UNCOUNTABLE ADMISSIBLES I: FORCING

BY

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Abstract. Assume \( V = L \). Let \( \kappa \) be a regular cardinal and for \( X \subseteq \kappa \) let \( \alpha(X) \) denote the least ordinal \( \alpha \) such that \( L_\alpha[X] \) is admissible. In this paper we characterize those ordinals of the form \( \alpha(X) \) using forcing and fine structure of \( L \) techniques. This generalizes a theorem of Sacks which deals with the case \( \kappa = \omega \).

Forcing has proved to be an extremely valuable tool for the recursion-theorist. Generic sets are used to construct a minimal hyperdegree (Gandy and Sacks [1967]), establish the plus-one and plus-two theorems in higher types (Sacks [1974], Harrington [1973]), refute the (relativized) McLaughlin conjecture (Steel [1978]) and characterize the countable admissible ordinals (Sacks [1976]). This is because genericity provides a natural way to control the definability properties of sets.

In the countable case generic set existence is not a problem. (This suffices for most applications of forcing in set theory.) However the existence of generic sets in the uncountable case can be an obstacle. One way around it is to exploit strong closure properties of the ground model in question.

This last technique can be illustrated as follows: Suppose \( \kappa \) is regular and \( M \) is a transitive set of cardinality \( \kappa \) which is \( < \kappa \)-closed; i.e., any sequence from \( M \) of length \( < \kappa \) belongs to \( M \). Assume that \( \mathcal{P} \) is a partial ordering of a subset of \( M \) which is also \( < \kappa \)-closed; i.e., for any sequence \( p_0 \geq p_1 \geq p_2 \geq \ldots \) from \( \mathcal{P} \) of length \( < \kappa \) there is \( p \in \mathcal{P} \) s.t. \( p \leq p_\gamma \) for each \( \gamma \). Then one can easily construct (in \( \kappa \) steps) sets which are \( \mathcal{P} \)-generic over \( M \).

The main result of the present paper implies that the preceding paragraph describes the best existence theorem for generic sets in the uncountable case. We present a simple forcing problem for uncountable \( \La(X) \)'s which can be solved positively only in cases where \( \La[X] \) possesses the strong closure property stated above.

The problem in question is a generalization of Sacks' characterization of countable admissible ordinals. Assume \( V = L \) and let \( \kappa \) be an uncountable cardinal. For \( X \subseteq \kappa \) we let \( \alpha(X) \) denote the least ordinal \( \alpha > \kappa \) s.t. \( L_\alpha(X) \) is admissible.

**Question (\( V = L \)).** Which admissible ordinals are of the form \( \alpha(X) \) for some \( X \subseteq \kappa \)?

We deal in this paper with the case where \( \kappa \) is regular. (The singular case will be treated in Friedman [1981a].) The answer to this Question is best phrased in terms of a strong form of \( < \kappa \)-closure which we call \( < \kappa \)-admissibility.

Received by the editors June 20, 1980. The results were presented at the Greek Logic Symposium, University of Patras, 1980.

1980 Mathematics Subject Classification. Primary 03D60, 03E45; Secondary 03E40.

1 The preparation of this paper was supported by a National Science Foundation grant.
DEFINITION. \( L_\alpha \) is \(< \kappa\)-admissible if cofinality (\( \alpha \)) \( \geq \kappa \) and \( L_\alpha \) is both closed under and admissible relative to the function \( y \mapsto [y]^{< \kappa} \), where \( [y]^{< \kappa} = \{ x \subseteq y \mid x \text{ has cardinality} < \kappa \} \). \(^2\)

Notice that \(< \kappa\)-admissibility implies \(< \kappa\)-closure. It is natural to attempt to build \( X \) as above by forcing over \( L_\alpha \). If \( L_\alpha \) is \(< \kappa\)-admissible then in §1 we construct a \(< \kappa\)-closed partial ordering \( \mathbb{P} \) such that \( \alpha = \alpha(X) \) whenever \( X \) is \( \mathbb{P} \)-generic over \( L \). The \(< \kappa\)-closure of \( L_\alpha \) implies the existence of such \( X \) and thus the Question is answered positively in this case.

The purpose of §2 is to provide a negative answer in all other cases.

**Theorem.** \( \alpha = \alpha(X) \) for some \( X \subseteq \kappa \) iff

(i) \( \kappa < \alpha < \kappa^+ \),

(ii) \( L_\alpha \) is \(< \kappa\)-admissible.

Thus even basic forcing results extend to the uncountable only in very special cases. In §2 we will also give examples where \( \alpha < \kappa^+ \) is admissible of cofinality \( \kappa \) but \( L_\alpha \) is not \(< \kappa\)-closed and also where \( \alpha < \kappa^+ \) is admissible, \( L_\alpha \) is \(< \kappa\)-closed but \( L_\alpha \) is not \(< \kappa\)-admissible. The former example corrects an assertion made following Problem 48 in H. Friedman's list of problems in logic (H. Friedman [1975]). Our methods can be used to give a negative solution to that problem in ZF (see Example 3 in §2).

The “only if” direction of our Theorem (proved in §2) is established via a combination of techniques from \( \beta\)-recursion theory and the fine structure of \( L \). We show that if \( \alpha = \alpha(X) \) for some \( X \) then not only \( \alpha \) but also \( \Sigma_n \) projectum (\( \beta \)) must have cofinality \( \kappa \) for various ordinals \( \beta \) closely related to \( \alpha \) (for certain \( n \in \omega \)). This condition in turn implies the \(< \kappa\)-closure of \( L_\alpha \) by an induction argument. The proof of \(< \kappa\)-admissibility uses related methods.

The “if” direction is proved by a forcing technique which extends the method of Jensen [1972] to the uncountable. The proof is in two parts. First a predicate \( B \subseteq L_\alpha \) is constructed so that \( \langle L_\gamma \cup B, B \rangle \) is admissible but \( \langle L_\gamma \cup [B \cap L_\gamma], B \cap L_\gamma \rangle \) is inadmissible for each \( \gamma < \alpha \). This uses (and in fact necessitates) the \(< \kappa\)-admissibility of \( L_\alpha \). Second the predicate \( B \) is coded using almost disjoint set forcing by \( X \in \kappa \). Thus \( \alpha = \alpha(X) \).

1. **Killing admissibles.** Our goal in this section is to establish

**Theorem 1 (\( V = L \)).** If \( \alpha \) has cardinality \( \omega_1 \), \( L_\alpha \) is \( \omega\)-admissible, then \( \alpha = \alpha(X) \) for some \( X \subseteq \omega_1 \).

Thus we consider only the case \( \kappa = \omega_1 \). We write “\( \omega\)-admissible” for “\(< \omega_1\)-admissible”. There are obvious modifications of what is described below when \( \kappa \) is any other regular \( L \)-cardinal.

Our proof is an extension to the uncountable of a forcing method developed by Jensen (see Jensen [1972]). He used a combination of unbounded Lévy forcing, a

\(^2\)Lemma 4 below shows that if \( \alpha \) is admissible then the words “and admissible relative to” can be deleted in this definition.
special forcing for destroying recursive Mahloness and almost disjoint set forcing to prove a strengthening of Sacks’ theorem that any countable admissible $\alpha > \omega$ is the first admissible relative to some real $R \subseteq \omega$. There are several new problems which arise when attempting to adapt these techniques to the present context. To demonstrate that admissibility is preserved by unbounded Lévy forcing one must use the $\omega$-admissibility of $L_{\alpha}$. A Skolem hull argument is needed then to construct a predicate $B \subseteq L_{\alpha}$ such that $\langle L_{\alpha}[B], B \rangle$ is admissible but for each $\gamma < \alpha \langle L_{\alpha}[B \cap L_{\gamma}], B \cap L_{\gamma} \rangle$ is inadmissible. Finally we make use of a technical lemma from Jensen [1975] to perform almost disjoint set coding in an admissibility-preserving way.

An $\omega_1$-closed unbounded set of inadmissibles. If $\alpha$ is countable and admissible then a closed unbounded $C \subseteq \alpha$ is constructed in Jensen [1972] such that

(a) $L_{\alpha}[C] = L_{\alpha}$,
(b) $\langle L_{\alpha}, C \rangle$ is admissible, and
(c) $\gamma \in C \rightarrow \gamma$ is inadmissible.

