in A^{t-1}, b \in A^{t+1-1} \) represent \( \tilde{a} \) and \( \tilde{b} \) respectively. Then \( d_1 a = p g \) for \( g \in A^{t-1+t}, \) and \( d_1 a + d_2 b - ph = 0 \) for \( h \in A^{3t-1} \). Therefore, \( d_1 a + d_2 b - ph = 0 \) for \( h \in A^{3t-1} \). Clearly, \( \tilde{g} + \tilde{h} \) represents \( \delta \alpha \in H^t_{\bar{A}}(H_t^1(\bar{A})) \), and \( d_2 \tilde{h} \) represents \( d_2 \alpha \). However, from \( d_2 \tilde{a} + d_1 b - ph = 0 \), we also obtain \( d_2 \tilde{a} + d_1 b = \tilde{a} + \tilde{b} \) or \( -d_1 d_2 \tilde{b} = d_2 \tilde{h} \). Since \( d_2 \tilde{a} \in A^{t+2-1} \) represents \( d_2 \tilde{a} \), it follows that \( -d_1 d_2 \tilde{b} \) represents \( d_2 \alpha \) as required. □

**Proposition 11.6.** For \( p > 3 \),

\[
d_2^{0.3} (\alpha_1 \delta \alpha_1) = 0, \quad d_2^{0.3} (\alpha_1 \delta \alpha_2 - \alpha_2 \delta \alpha_1) = 0.
\]
PROOF. The first result follows since $E_2$ is a differential ring and $d_2\alpha_1 = d_2\delta\alpha_1 = 0$. The second follows by Lemma 11.5 since $\delta(\alpha_1\alpha_2) = (\delta\alpha_1)\alpha_2 - \alpha_1(\delta\alpha_2)$ and $d_2(\alpha_1\alpha_2) = 0$. □

**Proposition 11.7.** For $p > 3$,

$$d_2^{0,3}(\alpha_1\alpha_2\alpha_3) = 0.$$ 

**Proof.**

$$(y - 1)(a_1a_2a_3) = a_1(a_1 + a_2)(a_2 + a_3) - a_1a_2a_3$$

$$= a_1^2a_2 + a_1a_3^2 + a_1a_2^2.$$ 

However, $a_k^2 = d\nu u_k$ where $u_k(v) = -\frac{1}{2}i_k^2$ for $v = i_1x_1 + i_2x_2 + i_3x_3$. Hence,

$$(y - 1)(a_1a_2a_3) = (d\nu u_1)a_2 + (d\nu u_1)a_3 + a_1d\nu u_2$$

$$= d\nu(u_1a_2 + u_1a_3 - a_1u_2).$$ 

Call the expression in parentheses $c$. Since $yu_1 = u_1$, $ya_1 = a_1$, we have $Tc = u_1Ta_2 + u_1Ta_3 - a_1Tu_2$. Because $T\hat{V} = 0$, $Ta_2 = Ta_3 = 0$, and $Tc = -a_1Tu_2$. From (9.2), we see that $(yu_2)(v) = -\frac{1}{2}(i_1 + i_2)^2$. Hence,

$$(Tu_2)(v) = -\frac{1}{2}\sum_{j=0}^{p-1} (ji_1 + i_2)^2$$

$$= -\frac{1}{2}\left[\left(\sum_{j=0}^{p-1} j^2\right)i_1 + 2\left(\sum_{j=0}^{p-1} j\right)i_1i_2 + pi_2^2\right] = 0 \text{ for } p > 3.$$ 

Thus $Tc = 0$ as required. □

**Proposition 11.8.** For $p > 3$,

$$d_2^{0,3}(\alpha_1\delta\alpha_3 + \alpha_3\delta\alpha_1 - \alpha_2\delta\alpha_2 + \alpha_2\delta\alpha_1) = 2\alpha_1\alpha_2.$$ 

**Proof.** Recall that $\delta\alpha_k$ is represented by $\delta a_k$ defined by

$$\delta a_k(v, w) = \begin{cases} 0, & 0 \leq i_k + j_k < p, \\
1, & p \leq i_k + j_k < 2p,
\end{cases}$$

where $v = i_1x_1 + i_2x_2 + i_3x_3$, $w = j_1x_1 + j_2x_2 + j_3x_3$, $0 \leq i_k, j_k < p$.

