RELATIONS BETWEEN \( H^p_u \) AND \( L^p_u \) WITH POLYNOMIAL WEIGHTS

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Abstract. Relations between \( L^p_u \) and \( H^p_u \) of the real line are studied in the case when \( p > 1 \) and \( u(x) = |q(x)|^p w(x) \), where \( q(x) \) is a polynomial and \( w(x) \) satisfies the \( A_p \) condition. It turns out that \( H^p_u \) and \( L^p_u \) can be identified when all the zeros of \( q \) are real, and that otherwise \( H^p_u \) can be identified with a certain proper subspace of \( L^p_u \). In either case, a complete description of the distributions in \( H^p_u \) is given. The questions of boundary values and of the existence of dense subsets of smooth functions are also considered.

1. Introduction. In this paper, we study relations between \( L^p_u \) and \( H^p_u \) of the real line, where \( 1 < p < \infty \), \( u = u(x) \) is a nonnegative weight function,

\[
L^p_u = \left\{ f : \left( \int_{\mathbb{R}} |f(x)|^p u(x) \, dx \right)^{1/p} < \infty \right\},
\]

and \( H^p_u \) is the corresponding Hardy space. To define \( H^p_u \) precisely, let \( \mathcal{S} \) be the Schwartz space of rapidly decreasing functions, \( \mathcal{S}' \) be the space of tempered distributions, and for \( l \in \mathcal{S}' \), let \( ML \) denote the nontangential maximal function defined by

\[
ML(x) = \left( M_{\gamma, \phi} l \right)(x) = \sup_{(\xi, t) \in \Gamma'_{\mathcal{S}}(x)} \left| \langle l, \phi_t(\xi - \cdot) \rangle \right|,
\]

where \( \Gamma'_{\mathcal{S}}(x) \) is the cone in \( \mathbb{R}^2_+ \) of points \( (\xi, t) \) with \( |x - \xi| < \gamma t \), \( \gamma > 0 \), \( \phi \in \mathcal{S} \), \( \phi_t(x) = t^{-1} \phi(x/t) \), \( t > 0 \), and \( \langle l, \psi \rangle \) denotes the action of \( l \) on \( \psi \). Then \( H^p_u \) is defined to be the collection of \( l \in \mathcal{S}' \) such that \( M_{\gamma, \phi} l \in L^p_u \) for some \( \gamma > 0 \) and some \( \phi \) with \( \int_{\mathbb{R}} \phi \, dx \neq 0 \). If \( u \) merely satisfies the doubling condition \( \int_I u \, dx \leq c \int_{2I} u \, dx \), where \( I \) is an interval, \( 2I \) is its double and \( c \) is a constant independent of \( I \), then the condition that \( l \in H^p_u \) is known to be independent of any particular \( \gamma > 0 \) or \( \phi \in \mathcal{S} \) with \( \int_{\mathbb{R}} \phi \, dx \neq 0 \) (see [9]). We will use the notations

\[
\|f\|_{L^p_u} = \|f\|_{p, u} = \left( \int_{\mathbb{R}} |f|^p u \, dx \right)^{1/p},
\]
\[
\|f\|_{H_u^p} = \|M_{\gamma, \phi} f\|_{p, u},
\]
for some fixed choice of \( \gamma \) and \( \phi \), \( \int_R \phi \, dx \neq 0 \).

If \( u \) satisfies the \( A_p \) condition
\[
\left( \frac{1}{|I|} \int_I u(x) \, dx \right) \left( \frac{1}{|I|} \int_I u(x)^{-1/(p-1)} \, dx \right)^{p-1} \leq c, \quad 1 < p < \infty,
\]
where \( c \) is independent of \( I \), then \( H_u^p \) and \( L_u^p \) can be identified in the sense that every \( f \in L_u^p \) generates a distribution \( l_f \in H_u^p \) defined by \( \langle l_f, \phi \rangle = \int_R f(x) \phi(x) \, dx, \phi \in \mathcal{S} \), and to every \( l \in H_u^p \), there corresponds a unique \( f \in L_u^p \) such that \( l = l_f \). Moreover, in this correspondence, \( \|f\|_{L_u^p} \) and \( \|l\|_{H_u^p} \) are equivalent. These results are well-known corollaries of the fact (see [6]) that the transformation \( f \to f^* \), where \( f^* \) denotes the Hardy-Littlewood maximal function of \( f \), is bounded on \( L_p^p \), \( 1 < p < \infty \), if \( u \in A_p \).

The weights \( u \) considered in this paper for \( L_u^p \) and \( H_u^p \) will belong to \( A_r \) for some \( r > p \) but will not belong to \( A_p \). Their specific form is \( u(x) = |q(x)|^p w(x) \) where \( q \) is a polynomial and \( w \in A_p \). Thus, \( u \) may have zeros of large orders, and consequently, functions in \( L_u^p \) are not generally locally integrable. The motivation for considering such \( u \) stems from several places. First, they arise in [8] in multiplier questions for \( L_p^p \). Since multiplier results are also derived in [9] for \( H_u^p \), we wished to relate \( L_p^p \) and \( H_u^p \) for this type of \( u \).

As further motivation for studying such \( u \), let us consider a variant of the Hilbert transform. We introduce this variant only for illustration since our proofs do not require any facts about Hilbert transforms. Let \( u = |q|^p w \) where \( q \) is a polynomial, \( 1 < p < \infty \), and \( w \in A_p \), and define
\[
H_q f(x) = \text{p.v.} \int_R f(z) \left[ \frac{1}{x - z} - \frac{1}{q(x)} \frac{q(x) - q(z)}{x - z} \right] \, dz.
\]

The identity
\[
\frac{1}{x - z} - \frac{1}{q(x)} \frac{q(x) - q(z)}{x - z} = \frac{1}{q(x)} \frac{q(z)}{x - z}
\]
shows that \( H_q f(x) = (fq)^- (x)/q(x) \), where \( "^- " \) denotes the ordinary Hilbert transform. Hence, by the principal result of [4], since \( w \in A_p \), we have
\[
\|H_q f\|_{p, u} = \|(fq)^-\|_{p, w} \leq c\|f\|_{p, u} = c\|f\|_{p, u},
\]
that is, \( H_q \) is bounded on \( L_p^p \).

Next note that if \( q \) has degree \( M \), then \( (q(x) - q(z))/(x - z) \) is a polynomial in \( z \) of degree \( M - 1 \). Hence, if the first \( M \) moments of \( f \) exist and equal zero, i.e., if
\[
\int_R f(z) z^i \, dz = 0, \quad i = 0, \ldots, M - 1,
\]
it follows that \( H_q f = \tilde{f} \). Condition (1.2) is clearly satisfied for any \( M \) if \( f \) belongs to the space \( \mathcal{S}_{0,0} \) of functions in \( \mathcal{S} \) whose Fourier transforms have compact support not containing the origin. Conversely, a result of E. Adams [1] states that if \( u \) satisfies
the doubling condition and the mapping \( f \to \tilde{f} \) satisfies \( \| \tilde{f} \|_{p,u} \leq c \| f \|_{p,u} \) for all \( f \in \mathbb{S}_{0,0} \), then \( u \) must have the form \( u = |q|^p w \) where \( q \) is a polynomial and \( w \in A_p \). Thus, in a sense, such \( u \) are the only weights for which the Hilbert transform is bounded. This can be used to show that if \( u \) satisfies \( A_r \) for any \( r \) and the mapping \( f \to Mf \) satisfies \( \| Mf \|_{p,u} \leq c \| f \|_{p,u} \) for all \( f \in \mathbb{S}_{0,0} \), i.e., if
\[
(1.3) \quad \| f \|_{H_p^u} \leq c \| f \|_{L_p^u}, \quad f \in \mathbb{S}_{0,0},
\]
then \( u \) must have the form above. In fact, the norm boundedness of \( Mf \) implies that of the usual (harmonic) Lusin area integral \( Sf \), and since \( Sf = S\tilde{f} \), also of \( M\tilde{f} \) (see [3, 9]). Since \( M\tilde{f} \) pointwise exceeds \( |\tilde{f}| \), it follows that the assumptions of Adams’ result hold, and therefore that \( u \) has the desired form.

The fact that \( H_q \) is bounded on \( L_p^u \) for such \( u \) makes it reasonable to conjecture that \( L_p^u \) and \( H_p^u \) should coincide. This, however, is not generally the case. The key point to consider turns out to be whether or not all the zeros of \( q \) are real. We will systematically use the notation \( Q(x) \) to denote a polynomial all of whose zeros are real. The degree of \( Q \) will always be denoted \( N \) and its distinct zeros will be \( \{a_k\}_{k=1}^N \).

We normalize \( Q \) so that
\[
Q(x) = \prod_{k=1}^n (x - a_k)^{\mu_k},
\]
\( \mu_k \) being the order (multiplicity) of the zero at \( a_k \). Associated with the partial fraction decomposition of \( 1/Q \), namely
\[
(1.4) \quad \frac{1}{Q(x)} = \sum_{k=1}^n \sum_{l=1}^{\mu_k} \frac{A_{k,l}}{(x - a_k)^l},
\]
will be the distribution
\[
(1.5) \quad \Theta = \Theta Q = \sum_{k=1}^n \sum_{l=1}^{\mu_k} \frac{A_{k,l}}{(l - 1)!} \delta_{a_k}^{(l-1)},
\]
where \( \delta_{a_k}^{(i)} \) denotes the \( i \)th derivative of the delta function at \( a_k \). \( \Theta \) is of course supported at the \( a_k \)’s. If \( F(x, y) \) is a function, \( \Theta \), \( F \) will denote the action of \( \Theta \) on \( F \) as a function of \( y \). It follows directly from (1.4) and (1.5) that
\[
\Theta Q(1/ (x - y)) = 1/Q(x).
\]
If \( \phi \) is a function whose derivatives of order \( \mu_k - 1 \) exist at \( a_k \), \( k = 1, \ldots, n \), define
\[
(1.6) \quad \Theta Q(\phi) = \Theta Q(\phi(x)) = Q(x) \Theta Q(\phi(y)/ (x - y)).
\]
We will discuss the principal properties of \( \Theta \) in the next two sections. Here, we mention that \( \Theta \) is a polynomial in \( x \) whose first \( \mu_k - 1 \) derivatives at \( a_k \) coincide with the corresponding derivatives of \( \phi \) at \( a_k \). Such a polynomial is called an interpolating polynomial; rather than trying to deduce its properties from known facts, we will give direct derivations based on (1.6).

We introduce \( \Theta \) in order to consider modified convolution operators
\[
\int_{\mathbb{R}} f(z) \left[ \phi(x - z) - \Theta Q(\phi(x - z))(z) \right] dz.
\]
Our motivation again comes from the Hilbert transform. In fact, the term subtracted in (1.1) from the usual kernel \(1/(x-z)\) of the Hilbert transform may be written

\[
\frac{1}{Q(x)} \frac{Q(x) - Q(z)}{x - z} = \frac{Q(z)}{x - z} \left[ \frac{1}{Q(z)} - \frac{1}{Q(x)} \right]
\]

\[
= \frac{Q(z)}{x - z} \left[ \frac{1}{Q(x)} - \frac{1}{Q(z)} \right]
\]

\[
= \frac{Q(z)}{x - z} \left[ \frac{1}{Q(x)} - \frac{1}{Q(z)} \right]
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\]

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\]

Our main results for the case when \(q\) has only real zeros are given in the following two theorems.

**Theorem 1.** Let \(1 < p < \infty\), \(Q\) be a polynomial of degree \(N\) with all real zeros, and

\[
f(x, t) = \int_{\mathbb{R}} f(z) \left[ \phi_t(x - z) - \frac{Q}{\phi_{\lambda}(x - z)}(z) \right] dz,
\]

where \((1 + |x|)^{\alpha + 1} |\phi^{(\alpha)}(x)|\) is bounded for \(\alpha = 0, \ldots, N\). If

\[
N^\lambda f(x) = \sup_{(\xi, t) \in \mathbb{R}^2} \left( \frac{t}{t + |x - \xi|} \right)^\lambda |f(\xi, t)|,
\]

then for \(\lambda \geq N + 1, \lambda > 1\), there is a constant \(c\) independent of \(f\) such that \(\|N^\lambda f\|_{p, u} \leq c \|f\|_{p, w}\), where \(u = |Q|^p w, w \in A_p\). In particular, the nontangential maximal function \(Mf\) satisfies \(\|Mf\|_{p, u} \leq c \|f\|_{p, u}\).

We will also prove a weak-type version of this result when \(p = 1\).

**Theorem 2.** Let \(1 < p < \infty\) and \(u = |Q|^p w\) where \(Q\) is a polynomial with all real zeros and \(w \in A_p\). Then \(H^p_u\) and \(L^p_u\) can be identified in the following sense: there is a unique correspondence between distributions \(l \in H^p_u\) and functions \(f \in L^p_u\) given by

\[
\langle l, \phi \rangle = \int_{\mathbb{R}} f(z) \left[ \phi(z) - \frac{Q}{\phi}(z) \right] dz, \quad \phi \in \mathcal{S}.
\]

Moreover, in this correspondence \(\|l\|_{H^p_u}\) and \(\|f\|_{L^p_u}\) are equivalent.

As a corollary of Theorem 2, we see that the \(H^p_{|Q|^p w}\) norm of \(f\) is equivalent to the \(H^p_u\) norm of \(fQ\). We will also show directly in §6 that the function

\[
f(x, t) = \langle l_f, \phi_t(x - \cdot) \rangle
\]

converges pointwise almost everywhere and in \(L^p_u\) norm to \(f\) as \(t \to 0\).