Now suppose $\alpha$ is as in the hypothesis of Theorem 1. Then we construct a closed unbounded $C \subseteq \alpha$ such that

(a) $L_{\alpha}[C] = L_{\alpha}$,
(b) $\langle L_{\alpha}, C \rangle$ is admissible, and
(c) $\gamma \in C$, cofinality $(\gamma) > \omega \rightarrow \gamma$ is inadmissible.

The extra hypothesis in (c) is necessary: For, there may be a closed unbounded $D \subseteq \alpha$ consisting only of admissibles. But then $C \cap D \neq \emptyset$ since $C, D$ are closed unbounded subsets of an ordinal of uncountable cofinality. We arrange that $C \cap D$ consists only of ordinals of cofinality $\omega$.

A condition $p$ is a closed subset of $\alpha$ such that

(i) $p$ has a greatest element,
(ii) $\gamma \in p$, cofinality $(\gamma) > \omega \rightarrow \gamma$ inadmissible, and
(iii) $p \in L_{\alpha}$.

If $p, q$ are conditions then $p$ is stronger than $q$, $p \leq q$, if $p \cap (\text{max}(q) + 1) = q$. Let $\mathcal{P}_\beta$ denote the collection of all conditions. $\mathcal{P}_\beta$ is countably closed. If $\phi(G)$ is a ranked sentence of $L_{\alpha}$, rank($\phi$) = $\beta$, then we define

$$p \models \phi \iff \beta < \text{max}(p) \quad \text{and} \quad L^p \models \phi(p).$$

So forcing for ranked sentences is a $\Delta_1$ relation. Forcing for unranked sentences is defined in the usual way. Thus forcing for $\Sigma_1$ sentences is $\Delta_1$ over $L_{\alpha}$. The existence of sets $\mathcal{P}_\beta$-generic for sentences of $L_{\alpha}$ is guaranteed by the countable closure of $L_{\alpha}$, the countable closure of $\mathcal{P}_\beta$ and the next lemma.

**Lemma 2.** If $p$ is a condition and $\phi$ is ranked then $\exists q \leq p \ (q \models \phi \lor q \models \neg \phi)$.

**Proof.** Choose $\beta < \alpha$ to be inadmissible and greater than rank($\phi$), max($p$). Then $p \cup \{\beta\} \models \phi$ or $p \cup \{\beta\} \models \neg \phi$. Also $p \cup \{\beta\} \leq p$. □

If $G$ is $\mathcal{P}_\beta$-generic for sentences of $L_{\alpha}$ then $C = \cup G$ is a closed unbounded subset of $\alpha$ such that $L_{\alpha}[C] = L_{\alpha}$. The next lemma guarantees that $\langle L_{\alpha}, C \rangle$ is admissible.
Lemma 3. Suppose $\beta < \alpha$ and $f: \beta \to \alpha$ is $\Sigma_1(L_\alpha, C)$. Then $f \in L_\alpha$.

Proof. Let $\phi(x, y, z, C)$ be a $\Delta_0$ formula ($C$ is a name for $C$). Suppose $p \models \forall y < \beta \exists \delta \exists x \phi(x, y, \delta, C)$ and $\max(p) > \beta$, $\operatorname{rank}(\phi)$. Define a sequence of conditions as follows: $p_0 = p$. $p_{\gamma + 1} = \text{least } p \leq p_\gamma \text{ s.t. } p \models \exists x \exists \delta \phi(x, y, \delta, C)$ and $p_\gamma \in L_{\max(p)}$. $p_\lambda = \bigcup_{\gamma < \lambda} p_\gamma \cup \{\sup(\bigcup_{\gamma < \lambda} p_\gamma)\}$. Note that $\langle p_\gamma \mid \gamma < \lambda \rangle$ is $\Sigma_1(L_{\max(p), \gamma})$ for each limit $\lambda$ as forcing for $\Sigma_1$ sentences is $\Delta_1$. So $\max(p_\lambda)$ is inadmissible for each limit $\lambda < \beta$. Thus $p_\beta$ is an extension of $p$ and $p_\beta \models \exists x \exists \delta \phi(x, y, z, C)$ defines a function in $L_\alpha$, where $\models^*$ denotes weak forcing. If $f \in \Sigma_1(L_\alpha, C)$ is defined by $\exists x \exists \delta \phi(x, y, z, C)$ then $\{p \mid p \models \text{"If } \exists x \exists \delta \phi(x, y, z, C) \text{ defines a function then this function is in } L_\alpha\} \text{ is dense and therefore } \exists p \in G \models^* \text{"The function defined by } \exists x \exists \delta \phi(x, y, z, C) \text{ is in } L_\alpha\}$. So $f \in L_\alpha$. □

For the application below we will need a relativized version of the preceding. If $L_\alpha[S] = L_\alpha$ is countably closed and $\langle L_\alpha, S \rangle$ is admissible then there is a closed unbounded $C \subseteq \alpha$ such that

(a) $L_\alpha[S, C] = L_\alpha$,
(b) $\langle L_\alpha, S, C \rangle$ is admissible, and
(c) $\gamma \in C$, cofinality $(\gamma) > \omega \to \langle L_\gamma, S \cap L_\gamma \rangle$ is amenable and inadmissible.

This is proved exactly as above, replacing “admissible” by “$S$-admissible” and “$\Sigma_1$” by “$\Sigma_1$ in $S$” throughout. Thus one can define a countably closed forcing $\mathcal{G}_j^S$ which is $\Delta_1$ over $\langle L_\alpha, S \rangle$ and such that any $C$ which is $\mathcal{G}_j^S$-generic over $\langle L_\alpha, S \rangle$ satisfies (a), (b) and (c) above.

Unbounded Lévy forcing. Again let $\alpha$ be as in the hypothesis of Theorem 1. We will choose $C \subseteq \alpha$ to be $\mathcal{G}_j^S$-generic over $L_\alpha$ where $S$ is defined as follows. $S = \{(x, y) \mid x, y \in L_\alpha \text{ and } y = [x]^{\omega}\}$. But first we establish some properties of $S$ which will be of use to us in §2.

Note that $L_\alpha$ is $\omega$-admissible if and only if cofinality $(\alpha) \geq \omega_1$, $L_\alpha$ is closed under $y \mapsto [y]^{\omega}$ and $\langle L_\alpha, S \rangle$ is admissible. We first show that this last condition is redundant.

Lemma 4. Suppose $L_\alpha$ is admissible and closed under $y \mapsto [y]^{\omega}$. Then $\langle L_\alpha, S \rangle$ is admissible.

Proof. There are two cases.

Case 1. There is a largest $\alpha$-cardinal $\kappa$. Then $\kappa$ has uncountable cofinality as otherwise the elements of $[\kappa]^{\omega}$ are constructed cofinally in $\alpha$. But $[\kappa]^{\omega} \in L_\alpha$ by hypothesis.

We now show that $S$ is actually $\Delta_1$ over $L_\alpha$: Given $y \in L_\alpha$ let $c$ be the $<_L$-least injection of $y$ into $\kappa$ and let $z = c[y]$. Then $[y]^{\omega} = \{c^{-1}[s] \mid s \in [z]^{\omega}\}$ and $[z]^{\omega} = [z]^{\omega} \cap L_\alpha$. This gives a $\Sigma_1$ definition for $y \mapsto [y]^{\omega}$.

Case 2. There is no largest $\alpha$-cardinal. We show that if $\kappa$ is a regular $\alpha$-cardinal then $\langle L_\kappa^+, S \cap L_{\kappa}^+ \rangle$ is a $\Sigma_1$-elementary substructure of $\langle L_\alpha, S \rangle$. This establishes the admissibility of $\langle L_\alpha, S \rangle$.

Suppose $\kappa$ is a regular $\alpha$-cardinal. Cofinality $(\kappa) > \omega$ as $L_\alpha$ is countably closed. Now suppose $\langle L_\kappa^+, S \rangle \models \exists \phi(x, y, S)$ where $\phi$ is $\Delta_0$ and $x \in L_{\kappa^+}$. We must show that
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\[ \langle L_{\kappa^+}, S \cap L_{\kappa^+} \rangle \models \exists y \phi(x, y, S, S \cap L_{\kappa^+}). \]

Choose \( y \in L_\alpha \) so that \( \langle L_\alpha, S \rangle \models \phi(x, y, S) \) and let \( \lambda \) be a regular \( \alpha \)-cardinal so that \( y \in L_\lambda \). Then \( \langle L_\lambda, S \cap L_\lambda \rangle \) is amenable so we may choose a \( \Sigma_1 \) Skolem function \( h \) for this structure. As \( \kappa \) is a regular \( \alpha \)-cardinal we may choose \( \delta < \kappa^+ \) so that \( L_\delta \) is countably closed and \( x \in L_\delta \). Then let \( H = \Sigma_1 \) Skolem hull of \( L_\delta \cup \{ y \} \) inside \( L_\lambda \).