We introduce some notation from computer programming. If $k = pq + r$, $0 \leq r < p$, write $k/p = q$ and $\text{mod}(k) = r$. Then, $\delta a_k(v, w) = (i_k + j_k)/p. (k/p \text{ and } \text{mod}(k) \text{ are integers, but we shall abuse notation by viewing them in } \mathbb{Z}/p\mathbb{Z} = F. \text{ The context will make clear which is meant.})$

**Lemma 11.9.** For $k = 2, 3$,

$$y\delta a_k = \delta a_{k-1} + \delta a_k + d\nu f_k$$

where $f_k(v) = -(i_{k-1} + i_k)/p$. Also, $y\delta a_1 = \delta a_1$. 

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Proof. We do the case $k = 2$. From (9.2), $y^{-1}(v) = i_1x_1 + (i_1 + i_2)x_2 + (i_2 + i_3)x_3$. Hence, $(y\delta a_2)(v, w) = \delta a_2(v', w')$ where

$$v' = i_1x_1 + \text{mod}(i_1 + i_2)x_2 + \text{mod}(i_2 + i_3)x_3,$$
$$w' = j_1x_1 + \text{mod}(j_1 + j_2)x_2 + \text{mod}(j_2 + j_3)x_3.$$ 

Hence, $(y\delta a_2)(v, w) = (\text{mod}(i_1 + i_2) + \text{mod}(j_1 + j_2))/p$. Hence,

$$(y\delta a_2 - \delta a_2)(v, w) = (*)$$

where

$$(*) = (\text{mod}(i_1 + i_2) + \text{mod}(j_1 + j_2))/p - (i_1 + j_1)/p - (i_2 + j_2)/p.$$ 

On the other hand, $d\nu f_2(v, w) = (**) where

$$(**) = - (j_1 + j_2)/p + (\text{mod}(i_1 + j_1) + \text{mod}(i_2 + j_2))/p - (i_1 + i_2)/p.$$ 

However,

$$i_1 + j_1 = p((i_1 + j_1)/p) + \text{mod}(i_1 + j_1),$$
$$i_2 + j_2 = p((i_2 + j_2)/p) + \text{mod}(i_2 + j_2),$$ 

so,

$$i_1 + j_1 + i_2 + j_2 = p[(i_1 + j_1)/p + (i_2 + j_2)/p]$$
$$+ \text{mod}(i_1 + j_1) + \text{mod}(i_2 + j_2),$$ 

and

$$(i_1 + j_1)/p + (i_2 + j_2)/p = (i_1 + j_1 + i_2 + j_2)/p$$
$$- [\text{mod}(i_1 + j_1) + \text{mod}(i_2 + j_2)]/p.$$ 

Hence,

$$(*) = [\text{mod}(i_1 + i_2) + \text{mod}(j_1 + j_2)]/p + [\text{mod}(i_1 + j_1) + \text{mod}(i_2 + j_2)]/p$$
$$- (i_1 + j_1 + i_2 + j_2)/p.$$ 

Similarly,

$$(i_1 + i_2)/p + (j_1 + j_2)/p = (i_1 + i_2 + j_1 + j_2)/p$$
$$- (\text{mod}(i_1 + i_2) + \text{mod}(j_1 + j_2))/p,$$ 

so

$$(***) = (\text{mod}(i_1 + i_2) + \text{mod}(j_1 + j_2))/p$$
$$+ (\text{mod}(i_1 + i_2) + \text{mod}(j_1 + j_2))/p - (i_1 + i_2 + j_1 + j_2)/p.$$ 

Hence, $(***) = (*)$ as required.  \qed

To continue the proof of the proposition, calculate as follows using (11.3).

$$(y - 1)(a_1\delta a_3 + a_3\delta a_1 - a_2\delta a_2 + a_2\delta a_1)$$
$$= a_1(\delta a_2 + \delta a_3 + d\nu f_1) + (a_2 + a_3)\delta a_1 - (a_1 + a_2)(\delta a_1 + \delta a_2 + d\nu f_2)$$
$$+ (a_1 + a_2)\delta a_1 - a_1\delta a_3 - a_3\delta a_1 + a_2\delta a_2 - a_2\delta a_1$$
$$= a_1d\nu f_2 - (a_1 + a_2)d\nu f_2$$
$$= d\nu (-a_1f_3 + (a_1 + a_2)f_3).$$
We need to calculate $Tc$ where $c = (a_1 + a_2)f_2 - a_1 f_3$. We have

$$Tc = T(a_1 f_2) + T(a_2 f_2) - T(a_1 f_3) = a_1 T f_2 + T(a_2 f_2) - a_1 T f_3.$$ 