If \(q\) has \(d\) complex roots, counting multiplicities, then

\[
c_1 \leq |q(x)| / \left(1 + |x|^2\right)^{d/2} |Q(x)| \leq c_2
\]

for positive constants \(c_1\) and \(c_2\), where \(Q\) contains all the real zeros of \(q\). Hence, we may assume without loss of generality that

\[
u(x) = \left(1 + |x|^2\right)^{d/2} |Q(x)|^p w(x).
\]
Again using the Hilbert transform for motivation, note that if \( q = q_1Q \) where \( q_1 \) has degree \( d > 1 \), then

\[
\frac{1}{q(x)} \frac{q(x) - q(z)}{x - z} - \frac{1}{Q(x)} \frac{Q(x) - Q(z)}{x - z} + \frac{1}{q(x)} \frac{q_1(x) - q_1(z)}{x - z}.
\]

Since \( (q_1(x) - q_1(z))/(x - z) \) is a polynomial in \( z \) of degree \( d - 1 \), the second term on the right is the sum of terms of the form \( c_i(x)Q(z)z^i \), \( i = 0, \ldots, d - 1 \). It follows that if

\[
\int f(z)Q(z)z^i \, dz = 0, \quad i = 0, \ldots, d - 1,
\]

then

\[
\frac{1}{Q(x)} \frac{Q(x) - Q(z)}{x - z} = \text{p.v.} \frac{1}{Q(z)} \frac{Q(z) - Q(x)}{x - z} \int f(z)Q(z)z^i \, dz = H_Q f(x).
\]

Thus, \( H_Q \) is bounded on the subset of \( L^p_u \) of functions \( f \) satisfying (1.7). This subset can be identified with \( H^p_u \) as the following result states.

**Theorem 3.** Let \( 1 < p < \infty \), \( d \) be a positive integer and \( u = (1 + |x|^2)^{d_p/2} |Q|^p \), where \( Q \) is a polynomial with all real zeros and \( w \in A_p \). Then \( H^p_u \) can be identified with the subspace of \( L^p_u \) of \( f \) with \( \int fQx^i \, dx = 0 \) for \( i = 0, \ldots, d - 1 \). The identification is given by

\[
\langle l, \phi \rangle = \int f(z) \left[ \phi(z) - Q_Q(z) \right] \, dz,
\]

\( l \in H^p_u, f \in L^p_u, \) and \( \|l\|_{H^p_u} \) is equivalent to \( \|f\|_{L^p_u} \).

As a corollary of Theorem 3, we will see that the \( H^p \) norm of \( f \) with weight \( u' = (1 + |x|^2)^{d_{p'}/2} |Q|^{p'} w \) is equivalent to the \( H^p \) norm of \( fQ \) with weight \( (1 + |x|^2)^{d_{p'}/2} w \). We will also derive the general form for embeddings of \( L^p_u \) in \( H^p_u \) which are the identity on \( H^p_u \).

In the last section of the paper, we prove several results about the density of \( S_{0,0} \) in \( H^p_u \). While our proofs are direct, we mention that some of the results can be obtained indirectly as corollaries of Theorems 2 and 3 and the density results in \([9]\).

Finally, we list some notation and basic facts. From the definition of \( A_p \), we have that \( w \in A_p, 1 < p < \infty \), if and only if \( w^{-1/(p-1)} \in A_{p'}, 1/p + 1/p' = 1 \). Also, from \([4]\), if \( w \in A_p \) then

\[
\int_{|x| > r} \frac{w(x)}{|x|^{d_p}} \, dx \leq \frac{c}{r^p} \int_{|x| < r} w(x) \, dx, \quad r > 0.
\]

In particular, both \( w(x)(1 + |x|)^{-p} \) and \( w(x)^{-1/(p-1)}(1 + |x|)^{-p'} \) are integrable over \( \mathbb{R} \) if \( w \in A_p \). If \( I \) is an interval and \( s > 0 \), let \( sI \) denote the interval concentric with \( I \) whose length is \( s |I| \). A weight \( u \) is said to satisfy the doubling condition of order \( \beta > 0 \) if there is a constant \( c \) independent of \( s \) such that \( \int_I u \, dx \leq cs^\beta \int_I u \, dx \), \( s > 1 \). It follows easily that any \( w \in A_p \) satisfies this condition with \( \beta = p \), and that any \( u \) of the form \( u = |q|^p w \) where \( q \) has degree \( M \) and \( w \in A_p \) satisfies it with \( \beta = (M + 1)p \).
About $H^p_u$, we will use the fact that $\|N_\lambda f\|_{L^p_u} \leq c \|f\|_{L^p_u}$ if $u$ satisfies the doubling condition of order $\beta$ and $\lambda > \beta/p$. In particular, if $u \in A_p$, $1 < p < \infty$, the fact that $H^p_u$ and $L^p_u$ can be identified then shows that $\|N_\lambda f\|_{L^p_u} \leq c \|f\|_{L^p_u}$ if $\lambda > 1$. Finally, we will use the fact that if $u$ satisfies any doubling condition, then a distribution $l$ whose radial maximal function

$$M_0l(x) = \sup_{t>0} |\langle l, \phi_t(x - \cdot)\rangle|$$

belongs to $L^p_u$ for some $\phi \in \Delta$ with $\int_{\mathbb{R}} \phi \, dx \neq 0$ also belongs to $H^p_u$. Proofs of these results may be found in [9].

2. Preliminaries. Let $Q$ and $\mathcal{O}$ be as in the introduction: that is, let $Q(x) = \prod_{k=x}^n (x - a_k)^{\mu_k}, a_k$ real and distinct, $\Sigma_{k=1}^n \mu_k = N$, and let

$$\mathcal{O} = \mathcal{O} Q = \sum_{k=1}^{\mu_1} \sum_{l=1}^N \frac{A_{k,l}}{(l-1)!} \delta_{a_k}^{(l-1)},$$

where $Q(x)^{-1} = \sum_{k=1}^n \sum_{l=1}^{\mu_k} (A_{k,l}/(x - a_k)^l)$. As noted in the introduction, we have (2.1)

$$\mathcal{O}_x((x - y)^{-1}) = (Q(x))^{-1}.$$

Lemma (2.2).

(i) If $P$ is a polynomial of degree $M$, then $\mathcal{O}_y(P(x) - P(y))/(x - y)$ is a polynomial of degree at most $M - 1$.

(ii) If $\phi \in C^\infty$, then $\mathcal{O}_y[(\phi(x) - \phi(y))/(x - y)] \in C^\infty$.

(iii) If $\phi \in C^\infty$ and $\mathcal{B}$ is any distribution of the form $\mathcal{B} = \sum_{k=x}^n \sum_{l=1}^{\mu_k} B_{k,l} \delta_{a_k}^{(l-1)}$, then $\mathcal{B}(\phi Q) = 0$.

Proof. (i) $(P(x) - P(y))/(x - y)$ is a polynomial in $x, y$ of degree $M - 1$. Applying $\mathcal{O}_y$ to it produces a polynomial in $x$ of degree at most $M - 1$.

(ii) It is enough to show that $(\phi(x) - \phi(y))/(x - y)$ is infinitely differentiable on $\mathbb{R} \times \mathbb{R}$. This follows easily from the formula

$$\frac{\phi(x) - \phi(y)}{x - y} = \int_0^1 \phi'(y + s(x - y)) \, ds.$$

(iii) It is enough to show each $\delta_{a_k}^{(l-1)}(\phi Q) = 0$, $1 \leq l \leq \mu_k$. By Leibniz’s rule,

$$\delta_{a_k}^{(l-1)}(\phi Q) = \sum_{j=0}^{l-1} \binom{l-1}{j} \phi^{(l-1-j)}(a_k) Q^{(j)}(a_k) = 0$$

since $Q^{(j)}(a_k) = 0$ for $0 \leq j \leq \mu_k - 1$.

Lemma (2.3). If $\phi \in C^\infty$, then $Q(x) \mathcal{O}_y[\phi(y)/(x - y)]$ is a polynomial of degree at most $N - 1$.

Proof. It is enough to show that each $Q(x) \delta_{a_k}^{(l-1)}[\phi(y)/(x - y)]$, $1 \leq l \leq \mu_k$, $1 \leq k \leq n$, is a polynomial of degree at most $N - 1$. But $\delta_{a_k}^{(l-1)}[\phi(y)/(x - y)]$ is a linear combination of terms $(x - a_k)^{-i}, 1 \leq i \leq l$, and since $Q(x)$ contains the factor $(x - a_k)^{\mu_k}$, the conclusion follows.
We use the notation \( \mathcal{P}_\phi(x) = \mathcal{P}_\phi^Q(x) = Q(x) \mathcal{P}_\phi^Q[\phi(y)/(x - y)] \) for the polynomial in Lemma (2.3). The next lemma shows that \( \mathcal{P}_\phi^Q \) is the interpolating polynomial for \( \phi \) based on \( Q \).

**Lemma (2.4).** Let \( \mathcal{B} \) be any distribution of the form \( \mathcal{B} = \sum_{k=1}^n \sum_{l=1}^d B_{k,l} \delta^{(l-1)}_a \). If \( \phi \in C^\infty \), then \( \mathcal{B}(\phi - \mathcal{P}_\phi^Q) = 0 \).

**Proof.** We have
\[
\phi(x) - \mathcal{P}_\phi^Q(x) = \phi(x) - Q(x) \mathcal{P}_\phi^Q[\phi(y)/(x - y)] = Q(x) \mathcal{P}_\phi^Q[(\phi(x) - \phi(y))/(x - y)] = Q(x) \psi(x)
\]
for some \( \psi \in C^\infty \), by (2.1) and Lemma (2.2)(ii). By Lemma (2.2)(iii), \( \mathcal{B}(Q \psi) = 0 \), and the proof is complete.

Note that since \( \mathcal{P}_\phi^Q \) has degree at most \( N - 1 \), it is the only polynomial of degree at most \( N - 1 \) with \( \mathcal{B}(\phi - \mathcal{P}_\phi^Q) = 0 \) for all \( \mathcal{B} \) as above.

We now list properties of \( \mathcal{P} \) and \( \mathcal{P}^Q \) which will be useful.

**Lemma (2.5).**

(i) \( \mathcal{P}_\phi^Q \) is a polynomial of degree at most \( N - 1 \), then \( \mathcal{P}_\phi^Q[P(y)/(x - y)] = P(x)/Q(x) \). Equivalently, \( \mathcal{P}_\phi^Q(x) = P(x) \) for such \( P \).

(ii) \( \mathcal{P}_\phi^Q(x) = 0 \) if \( \mathcal{B} \in C^\infty \).

**Proof.** Part (i) follows immediately from the uniqueness of \( \mathcal{P} \) mentioned above. Part (ii) is similar, using Lemma (2.2)(iii).

**Lemma (2.6).** If \( \phi \in \mathcal{S} \) and \( Q \) has degree \( N \), then for any \( z \)
\[
\int_{\mathbb{R}} \left( \frac{\phi(x - z) - \mathcal{P}_\phi^Q}{x^j} \right) x^j \, dx = 0, \quad j = 0, \ldots, N - 1.
\]

**Proof.** The integral may be written
\[
Q(z) \mathcal{P}_\phi^Q \left\{ \int_{\mathbb{R}} \left[ \frac{\phi(x - z) - \phi(x - y)}{y} \right] x^j \, dx/ (z - y) \right\}.
\]
We have \( \int_{\mathbb{R}} \phi(x - z)x^j \, dx = \int_{\mathbb{R}} \phi(x)(x - z)^j \, dx = P(z) \), where \( P \) is a polynomial in \( z \) of degree at most \( j \). The expression above then equals
\[
Q(z) \mathcal{P}_\phi^Q \left\{ \left( \frac{P(z) - P(y)}{z - y} \right) \right\} = 0 \quad \text{if} \quad j \leq N - 1
\]
by Lemma (2.5)(i).

Next, we list some relations between the \( \mathcal{P} \)'s and \( \mathcal{P}^Q \)'s associated with \( Q(x) \) and \( xQ(x) \).

**Lemma (2.7).** (i) If \( \phi \in C^\infty \), then \( \mathcal{P}_\phi^{xQ}(\phi) = \mathcal{P}_\phi^Q \left[ ((\phi(y) - \phi(0))/y \right] \).

(ii) If \( \phi \in C^\infty \), then \( \mathcal{P}_\phi^{xQ} = \mathcal{P}_\phi^Q + \mathcal{P}_\phi^{xQ}(\phi) \cdot Q \).

**Proof.** (i) Note that \( (\phi(y) - \phi(0))/y = \int_0^1 \phi'(sy) \, ds \), so that \( (\phi(y) - \phi(0))/y \) is also smooth and equals \( \phi'(0) \) at \( y = 0 \). Let \( \mathcal{P}_\phi^Q(\phi) = \mathcal{P}_\phi^Q \left[ ((\phi(y) - \phi(0))/y \right] \). Then both \( \mathcal{P}_\phi^Q \) and \( \mathcal{P}_\phi^{xQ} \) are linear combinations of derivatives of delta functions. To show that \( \mathcal{P}_\phi(\phi) = \mathcal{P}_\phi^{xQ}(\phi) \), it is then enough, using the uniqueness of partial fraction
decompositions, to show that \( \partial_y[1/(x - y)] = \partial_y x^Q[1/(x - y)] \). However, by (2.1), 
\[ \partial_y x^Q[1/(x - y)] = 1/xQ(x) \], while 
\[ \partial_y \left( \frac{1}{x - y} \right) = \partial_y \left[ \frac{1}{(x - y)(x - y)} \right] \]
by (2.1) again.