We claim that \( H \) is countably closed. For, if \( x_1, x_2, \ldots \in H \) then \( x_i = h(n_i, \gamma_i, y) \) for some \( n_i \in \omega \), \( \gamma_i < \delta \). Then \( \langle (n_i, \gamma_i) \mid i \in \omega \rangle \in L_\delta \) and as the \( \Sigma_1 \) sentence \( \exists z \forall i (z(i) = h(n_i, \gamma_i, y)) \) is true in \( L_\lambda \) (by the countable closure of \( L_\lambda \)), it is true in \( H \). So \( \langle x_i \mid i \in \omega \rangle \in H \).

Now collapse \( H \) isomorphically to \( L_\gamma \), \( \delta \leq \gamma < \kappa^+ \). Then \( L_\gamma \) is countably closed and \( S \cap H \) collapses to \( S \cap L_\gamma \). So as \( \langle L_\lambda, S \cap L_\lambda \rangle \models \phi(x, y, S \cap L_\lambda) \) we have \( \langle L_\lambda, S \cap L_\lambda \rangle \models \phi(x, \bar{\gamma}, S \cap L_\gamma) \) where \( \bar{\gamma} = \text{image}(\gamma) \) under the collapse. Hence \( \langle L_{\kappa^+}, S \cap L_{\kappa^+} \rangle \models \phi(x, \bar{\gamma}, S \cap L_{\kappa^+}) \) since \( \phi \) is \( \Delta_0 \). \( \square \)

This now gives us an explicit characterization of \( \omega \)-admissibility in terms of countable closure. Countable closure is in turn characterized by a "fine structure" condition in the next section.

**Lemma 5.** \( L_\alpha \) is \( \omega \)-admissible iff \( L_\alpha \) is countably closed and

\[ (*) \] either there is no largest \( \alpha \)-cardinal or the largest \( \alpha \)-cardinal has uncountable cofinality.

**Proof.** The "only if" direction was covered in the proof of Lemma 4, Case 1. By Lemma 4, to prove the "if" direction it suffices to show that \( L_\alpha \) is closed under \( y \mapsto [y]^\alpha \) if \( L_\alpha \) is countably closed and obeys \((*)\). But this is clear by Gödel unless \( \gamma \) has cardinality = greatest \( \alpha \)-cardinal. But then if \( c \in L_\alpha \) is a bijection between \( y \) and \( gc_a \) then \( [y]^\omega = \{ c^{-1}[s] \mid s \in [gc_a]^\omega \} \) and \( [gc_a]^\omega = [gc_a]^\omega \in L_{gc_a} \) by \((*)\). \( \square \)

Now let \( C \) be \( \mathcal{P}_S \)-generic over \( L_\alpha \). Then \( L_\alpha[S, C] = L_\alpha \) is admissible. We seek to build a class \( A \subseteq \alpha \) such that \( \langle L_\alpha[A], A, C \rangle \) is admissible and \( L_\alpha[A] \models \omega_1 \) is the largest cardinal. We construct \( A \) to be generic for a countably closed forcing for then the countable closure of \( L_\alpha \) guarantees the existence of generic classes. In the countable case this type of forcing was done in Sacks [1976] with finite conditions. He exploits the fact that \( \beta < \alpha \) implies \([\beta]^\omega \subseteq L_\alpha \) in order to help reduce this class forcing to set forcing. The analogous property in our present context, \( \beta < \alpha \) implies \([\beta]^\omega \subseteq L_\alpha \), is guaranteed by the \( \omega \)-admissibility of \( L_\alpha \). Moreover by working relative to \( S \) we can assume that the operation \( \beta \mapsto [\beta]^\omega \) is effective.

We now define the countably closed forcing \( \mathcal{P}_L \) for making \( \omega_1 \), the largest cardinal. A condition \( p \) is a countable partial function \( \alpha \times \omega_1 \to \alpha \) such that \( p(\beta, \gamma) < \beta \) for each \( (\beta, \gamma) \in \text{Domain}(p) \). The condition \( p \) is stronger than the condition \( q, p \leq q, \) if \( p \) extends \( q \) as a partial function. Thus if \( A \) is \( \mathcal{P}_L \)-generic over \( L_\alpha \) then \( L_\alpha[A] \models \omega_1 \) is the largest cardinal. We in fact want \( A \) to be \( \mathcal{P}_L \)-generic over \( \langle L_\alpha, S, C \rangle \) in order to guarantee that \( \langle L_\alpha[A], A, C \rangle \) is admissible. The countable closure of \( L_\alpha \) guarantees the existence of such an \( A \).

The proof of admissibility is much as in the countable case. \( \omega \)-admissibility is used to "bound the forcing relation": For \( \beta < \alpha \) and \( p \in \mathcal{P}_L \) let \( p^<\beta = p \cap (\beta \times \omega_1) \times \beta \). Then \( \mathcal{P}_L^{<\beta} = \{ p^{<\beta} \mid p \in \mathcal{P}_L \} \) is a member of \( L_\alpha \) and the function \( \beta \mapsto \mathcal{P}_L^{<\beta} \) is
A simple induction shows

\[ (*) \text{ if } p \vdash \phi, \text{ rank}(\phi) < \beta \text{ then } p^{<\beta} \vdash \phi. \]

Forcing is defined in the usual way by induction. All instances of this induction for ranked \( \phi \) are clearly \( S_1(\langle L_\alpha, S, C \rangle) \) except the negation case which is \( S_1(\langle L_\alpha, S, C \rangle) \) due to \((*)\).

\[ \vdash \phi \text{ iff } \forall q \leq p (\neg q \vdash \phi) \text{ iff } \exists \beta \left( \text{rank}(\phi) < \beta \text{ and } \forall q \in \mathcal{P}^{<\beta}(q \leq p \rightarrow \neg q \vdash \phi) \right). \]

Thus the relation \( p \vdash \phi \) of \( \langle L_\alpha, S, C \rangle \) is \( S_1(\langle L_\alpha, S, C \rangle) \) for ranked \( \phi \).

It is now easy to establish the admissibility of \( L_\alpha[A] \). Suppose \( \phi \) is ranked and \( p_0 \vdash \forall x \exists y \phi(x, y) \). Thus for each \( p \leq p_0 \), constant \( c \) there is \( q(p, c) \leq p \), constant \( d(p, c) \) such that \( q(p, c) \vdash \phi(c, d(p, c)) \). The functions \( q, d \) can be assumed to be \( \langle L_\alpha, S, C \rangle \) as the forcing relation is \( S_1(\langle L_\alpha, S, C \rangle) \) for ranked sentences. Let \( \beta < \alpha \) be a fixed point so that \( L_\beta \) is countably closed; i.e.,

\[
p_0 \in L_\beta,
\]

\[
p, c \in L_\beta \rightarrow q(p, c), \quad d(p, c) \in L_\beta,
\]

\[
\left[ L_\beta \right]^\omega \subseteq L_\beta.
\]

\( \beta \) exists by admissibility of \( \langle L_\alpha, S, C \rangle \).

**Claim.** \( p_0 \vdash \forall x \exists y \phi(x, y) \).

**Proof of Claim.** Suppose \( p \leq p_0 \), \( c \in L_\beta \). Then \( p^{<\beta} \in L_\beta \) since \( L_\beta \) is countably closed. As \( p^{<\beta} \leq p_0 \) we see that \( q = q(p^{<\beta}, c), \quad d = d(p^{<\beta}, c) \) are defined and belong to \( L_\beta \). Thus \( q \cup p \vdash \phi(c, d) \). So we have shown

\[
\forall p \leq p_0 \forall c \in L_\beta \exists r \leq p \exists d \in L_\beta \vdash \phi(c, d).
\]

This is exactly the statement of the Claim.