However,

$$T(a_2 f_2) = \sum_{k=0}^{p-1} (y^k a_2)(y^k f_2) = \sum_{k=0}^{p-1} (k a_1 + a_2)(y^k f_2) = a_1 \left( \sum_{k=0}^{p-1} k y^k f_2 \right) + a_2 (T f_2).$$

Thus, $Tc = (a_1 + a_2)T f_2 + a_1 (T' f_2 - T f_3)$ where $T' f_2 = \sum_{k=0}^{p-1} k y^k f_2$. We shall show that $T f_2 = -a_1$ and $T' f_2 - T f_3 = \frac{1}{2} a_1 + a_2$. From this it follows that

$$Tc = - (a_1 + a_2) a_1 + a_1 (\frac{1}{2} a_1 + a_2) = a_1 a_2 - a_2 a_1 - \frac{1}{2} d_f u_1.$$

Thus, $Tc$ represents $a_1 a_2 - a_2 a_1 = 2a_1 a_2$ as required.

Thus, we need only establish

**Lemma 11.10.** $T f_2 = -a_1$, $T' f_2 - T f_3 = \frac{1}{2} a_1 + a_2$.

**Proof.** Define $i_2^{(k)}$, $k = 0, 1, \ldots, p$, inductively by $i_2^{(0)} = i_2$, $i_1 + i_2^{(k)} = pq_k + i_2^{(k+1)}$, $0 \leq i_2^{(k+1)} < p$. ($i_2^{(k)}$ is the coefficient of $x_2$ in $y^{-k} v$ for $v = i_1 x_1 + i_2 x_2 + i_3 x_3$.)

Then adding yields

$$(\dagger) \quad pi_1 + \sum_{k=0}^{p-1} i_2^{(k)} = p \sum_{k=0}^{p-1} q_k + \sum_{k=1}^{p} i_2^{(k)}.$$ 

From this we conclude $i_2^{(p)} \equiv i_2^{(0)} \mod p$ so $i_2^{(p)} = i_2$. Again, from (\dagger) we get $\sum_{k=0}^{p-1} q_k = i_1$. However, $(y^k f_2)(v) = f_2(y^{-k} v) = -(i_1 + i_2^{(k)})/p = -q_k$ so $(T f_2)(v) = -\sum_{k=0}^{p-1} q_k = -i_1 = -a_1(v)$, and $T f_2 = -a_1$.

Furthermore, multiplying $i_1 + i_2^{(k)} = pq_k + i_2^{(k+1)}$ by $k$ and adding yields

$$\left( \sum_{k=0}^{p-1} k \right) i_1 + \sum_{k=0}^{p-1} k i_2^{(k)} = p \sum_{k=0}^{p-1} kq_k + \sum_{k=0}^{p-1} k i_2^{(k+1)}.$$ 

But,

$$\sum_{k=0}^{p-1} k i_2^{(k+1)} = \sum_{k=1}^{p} (k - 1) i_2^{(k)} = \sum_{k=1}^{p} k i_2^{(k)} - \sum_{k=1}^{p} i_2^{(k)} = \sum_{k=0}^{p-1} k i_2^{(k)} + pi_2 - \sum_{k=0}^{p-1} i_2^{(k)}$$

(use $i_2^{(p)} = i_2^{(0)} = i_2$). Putting this in the above equation yields

$$p \frac{p-1}{2} i_1 = p \sum_{k=0}^{p-1} kq_k + pi_2 - \sum_{k=0}^{p-1} i_2^{(k)}.$$
Hence,

\[
(T'f_2)(v) = \sum_{k=0}^{p-1} kf_2(y^{-k}v) = -\sum_{k=0}^{p-1} kq_k
\]

\[
= -\frac{p-1}{2} i_1 + i_2 - \frac{1}{p} \left( \sum_{k=0}^{p-1} i_2^{(k)} \right).
\]

To calculate \(Tf_3\), define \(i_3^{(k)}\) inductively by \(i_3^{(0)} = i_3\), \(i_2^{(k)} + i_3^{(k)} = pr_k + i_3^{(k+1)}\), \(0 \leq i_3^{(k+1)} < p\). (\(i_3^{(k)}\) is the coefficient of \(x_3\) in \(y^{-kv}\) for \(v = z'x_1 + z'x_2 + z'x_3\).) Adding yields

\[
\sum_{k=0}^{p-1} i_2^{(k)} + \sum_{k=0}^{p-1} i_3^{(k)} = p \sum_{k=0}^{p-1} r_k + \sum_{k=0}^{p-1} i_3^{(k)}.
\]