To prove (ii), write \( x = (x - y) + y \) to obtain
\[ x \partial_y x^Q \left( \frac{\phi(y)}{x - y} \right) = \partial_y x^Q \left( \frac{x\phi(y)}{x - y} \right) = \partial_y x^Q(\phi) + \partial_y x^Q \left( \frac{y\phi(y)}{x - y} \right). \]
By (i), the last term on the right equals \( \partial_y x^Q[\phi(y)/(x - y)] \), and (ii) follows immediately by multiplying the resulting identity by \( Q(x) \).

By repeated application of formula (ii), we obtain for any positive integer \( d \)
\[ (2.8) \partial_\phi x^Q(z) = \partial_\phi Q(z) + \sum_{j=0}^{d-1} \partial_\phi x^{j+1}Q(\phi) \cdot z^jQ(z). \]

Finally, we list some relations between \( \partial_\phi \) and ordinary Taylor polynomials.

**Lemma (2.9).** With the usual notation, the following formulas hold:

(i) \[ \partial_\phi (x-a)^{\gamma}(z) = \sum_{i=0}^{l-1} \frac{\phi^{(i)}(a)}{i!}(z-a)^i; \]

(ii) \[ \partial_\phi Q(z) = Q(z) \sum_{k=1}^{n} \sum_{l=1}^{\mu_k} \frac{A_{k,l}}{(z-a_k)^l} \partial_\phi (x-a_k)^{\gamma}(z); \]

(iii) \[ \partial_\phi Q(z) = P(z) + Q(z) \sum_{k=1}^{n} \sum_{l=1}^{\mu_k} \frac{A_{k,l}}{(z-a_k)^l} \partial_\phi (x-a_k)^{\gamma}(z), \]

where in (iii), \( P(z) \) is any polynomial of degree at most \( N - 1 \).

**Proof.** By definition,
\[ \partial_\phi Q(z) = Q(z) \partial_y \left( \frac{\phi(y)}{z-y} \right) = Q(z) \sum_{k=1}^{n} \sum_{l=1}^{\mu_k} \frac{A_{k,l}}{(l-1)!} \delta_{a_k,y}^{(l-1)} \left( \frac{\phi(y)}{z-y} \right). \]

Hence,
\[ \partial_\phi (x-a)^{\gamma}(z) = (z-a)^l \cdot \frac{1}{(l-1)!} \delta_{a,y}^{(l-1)} \left( \frac{\phi(y)}{z-y} \right), \]
and (i) follows by direct computation with Leibniz’s rule. Part (ii) follows from the last two formulas by substituting the second into the first. To prove (iii), write \( \phi = P + (\phi - P) \) to obtain \( \partial_\phi Q = \partial_\phi P + \partial_\phi Q_{\phi-P} = P + \partial_\phi Q_{\phi-P} \), since \( P \) has degree at most \( N - 1 \). Applying (ii) to the last term on the right gives (iii).
Formula (iii) may be used to compare $\Phi^Q_\phi$ to any $P$ of degree $\leq N - 1$. We will choose $P = \Phi^x_\phi$, the Taylor polynomial of $\phi$ of order $N - 1$ around the origin. The derivatives of a Taylor polynomial satisfy

$$
(\Phi^x_\phi)^{(j)} = \Phi^{x-N-j}_\phi, \quad j = 0, \ldots, N - 1,
$$

which follows easily for $j = 1$ from (i) of the last lemma, and for $j > 1$ by repeated application of the case $j = 1$. Using this and (i) of the lemma, we have

$$
\begin{align*}
\Phi^{x-a_k y}_{\Phi^x_\phi}(z) &= \sum_{j=0}^{N-1} \frac{1}{j!} (\phi - \Phi^x_\phi)^{(j)}(a_k)(z - a_k)^j \\
&= \sum_{j=0}^{N-1} \frac{1}{j!} \left( \Phi^{(j)}(a_k) - \Phi^{x-N-j}_\phi(a_k) \right) (z - a_k)^j \\
&= \sum_{j=0}^{N-1} \frac{1}{j!} \left( \Phi^{(j)}(a_k) - \sum_{i=0}^{N-j-1} \frac{1}{i!} \phi^{(j+i)}(0)a_k^i \right) (z - a_k)^j.
\end{align*}
$$

Hence, by (iii) of the lemma,

$$
\Phi^Q_\phi(z) = \sum_{j=0}^{N-1} \frac{\phi^{(j)}(0)}{j!} z^j + Q(z) \sum_{k=1}^{n} \sum_{l=1}^{\mu_k} \frac{A_{k,l}}{(z - a_k)^l}
$$

(2.10)

$$
\cdot \sum_{j=0}^{N-1} \frac{1}{j!} \left( \Phi^{(j)}(a_k) - \sum_{i=0}^{N-j-1} \frac{1}{i!} \phi^{(j+i)}(0)a_k^i \right) (z - a_k)^j.
$$

3. Basic estimates. The main results of this section concern estimates for the interpolating polynomial $\Phi_{\phi(x,y)}(z)$ and the remainder $\phi(x - z) - (\Phi_{\phi(x,y)}(z)$ which are uniform in $t > 0$. These estimates are given in Lemma (3.3). We will obtain tempered tangential (and therefore also nontangential) estimates as corollaries. Throughout this section, $Q$ will denote a polynomial of degree $N$ with only real zeros $\{a_k\}$ and $\Phi_\phi = \Phi^Q_\phi$. Moreover, by $A_1$, we mean the class of weights $w$ with $|I|^{-1/2} w dx \leq c \text{ ess inf } w$.

The following simple preliminary lemma will be useful later.

**Lemma (3.1).** If $\phi$ has $N$ bounded derivatives and $c_\phi = \max_{j=0, \ldots, N} ||\phi^{(j)}||_\infty$, then

$$
|\phi(x) - \Phi_\phi(x)| \leq c_\phi |Q(x)|/(1 + |x|).
$$

Moreover, if $1 \leq p < \infty$, $w \in A_p$ and $u = |Q|^p w$, then $\phi - \Phi_\phi$ belongs to $L^p_u$, with norm bounded by a constant depending on $w$ times $c_\phi$. (When $p = 1$, we interpret this to mean that $(\phi - \Phi_\phi)/Qw$ is bounded by a multiple of $c_\phi$.)

**Proof.** We have $\phi(x) - \Phi_\phi(x) = Q(x)\psi(x)$, where

$$
\psi(x) = \Phi_\phi \left( \frac{\phi(x) - \phi(y)}{x - y} \right).
$$


As noted in Lemma (2.2)(ii) and its proof, \( \psi \in C^\infty \) and
\[
\frac{\phi(x) - \phi(y)}{x - y} = \int_0^1 \phi'(sx + (1 - s)y) \, ds.
\]
Applying \( \partial_x \) to both sides and recalling that \( \partial_x \) has order \( N - 1 \), we see \( |\psi(x)| \leq c_\phi \).
Moreover, for large \( |x| \), formula (3.2) gives \( |\psi(x)| \leq c_\phi |x|^{-1} \). Combining estimates, we obtain the first statement of the lemma.

For the second statement, as noted in the introduction,
\[
\int_\mathbb{R} \frac{w(x)^{-1/(p-1)}}{(1 + |x|)^p} \, dx = c_p < \infty
\]
if \( w \in A_p, 1 < p < \infty \). The analogue for \( p = 1 \) is \( w(x)^{-1}/(1 + |x|) \leq c < \infty \).
These easily imply that \( |Q(x)|/(1 + |x|) \in L^p_{\mathbb{R}} \) with norm bounded by \( c \), and the second statement follows from the first.

**Lemma (3.3).** Let \( z \) and \( a_{k_0} \) satisfy \( |z - a_{k_0}| = \min_k |z - a_k| \). Then

(i) \( \sup_{t \geq 0} |\psi_t(z - x) - \psi_t(z)| \leq c \frac{|Q(z)|}{|Q(x)|} \frac{1}{|x - a_{k_0}|} \),

if \( |z - a_{k_0}| \leq \frac{1}{2} |x - a_{k_0}| \);

(ii) \( \sup_{t \geq 0} |\partial_t \psi_t(z)| \leq c \frac{|Q(z)|}{|Q(x)|} \frac{1}{|z - a_{k_0}|} \),

if \( |z - a_{k_0}| \geq \frac{1}{2} |x - a_{k_0}| \);

(iii) \( |Q(x)| \leq c |Q(z)| \),

if \( |z - a_{k_0}| \geq \frac{1}{2} |x - a_{k_0}| \).

In all cases, the constant \( c \) is independent of \( x \) and \( z \); in fact, in (iii), \( c \) depends only on the degree of \( Q \), and in (i) and (ii) it depends only on \( Q \) and the bounds on \( (1 + |x|)^{\alpha + 1} |\phi^{(\alpha)}(x)|, \alpha = 0, \ldots, N \).

**Proof.** For (iii), note \( |x - z| \leq |x - a_{k_0}| + |a_{k_0} - z| \leq 3 |z - a_{k_0}| \leq 3 |z - a_k| \) for any \( k \). Hence, \( |x - a_k| \leq |x - z| + |z - a_k| \leq 4 |z - a_k| \). Thus, each factor of \( |Q(x)| \) is bounded by \( 4 \) times the corresponding factor of \( |Q(z)| \).

It is enough to prove (i) and (ii) in case \( a_{k_0} = 0 \); that this is so follows by denoting \( \tau_\alpha \phi(x) = \phi(x - a) \) and observing that
\[
(3.4) \quad \partial_t \tau_\alpha \phi(Q(z)) = (\tau_{\alpha} \partial_t \phi)(z).
\]

Choose \( c_0 \) so that \( |\phi^{(\alpha)}(x)| \leq c_0 (1 + |x|)^{-\alpha - 1} \) for \( \alpha = 0, \ldots, N \). Then \( |\phi_t^{(\alpha)}(x)| = t^{-\alpha - 1} |\phi^{(\alpha)}(x/t)| \leq c_0 (t + |x|)^{-\alpha - 1} \leq c_0 |x|^{-\alpha - 1} \) for such \( \alpha \). We will first prove (ii) for \( x \) in a bounded set, say \( |x| \leq A = 10 \max |a_k| + 1 \). From the definition of \( \partial_t \),
\[
(\partial_t \phi)(x - y) = \left| \begin{aligned}
|\phi_t(x - y)|^{1/(1 + |x|)} & \leq c \max_k \max_{0 \leq i + j \leq \mu_k} |\phi_t^{(i)}(x - a_k)| \\
& \leq c c_0 \max_k \max_{0 \leq i + j \leq \mu_k} |z - a_k|^{-\alpha - 1} |x - a_k|^{\alpha + 1}.
\end{aligned} \right.
\]
Since \( a_{k_0} = 0 \) is the closest \( a_k \) to \( z \), \(|z - a_k| \geq |z|\) for all \( k \). For the same reason, there is a constant \( c > 0 \) depending on \( Q \) so that if \( k \neq k_0 \), \(|z - a_k| \geq c > 0\). Moreover, if \( k = k_0 \), \(|z - a_{k_0}| = |z| > |x|/2\) by hypothesis. Hence, the last expression is at most

\[
\frac{cc_0}{|z|} \left( \max_{k \neq k_0} \max_{0 \leq i < \mu_k} \frac{1}{|x - a_k|^{i+1}} + \max_{0 \leq i + j < \mu_{k_0}} \frac{1}{|x|^{i+j+1}} \right).
\]

Since \(|x| \leq A\), this is bounded by

\[
\frac{cc_0}{|z|} \max_k \frac{1}{|x - a_k|^{\mu_k}} \leq \frac{cc_0}{|z| |Q(x)|}.
\]

Using the definition of \( \mathcal{P} \), we then obtain (ii) for \(|x| \leq A\).

To prove (ii) for large \(|x|\), i.e., \(|x| > A\), we apply formula (2.10) to the function \( \phi_t(x - \cdot) \), \( x \) fixed, to obtain

\[
\mathcal{P}_{\phi_t(x - \cdot)}(z) = \sum_{j=0}^{N-1} \frac{(-1)^j}{j!} \phi^{(j)}(x) z^j + Q(z) \sum_{k=1}^n \sum_{l=1}^{\mu_k} \frac{A_{k,l}}{(z - a_k)^l}
\]

(3.5)

\[
\cdot \sum_{j=0}^{l-1} \frac{(-1)^j}{j!} \left( \phi^{(j)}(x - a_k) - \sum_{i=0}^{N-j-1} \frac{(-1)^i}{i!} \phi^{(j+i)}(a_k)(x-a_k)^i \right) (z-a_k)^j.
\]

Now \(|z|\) is also large since \(|z| > |x|/2\). The first sum on the right of (3.5) is bounded in absolute value by \(\sum_{j=0}^{N-1} c_0 \frac{|x|^{-j-1}}{|z|} \). However,

\[
|x|^{-j-1} |z|^j = \left( \frac{|z|}{|x|} \right)^{j+1} \frac{1}{|z|} = c \left( \frac{|z|}{|x|} \right)^N \frac{1}{|z|}
\]

since \( j + 1 \leq N \) and \(|z| > |x|/2\). Since \(|x|\) is large, \(|Q(x)| \sim |x|^N\), that is, \(|Q(x)|\) is bounded above and below by positive multiples of \(|x|^N\). Also \(|z|^N \sim |Q(z)|\), and therefore the first sum on the right of (3.5) is bounded by \(cc_0 |Q(z)|/|Q(x)||z|\), as desired.

Consider the second term on the right in (3.5). Since \(|x|\) is large, Taylor's theorem shows that each term in curly brackets is bounded by a multiple of \( |\phi^{(N)}| \) evaluated at a point whose absolute value is comparable to \(|x|\). Thus, each term in curly brackets is bounded by \(cc_0 |x|^{-N-1}\), and so by \(cc_0 |x|^{-N}\). Also, since \(|z|\) is large and \( j \leq l - 1 \), \(|z - a_k|^{-j} \leq c |z|^{-l+j} \leq c |z|^{-1}\). Combining these facts, we see that for \(|x| > A\) and \(|z| > |x|/2\) the second term on the right of (3.5) is bounded in absolute value by

\[
|Q(z)| |z|^{-1} cc_0 |x|^{-N} \leq cc_0 |Q(z)|/|Q(x)||z|.
\]

This completes the proof of (ii).