Now if \( \langle L_\alpha[A], A, S, C \rangle \vdash \forall x \exists y \phi(x, y) \), \( \phi \) ranked then for some \( p \in A \) \( p \vdash \forall x \exists y \phi(x, y) \). By the claim, for some \( \beta < \alpha \) \( p \vdash \forall x \exists y \phi(x, y) \) so \( L_\alpha[A] \vdash \forall x \exists y \phi(x, y) \). We have demonstrated \( \Pi_\alpha \)-reflection for \( \langle L_\alpha[A], A, S, C \rangle \).

**Almost disjoint set forcing.** We are now in the following situation: There are predicates \( A, S, C \subseteq L_\alpha \) such that

(a) \( \langle L_\alpha[A], A, S, C \rangle \) is admissible,

(b) \( L_\alpha[A] \vdash \omega_1 \) is the largest cardinal,

(c) \( C \) is closed unbounded in \( \alpha \), and

(d) \( \gamma \in C \rightarrow \text{cof}(\gamma) = \omega \text{ or } \langle L_\gamma, S \cap L_\gamma \rangle \) is inadmissible and amenable.

It is now fairly easy to construct a predicate \( B \subseteq L_\alpha \) such that \( \langle L_\gamma[B \cap L_\gamma], B \cap L_\gamma \rangle \) is inadmissible for each \( \gamma < \alpha \) and \( B \) is \( \Delta_1 \langle L_\alpha[A], A, S, C \rangle \). Then the idea will be to code \( B \) by a subset \( X \) of \( \omega_1 \) in a very efficient way, so that \( B \cap L_\gamma \) is \( \Delta_1 \) over \( L_\gamma[X] \) for each \( \gamma < \alpha \) and \( L_\alpha[X] \) is admissible.

The construction of \( B \) is as follows: For each \( \gamma \in C \) let \( w_\gamma \) be the \( L_\alpha[A] \)-least wellordering of \( \omega_1 \) of ordertype \( \gamma' \), where \( \gamma' = \text{least member of } C \text{ greater than } \gamma \). We identify \( w_\gamma \) with a subset of \( \omega_1 \). Let \( B' = \{ \gamma + \delta \mid \gamma \in C \text{ and } \delta \in w_\gamma \} \). Then \( B = \text{join}(S, B') \). (Thus \( B = \{ 2\mu \mid \mu \in S \} \cup \{ 2\mu + 1 \mid \mu \in B' \} \).) Certainly \( \langle L_\gamma[B \cap L_\gamma], B \cap L_\gamma \rangle \) is inadmissible when \( \gamma \) is not the limit of elements of \( C \) as this structure contains a wellordering of \( \omega_1 \) of ordertype \( \geq \gamma \) in this case. If \( \gamma < \alpha \) is the limit of elements of \( C \) and has uncountable cofinality then again this structure is inadmissible since \( \langle L_\gamma, S \cap L_\gamma \rangle \) is.
We now show that \( \langle L_\gamma[B \cap L_\gamma], B \cap L_\gamma \rangle \) is inadmissible for every \( \gamma < \alpha \). Otherwise let \( \gamma \) be the least ordinal such that \( \langle L_\gamma[B \cap L_\gamma], B \cap L_\gamma \rangle \) is admissible. We show that \( \gamma \) has uncountable cofinality, contradicting the previous paragraph. Note that as \( \gamma \) is the limit point of elements of \( C, L_\gamma[B \cap L_\gamma] \models \omega_1 \) is the largest cardinal.

Let \( h \) be a \( \Sigma_1 \) Skolem function for \( \langle L_\gamma[B \cap L_\gamma], B \cap L_\gamma \rangle \). Then for each \( \delta < \gamma \) \( h[\omega \times L_\delta] \) is transitive as \( L_\delta[B \cap L_\delta] \models \omega_1 \) is the largest cardinal. Thus \( h[\omega \times L_{\omega_1 + 1}] \) is of the form \( \langle L_\gamma[B \cap L_\gamma], B \cap L_\gamma \rangle \) and is admissible. The leastness of \( \gamma \) implies that \( \gamma' = \gamma \) and so there is a \( \Sigma_1 \langle L_\gamma[B \cap L_\gamma], B \cap L_\gamma \rangle \) injection \( g \) of \( \gamma \) into \( \omega_1 \). For each \( \delta < \omega_1, g^{-1}[\delta] \) is bounded in \( \gamma \) by the admissibility of \( \langle L_\gamma[B \cap L_\gamma], B \cap L_\gamma \rangle \). The function \( \delta \mapsto \sup g^{-1}[\delta] \) is a nondecreasing unbounded function from \( \omega_1 \) into \( \gamma \) so cofinality \( (\gamma) = \omega_1 \).

**Lemma 6.** For each \( \gamma \geq \omega_1, L_{\gamma + 1}[B \cap L_\gamma] \models \gamma \) has cardinality \( \omega_1 \).

**Proof.** Otherwise let \( \gamma \) be the least \( \delta \geq \omega_1 \) s.t. \( L_{\delta + 1}[B \cap L_\delta] \models \delta \) has cardinality \( > \omega_1 \). Then \( L_{\gamma + 1}[B \cap L_\gamma] \models \gamma = \omega_2 \) and so certainly \( L_\gamma[B \cap L_\gamma] \) is admissible. But this contradicts the above established property of \( B \). \( \square \)

Now we must code \( B \) by a subset of \( \omega_\alpha \). The standard way to do this is with almost disjoint set forcing, invented in Jensen and Solovay [1970] and perfected in Jensen [1972, 1975]. This method can be described as follows: First choose a \( \Delta_1 \langle L_\alpha[B], B \rangle \) sequence \( \langle X_\gamma | \gamma < \alpha \rangle \) of subsets of \( \omega_1 \) such that \( \gamma \neq \gamma' \rightarrow X_\gamma \cap X_{\gamma'} \) is countable \((X_\gamma, X_{\gamma'}) \) are almost disjoint). This is easily done. Also define \( B* = \{ \omega_1 + \gamma | \gamma \text{th set in } \langle L_\alpha \rangle \) belongs to \( B \} \). Then consider the forcing \( \mathcal{P}_3 \) where a typical condition is of the form \( (s, Y), s \) a countable subset of \( \omega_1 \) and \( Y \) a countable subset of \( \{ X_\gamma | \gamma \in B* \} \). We write \( (s, Y) \leq (t, Z) \) if

\[
S \supseteq T, \quad Y \supseteq Z, \quad X_\gamma \in Z \rightarrow (s - t) \cap X_\gamma = \emptyset.
\]

We identify a generic object \( G \) with \( X = \bigcup \{ s | \exists Y(s, Y) \in G \} \). Thus if \( X \) is \( \mathcal{P}_3 \)-generic over \( \langle L_\alpha[B], B \rangle \) then \( \omega_1 + \gamma \in B^* \leftrightarrow X_\gamma \) is almost disjoint from \( X \). So \( B \cap L_\gamma \) is \( \Delta_1 \) over \( L_\gamma[X, \langle X_\gamma | \gamma' < \gamma \rangle] \) provided \( \gamma > \omega_1 \) and is (say) primitive recursively closed. Conversely we arrange that \( \langle X_\gamma | \gamma' < \gamma \rangle \) is \( \Delta_1 \) over \( L_{\gamma'}[B \cap L_\gamma] \) where \( \gamma = \text{least p.r. closed ordinal } \gamma \supset \omega_1 \). Thus knowing \( B \cap L_\gamma \) allows one to determine \( \langle X_\gamma | \gamma' < \gamma \rangle \) which in turn allows one to recover \( B \cap L_\gamma \). This “bootstrap” idea is key to the decoding process as the proof of the next lemma indicates.