If \(i_1 = 0\), \(\sum_{k=0}^{p-1} i_2^{(k)} = pi_2\); if \(i_1 \neq 0\), \(i_2^{(0)}, i_2^{(0)}, \ldots, i_2^{(p-1)}\) is a permutation of \(0, 1, 2, \ldots, p - 1\), and \(\sum_{k=0}^{p-1} i_2^{(k)} = p(p - 1)/2\). In either case, the sum is \(\equiv 0 \mod p\). As above, it follows that \(i_3^{(p)} = i_3^{(0)} = i_3\), and

\[
(Tf_3)(v) = -\sum_{k=0}^{p-1} r_k = -\frac{1}{p} \sum_{k=0}^{p-1} i_2^{(k)}.
\]

Hence,

\[
(T'f_2)(v) - (Tf_3)(v) = -\frac{(p-1)}{2} i_1 + i_2 = \frac{1}{2} i_1 + i_2.
\]

Thus \(T'f_2 - Tf_3 = \frac{1}{2} a_1 + a_2\) as claimed. \(\square\)

12. \(d_2^0\) for \(GL(2, p)\). We may summarize the results of §§10 and 11 in a table.

**Proposition 12.1.** For \(p > 3\), \(d_2^0\) is given in Table 2 for \(t = 1, 2, 3\).

**Table 2**

<table>
<thead>
<tr>
<th>(\xi)</th>
<th>Split case</th>
<th>Nonsplit case (\bar{U})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_1)</td>
<td>0</td>
<td>-(\frac{1}{2} a_1 a_2)</td>
</tr>
<tr>
<td>(a_1 a_2)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\delta a_1)</td>
<td>0</td>
<td>(\delta a_1)</td>
</tr>
<tr>
<td>(a_1 a_2 a_3)</td>
<td>0</td>
<td>(\frac{1}{2} a_1 a_2)</td>
</tr>
<tr>
<td>(a_1 \delta a_2 - \alpha_2 \delta a_1)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(a_1 \delta a_3 + \alpha_3 \delta a_1 - \alpha_2 \delta a_2 + \alpha_2 \delta a_1)</td>
<td>2(a_1 a_2)</td>
<td>2(a_1 a_2) - (\frac{1}{2} \delta a_1)</td>
</tr>
</tbody>
</table>

We now make the transition to \(GL(2, p)\) and then to homology. We need only concern ourselves with \(t = 3\). Using additive notation, we have \(z_1 = e_{21}, z_2 = h_1,\) and \(z_3 = e_{12}\) and we denote the dual basis by \(e^{21}, h^1,\) and \(e^{12}\). Referring to §9, we see easily that \(e^{21} = a_1, h^1 = a_2, e^{12} = -a_2 - 2a_3\). We know (by Proposition 3.2) that \(H^0(GL(2, p), H^3(V, F))\) (dual to \(H_0(GL(2, p), H_3(V, F))\)) is 2 dimensional, and it is easy to see that it is the subspace of \(H^3(V, F)\) spanned by \(\chi_1 = e^{21} h^1 e^{12} = -2a_1 a_2 a_3\) and \(\chi_2 = e^{21} h^1 e^{12} + e^{12} h^1 + 2h^1 h^1 = -a_1 a_2 - 2a_1 a_3 - \alpha_2 a_1 - 2a_2 \delta a_1 + 2a_2 \delta a_2\) or

\[
\chi_2 = 2(a_1 \delta a_3 + a_3 \delta a_1 - a_2 \delta a_2 + a_2 \delta a_1) - (a_1 a_2 - a_2 \delta a_1).\]
From Table 2, we have

**Proposition 12.2.** For \( p > 3 \), \( d_2^{0,3}(\xi) \) is given by

\[
\begin{array}{c|c|c}
\xi & \text{Split case} & \text{Nonsplit case} \\
\hline
\chi_1 & 0 & \alpha_1 \alpha_2 \\
\chi_2 & -4 \alpha_1 \alpha_2 & -4 \alpha_1 \alpha_2 + \delta \alpha_1 \\
\end{array}
\]

Note. \( d_2(\xi) \in H^2(GL(2, p), H^2(V, F)) \) but we may identify this group with \( H^2(P, H^2(V, F)) = (\bigwedge^2 \tilde{V})^p \oplus \tilde{V}^p \) through restriction. For, by Proposition 5.2,

\[
H^2(GL(2, p), \tilde{V}) = H^2(GL(2, p), \tilde{V}) = \mathbb{Z}/p\mathbb{Z}
\]

so \( \text{res}: H^2(GL(2, p), \tilde{V}) \to H^2(P, \tilde{V}) = \mathbb{Z}/p\mathbb{Z} \) is an isomorphism. Since \( V \wedge V \cong V \) also, the above contention follows.