To prove (i) for \(|x| < A\), write

\[
\phi_t(x - z) - \mathcal{P}_{\phi_t(x - \cdot)}(z) = Q(z) \mathcal{P}_y \left( \phi_t(x - z) - \phi_t(x - y) \right) \frac{z - y}{z - y}.
\]
Split $\mathcal{Q}_y = \mathcal{Q}_y^0 + \mathcal{Q}_y^1$ where $\mathcal{Q}_y^0$ is supported at $a_{k_0} = 0$ and $\mathcal{Q}_y^1$ at $\{a_k\} \setminus \{0\}$. Since

$$
\phi_t(x - z) - \phi_t(x - y) = \frac{-\int_0^1 \phi'_t(x - y - s(z - y)) \, ds}{z - y},
$$

we have

$$
\left| \mathcal{Q}_y^0 \left( \frac{\phi_t(x - z) - \phi_t(x - y)}{z - y} \right) \right| \leq c \max_{0 < \alpha < \mu_{k_0}} \int_0^1 \left| \phi_{\alpha + 1}(x - sz) \right| \, ds
$$

(3.6)

where $\alpha < \mu_{k_0}$, and

$$
\left| \mathcal{Q}_y^1 \left( \frac{\phi_t(x - z) - \phi_t(x - y)}{z - y} \right) \right| \leq c \max_{0 < \alpha < \mu_{k_0}} \left| \phi_{\alpha + 1}(x - u) \right|.
$$

By assumption, $|u| \leq |z| < |x|/2$, and formula (3.6) is therefore bounded by $c c_0 \max_{0 < \alpha < \mu_{k_0}} |x|^{-\alpha - 1}$. For $|x| < A$, this is at most

$$
c c_0 \left| x \right|^{-k_0 - 1} \leq c c_0 \left| x \right|^{-1} \left| Q(x) \right|^{-1},
$$

as desired. Also,

$$
\left| \mathcal{Q}_y^0 \left( \frac{\phi_t(x - z) - \phi_t(x - y)}{z - y} \right) \right| \leq c \left\{ \left| \phi_t(x - z) \right| + \max_{0 < \alpha < \mu_k} \left| \phi_{\alpha + 1}(x - a_k) \right| \right\}
$$

(3.7)

since $|z - a_k| \geq c > 0$ for $k \neq k_0$. This is at most

$$
c c_0 \left\{ \left| x - z \right|^{-1} + \max_{k; 0 < \alpha < \mu_k} \left| x - a_k \right|^{-\alpha - 1} \right\}.
$$

We have $|x - z| \geq |x| - |z| > |x|/2$. Hence, since $x$ is bounded, the last estimate is at most $c c_0 \max_k |x - a_k|^{-\mu_{k_0}}$, so that

$$
\left| \mathcal{Q}_y^1 \left( \frac{\phi_t(x - z) - \phi_t(x - y)}{z - y} \right) \right| \leq c c_0 \frac{1}{\left| Q(x) \right|}, \quad |x| < A.
$$

This is bounded by $c c_0 \left| Q(x) \right|^{-1} |x|^{-1}$ since $|x| < A$, and (i) follows for $|x| < A$.

To prove (i) for $|x| > A$, we use (3.5). Recall that each term there in curly brackets is bounded by $c \left| \phi_{k(N)}(\xi) \right| \left| \xi \right|^{-1} \left| x \right|$, and so is bounded by $c c_0 \left| x \right|^{-N - 1}$. Furthermore, note that the term in curly brackets corresponding to $k = k_0$ is zero. For $k \neq k_0$, $|z - a_k| \geq c > 0$ and therefore $|z - a_k|^{-i+j}$ is bounded since $j \leq l - 1$. Hence, for $|x| > A$, the second term on the right of (3.5) is majorized in absolute value by $c c_0 \left| Q(z) \right| \left| x \right|^{-N - 1} \leq c c_0 \left| Q(z) \right| \left| Q(x) \right| \left| x \right|^{|x|}.

To complete the proof of (i) for $|x| > A$, it is then enough by (3.5) to estimate the difference between $\phi_t(x - z)$ and the first term on the right of (3.5). Taylor's theorem shows this difference is in absolute value at most $\left| x \right|^{-N} \left| \phi_t^{(N)}(\xi) \right| \left| x \right|^{|x|}$ for some $\xi$ between $x$ and $x - z$. This is bounded by $c c_0 \left| x \right|^{-N} \left| x \right|^{-N - 1}$ since $|z| < |x|/2$. We have $|x|^{-N} \sim \left| Q(x) \right|^{-1}$ since $|x|$ is large. Finally, $\left| x \right|^{-N} \leq c \left| Q(z) \right|$ since if $|z|$ is large, $\left| z \right|^N \sim \left| Q(z) \right|$, while if $|z|$ is bounded, $|z|^{-N} \leq c \left| z \right|^{|\mu|_{k_0}} \leq c \left| Q(z) \right|$ since the other factors of $\left| Q(z) \right|$ are all bounded below by positive constants. This completes the proof of the lemma.

As a corollary, we obtain the following estimates.
LEMMA (3.8). If $z$ and $a_{k_0}$ satisfy $|z - a_{k_0}| = \min_k |z - a_k|$, then

\begin{align*}
\text{(i)} & \quad \sup_{(\xi, t) \in \mathbb{R}^2_+} \left( \frac{t}{t + |\xi - x|} \right)^{N+1} \left| \phi_i(\xi - z) - \varphi_{\phi_i(\xi - z)}(z) \right| \leq c \frac{|Q(z)|}{|Q(x)|} \frac{1}{|x - a_{k_0}|} \\
\text{if} & \quad |z - a_{k_0}| \leq \frac{1}{2} |x - a_{k_0}|;
\end{align*}

\begin{align*}
\text{(ii)} & \quad \sup_{(\xi, t) \in \mathbb{R}^2_+} \left( \frac{t}{t + |\xi - x|} \right)^{N+1} \left| \varphi_{\phi_i(\xi - z)}(z) \right| \leq c \frac{|Q(z)|}{|Q(x)|} \frac{1}{|z - a_{k_0}|} \\
\text{if} & \quad |z - a_{k_0}| > \frac{1}{2} |x - a_{k_0}|. \quad \text{The constant} \ c \ \text{depends only on} \ Q \ \text{and the bounds on} \ \left(1 + |u|\right)^{\alpha + 1} |\phi^{(\alpha)}(u)|, \ \alpha = 0, \ldots, N.
\end{align*}

PROOF. Given $\phi$, let $\psi(u) = \phi(u + (\xi - x)/t)$. Note that

\[ \left(1 + |u|\right)^{\alpha + 1} |\psi^{(\alpha)}(u)| = \left(1 + |u|\right)^{\alpha + 1} |\phi^{(\alpha)}(u + (\xi - x)/t)|, \]

so that the sup over $u$ of the expression on the left is at most $\left(1 + |\xi - x|/t\right)^{\alpha + 1}$ times a multiple of the sup over $u$ of $\left(1 + |u|\right)^{\alpha + 1} |\phi^{(\alpha)}(u)|$. Since $\psi(x - z) = \phi_i(\xi - z)$, it then follows by applying Lemma (3.3)(i) to $\psi$ that if $|z - a_{k_0}| = \min_k |z - a_k| < \frac{1}{2} |x - a_{k_0}|$,

\[ \left| \phi_i(\xi - z) - \varphi_{\phi_i(\xi - z)}(z) \right| \leq c \left(1 + \frac{|\xi - x|}{t}\right)^{N+1} \frac{|Q(z)|}{|Q(x)|} \frac{1}{|x - a_{k_0}|}. \]

Here $c$ depends only on $Q$ and the bounds on $\left(1 + |u|\right)^{\alpha + 1} |\phi^{(\alpha)}(u)|, \ \alpha = 0, \ldots, N$. This proves part (i) of the lemma. Part (ii) follows in the same way from Lemma (3.3)(ii).

4. Proof of Theorem 1. Let $1 < p < \infty$, $w \in A_p$, $Q$ be a polynomial of degree $N$ with all real zeros and $u = |Q|^P w$. Let

\[ f(x, t) = \int_{\mathbb{R}} f(z) \left[ \phi_i(x - z) - \varphi_{\phi_i(x - z)}(z) \right] dz \]

and

\[ N_h f(x) = \sup_{(\xi, t) \in \mathbb{R}^2_+} \left( \frac{t}{t + |\xi - x|} \right)^\lambda |f(\xi, t)|. \]

Note that if $f \in L^p_u$, then by Lemma (3.1), the integral defining $f(x, t)$ converges absolutely.

For each zero $a_{k_0}$ of $Q$, define

\[ f_{k_0}(z) = \begin{cases} f(z) & \text{if} \ |z - a_{k_0}| = \min_k |z - a_k|, \\ 0 & \text{otherwise.} \end{cases} \]

By summation, it is enough to prove the theorem for $f = f_{k_0}$. Thus, the estimates of Lemma (3.3) are valid for any $z$ where $f(z) \neq 0$. We may also assume for simplicity that $a_{k_0} = 0$; that we may do so follows by translating variables, using the identity (3.4), and observing that the condition $w(x) \in A_p$ is independent of a translation of $x$.  

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We have
\[
|f(\xi, t)| \leq \int_{|z|<|x|/2} |f(z)| \left| \phi_1(\xi - z) - \phi_1(\xi - \cdot)(z) \right| \, dz \\
+ \int_{|z|>|x|/2} |f(z)| \left| \phi_1(\xi - z) \right| \, dz + \int_{|z|>|x|/2} |f(z)| \left| \phi_1(\xi - \cdot)(z) \right| \, dz.
\]
Multiply both sides of this inequality by \( t^\lambda / (t + |x - \xi|)^\lambda \), and let \( A(x), B(x) \) and \( C(x) \) denote the suprema over \((\xi, t) \in \mathbb{R}^n_+ \) of the resulting three terms on the right, resp. Thus, \( N_\lambda f(x) \leq A(x) + B(x) + C(x) \), and it is enough to estimate the norms of \( A, B \) and \( C \).

By Lemma (3.8)(i), if \( \lambda \geq N + 1 \),
\[
A(x) \leq c \left( \frac{1}{|Q(x)|} \right) \int_{|z|<|x|/2} |f(z)| |Q(z)| \, dz
\]
Therefore,
\[
\|A\|_{p,q} \leq c \left( \int_{|z|<|x|} |f(z)| |Q(z)| \, dz \right)^{1/p} \left( \int_{|z|<|x|} |Q(z)| \, dz \right)^{1/q}
\]
Applying Hardy’s inequality in the form
\[
\left( \int_{|z|<|x|} g(z) \, dz \right)^{1/p} \leq c \|g\|_{p,w}, \quad w \in A_p
\]
(see, e.g., [5]), we obtain \( \|A\|_{p,w} \leq c \|fQ\|_{p,w} = c \|f\|_{p,u} \) as desired.

By Lemma (3.8)(ii), if \( \lambda \geq N + 1 \),
\[
C(x) \leq c \left( \frac{1}{|Q(x)|} \right) \int_{|z|>|x|/2} |f(z)| |Q(z)| \left( \frac{1}{|z|} \right) \, dz,
\]
\[
\|C\|_{p,u} \leq c \left( \int_{|z|>|x|/2} |f(z)| |Q(z)| \left( \frac{1}{|z|} \right) \, dz \right)^{1/p} \left( \int_{|z|>|x|/2} |Q(z)| \, dz \right)^{1/q}
\]
Applying the dual version of Hardy’s inequality, that is,
\[
\left( \int_{|z|>|x|/2} g(z) \, dz \right)^{1/p} \leq c \|g\|_{p,w(x)|x|^p}, \quad w \in A_p
\]
(see, e.g., [5]), we get the desired estimate
\[
\|C\|_{p,u} \leq c \|f(x)Q(x)/x\|_{p,w(x)|x|^p} = c \|f\|_{p,u}.
\]
By Lemma (3.3)(iii),
\[
B(x) \leq c \left( \frac{1}{|Q(x)|} \right) \sup_{(\xi, t) \in \mathbb{R}^n_+} \left( \frac{t}{t + |\xi - x|} \right)^\lambda \int_{|z|>|x|/2} |f(z)Q(z)| |\phi_1(\xi - z)| \, dz.
\]
Enlarging the domain of integration on the right to all of \( \mathbb{R} \), it follows that the sup on the right is at most the usual tempered tangential maximal function of order \( \lambda \) of \( |fQ| \) formed with \( |\phi_1| \) as the approximation of the identity. As noted in the introduction, this maximal function is bounded on \( L^p_w \) if \( \lambda > 1 \) and \( w \in A_p \). Hence,
\[
\|B\|_{p,w} \leq c \|fQ\|_{p,w} = c \|f\|_{p,u}.
\]
This completes the proof of the theorem.

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Theorem 1 has a weak-type analogue when \( p = 1 \). To state it, we will use the notation \( m_w(E) = \int_E w(x) \, dx \) for the \( w \)-measure of a set \( E \).

**Theorem (4.1).** Let \( Q(x) \), \( f(x, t) \) and \( N_\lambda f(x) \) be as in Theorem 1, and let \( w \in A_1 \).

If \( \lambda \geq N + 1 \) and \( \lambda > 1 \), then

\[
m_w \left\{ x : N_\lambda f(x)|Q(x)| > s \right\} \leq c s^{-1} \| f \|_{L^1|Q|w}, \quad s > 0,
\]

where \( c \) is independent of \( f \) and \( s \).