**Lemma 7.** \( L_\gamma[X] \) is inadmissible for each \( \gamma < \alpha \).

**Proof.** It is enough to show by induction that \( B \cap L_\gamma \) is \( \Delta_1 \) over \( L_\gamma[X] \) uniformly, for p.r. closed \( \gamma > \omega_1 \). If \( \gamma = \omega_1 \) then \( \langle X_\gamma | \gamma' < \gamma \rangle \) is \( \Delta_1 \) over \( L_\gamma \) so \( B \cap L_\gamma \) is \( \Delta_1 \) over \( L_\gamma[X] \). The uniformity renders trivial the case of \( \gamma \) being a limit of p.r. closed ordinals. Otherwise \( \gamma = \delta \) where \( \delta > \omega_1 \) is p.r. closed and hence by induction \( B \cap L_\delta \) is \( \Delta_1 \) over \( L_\delta[X] \). But then \( \langle X_\gamma | \gamma' < \gamma \rangle \) is \( \Delta_1 \) over \( L_\gamma[X] \) as it is \( \Delta_1 \) over \( L_\gamma[B \cap L_\delta] \). Thus \( B \cap L_\gamma \) is \( \Delta_1 \) over \( L_\gamma[X] \) as it is \( \Delta_1 \) over \( L_\gamma[X, \langle X_\gamma | \gamma' < \gamma \rangle] \). \( \square \)
The proof of Lemma 7 depends on the fact that the sequence of codes \( \langle X_\gamma | \gamma < \alpha \rangle \) can be chosen so that \( \langle X_{\gamma'} | \gamma' < \gamma' \rangle \) is \( \Delta_1 \) over \( L_\gamma[B \cap L_\gamma] \) for each \( \gamma \). This is possible thanks to Lemma 6.

Guaranteeing the admissibility of \( L_a[X] \) for \( \mathcal{P}_S \)-generic \( X \) requires a further restriction on the sequence \( \langle X_\gamma | \gamma < \alpha \rangle \). As in Jensen [1972] one must choose the \( X_\gamma \)’s to be generic. In Jensen [1972] it suffices to arrange that for each p.r. closed \( \gamma \), \( X_\gamma \) is Cohen generic over \( L_{\gamma+1}[B \cap L_\gamma] \) and belongs to \( L_\gamma[B \cap L_\gamma] \). For our purposes we must use the more sophisticated technology of Jensen [1975, p. 13], where it is arranged that for each p.r. closed \( \gamma \) and each 1-1 function \( f : \omega \to (\alpha - \gamma) \), the sequence \( \langle X_{f(0)}, X_{f(1)}, \ldots \rangle \) is Cohen generic (as an \( \omega \)-sequence) over \( L_{\gamma+1}[B \cap L_\gamma] \) and in addition \( \langle X_\gamma | \gamma < \gamma' \rangle \) is uniformly \( \Sigma_1 \) definable over \( L_\gamma[B \cap L_\gamma] \). We now assume that the sequence \( \langle X_\gamma | \gamma < \alpha \rangle \) has these properties and in addition for each p.r. closed \( \gamma \) and each Cohen condition \( p \) there is \( X_p \) extending \( p, \gamma < \delta < \gamma' \).

**Lemma 8.** \( L_a[X] \) is admissible when \( X \) is \( \mathcal{P}_S \)-generic over \( \langle L_a[B], B \rangle \).

**Proof.** For each \( \gamma \) let \( \mathcal{P}_S^\gamma = \{ \langle s, Y \rangle \in \mathcal{P}_S | Y \subseteq \{ X_\delta | \delta < \gamma \} \} \). We claim that if \( \omega_1 < \gamma \) is p.r. closed and has uncountable cofinality, \( M \subseteq \mathcal{P}_S^\gamma \) is a maximal antichain in \( \mathcal{P}_S^\gamma \) and \( M \in L_{\gamma+1}[B \cap L_\gamma] \) then \( M \) is a maximal antichain in \( \mathcal{P}_S^\gamma \). From this it will be fairly easy to establish the admissibility of \( L_a[X] \). But first we establish this claim.

Suppose not and let \( \langle s, Y \rangle \in \mathcal{P}_S^\gamma \) be a condition incompatible with each element of \( M \). Write \( Y = Y_1 \cup Y_2 \) where \( Y_1 \subseteq \{ X_\delta | \delta < \gamma \} \) and \( Y_2 \subseteq \{ X_\delta | \delta \geq \gamma \} \). List the elements of \( Y_2 \) in an \( \omega \)-sequence \( X_{f(0)}, X_{f(1)}, \ldots \) and then since \( \langle X_{f(0)}, X_{f(1)}, \ldots \rangle \) is \( \mathcal{P}_S^\gamma \)-generic over \( L_{\gamma+1}[B \cap L_\gamma] \) (\( \mathcal{C} = \text{Cohen forcing} \)) there is a condition \( \langle p_0, p_1, \ldots \rangle \in \mathcal{C}^\omega \) such that \( X_{f(i)} \) extends \( p_i \) for each \( i \) and

\[
\langle p_0, p_1, \ldots \rangle \upharpoonright \langle \langle s, Y_1 \cup \{ G(n) | n \in \omega \} \rangle \quad \text{is incompatible with each element of } M.\]

Now \( \langle s, Y_1 \rangle \in \mathcal{P}_S^{\delta} \) for some \( \delta < \gamma \) since \( \gamma \) has uncountable cofinality. For each \( i \) choose \( X_{g(i)}, \delta < g(i) < \delta', \) such that \( X_{g(i)} \) extends \( p_i \) (see the remark immediately before the statement of Lemma 8). As \( \langle s, Y_1 \cup \{ X_{g(i)} | i \in \omega \} \rangle \) is a condition in \( \mathcal{P}_S^{\gamma} \) it is compatible with some \( \langle t, Z \rangle \in M \). Now let \( \beta = \sup(t) \) and consider the condition \( \langle X_{g(0)}^{f(i)}, \beta, X_{f(1)}^{g(i)}, \beta, \ldots \rangle \in \mathcal{C}^\omega \). This condition extends \( \langle p_0, p_1, \ldots \rangle \) and (weakly) forces \( \langle \langle s, Y_1 \cup \{ G(n) | n \in \omega \} \rangle \) is compatible with \( \langle t, X \rangle \), contradicting the choice of \( \langle p_0, p_1, \ldots \rangle \).

It now follows that if \( \omega_1 < \gamma \) is p.r. closed and of uncountable cofinality and \( G \) is \( \mathcal{P}_S \)-generic over \( L_a[B], B \) then \( G \cap \mathcal{P}_S^\gamma \) is \( \mathcal{P}_S^\gamma \)-generic over \( L_{\gamma+1}[B \cap L_\gamma] \). For if \( M \in L_{\gamma+1}[B \cap L_\gamma] \) is a maximal antichain in \( \mathcal{P}_S^\gamma \) then \( M \) is maximal in \( \mathcal{P}_S \) and thus \( G \cap M = (G \cap \mathcal{P}_S^\gamma) \cap M \neq \emptyset \). As any dense open subset of \( \mathcal{P}_S^\gamma \) contains a maximal antichain, this proves that \( G \cap \mathcal{P}_S^\gamma \) is \( \mathcal{P}_S^\gamma \)-generic over \( L_{\gamma+1}[B \cap L_\gamma] \). We now show by induction on \( \phi \in L_\gamma[B \cap L_\gamma] \) that for all \( \langle s, Y \rangle \in \mathcal{P}_S^\gamma \)

\[
\langle s, Y \rangle \vdash \varphi \iff \langle s, Y \rangle \vdash \varphi.
\]
Each step of the induction is clear save the negation case. If \( \langle s, Y \rangle \Vdash \varphi \) then certainly \( \langle s, Y \rangle \Vdash \varphi \) follows by induction. Conversely suppose \( \langle s, Y \rangle \Vdash \varphi \). Then for every \( \mathbb{P}_s \)-generic \( H \) over \( L_{\gamma+1}[B \cap L_{\gamma}] \) extending \( \langle s, Y \rangle \) we have \( \langle L_{\gamma}[B \cap L_{\gamma}, H], B \cap L_{\gamma} \rangle \Vdash \varphi \) by the truth lemma. But then for every \( \mathbb{P}_s \)-generic \( G \) over \( \langle L_{\alpha}[B], B \rangle \) extending \( \langle s, Y \rangle \) we have \( \langle L_{\alpha}[B, G], B \rangle \Vdash \varphi \) since in this case \( G \cap \mathbb{P}_s \) is \( \mathbb{P}_s \)-generic over \( L_{\gamma+1}[B \cap L_{\gamma}] \). Thus \( \langle s, Y \rangle \Vdash \varphi \) (since we are using weak forcing).