It now follows that \( d_2^{0,3}: H^0(GL(2, p), H^3(V_2, F)) \to H^2(GL(2, p), H^2(V_2, F)) \) has rank one in the split case and is an isomorphism in the nonsplit case. By duality, we have

**Proposition 12.3.** For \( p > 3 \), we have that (a) \( d_2^2: H_2(GL(2, p), H_2(V_2, \mathbb{Z}/p\mathbb{Z})) \to H_0(GL(2, p), H_3(V_2, \mathbb{Z}/p\mathbb{Z}) \) is an isomorphism for the nonsplit extension \( 1 \to V_2 \to \text{SL}(2, p^2) \to GL(2, p) \to 1 \) and

(b) \( d_2^2 \) has rank 1 for the corresponding split extension.

To deal with integral homology, we consider dually the coefficient group \( \mathbb{Q}/\mathbb{Z} \).

The exact sequence \( 0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to 0 \) yields

\[
0 \to H^2(V_2, \mathbb{Q}/\mathbb{Z}) \to H^3(V_2, \mathbb{Z}/p\mathbb{Z}) \to H^3(V_2, \mathbb{Q}/\mathbb{Z}) \to 0.
\]

Since

\[
H^r(GL(2, p), H^2(V_2, \mathbb{Q}/\mathbb{Z})) \cong H^r(GL(2, p), H_2(V_2, \mathbb{Z})) = 0
\]

for \( r = 0, 1 \) (Proposition 5.2 and \( V_2 \wedge V_2 \cong V_2 \)), we have

\[
H^0(GL(2, p), H^3(V_2, \mathbb{Z}/p\mathbb{Z})) \cong H^0(GL(2, p), H^3(V_2, \mathbb{Q}/\mathbb{Z})).
\]

Similarly, we have \( 0 \to H^1(V_2, \mathbb{Q}/\mathbb{Z}) \to H^2(V_2, \mathbb{Z}/p\mathbb{Z}) \to H^2(V_2, \mathbb{Q}/\mathbb{Z}) \to 0 \), and

\[
H^3(GL(2, p), H^1(V_2, \mathbb{Q}/\mathbb{Z})) \cong H^3(GL(2, p), V_2) = 0
\]

(Proposition 5.2). Hence,

\[
H^2(GL(2, p), H^2(V_2, \mathbb{Z}/p\mathbb{Z})) \to H^2(GL(2, p), H^2(V_2, \mathbb{Q}/\mathbb{Z}))
\]

is an epimorphism. It now follows easily that for the nonsplit extension \( d_2^{0,3}: H^0(GL(2, p), H^3(V_2, \mathbb{Q}/\mathbb{Z})) \to H^2(GL(2, p), H^2(V_2, \mathbb{Q}/\mathbb{Z})) \) is an epimorphism. To deal with the split extension, consider

\[
0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0
\]

and

\[
0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to 0.
\]
It follows that $\delta \alpha_1 \in H^2(V_2, \mathbb{Z}/p\mathbb{Z})$ is in the image of $H^1(V_2, \mathbb{Q}/\mathbb{Z})$; hence, $\delta \alpha_1 \mapsto 0$ in $H^2(V_2, \mathbb{Q}/\mathbb{Z})$. Thus, under the epimorphism

$$H^2(\text{GL}(2, p), H^2(V_2, \mathbb{Z}/p\mathbb{Z})) \to H^2(\text{GL}(2, p), H^2(V_2, \mathbb{Q}/\mathbb{Z})),$$

the image of $\alpha_1 \alpha_2$ (viewed as an element on the left through $\text{res}^{-1}$) spans $H^2(\text{GL}(2, p), H^2(V_2, \mathbb{Q}/\mathbb{Z}))$. Since $d^3 H^2 \alpha_1 \alpha_2 = -4 \alpha_1 \alpha_2$, it follows that $d^3 H^2$ is an epimorphism for $\mathbb{Q}/\mathbb{Z}$. By duality, we have

**Proposition 12.4.** For $p > 3$, we have for both (a) split and (b) nonsplit extensions,

$$d^2_{2,2} : H_2(\text{GL}(2, p), H_2(V_2, \mathbb{Z})) \to H_0(\text{GL}(2, p), H_3(V_2, \mathbb{Z}))$$

is a monomorphism (so its image has dimension 1).

**References**


Department of Mathematics, Northwestern University, Evanston, Illinois 60201