**Proof.** As in the proof of Theorem 1, it is enough to show that \( A(x) \), \( B(x) \) and \( C(x) \) satisfy the estimate. For \( A \), we have

\[
A(x)|Q(x)| \leq c \frac{1}{|Q(x)|} \int_{|z| < |x|/2} |f(z)| |Q(z)| dz,
\]

so that

\[
A(x)|Q(x)| \leq c \frac{1}{|x|} \int_{|x-z| < 2|x|} |f(z)Q(z)| dz \leq c (fQ)^*(x),
\]

where "*" denotes the Hardy-Littlewood maximal function. Hence,

\[
m_w \left\{ x : A(x)|Q(x)| > s \right\} \leq m_w \left\{ x : (fQ)^*(x) > s/c \right\} \leq \frac{c}{s} \int |fQ| w \, dx,
\]

by [6], since \( w \in A_1 \). We also have

\[
C(x) \leq \frac{c}{|Q(x)|} \int_{|z| > |x|/2} |f(z)| |Q(z)| \frac{1}{|z|} \, dz.
\]

Thus,

\[
m_w \left\{ x : C(x) \mid Q(x) \mid > s \right\} \leq \frac{1}{s} \int_{\mathbb{R}} C(x)|Q(x)|w(x) \, dx
\]

\[
\leq \frac{c}{s} \int_{\mathbb{R}} |f(z)| |Q(z)| \left\{ \frac{1}{|z|} \int_{|x| < 2|z|} w(x) \, dx \right\} \, dz.
\]

Now using the estimate

\[
\frac{1}{|z|} \int_{|x| < 2|z|} w(x) \, dx \leq \frac{1}{|z|} \int_{|x-z| < 3|z|} w(x) \, dx \leq cw^*(z) \leq cw(z),
\]

since \( w \in A_1 \), we see the last expression is at most \( cs^{-1} \| f \|_{L^1|Q|w} \) as desired. Finally, recall that \( B(x) \mid Q(x) \mid \) is majorized by a multiple of the usual tempered tangential maximal function of order \( \lambda \) of \( |fQ| \) at \( x \), formed with \( |\phi_i| \) as approximation of the identity. Denoting this by \( N_{\lambda}^t(fQ)(x) \) and using the known fact (see [9]) that if \( \lambda > 1 \) and \( w \in A_1 \), then \( m_w \left\{ x : N_{\lambda}^t(g)(x) > s \right\} \leq (c/s) \| g \|_{L^1,w} \), we immediately obtain the desired estimate for \( B \). This completes the proof.

5. **Theorem 2.** In this section, we prove Theorem 2 and deduce several corollaries. As usual, we let \( u = |Q|^p w \) where \( 1 < p < \infty \), \( Q \) has only real zeros and \( w \in A_p \).

We will first show if \( f \in L_u^p \) and \( l_f \) is defined by

\[
\langle l_f, \phi \rangle = \int_{\mathbb{R}} f(z) \left[ \phi(z) - \Phi_p(z) \right] \, dz, \quad \phi \in \mathcal{S},
\]

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% = %ö>tfien '/ is a tempered distribution in \( H^p \) and
\[
\|f\|_{H^p} \leq c\|f\|_{L^p}
\]
with \( c \) independent of \( f \). In fact, by the second statement in Lemma (3.1), it follows immediately that \( f \) defines a tempered distribution. Moreover, since the \( H^p \) norm of \( f \) is equivalent to the \( L^p \) norm of the nontangential maximal function of
\[
\int \frac{f(z)}{\Lambda(\varphi, (x - z))} \, dz = fRf(z)\left[\varphi, (x - z) - \varphi(x)\right] \, dz,
\]
we obtain (5.1) from Theorem 1.

Now let \( f \) be any distribution in \( H^p \), and let \( l(z, s) = \langle l, \psi_s(z - \cdot) \rangle = (l * \psi_s)(z) \) for a fixed \( \psi \in \mathcal{S} \) with \( \int_R \psi(z) \, dz = 1 \). By hypothesis, the \( L^p \) norm of \( l(z, s) \) as a function of \( z \) is bounded in \( s \) by \( \|l\|_{\mathcal{L}^p} \). Hence, since \( p > 1 \), there exist \( s_m \to 0 \) and \( f \in L^p \) so that \( l(z, s_m) \) converges weakly in \( L^p \) to \( f \) and \( \|f\|_{\mathcal{L}^p} \leq \|f\|_{H^p} \). Our goal is to show that \( l = f \).

By Lemma (3.1), \( \psi - \mathscr{P}_\phi \) belongs to the dual of \( H^p \), so by weak convergence,
\[
\int_R l(z, s_m)\left[\phi(z) - \mathscr{P}_\phi(z)\right] \, dz \to \langle f, \phi \rangle
\]
as \( m \to \infty \). Let us denote
\[
\langle l(s), \phi \rangle = \int_R l(z, s)\left[\phi(z) - \mathscr{P}_\phi(z)\right] \, dz,
\]
\[
\langle \tilde{l}(s), \phi \rangle = \langle l * \psi_s, \phi \rangle = \int_R l(z, s)\phi(z) \, dz.
\]
Since \( l(z, s) \in L^p \), \( l(s) \) defines a distribution in \( H^p \) by what has already been proved. As we will now show, the same is true for \( \tilde{l}(s) \). That \( \tilde{l}(s) \) defines a distribution follows from the fact that, for \( s > 0 \), \( l(z, s) \) is a locally bounded function of \( z \) which is also in \( L^p \); thus, \( l(z, s) \in L^p_{1 + |x|} \), while any \( \psi \in \mathcal{S} \) belongs to the dual space \( L^p_{1 + |x|} \mathcal{S}^* \) (see the end of the introduction). To see that \( \tilde{l}(s) \in H^p \), we will show that its radial maximal function
\[
M_0(x) = \sup_{t > 0} \left| \langle \tilde{l}(s), \phi_t(x - \cdot) \rangle \right|, \quad \phi \in \mathcal{S},
\]
is pointwise less than a constant times the sum of the tangential maximal functions of \( l \) formed from \( \phi \) and from \( \psi \). In fact,
\[
\langle \tilde{l}(s), \phi_t(x - \cdot) \rangle = (l * \psi_s * \phi_t)(x) = \int_R l(z, s)\phi_t(x - z) \, dz,
\]
so that given \( M > 0 \), there is a constant \( c \) with
\[
\left| \langle \tilde{l}(s), \phi_t(x - \cdot) \rangle \right| \leq c \frac{1}{t} \int_R l(z, s) \left(1 + \frac{|x - z|}{t}\right)^{-M} \, dz
\]
\[
\leq c \left[ \sup_{(z, s)} \left(1 + \frac{|x - z|}{s}\right)^{-\lambda} |l(z, s)| \right] \cdot \frac{1}{t} \int_R \left(1 + \frac{|x - z|}{t}\right)^{\lambda} \left(1 + \frac{|x - z|}{t}\right)^{-M} \, dz.
\]

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If \( t \leq s \) and \( \lambda > 0 \),
\[
\frac{1}{t} \int \left( 1 + \frac{|x-z|}{s} \right)^\lambda \left( 1 + \frac{|x-z|}{t} \right)^{-M} \, dz \leq \frac{1}{t} \int \left( 1 + \frac{|x-z|}{t} \right)^{-\lambda-M} \, dz,
\]
which is a constant independent of \( x \) and \( t \) if \( M > \lambda + 1 \). This shows that if \( M > \lambda + 1, \lambda > 0 \), then
\[
\sup_{t<s} \left| \langle \tilde{l}(s), \phi_x(x - \cdot) \rangle \right| \leq c \sup_{(z,s)} \left( 1 + \frac{|x-z|}{s} \right)^{-\lambda} |l(z,s)|.
\]
The expression on the right is the tangential maximal function of \( l \) formed from \( \psi \).
To obtain an estimate for \( t > s \), we only need to note that \( l \ast \psi_s \ast \phi_x = (l \ast \phi_x) \ast \psi_s \) and repeat the argument above, obtaining for \( M > \lambda + 1, \lambda > 0 \),
\[
\sup_{t>s} \left| \langle \tilde{l}(s), \phi_x(x - \cdot) \rangle \right| \leq c \sup_{(z,t)} \left( 1 + \frac{|x-z|}{t} \right)^{-\lambda} |(l \ast \phi_x)(z)|.
\]
Combining inequalities proves the desired estimate for \( M_0(x) \).
Next, we will show that \( l(s) = l(s) \). This will prove that \( l = l_f \) since the fact that \( f \psi = 1 \) implies \( \langle \tilde{l}(s), \phi \rangle \to \langle l, \phi \rangle \) as \( s \to 0 \), while by (5.2), \( \langle l(s_m), \phi \rangle \to \langle l_f, \phi \rangle \). Let \( l_1(s) = \tilde{l}(s) - l(s) \), so that \( l_1 \in H^p_u \) and
\[
\langle l_1(s), \phi \rangle = \int l(z,s) \mathcal{P}_\phi(z) \, dz.
\]
We want to show that \( l_1(s) \equiv 0 \). Write
\[
\mathcal{P}_\phi(z) = Q(z) \prod_{j=1}^n \frac{\phi(y)}{(z-y)} = \sum_{k=1}^n \sum_{l=1}^{\mu_k} B_{k,l}(z) \delta_{-1}^{(l-1)} \phi,
\]
where the \( B_{k,l}(z) \) are polynomials of degree at most \( N - 1 \). Since any such polynomial is in \( L^p_u(1+|x|)^{-\rho - w} \), we may integrate term-by-term to get
\[
l_1(s) = \sum_{k=1}^n \sum_{l=1}^{\mu_k} B_{k,l} \delta_{-1}^{(l-1)}, \quad \text{where} \quad B_{k,l} = \int l(z,s) B_{k,l}(z) \, dz.
\]
In particular, \( l_1 \) has compact support and also, by Lemma (2.4), \( \langle l_1(s), \phi - \mathcal{P}_\phi \rangle = 0 \).
Therefore, \( \langle l_1(s), \phi \rangle = \langle l_1(s), \phi - \mathcal{P}_\phi \rangle + \langle l_1(s), \mathcal{P}_\phi \rangle = 0 + \langle l_1(s), \mathcal{P}_\phi \rangle \), and we immediately obtain \( l_1(s) = 0 \) by applying the following lemma.

**Lemma (5.4).** Let \( 1 < p < \infty \) and \( u(x) = |Q(x)|^p w(x) \) where \( Q \) is a polynomial of degree \( N \) with all real zeros and \( w \in A_p \). If \( l_1 \) is a distribution with compact support such that \( l_1 \in H^p_u \), then \( \langle l_1, x^k \rangle = 0 \) for \( 0 \leq k \leq N - 1 \).

**Proof.** Assume conversely that \( \langle l_1, x^k \rangle = c_k \neq 0 \) for some \( 0 \leq k \leq N - 1 \). Let \( \rho(x) \in C^\infty \) with \( \rho(x) = 1 \) for \( |x| < 1 \), \( \rho(x) = 0 \) for \( |x| > 2 \) and \( 0 \leq \rho(x) \leq 1 \). Set \( \phi(x) = x^k \rho(x) \). Then \( \phi_t(x) = t^{-1} \phi(x/t) = t^{-k-1} x^k \rho(x/t) \), and for \( t \) large we have
\[
\langle l_1, \phi_t \rangle = t^{-k-1} \langle l_1, x^k \rho(x/t) \rangle = t^{-k-1} \left[ \langle l_1, x^k \rangle - \langle l_1, x^k (1 - \rho(x/t)) \rangle \right]
\]
\[
= t^{-k-1} c_k.
\]
using the fact that \( x^k(1 - \rho(x/t)) = 0 \) on the support of \( l_1 \) when \( t \) is large. Setting 
\[ \psi(x) = \phi(-x), \]
we see that the function \( l_1(z, t) = \langle l_1, \psi(z - \cdot) \rangle \) satisfies \( l_1(0, t) = t^{-k-1}c_k \) for large \( t \). Thus,
\[
\sup_{\Gamma_\gamma(x)} |l_1(z, t)| \geq |c_k| \left( \frac{|x|}{\gamma} \right)^{-k-1} \quad \text{for } |x| \text{ large.}
\]
This, however, contradicts the fact that \( l_1 \in \mathcal{H}^p_u \) since for large \( |x| \) and \( k \leq N - 1 \),
\[
(|x|^{-k-1})^p u(x) \equiv c |x|^{(N-k-1)p} w(x) \equiv cw(x),
\]
and \( w(x) \) is not integrable at infinity. This completes the proof of the lemma.

The only part of Theorem 2 which remains to be proved concerns the uniqueness of the correspondence between \( l \) and \( f \). However, if \( g \in \mathcal{L}^p \) and
\[
\int f(z) [\phi(z) - \phi(x)] dz = \int g(z) [\phi(z) - \phi(x)] dz,
\]
for \( \phi \) which is zero in a small neighborhood of each \( a_k \),
\[
\int f(z) \phi(z) dz = \int g(z) \phi(z) dz.
\]
Since \( f \) and \( g \) are locally integrable away from the \( a_k \)'s, it follows that \( f = g \) a.e. This completes the proof of the theorem.