We have proved that the relation \( \langle s, Y \rangle \Vdash \varphi \) restricted to ranked \( \varphi \) is \( A_s \), over \( (L_{\alpha}[B], B) \). We can now prove the lemma. Suppose \( \langle s, Y \rangle \Vdash \forall x \exists y \varphi(x, y) \) where \( \varphi \) is \( A_0 \). We shall produce an ordinal \( \gamma < \alpha \) such that \( \langle s, Y \rangle \Vdash \forall x \exists y \varphi(x, y) \) (where \( x, y \) range over elements of \( L_{\gamma}[B \cap L_{\gamma}] \)). Our hypothesis and the \( \Delta_1 \)-ness of the forcing relation imply the existence of a \( \Sigma_1 \langle L_{\alpha}[B], B \rangle \) function \( f: \alpha \rightarrow \alpha \) such that \( \forall x \forall \langle t, Z \rangle \leq \langle s, Y \rangle [(x \in L_{\beta}[B \cap L_{\beta}] \land \langle t, Z \rangle \in \mathbb{P}_s) \rightarrow \exists \langle t', Z' \rangle \leq \langle t, Z \rangle \exists y \in L_{f(\delta)}[B \cap f(\delta)], \langle t', Z' \rangle \in \mathbb{P}_s(\delta) \mathrm{and} \langle t', Z' \rangle \Vdash \varphi(x, y)] \). Let \( \omega_1 < \gamma \) be a p.r. closed fixed point for \( \varphi \) of uncountable cofinality. Then for each

\[
D = \{ \langle t, Z \rangle \in \mathbb{P}_s \mid \exists y \in L_{\beta}[B \cap L_{\beta}] \langle t, Z \rangle \Vdash \varphi(x, y) \}
\]

is dense in \( \mathbb{P}_s \) below \( \langle s, Y \rangle \) and belongs to \( L_{\gamma+1}[B \cap L_{\gamma}] \). Thus if \( \langle t, Z \rangle \leq \langle s, Y \rangle \) then \( \langle t, Z \rangle \) is compatible with some element of \( D \) and hence

\[
\exists \langle t', Z' \rangle \leq \langle t, Z \rangle \exists y \in L_{\gamma}[B \cap L_{\gamma}] \langle t', Z' \rangle \Vdash \varphi(x, y).
\]

We have just proved \( \langle s, Y \rangle \Vdash \forall x \exists y \varphi(x, y) \). Thus \( \Pi_2 \)-reflection holds for \( L_{\alpha}[X] \) and so \( L_{\alpha}[X] \) is admissible. \( \square \)

2. Converse to Theorem 1. We make use of Jensen’s theory of projecta (see Jensen [1972A]) to characterize ordinals of the form \( \alpha(X) \) for \( X \subseteq \omega_1 \). In fact we show that unless \( L_{\alpha} \) is \( \omega \)-admissible there is no \( X \subseteq \omega_1 \) such that \( L_{\alpha}[X] \) is admissible and \( L_{\alpha}[X] \Vdash \omega_1 \) is the largest cardinal. Our proof is best motivated by considering the following example which provides an admissible \( \alpha \) of cardinality and cofinality \( \omega_1 \) such that \( L_{\alpha}[X] \) is inadmissible whenever \( L_{\alpha}[X] \Vdash \omega_1 \) is the largest cardinal.

**Example 1.** Choose \( \alpha < \omega_2 \) to be admissible of cofinality \( \omega_1 \) and such that \( \sigma 2 \alpha = \Sigma_2 \) projectum of \( \alpha \) has cofinality \( \omega_1 \) in \( L_{\alpha} \). Such an \( \alpha \) is obtained by choosing \( L_{\alpha} \) = transitive collapse (M), \( M \) is an elementary submodel of \( L_{\gamma} \) of cardinality \( \omega_1 \), where \( \gamma = \omega_1 \)st stable past \( L_{\omega} \). Thus \( \sigma 2 \alpha = (L_{\omega})^{L_{\alpha}} \mathrm{and} \Sigma_2 \) cofinality (\( \alpha \)) = \( \omega_1 \).

Now let \( \beta = \sigma 2 \alpha = (L_{\omega})^{L_{\alpha}} \). We can obtain a wellordering \( R \) of a subset of \( \beta \) of ordertype \( \alpha \) which is \( \Sigma_2 \langle L_{\alpha} \rangle \) as follows. Choose a \( \Sigma_2 \langle L_{\alpha} \rangle \) injection \( f: \alpha \rightarrow \beta \) and let \( R(x, y) \leftrightarrow f^{-1}(x) < f^{-1}(y), x, y \in \text{Range}(f) \). Jensen’s fundamental theorem about \( \Sigma_\alpha \) projecta states that any \( \Sigma_\alpha \langle L_{\alpha} \rangle \) bounded subset of \( \sigma \alpha \Sigma_\alpha = \Sigma_\alpha \) projectum (\( \alpha \)) is a member of \( L_{\alpha \omega \alpha} \). Thus if we let \( \beta_n = (L_{\omega})^{L_{\alpha \omega \alpha}} \) we see that \( R \cap (\beta_n \times \beta_n) \subseteq L_{\beta} \) for each \( n \). We can thereby “code” \( R \) by the \( \omega \)-sequence \( s: \omega \rightarrow L_{\beta} \) defined by \( s(n) = R \cap (\beta_n \times \beta_n) \).

Finally suppose \( X \subseteq \omega_1 \) and \( L_{\alpha}[X] \Vdash \omega_1 \) is the largest cardinal. Then there exists an injection \( c: L_{\beta} \rightarrow \omega_1 \) such that \( c \in L_{\alpha}[X] \). The composition \( c \circ s \) belongs to \( L_{\omega_1} \), as \( c \circ s \) is a function from \( \omega \) to \( \omega_1 \) and we have assumed \( V = L \). Thus \( s = c^{-1} \circ (c \circ s) \subseteq L_{\alpha}[X] \). We now have \( \bigcup \text{Range}(s) = R \subseteq L_{\alpha}[X] \) and thus \( L_{\alpha}[X] \) contains a wellordering of ordertype \( \alpha \). So \( L_{\alpha}[X] \) is inadmissible.
The idea of this example can be used to show that not only \( \Sigma_n \) projectum (\( \alpha \)) but also many other “projecta” associated to \( \alpha \) must have uncountable cofinality as well. By establishing the uncountable cofinality of a sufficient number of these related projecta we are able to ultimately show that \( L_\alpha \) is countably closed. Thus we have also obtained a “fine structure” characterization of countable closure.

For the sake of the definition below recall the \( S \)-hierarchy of Jensen, defined and discussed in Devlin \[1973, \text{p. 82}\]. This hierarchy is a more convenient way of generating \( L \) than the usual \( L \)-hierarchy. \( S_\beta \) has very nice closure properties for limit \( \beta \). \( S_\beta \cap OR = \beta \) for limit \( \beta \). In what follows \( \beta \) always denotes a limit ordinal.

**Definition.** Let \( \alpha \leq \beta \) and \( n \) a positive integer. The \((n, \beta)\) projectum of \( \alpha \) = least \( \gamma \) s.t. there is a \( \Sigma_n(S_\beta) \) injection of \( \alpha \) into \( \gamma \). We write \((n', \beta') < (n, \beta)\) if \( \beta' < \beta \) or \((\beta' = \beta \) and \( n' < n \)). Then \((n, \beta)\) is an \( \alpha \)-critical pair if \((n', \beta')\) projectum \((\alpha) > (n, \beta)\) projectum \((\alpha)\) whenever \((n', \beta') < (n, \beta)\). Notice that there are only finitely many \( \alpha \)-critical pairs, beginning with \((1, \alpha)\). Let \((1, \alpha) = (n_1, \beta_1) < (n_2, \beta_2) < \cdots < (n_k, \beta_k)\) be a list of all \( \alpha \)-critical pairs and let \( \rho_1 > \rho_2 > \cdots > \rho_k = \omega_1 \) be a list of the corresponding projecta; i.e., \( \rho_i = (n_i, \beta_i) \) projectum \((\alpha)\).

**Lemma 9.** If \((n, \beta)\) is \( \alpha \)-critical then \((n, \beta)\) projectum \((\alpha) = \Sigma_n\) projectum \((\beta)\).