**Corollary (5.5).** Let \( 1 \leq p < \infty \) and \( u = |Q|^p w \) where \( Q \) has all real zeros and \( w \in A_p \). If \( l \in H^p_u \) and \( \ell(x, s) = (l*x^p_s)(x) \), \( s > 0 \), then
\[
\int l(x, s)x^j dx = 0, \quad j = 0, \ldots, N - 1.
\]

**Proof.** As shown in the proof of Theorem 2 (see (5.3) and use the fact that \( l_1(s) = 0 \)),
\[
\int l(x, s) \phi(x) dx = 0.
\]
Picking \( \phi(x) = x^j \rho(x) \), where \( j = 0, \ldots, N - 1 \) and \( \rho \) is a function in \( \mathcal{S} \) which equals 1 on the support of \( \partial_{\gamma} \), we have \( \partial_{\gamma} \phi(x) = \partial_{\gamma}^x \phi(x) = x^j \), and the corollary follows.

We remark that Corollary (5.5) can also be derived from Theorem 2 by applying Fubini's theorem and Lemma (2.6).

**Corollary (5.6).** Let \( 1 \leq p < \infty \), \( Q \) be a polynomial with all real zeros and \( w \in A_p \). If \( f \in \mathcal{L}^p_{|Q|^p w} \), \( \phi \in \mathcal{S} \) and \( \int \phi = 1 \), then
\[
\left\| \sup_{\Gamma_\gamma(x)} \left| \int f(z) \left[ \phi(y - z) - \partial_{\gamma}^Q \phi(y - \cdot)(z) \right] dz \right| \right\|_{L^p_{|Q|^p w}}
\]
and
\[
\left\| \sup_{\Gamma_\gamma(x)} \left| \int f(z) Q(z) \phi(y - \cdot)(z) dz \right| \right\|_{L^p}
\]
are equivalent.

**Proof.** We have \( f \in \mathcal{L}^p_{|Q|^p w} \) if and only if \( fQ \in L^p_{|Q|^p w} \). Moreover, \( \|f\|_{p, |Q|^p w} = \|fQ\|_{p, w} \). The corollary then follows immediately from Theorem 2 and the fact that \( \|fQ\|_{p, w} \) is equivalent to the second norm in the statement.
6. Convergence of \( f(x, t) \). In this section, we give a proof based on the estimates in §3 of the pointwise convergence a.e. of \( f(x, t) \) as \( t \to 0 \). We also study norm convergence.

**Theorem (6.1).** Let \( 1 < p < \infty \) and \( u = |Q|^p w \) where \( Q \) is a polynomial of degree \( N \) with all real zeros and \( w \in A_p \). Let \( \phi \) satisfy

\[
\varphi = \max_{|a| \leq N} \left\| (1 + |z|)^{a+1+\varepsilon} \phi(z) \right\|_{\infty} < \infty
\]

for some \( \varepsilon > 0 \). If \( f \in L^p_u \) and

\[
f(x, t) = \int_{\mathbb{R}} f(z) \left[ \phi_t(x - z) - \varphi_{\phi_t(x - \cdot)}(z) \right] dz,
\]

then

\[
\lim_{t \to 0} f(x, t) = \left( \int_{\mathbb{R}} \varphi(z) \right) f(x_0)
\]

at any Lebesgue point \( x_0 \) of \( f \) where \( Q(x_0) \neq 0 \).

**Proof.** The proof is a modified version of the usual Lebesgue point proof of the convergence of approximations of the identity. As usual, we may assume that \( x_0 = 0 \) and \( f \phi = 1 \).

First, consider the case when \( \phi \) is bounded and has compact support. Then if \( t \) is small enough and \( |x| < \gamma t \), \( \phi_t(x - \cdot) \) is supported away from the zeros of \( Q \). Thus, \( \varphi_{\phi_t(x - \cdot)} = 0 \) and

\[
\left| f(x, t) - f(0) \right| = \left| \int_{\mathbb{R}} [f(z) - f(0)] \phi_t(x - z) dz \right|.
\]

If \( |x| < \gamma t \), there exists a constant \( c \) so that \( |\phi_t(x - z)| \leq c t^{-1} \chi_{(-ct, ct)}(z) \), and therefore

\[
\left| f(x, t) - f(0) \right| \leq c \frac{1}{t} \int_{|z| < ct} |f(z) - f(0)| dz,
\]

which tends to zero with \( t \) since \( x_0 = 0 \) is a Lebesgue point of \( f \). This completes the proof for this case.

For the general case, write \( \phi(x) = \rho(r x) \phi(x) + (1 - \rho(r x)) \phi(x) = \phi_1 + \phi_2 \), where \( \rho \) is a smooth truncation defined by \( \rho(x) = 1 \) if \( |x| < 1 \), \( \rho(x) = 0 \) if \( |x| > 2 \), and \( \rho \in C^\infty \). For fixed \( r \), \( \phi_1 \) has compact support and the corresponding extension \( f_1(x, t) \) converges to \( \int f_1 dx f(0) \) as shown above. Furthermore, as \( r \to 0 \), \( \int f_1 dx \to \int f dx \) and the constant \( n_{\phi_2} \) defined by (6.2) with \( \phi = \phi_2 \) tends to zero. Hence, to prove the theorem, it is enough to show there is a constant \( A \) depending on \( f \) and \( x_0 \) (= 0) such that

\[
\sup_{|x| < \gamma t, t < 1} |f(x, t)| \leq A n_{\phi_2}.
\]

Since \( n_{\phi_2} \leq c n_{\phi} \) if \( |\alpha| < \gamma \), it is enough to show that \( \sup_{t < 1} |f(0, t)| \leq A n_{\phi} \).
Combining the estimates in Lemma (3.3), we get
\[|\phi_t(z) - Q_{\phi_t(x-z)}(z)| \leq c_0|Q(z)|\left[\frac{n_\phi}{1 + |z|} + |\phi_t(-z)|\right],\]
where \(c_0\) depends on \(Q\) and \(x_0\). We also have
\[|\phi_t(-z)| \leq n_\phi t^{-1}(1 + |z|/t)^{-1-\epsilon}.
\]
Therefore,
\[\frac{|f(0, t)|}{n_\phi} \leq c_0\int_{\mathbb{R}} |fQ|(1 + |z|)^{-1} dz + c_0\int_{\mathbb{R}} |fQ|t^{-1}(1 + |z|/t)^{-1-\epsilon} dz.
\]
By Hölder’s inequality, the first integral on the right is at most
\[\|f\|_{p,w}\|w(z)^{-1/p}(1 + |z|)^{-1}\|_{p'} \leq A.
\]
Since \(t^{-1}(1 + |z|/t)^{-1-\epsilon} \leq c_\delta(1 + |z|)^{-1}\) if \(|z| > \delta > 0\), the second is at most a multiple of
\[\int_{|z| < \delta} |fQ|t^{-1}(1 + |z|/t)^{-1-\epsilon} dz + \int_{|z| > \delta} |fQ|(1 + |z|)^{-1} dz.
\]
Of these two integrals, the second was shown above to be bounded by \(A\). The first, since \(\epsilon > 0\), is by well-known facts about approximations of the identity bounded by a constant times the Hardy-Littlewood maximal function of \(fQ\chi_{(|z|<\delta)}\) evaluated at 0. This is bounded by
\[c \sup_{s < \delta} \frac{1}{s} \int_{|z| < s} |fQ|dz \leq c \sup_{s < \delta} \frac{1}{s} \int_{|z| < s} |f|dz \leq A\]
for small \(\delta\), and the proof is complete.

**Theorem (6.3).** With the same assumptions and notation as in Theorem (6.1), \(f(x, t)\) converges in \(L^p_u\) norm to \((\int_{\mathbb{R}} \phi dz)f(x)\) as \(t \to 0\).

**Proof.** For \(1 < p < \infty\), \(\sup_{t > 0} |f(x, t)| \in L^p_u\) by Theorem 1, and the result follows from Theorem (6.1) by the Lebesgue dominated convergence.

For \(p = 1\), the proof has two steps, the first being to show that
\[\|f(x, t)\|_{1,u} \leq c\|f\|_{1,u}\]
with \(c\) independent of \(f\) and \(t > 0\), \(u = |Q|w\), \(w \in A_1\). This is based on estimates very much like those in §3. We may assume that \(f(z) = 0\) unless \(|z - a_{k_0}| = \min_k |z - a_k|\) and that \(a_{k_0} = 0\). Split the integral defining \(f(x, t)\) into parts with \(|z| > |x|/2\) and \(|z| < |x|/2\). For the part with \(|z| > |x|/2\),
\[\left|\phi_t(x - z) - Q_{\phi_t(x-z)}(z)\right| \leq |\phi_t(x - z)| + |Q_{\phi_t(x-z)}(z)| \leq c \frac{|Q(z)|}{|Q(x)|}\left\{\left|\phi_t(x - z)\right| + \frac{1}{|z|}\right\}
\]
by Lemma (3.3)(ii) and (iii). The corresponding part of \( \| f(x, t) \|_{1,u} \) is then at most

\[
c \int_{|z|>|x|/2} |f(z)| |Q(z)| \left( |\phi_i(x-z)| + \frac{1}{|z|} \right) dz \w(x) dx.
\]

By Fubini’s theorem and the fact that \( |x-z|<3|z| \) if \( |x|>|z| \), this is at most

\[
c \int_{R} |f(z)| |Q(z)| \left[ \int_{|z|<|x-z|<3|z|} |\phi_i(x-z)| w(x) dx + \frac{1}{|z|} \int_{|x-z|<3|z|} w(x) dx \right] dz
\]

\[
\leq c \int_{R} |f(z)| |Q(z)| w^*(z) dz.
\]

Since \( w \in A_1 \), we have \( w^*(z) \leq cw(z) \) for a.e. \( z \), and therefore the last integral is at most \( \| f \|_{1,u} \).

For the part of \( f(x, t) \) with \( |z|<|x|/2, |x|<A \), we claim

\[
|\phi_i(x-z) - \Phi_{\phi_i(x-z)}(z)| \leq c( |Q(z)|/|Q(x)| )[\psi_i(x-z) + 1],
\]

where \( \psi_i(x) = (1 + |x|)^{-1-\epsilon} \). In fact, combining (3.6) and (3.7) we get the bound

\[
c |Q(z)| \max \left[ |x|/t \right]^{a-2} \left( 1 + \frac{|x|}{t} \right)^{-a-2-\epsilon} + \frac{1}{|Q(x)|}.
\]

However, since \( (1 + |x|/t)^{-a-1} \leq (|x|/t)^{-a-1} \),

\[
t^{-a-2}(1 + |x|/t)^{-a-2-\epsilon} \leq \psi_i(x) |x|^{-a-1} \leq \psi_i(x) |Q(x)|^{-1},
\]

since \( |Q(x)| \leq c|x|^{a_0} \leq c|x|^{a+1} \) if \( \alpha + 1 < \mu_k \) and \( |x|<A \). The claim above now follows immediately from the fact that \( \psi_i(x) \) and \( \psi_i(x-z) \) are comparable for \( |z|<|x|/2 \). The corresponding part of \( \| f(x, t) \|_{1,u} \) is then at most

\[
c \int_{|x|<A} \left[ \int_{|z|<|x|/2} |f(z)| |Q(z)| (\psi_i(x-z) + 1) dz \right] w(x) dx
\]

\[
\leq c \int_{|z|<A/2} |f(z)| |Q(z)| \left[ \int_{R} \psi_i(x-z) w(x) dx + \int_{|x|<A} w(x) dx \right] dz.
\]

The two inner integrals are bounded by \( cw^*(z) \) for \( |z|<A/2 \), and the entire expression is at most \( \| f \|_{1,u} \) since \( w \in A_1 \).

Finally, for the part of \( f(x, t) \) with \( |z|<|x|/2, |x|>A \), we claim

\[
|\phi_i(x-z) - \Phi_{\phi_i(x-z)}(z)| \leq c( |Q(z)|/|Q(x)| )\psi_i(x-z),
\]

where \( \psi_i \) as above. In fact, as shown in §3, we have the bound \( c |Q(z)| |\phi_i^N(\xi)| \) where \( |\xi| \sim |x| \). Thus

\[
|\phi_i^N(\xi)| \leq ct^{-N-1}(1 + |x|/t)^{-N-1-\epsilon}
\]

\[
\leq c\psi_i(x)|Q(x)|^{-1},
\]

since \( |Q(x)| \sim |x|^N \) for large \( |x| \). This implies the claim, and the corresponding part of \( \| f(x, t) \|_{1,u} \) can be easily estimated as before. This completes the first step of the proof.
Pick \( g \in C^\infty \) with compact support away from the zeros of \( Q \) and \( \| f - g \|_{1,u} \) small. That such \( g \) exists follows from the local integrability of \( u \): in fact, we may successively approximate \( f \) by bounded functions with compact support away from the zeros of \( Q \), continuous functions with such support, functions that are polynomials on the intervals on which they are supported, and finally functions \( g \) of the type desired. To complete the proof, we may assume \( f = 1 \) and show that \( f(x, t) \to f(x) \) in \( L^1_u \). We have

\[
\| f(x, t) - f(x) \|_{1,u} \\
\leq \| f(x, t) - g(x, t) \|_{1,u} + \| g(x, t) - g(x) \|_{1,u} + \| g(x) - f(x) \|_{1,u} \\
\leq 2 \| f(x) - g(x) \|_{1,u} + \| g(x, t) - g(x) \|_{1,u}
\]

by (6.4), and it is enough to show \( g(x, t) \to g(x) \) in norm.

We will first show that \( g(x, t) \) converges uniformly to \( g(x) \) on compact sets away from the zeros of \( Q \). Write

\[
g(x, t) = \int \phi(x - z) dz - \int \phi(x - z) dz.
\]

The first term on the right converges uniformly to \( g(x) \) on compact sets. In the second term,

\[
\left| \phi(x - z) \right| \leq \max_{k: 0 < a < \mu_k} \left| \phi_{\gamma_i}(x - a_k) \right|
\]

since \( |z - a_k| \geq c > 0 \) for all \( k \) if \( g(z) \neq 0 \). If \( x \) is also bounded away from every \( a_k \),

\[
\left| \phi(x - z) \right| \leq \max_{0 < a < \mu_k} t^{-a-1}(1 + t^{-1})^{-a-1-t} \leq ct^t,
\]

and therefore the second term above converges uniformly to zero away from the \( a_k \)'s as \( t \to 0 \). It follows that the part of \( \| g(x, t) - g(x) \|_{1,u} \) with \( x \) in a compact set away from the \( a_k \)'s tends to zero with \( t \).

The part of \( \| g(x, t) - g(x) \|_{1,u} \) with \( x \) in a small neighborhood of some \( a_k \) is at most

\[
\int_{|x - a_k| < \delta} |g(x, t)| |Q(x)| w(x) \, dx + \int_{|x - a_k| < \delta} |g(x)| |Q(x)| w(x) \, dx.
\]

The second term is independent of \( t \) and small with \( \delta \). To estimate the first term, note that if \( g(z) \neq 0 \) then \( z \) is bounded and away from all the \( a_k \)'s. Since \( x \) is very near \( a_k \),

\[
\left| \phi(x - z) - \phi_{\gamma_i}(x - z) \right| \leq c \left[ 1 + \max_{0 < a < \mu_k} \left| \phi_{\gamma_i}(x - a_j) \right| \right]
\]

\[
\leq c \max_{0 < a < \mu_k} |x - a_k|^{-a-1} \leq c |Q(x)|^{-1}.
\]

Hence, \( |g(x, t)| \leq c \| g \|_1 |Q(x)|^{-1} \), and the first term above is bounded uniformly in \( t \) by \( c \int_{|x - a_k| < \delta} w(x) \, dx \), which is small with \( \delta \).

Finally, for \( |x| \) large and \( |z| \) bounded, as noted in the first half of the proof,

\[
\left| \phi(x - z) - \phi_{\gamma_i}(x - z) \right| \leq c \phi_i(x) |Q(x)|^{-1}.
\]
Hence, \(|g(x, t)| \leq c\psi_t(x)|Q(x)|^{-1}\) for large \(|x|\). Since \(g(x) = 0\) for \(|x| > M\), the part of \(\|g(x, t) - g(x)\|_{L^1, u}\) with \(|x| > M\) is at most
\[
c\int_{|x| > M} \psi_t(x) w(x) \, dx \leq ct \int_{|x| > M} \frac{w(x)}{|x|^{1+\epsilon}} \, dx.
\]
This tends to zero with \(t\), and the proof is complete.

7. Theorem 3. In this section, we prove Theorem 3 and identify the embeddings of \(L^p_u\) in \(H^p_u\) which are the identity on \(H^p_u\). Let \(1 < p < \infty\), \(d\) be a positive integer and
\[
u = (1 + x^2)^{dp/2} Q w,\]
where \(Q\) has all real zeros and \(w \in A_p\).

First observe that \(\|fQx^i\|_{L^1} \leq c \|f\|_{p, u}, i = 0, \ldots, d-1\), since by Hölder’s inequality,
\[
\|fQx^i\|_{L^1} \leq \|f\|_{p, u} \|x^i w(x)^{-1/p} (1 + |x|^2)^{-d/2}\|_{p'},
\]
(7.1)
i.e., \(i = 0, \ldots, d-1\). This shows that the moments \(\int_R fQx^i \, dx, i = 0, \ldots, d-1\), are finite if \(f \in L^p_u\).

If \(f \in L^p_u\), then \(f\) belongs to both \(L^p_{Q^*w}\) and \(L^p_{x^dQ^*w}\). By Theorem 2, it follows that \(f\) corresponds to \(l_1 \in H^p_{Q^*w}\) and \(l_2 \in H^p_{x^dQ^*w}\):
\[
\langle l_1, \phi \rangle = \int_R f(z) \left[\phi(z) - \frac{Q}{Q'}(z)\right] \, dz,
\]
\[
\langle l_2, \phi \rangle = \int_R g(z) \left[\phi(z) - \frac{Q}{Q'}(z)\right] \, dz,
\]
\[
\|l_1\|_{H^p_{Q^*w}} \leq c \|f\|_{L^p_u}, \quad \|l_2\|_{H^p_{x^dQ^*w}} \leq c \|f\|_{L^p_u}.
\]
Thus,
\[
\langle l_1, \phi \rangle - \langle l_2, \phi \rangle = \int_R f(z) \left[\frac{Q}{Q'}(z) - \frac{Q}{Q'}(z)\right] \, dz
\]
\[
= \sum_{i=0}^{d-1} x^{i+1}Q(\phi) \int_R f(z) Q(z) z^i \, dz
\]
by (2.8). Assuming that the moments of \(fQ\) of order \(0, \ldots, d-1\) vanish gives \(l_1 = l_2\). Calling this common value \(l\), we see that \(l\) belongs to both \(H^p_{Q^*w}\) and \(H^p_{x^dQ^*w}\), and therefore to \(H^p_u\) with \(\|l\|_{H^p_u} \leq c \|f\|_{L^p_u}\).

Conversely, if \(l \in H^p_u\), the nontangential maximal function of \(l\) belongs to both \(L^p_{Q^*w}\) and \(L^p_{x^dQ^*w}\). By Theorem 2, there exist \(f\) and \(g\), \(\|f\|_{L^p_{Q^*w}} \leq c \|l\|_{H^p_u}, \|g\|_{L^p_{x^dQ^*w}} \leq c \|l\|_{H^p_u}\), such that
\[
\langle l, \phi \rangle = \int_R f(z) \left[\phi(z) - \frac{Q}{Q'}(z)\right] \, dz, \quad \langle l, \phi \rangle = \int_R g(z) \left[\phi(z) - \frac{Q}{Q'}(z)\right] \, dz.
\]
Choosing \(\phi\) to vanish near the zeros of \(x^dQ\), we obtain \(\frac{Q}{Q'} = 0\) and
\[
\int_R f(z) \phi(z) \, dz = \int_R g(z) \phi(z) \, dz.
\]
Therefore, \(f = g\) a.e., \(\|f\|_{L^p_u} \leq c \|l\|_{L^p_u}\), and if we take the difference of the two formulas for \(\langle l, \phi \rangle\) we obtain
\[
\int_R f(z) \left[\frac{Q}{Q'}(z) - \frac{Q}{Q'}(z)\right] \, dz = 0, \quad \phi \in S.
\]
Pick \( \rho \in S \), \( \rho(x) = 1 \) near the support of \( \phi^{Q,x} \), and let \( \phi(x) = x^iQ(x)\rho(x), \) \( i = 0, \ldots, d - 1 \). Then

\[
\phi^{Q,x}(z) - \phi^{Q,x}(z) = \phi^{Q,x}(z) - \phi^{Q,x}(z) = z^iQ(z) - 0 = z^iQ(z)
\]

by Lemma (2.5)(i) and (ii). Therefore, \( \int \phi(z)z^iQ(z)dz = 0, i = 0, \ldots, d - 1 \), and the theorem follows.

Theorem 3 is valid for \( Q \equiv 1 \) if we adopt the convention that the interpolating polynomial \( \phi^{Q} \) is zero in this case. This follows by checking the proof. Note that a function \( f \) satisfies \( f \in L^p_{(1+x^2)^{\epsilon\rho/2}}Q_{w}^{p/2} \) with the moments of \( fQ \) up to order at least \( d - 1 \) all zero if and only if the function \( g = fQ \) satisfies \( g \in L^p_{(1+x^2)^{\epsilon\rho/2}}w \) with the moments of \( g \) up to order at least \( d - 1 \) all equal to zero. Hence, we obtain the following (cf. Corollary (5.6)).

**Corollary (7.2).** Let \( 1 < p < \infty \), \( d \) be a positive integer, \( Q \) be a polynomial with all real zeros and \( w \in A_p \). If \( f \in L^p_{(1+x^2)^{\epsilon\rho/2}}Q_{w}^{p/2} \) and the moments of \( fQ \) of order \( 0, \ldots, d - 1 \) all vanish, then

\[
\sup_{\Gamma_i(x)} \left| \int_{\mathbb{R}} f(z) \left[ \phi_i(y - z) - \phi_i^{Q}(y - z) \right] \, dz \right|_{L^p_{(1+x^2)^{\epsilon\rho/2}}Q_{w}^{p/2}}
\]

and

\[
\sup_{\Gamma_i(x)} \left| \int_{\mathbb{R}} f(z)Q(z)\phi_i(y - z) \, dz \right|_{L^p_{(1+x^2)^{\epsilon\rho/2}}w}
\]

are equivalent if \( \phi \in S \), \( \int \phi = 1 \).

**Corollary (7.3).** Under the same assumptions as in Theorem 3, there is a continuous linear embedding \( f \to f - k_i \) of \( L^p_u \) onto \( H^p_u \) which is the identity on \( H^p_u \) and which satisfies

\[
\int_{\mathbb{R}} k_i Q x^i \, dx = \int_{\mathbb{R}} Q x^i \, dx, \quad i = 0, \ldots, d - 1.
\]

Conversely, any continuous linear embedding of \( L^p_u \) in \( H^p_u \) which is the identity on \( H^p_u \) has this form.

**Proof.** As noted in (7.1), \( \| fQ x^i \|_{L^p} \leq c \| f \|_{L^p} \) for \( i = 0, \ldots, d - 1 \). Pick \( \eta_i \in S \), \( i = 0, \ldots, d - 1 \), with \( \int \eta_i(x)Q(x)x^i \, dx = \delta_{ij} \), where \( \delta_{ij} \) is the Kronecker \( \delta \). That such \( \eta_i \) exist can be seen as follows. First, we use Lemma 2.6, p. 182 of [2] to pick \( \nu_i \in C_0^\infty \) with support away from the zeros of \( Q \) such that \( \int \nu_i(x)x^j \, dx = \delta_{ij} \), and then we set \( \eta_i = \nu_i/Q \). Let

\[
g_f(x) = \sum_{i=0}^{d-1} \eta_i(x) \int_{\mathbb{R}} f(t) Q(t) t^i \, dt.
\]

Then \( \| g_f \|_{L^p_u} \leq c \| f \|_{L^p_u} \) and \( \int \phi f Q x^i \, dx = \int \phi Q x^i \, dx, \quad i = 0, \ldots, d - 1 \). Hence, \( f - g_f \in L^p_u \) with \( \| f - g_f \|_{L^p_u} \leq c \| f \|_{L^p_u}, \) and the first \( d - 1 \) moments of \( (f - g_f)Q \) vanish. Therefore, \( f - g_f \in H^p_u \) by Theorem 3, and since \( g_f = 0 \) if \( f \in H^p_u \) by
Theorem 3 again, the mapping $f \rightarrow f - g_f$ is an embedding of $L_u^p$ into $H_u^p$ which is the identity $H_u^p$.

Conversely, let $S$ be any embedding of $L_u^p$ in $H_u^p$ which is the identity on $H_u^p$. If $f \in L_u^p$, choose $g_f$ as above. Then $f - g_f \in H_u^p$, so $S(f - g_f) = f - g_f$, $Sf = f - (g_f - Sg_f)$. The function $g_f - Sg_f$ belongs to $L_u^p$ with norm bounded by a multiple of the norm of $f$, and for $i = 0, \ldots, d - 1$,

$$\int_R (g_f - Sg_f) Qx^i \, dx = \int_R g_f Qx^i \, dx - \int_R Sg_f Qx^i \, dx$$

$$= \int_R f Qx^i \, dx - 0 = \int_R f Qx^i \, dx,$$

by Theorem 3 since $Sg_f \in H_u^p$. This completes the proof.

8. Density theorems.

**Theorem (8.1).** Let $Q$ be a polynomial with all real zeros. If $1 \leq p < \infty$ and $w(x)(1 + |x|)^{-M}$ is integrable for some $M$, then the class of $g \in S$ with $\int_R g(x)x^j \, dx = 0$ for $j = 0, \ldots, N - 1$ is dense in $L_u^p$, $u = |Q|^p w$.

**Proof.** Since $w(x)(1 + |x|)^{-M}$ is integrable, $S \subset L_u^p$. Since $u$ is locally integrable, $S$ is dense in $L_u^p$ (see the related statement for $p = 1$ in the proof of Theorem (6.3)). Hence, it is enough to show that functions in $S$ can be approximated in $L_u^p$ by such $g$.

Let $P_{k,i}(x)$ be the polynomials of degree $N - 1$ with

$$\phi_{Q}^{\phi}(x) = Q(x) \phi(x)(x - y) = \sum_{k=1}^{n} \sum_{l=1}^{\mu_k} \phi(l-1) (a_k) P_{k,i}(x), \quad \phi \in S.$$  \hspace{1cm} (8.2)

The $P_{k,i}$ of course depend on $Q$ but are independent of $\phi$. Moreover, for $k' \neq k$, $P_{k,i}(x)$ has a zero of order $\mu_k$ at $x = a_k$. Let $\eta_{i}(x)$, $i = 1, \ldots, N$, be $C^\infty$ functions supported in $[-1,1]$ with $\int_R \eta_{i}(x)x^{j-1} \, dx = \delta_{ij}$, $j = 1, \ldots, N$, where $\delta_{ij}$ is the Kronecker $\delta$: that such $\eta_{i}$ exist was noted in the proof of Corollary (7.3). Given $f \in S$, let

$$c_{k,i} = \int_R f(x) P_{k,i}(x) \, dx, \quad k = 1, \ldots, n, \quad l = 1, \ldots, \mu_k,$$

$$f_r(x) = f(x) - \sum_{k=1}^{n} \sum_{l=1}^{\mu_k} c_{k,i} \eta_{i} \left( \frac{x - a_k}{r} \right)^{r-l}, \quad r > 0.$$ \hspace{1cm} (8.3)

We claim that $f_r \in S$, $\int_R f_r(x)x^j \, dx = 0$ for $j = 0, \ldots, N - 1$, and $f_r$ converges to $f$ in $L_u^p$ as $r \to 0$. That $f_r \in S$ is obvious. For the norm convergence, it is enough to show that

$$\int_R \left| \eta_{i} \left( \frac{x - a_k}{r} \right)^{r-l} Q(x) \right|^p w(x) \, dx \to 0$$

as $r \to 0$ for $k = 1, \ldots, n$ and $l = 1, \ldots, \mu_k$. But $\eta_{i}((x - a_k)/r)$ is supported in $|x - a_k| < r$, where $|Q(x)| \leq cr^{\mu_k}$. Hence, the last integral is at most $cr^{(\mu_k-l)} \int_{|x - a_k| < r} w(x) \, dx$, which tends to zero with $r$ since $\mu_k - l \geq 0$. 