**Proof.** Let \( \eta_1 = (n, \beta)\) projectum \((\alpha)\) and \( \eta_2 = \Sigma_n\) projectum \((\beta)\). As \( \alpha \leq \beta \) it follows that \( \eta_1 \leq \eta_2 \). Now choose a \( \Sigma_n(S_\beta) \) injection \( f: \alpha \to \eta_1 \). Then \( R = \{(x, y) \mid x, y \in \text{Range}(f) \text{ and } f^{-1}(x) < f^{-1}(y)\} \) is a \( \Sigma_n(S_\beta) \) subset of \( \eta_1 \times \eta_2 \). But \( R \not\in S_\beta \) as otherwise \((n', \beta')\) projectum \((\alpha) \leq \eta_1 \) for some \((n', \beta') < (n, \beta)\), contradicting the hypothesis that \((n, \beta)\) is \( \alpha \)-critical. Now by Jensen’s characterization of the \( \Sigma_n \) projectum we must have \( \eta_2 \leq \eta_1 \). \( \square \)

The idea used in Example 1 also establishes the next result.

**Lemma 10.** If \( \alpha = \alpha(X) \) for some \( X \) then \( \rho_i \) has uncountable cofinality for each \( i \).

**Proof.** Suppose not. Choose \( \iota: \omega \to \rho_i \) to be unbounded. As before choose a \( \Sigma_n(S_\beta) \) injection \( f: \alpha \to \rho_i \) and let \( R = \{(x, y) \mid x, y \in \text{Range}(f) \text{ and } f^{-1}(x) < f^{-1}(y)\} \). Then by Lemma 9 \( R \cap (t(n) \times t(n)) \in L_{\rho_i} \) for each \( n \). We let \( s(n) = R \cap (t(n) \times t(n)) \).

If \( L_\alpha[X] \models \omega_1 \) is the largest cardinal then as before \( s \in L_\alpha[X] \). But then \( L_\alpha[X] \) is inadmissible as \( R = \bigcup \text{Range}(s) \in L_\alpha[X] \) and \( L_\alpha[X] \) contains a wellordering of ordertype \( \alpha \). \( \square \)

Establishing the countable closure of \( L_\alpha \) (when \( \alpha = \alpha(X) \) for some \( X \)) necessitates consideration of other projecta closely related to those above. For each \( i, \ 1 \leq i \leq k \), we define

\[ \rho'_i = \Sigma_{n_i-1} \text{ projectum } (\beta_i). \]

Recall that \( \rho_i = \Sigma_{n_i} \text{ projectum } (\beta_i) \). If \( n_i = 1 \) we define \( \Sigma_{n_i-1} \text{ projectum } (\beta_i) = \Sigma_0 \text{ projectum } (\beta_i) = \beta_i \). The countable closure results will follow once we demonstrate that \( \rho'_i \) also has uncountable cofinality for each \( i \). The proof uses a combination of ideas from the theory of master codes and \( \beta \)-recursion theory.
Suppose \( n > 0 \), \( A \subseteq \Sigma_n \) projectum(\( \beta \)) is a \( \Sigma_n \) Master Code for \( \beta \) if \( A \in \Sigma_n(S_\beta) \) and for any \( B \subseteq \Sigma_n \) projectum (\( \beta \)) = \( \sigma n \beta \),

\[
B \text{ is } \Sigma_1 \left\langle L_{\sigma n \beta}, A \right\rangle \text{ iff } B \text{ is } \Sigma_{n+1}(S_\beta).
\]

Jensen defined and proved the existence of \( \Sigma_n \) Master Codes (Jensen [1972A]). If \( n = 0 \) we define \( \varnothing = \Sigma_0 \) Master Code for \( \beta \). Then for each \( i \) define

\[
A_i = a \Sigma_{n_i-1} \text{ Master Code for } \beta_i.
\]

Thus \( A_i \subseteq \rho_i \) and the structure \( \mathcal{A}_i = \langle S_{\rho_i}, A_i \rangle \) is amenable (that is, \( A_i \cap \gamma \in S_{\rho_i} \) for each \( \gamma < \rho_i \)). We can also think of \( \rho_i \) as the \( \Sigma_1 \) projectum of the structure \( \mathcal{A}_i \). Thus the introduction of master codes allows us to deal with \( \Sigma_1 \) predicates (over an appropriate amenable structure) where the methods of \( \beta \)-recursion theory apply.

**Lemma 11.** Suppose \( \rho_i' > \alpha \). Then \( \rho_{i-1} \) is a \( \rho_i' \)-cardinal.

**Proof.** Since \( \rho_i' > \alpha = \rho_1' \) we see that \( i > 1 \). Let \( (n, \beta) \) be the least pair such that there is a \( \Sigma_n(S_\beta) \) injection of \( \rho_{i-1} \) into a smaller ordinal. Then \( (n, \beta) \) is \( \alpha \)-critical.

Thus \( (n, \beta) = (n_i, \beta_i) \) and \( \rho_{i-1} \) is a \( \beta_i \)-cardinal. We are done since \( \rho_i' = \Sigma_{n_i-1} \) projectum (\( \beta_i \)) \( \leq \beta_i \). \( \square \)

We are now prepared to prove the key lemma toward establishing countable closure.

**Lemma 12.** If \( \alpha = \alpha(X) \) for some \( X \) then \( \rho_i' \) has uncountable cofinality for each \( i \).

**Proof.** If \( \rho_i' < \alpha = \rho_1' \) then the result follows from Lemma 10. For in this case \( \rho_i' = \rho_{i-1} \) as \( \rho_i' = (n_i - 1, \beta_i) \) projectum (\( \alpha \)) and \( (n_i, \beta_i) \) is \( \alpha \)-critical.

We must deal with the case \( \rho_i' > \alpha \). Suppose then that cofinality (\( \rho_i' \)) = \( \omega \). Choose \( f: \omega \rightarrow \rho_i' \) to be cofinal and increasing. Again there is a \( \Sigma_n(S_\beta) \) wellordering \( R \) of \( \rho_i \) of ordertype \( \alpha \). \( R \) is \( \Sigma_1(\mathcal{A}_i) \) so we can choose a \( \Sigma_1 \) formula \( \phi(x, y, A_i) \) which defines \( R \) over \( \mathcal{A}_i \). We let \( R_n \) be the \( f(n) \)th approximation to \( R \); i.e., \( (x, y) \in R_n \leftrightarrow \langle S_{f(n)}, A_i \cap f(n) \rangle \models \phi(x, y, A_i \cap f(n)) \). Thus \( R_n \in L_{\rho_i-1} \) for each \( n \) as \( R_n \in S_{\rho_i} \) and \( \rho_{i-1} \) is a \( \rho_i' \)-cardinal. \( \rho_{i-1} \) has uncountable cofinality. So there is a \( \gamma < \alpha \) s.t. \( R_n \in S_\gamma \) for all \( n \).

Now if \( L_\alpha[X] \models \omega_1 \) is the largest cardinal then as before \( \langle R_n \mid n \in \omega \rangle \in L_\alpha[X] \). So \( R \in L_\alpha[X] \) and \( L_\alpha[X] \) is inadmissible. \( \square \)

**Theorem 13.** If \( \alpha = \alpha(X) \) for some \( X \) then \( L_\alpha \) is countably closed.

**Proof.** Define \( \rho_0 = \alpha \). We show by induction on \( k - i \) that if \( f: \omega \rightarrow \rho_i \) then \( f \in L_\alpha \). If \( k - i = 0 \) then \( \rho_i = \rho_k = \omega_1 \) so the result is clear.

Now assume the result for \( k - (i + 1) \) and we demonstrate it for \( k - i \). Thus we are given \( f: \omega \rightarrow \rho_i \) and we know that \( g: \omega \rightarrow \rho_{i+1} \) implies \( g \in L_\alpha \). Note that \( \rho_{i+1} \geq \rho_i \) as either \( \rho_i' \geq \alpha \) or \( \rho_i' = \rho_i \) (see the proof of Lemma 12). Thus \( f: \omega \rightarrow \rho_{i+1} \). Also let \( h \) be a \( \Sigma_1(\mathcal{A}_{i+1}) \) injection of \( \rho_{i+1} \) into \( \rho_{i+1} \).

By induction \( h \circ f \in L_\alpha \). Moreover \( h \circ f \in L_{\rho_{i+1}} \) as \( \rho_{i+1} \) is an \( \alpha \)-cardinal of uncountable cofinality. Now \( f = h^{-1} \circ (h \circ f) \) is therefore \( \Sigma_1(\mathcal{A}_{i+1}) \). As cofinality \( (\rho_{i+1}) > \omega \) we see that \( f \in L_{\rho_{i+1}} \). If \( \rho_{i+1} \leq \alpha \) we are done. Otherwise \( f \in L_{\rho_i} \) as \( \rho_i \) is a \( \rho_{i+1} \)-cardinal of uncountable cofinality. As \( \rho_i \leq \alpha \) we have \( f \in L_\alpha \). \( \square \)
Now that we have dispensed with countable closure, $\omega$-admissibility can be easily dealt with.

**Theorem 14.** If $\alpha = \alpha(X)$ for some $X \subseteq \omega_1$ then $L_\alpha$ is $\omega$-admissible.

**Proof.** The theorem follows if we show that $L_\alpha$ is closed under $y \mapsto [y]^{\omega}$ (see Lemma 4). Fix $y \in L_\alpha$. As we know that $L_\alpha$ is countably closed we can define a $\Sigma_i(L_\alpha[X])$ function as follows: Given $z \in [y]^{\omega}$ let $h(z) =$ the ordinal at which $z$ is constructed in $L_\alpha$. Then Range($h$) is bounded by some $\beta < \alpha$ and so $[y]^{\omega} = [y]^{\omega} \cap L_\beta \in L_\alpha$. □

Lastly we show that $\omega$-admissibility may fail for $L_\alpha$ even when $L_\alpha$ is countably closed.

**Example 2.** Let $M$ be the $\Sigma_2$ Skolem Hull of $\omega_1$ inside $L_\gamma$ where $\gamma = \omega_1$st stable above $\aleph_0$. Let $L_\alpha$ be the transitive collapse of $M$. Then $\alpha$ has cofinality $\omega_1$, $L_\alpha$ has the largest cardinal and $\Sigma_2$ projectum $(\alpha) = \omega_1$. Also $\alpha^* = \alpha$. We show that $L_\alpha$ is countably closed in two steps: (a) If $f : \omega \rightarrow L_\alpha$ then $f$ is $\Sigma_2(L_\alpha)$. For, let $p : L_\alpha \rightarrow \omega_1$ be a $\Sigma_2(L_\alpha)$ injection. Then $p \circ f \in L_\omega$, and thus $f = p^{-1} \circ (p \circ f)$ is $\Sigma_2(L_\alpha)$. (b) If $f : \omega \rightarrow L_\alpha$ is $\Sigma_2(L_\alpha)$ then $f \in L_\alpha$. For, if $C$ is a $\Sigma_1$ Master Code for $L_\alpha$ then $f$ is $\Sigma_1(L_\alpha, C)$ (recall that $\alpha^* = \alpha$). But $\alpha$ has uncountable cofinality so $f \not\in L_\alpha$. Note that Lemma 5 implies that $L_\alpha$ is not $\omega$-admissible.

We can sum up our results as follows.

**Theorem 15 ($V = L$).** (a) $L_\alpha$ is countably closed iff $\rho_i, \rho_i'$ have uncountable cofinality for each $i$.

(b) $L_\alpha$ is $\omega$-admissible iff $L_\alpha$ is countably closed and in addition if $L_\alpha$ has a largest cardinal, it has uncountable cofinality.

(c) $\alpha = \alpha(X)$ for some $X \subseteq \omega_1$ iff $\omega_1 < \alpha < \omega_2$ and $L_\alpha$ is $\omega$-admissible.

**Example 3 ($ZF$).** There is an admissible $\alpha$, $\omega_1 < \alpha < \omega_2$, such that cofinality $(\alpha) = \omega_1$ and $\alpha \not= \alpha(X)$ for any $X \subseteq \omega_1$.

**Proof.** Let $\kappa$ denote true $\omega_1$. As in Example 1 let $\alpha < (\kappa^+)^L$ have $L$-cofinality $\kappa$ and be such that $\sigma_2 \rho \alpha$ has cofinality $\omega$ in $L_\alpha$. Then there is a wellordering $R$ of $\lambda = \sigma_2 \rho \alpha$ of ordertype $\alpha$ which is $\Sigma_2(L_\alpha)$. Moreover $R = \bigcup_n R_n$ where $R_n \subseteq L_\lambda$ for each $n$.

Suppose $\alpha = \alpha(X)$. Then $L_\alpha[X]$ contains an injection $g : L_\lambda \rightarrow \kappa$. But then $\langle g(R_n) \mid n \in \omega \rangle \in [\kappa]^{\omega} \cap L[X] \subseteq L_\kappa[X]$. So $R \subseteq L_\kappa[X]$, contradicting the admissibility of $L_\alpha[X]$. □

3. Further results and open questions. (1) There is a version of our result for $\Sigma_n$ admissibility, $n > 1$. Thus for regular $\kappa$, $\alpha$ is the least $\Sigma_n$ admissible relative to a subset of $\kappa$ iff $\kappa < \alpha < \kappa^+$, $\alpha$ has cofinality $\kappa$ and $L_\alpha$ is closed under the $\Sigma_n$ admissible relative to the function $y \mapsto [y]^{\omega}$. For example, let $n = 2$, $\kappa = \omega_1$. The necessity of this condition follows as in §2. For the sufficiency, one first adds a closed unbounded $C \subseteq \alpha$ so that $\gamma \in C$, cofinality $(\gamma) > \omega \rightarrow \langle L_\gamma, S \cap L_\gamma \rangle$ is $\Sigma_2$ inadmissible (where $S$ is defined as before). Moreover $\langle L_\alpha, S, C \rangle$ is amenable and $\Sigma_2$ admissible. Then generically add $A \subseteq L_\alpha$ so that $\langle L_\alpha[A], S, C, A \rangle$ is $\Sigma_2$ admissible and $L_\alpha[A]$ is the largest cardinal. There is a predicate $B \subseteq L_\alpha$ so that $B$ is
\(\Delta_1 \langle L_\alpha[A], S, C, A \rangle\) and for each \(\gamma < \alpha \langle L_\gamma[B \cap L_\gamma], B \cap L_\gamma \rangle\) is \(\Sigma_2\) inadmissible. Now code \(B\) by \(X \subseteq \omega_1\) as before. Then \(B \cap L_\gamma\) is \(\Delta_1\) over \(L_\gamma[X]\) for p.r. closed \(\gamma > \omega_1\) so \(L_\gamma[X]\) is \(\Sigma_2\) inadmissible for \(\gamma < \alpha\). The proof that \(L_\alpha[X]\) is \(\Sigma_2\) admissible proceeds as before, using the fact that maximal antichains when the forcing is restricted to p.r. closed \(\gamma\) of uncountable cofinality are maximal in the whole partial-ordering, and the resulting fact that forcing for \(\Sigma_2\) sentences is a \(\Sigma_2\) relation.

(2) Sacks' pointed perfect forcing (Sacks [1976]) can be adapted to the present context to prove: If \(\alpha\) has cardinality and cofinality \(\kappa\), \(\kappa\) regular, and \(L_\alpha\) is \(\Sigma_n\) admissible relative to \(y \mapsto [y]^{< \kappa}\) then there is \(X \subseteq \kappa\) s.t. \(\alpha\) is the least \(\Sigma_n\) admissible relative to \(X\) and if \(Y \in L_\alpha[X], Y \subseteq \kappa\), then either \(X \in L_\alpha[Y]\) or the least \(\Sigma_n\) admissible relative to \(Y\) is less than \(\alpha\). One must use here the full version of this forcing, including Sacks' "triple forcing," adapted to trees on \(\kappa\).

(3) In the countable case there is a model-theoretic proof of Sacks' characterization of countable admissibles. Of course compactness fails at regular cardinals > \(\omega\). But is there a model-theoretic proof of Theorem 1, say like that given of Sacks' theorem in Friedman [1981]? It would help to develop a countably closed version of Steel forcing.3

(4) R. David [1981] has shown that if \(\alpha\) is countable and \(L_\alpha \models ZF\) then for some \(R \subseteq \omega,\) \(\alpha\) is the least ordinal such that \(L_\alpha[R] \models ZF\). What is the uncountable version of this result?

(t) Is there a nice characterization of which sequences of admissibles \(\langle \alpha_\gamma \mid \gamma < \lambda \rangle,\) \(\lambda < \kappa^+\), between \(\kappa\) and \(\kappa^+\) are the first \(\lambda\) admissibles relative to some \(X \subseteq \kappa\) (as in Jensen [1972]) for the case \(\kappa = \omega\)?

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3This has now been accomplished: See our forthcoming paper *Model theory for L_{\omega_1^\omega}*. 

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