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To show $\int_{\mathbb{R}} f(x)x^j\,dx = 0$ for $j = 0, \ldots, N - 1$, note that $\{P_{k,i}(x)\}_{k,i}$ spans the space of polynomials $P(x)$ of degree at most $N - 1$: in fact, by Lemma (2.5)(i) and (8.2),

$$P(x) = \sum_{k=1}^{n} \sum_{l=1}^{\mu_k} \frac{P^{(l-1)}(a_k)}{(l-1)!} P_{k,i}(x)$$

for such $P$. Thus, it is enough to show that

$$\int_{\mathbb{R}} f(x)P_{k',i'}(x)\,dx = 0$$

for $k' = 1, \ldots, n, l' = 1, \ldots, \mu_{k'}$. We have

$$\int_{\mathbb{R}} f(x)P_{k',i'}(x)\,dx = \int_{\mathbb{R}} P_{k',i'}(x)\,dx - \sum_{k=1}^{n} \sum_{l=1}^{\mu_k} c_{k,i}\int_{\mathbb{R}} P_{k',i'}(x)\eta_l\left(\frac{x-a_k}{r}\right)^{r-l}\,dx$$

$$= c_{k',i'} - \sum_{k=1}^{n} \sum_{l=1}^{\mu_k} c_{k,i}\int_{\mathbb{R}} P_{k',i'}(x)\eta_l\left(\frac{x-a_k}{r}\right)^{r-l}\,dx.$$

We will show that for $k = 1, \ldots, n$ and $l = 1, \ldots, \mu_k$,

$$\int_{\mathbb{R}} P_{k',i'}(x)\eta_l\left(\frac{x-a_k}{r}\right)^{r-l}\,dx = \delta_{kk}\delta_{ll},$$

from which (8.5) follows. Let $\tau_{k,i}(x) = \eta_l((x - a_k)/r)^{r-l}$. Changing variables and using the definition of $\eta_l$, we obtain

$$\int_{\mathbb{R}} \tau_{k,i}(x)(x - a_k)^{r-l}\,dx = \delta_{kk}\delta_{ll},$$

$k = 1, \ldots, n, l = 1, \ldots, N, j = 1, \ldots, N$. As noted earlier, if $k' \neq k$, $P_{k',i'}$ has a zero of order $\mu_k$ at $a_k$, i.e.,

$$P_{k',i'}(x) = \sum_{j > \mu_k} b_{k',i',j}(x - a_k)^{j-1}, \quad k' \neq k.$$

Therefore, if $k' \neq k$ and $l = 1, \ldots, \mu_k$, (8.6) implies

$$\int_{\mathbb{R}} P_{k',i'}(x)\tau_{k,i}(x)\,dx = 0.$$

If $k' = k$, (8.4) gives

$$(x - a_k)^{r-1} = \sum_{k'' = 1}^{n} \sum_{l'' = 1}^{\mu_k} \alpha_{k'',i''} P_{k'',i''}(x) + P_{k,i}(x).$$
Thus,
\[
\int_{\mathbb{R}} P_{k,l}(x) \tau_{k,l}(x) \, dx
= \int_{\mathbb{R}} (x - a_k)^{l'-1} \tau_{k,l}(x) \, dx - \sum_{k'' \neq k, k'' \geq l''} \alpha_{k'',l''} \int_{\mathbb{R}} P_{k'',l''}(x) \tau_{k'',l''}(x) \, dx
= \delta_{l''} - 0 = \delta_{l''}
\]
by (8.6) and (8.7). This completes the proof of the theorem.

The conclusion of Theorem (8.1) can be obtained in another way under the stronger assumptions that \(1 < p < \infty\) and \(w \in A_p\). As noted at the beginning of the proof of (8.1), it is enough to show that functions in \(S\) can be approximated in \(L^p\) by functions in \(\check{S}\) whose moments up to order \(N - 1\) vanish. Consider the extension of \(f\) defined in Theorem 1:

\[
f_t(x) = \int_{\mathbb{R}} f(z) \left[ \phi_t(x - z) - \Phi_t(x) \right] \, dz.
\]

If \(f, \phi \in \mathcal{S}\), it follows from the definition of \(\Phi\) that \(f_t \in \mathcal{S}\). Moreover, by Corollary (5.5) (or Lemma (2.6) when \(p = 1\)), \(f_t\) satisfies the required moment condition, and if we choose \(\phi\) with \(\int \phi = 1\), then by Theorem (6.3), \(f_t\) converges to \(f\) in norm as \(t \to 0\). Thus, \(f_t\) satisfies all the requirements.

**Theorem (8.8).** Let \(Q\) be a polynomial with all real zeros. If \(1 \leq p < \infty\) and

\[\lim_{m \to \infty} \frac{1}{m^p} \int_{-m}^{m} w(x) \, dx = 0,\]

then \(S_{0,0}\) is dense in \(L^p_u\), \(u = |Q|^p w\).

For the proof we need the following fact.

**Lemma (8.10).** Let \(1 \leq p < \infty\), \(i\) be a nonnegative integer, and \(u\) be a nonnegative function with

\[\lim_{m \to \infty} \frac{1}{m^{i+2p}} \int_{-m}^{m} u(x) \, dx = 0.\]

If \(f \in \mathcal{S}\) and \(\int f(x)x^j \, dx = 0, j = 0, \ldots, i,\) then there is a sequence of functions in \(S_{0,0}\) which converges to \(f\) in \(L^p_u\).

This is a special case of Theorem 6.13 of [7].

To prove Theorem (8.8), first note (8.9) implies that for \(m \geq 0,\)

\[
\int_{2^m \leq |x| \leq 2^{m+1}} \frac{w(x)}{|x|^M} \, dx \leq c(2^m)^{p-M}.
\]

Adding these inequalities for \(m \geq 0\) and using the local integrability of \(w\) shows that \(w(x)(1 + |x|)^{-M}\) is integrable for \(M > p\). To complete the proof, we only need to
combine Theorem (8.1) with Lemma (8.10) for \( i = N - 1 \) and \( u = Q \| \mathcal{P} \| w \), noting that

\[
\frac{1}{m^{(N+1)p}} \int_{-m}^{m} |Q(x)|^p w(x) \, dx \leq \frac{c}{m^p} \int_{-m}^{m} w(x) \, dx \to 0
\]
as \( m \to \infty \) by (8.9).

**Corollary (8.11).** Let \( Q \) be a polynomial with all real zeros, \( 1 < p < \infty \), \( w \in A_p \) and \( u = Q \mathcal{P} \| w \). Then \( S_{0,0} \) is dense in \( L_p^u \).

**Proof.** Since \( w \in A_p \), it satisfies the doubling condition of order \( p \),

\[
\int_{-m}^{m} w(x) \, dx \leq c m^p \int_{-1}^{1} w(x) \, dx, \quad m > 1.
\]

If \( p > 1 \), the fact that \( w \in A_{p-\epsilon} \) for some \( \epsilon > 0 \) (see [6]) then implies that (8.9) holds, and the corollary follows.

The density theorems above have analogues for \( L_p^u \) when \( u \) has the form \( u = (1 + |x|^2)^{d/2} |Q \mathcal{P} \| w \) where \( d \) is a positive integer.

**Theorem (8.12).** Let \( 1 < p < \infty \), \( d \) be a positive integer and \( u = (1 + |x|^2)^{d/2} |Q \mathcal{P} \| w \), where \( w \) is a nonnegative function such that \( w(x)(1 + |x|)^{-M} \) is integrable for some \( M \) and \( w(x)^{-1/p}(1 + |x|)^{-1} \in L_p^u \). Then the class of \( g \) in \( S \) with \( \int_R g(x)x^{j} \, dx = 0 \) for \( j = 0, \ldots, N + d - 1 \) is dense in the subspace of \( L_p^u \) of \( f \) with \( \int_R Q x^{j} \, dx = 0 \) for \( i = 0, \ldots, d - 1 \).

We remark that the assumptions on \( w \) are true if \( w \in A_p \).

**Proof.** Recall from (7.1) that \( \| Q x^j \|_{L_p^u} \leq c \| f \|_{L_p^u}, \quad i = 0, \ldots, d - 1. \)

This shows that the moments \( \int_R f Q x^i \, dx, \quad i = 0, \ldots, d - 1 \), are finite if \( f \in L_p^u \) and that if \( f_n \to f \) in \( L_p^u \), then

\[
\int_R f_n Q x^i \, dx \to \int_R f Q x^i \, dx, \quad i = 0, \ldots, d - 1.
\]

We first show that if \( f \in L_p^u \) and \( \int_R f Q x^i \, dx = 0, \quad i = 0, \ldots, d - 1 \), then \( f \) can be approximated in \( L_p^u \) by \( g \in S \) with \( \int_R g Q x^i \, dx = 0, \quad i = 0, \ldots, d - 1 \). In fact, pick \( f_n \in S \) with \( f_n \to f \) in \( L_p^u \), and let

\[
g_n(x) = f_n(x) - \sum_{i=0}^{d-1} \eta_i(x) \int_R f_n(t) Q(t) t^i \, dt,
\]

where \( \{ \eta_i \} \) is chosen with \( \eta_i \in S \) and \( \int_R \eta_i(x) Q(x)x^j \, dx = \delta_{ij} \). (See the proof of Corollary (7.3).) Clearly, \( g_n \in S \) and

\[
\int_R g_n Q x^j \, dx = \int_R f_n Q x^j \, dx - \int_R f_n Q t^i \, dt = 0, \quad j = 0, \ldots, d - 1.
\]

Moreover, the facts that \( f_n \to f \) in \( L_p^u \), \( \eta_i \in L_p^u \) and \( \int_R f_n Q t^i \, dt = 0, \quad i = 0, \ldots, d - 1 \), show that \( g_n \to f \) in \( L_p^u \), as desired.

Thus, it is enough to prove the theorem for \( f \in S \). Let \( g_r \) be the function \( f \) defined in (8.3) but now formed using \( x^d Q(x) \) as generating polynomial. Then, as shown in the proof of Theorem (8.1), \( g_r \in S \), the moments of \( g_r \), of order \( \leq N + d - 1 \) vanish, and \( g_r \to f \) in \( L_p^u \) as \( r \to 0 \). Since \( u \leq c(\| Q \mathcal{P} \| w + |x^d Q \| w) \), the proof of the theorem will be complete if we show that \( g_r \) also converges to \( f \) in \( L_p^u \). Let \( f_r \)
denote the function in (8.3) formed with $Q$ as generator and the same $\{\eta_i\}$ as for $g_r$. Then $f_r \to f$ in $L^p_{[Q^p]}$, so we will be done if we show that $f_r = g_r$. However, it follows easily from (8.2) and (2.8) that those polynomials $P_{k,l}$ for $x^d Q$ with $k = 1, \ldots, n$, $l = 1, \ldots, \mu_k$, differ from the corresponding $P_{k,l}$ for $Q$ by linear combinations of the $x^j Q$, $j = 0, \ldots, d - 1$, while those $P_{k,l}$ for $x^d Q$ with $k > n$ or $l > \mu_k$ are linear combinations of the $x^j Q$. Hence, since $\int_R f x^j Q \, dx = 0$ for $j = 0, \ldots, d - 1$, we obtain $f_r = g_r$.

**Theorem (8.13).** Let $1 < p < \infty$, $d$ be a positive integer and

$$u = (1 + |x|^2)^{dp/2} |Q|^p w,$$

where $w$ is a nonnegative function such that $w(x)^{-1/p}(1 + |x|)^{-1}$ is in $L^{p'}$ and (8.9) holds. Then $S_{0,0}$ is dense in the subspace of $L^p_u$ of $f$ with $\int R f Q x^i \, dx = 0$ for $i = 0, \ldots, d - 1$.

**Proof.** This follows from the previous theorem by using Lemma (8.10), with $i$ there taken to be $N + d - 1$, in the same way that Theorem (8.8) follows from Theorem (8.1).

**Corollary (8.14).** If $1 < p < \infty$, $d$ is a positive integer and $u = (1 + |x|^2)^{dp/2} |Q|^p w$ with $w \in A_p$, then $S_{0,0}$ is dense in the subspace of $L^p_u$ of $f$ with $\int R f Q x^i \, dx = 0$ for $i = 0, \ldots, d - 1$. In particular, the closure of $S_{0,0}$ in $L^p_u$ is this subspace.

**Proof.** The first statement follows as in the proof of Corollary (8.11). For the second, note that if $f$ is in the closure of $S_{0,0}$, then there exist $f_n$ in $S_{0,0}$ such that $f_n \to f$ in $L^p_u$, and therefore, $\int R f_n Q x^i \, dx \to \int R f Q x^i \, dx$, $i = 0, \ldots, d - 1$. Since the integrals on the left all vanish, so does the one on the right.

**References**


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