

FREE LIE SUBALGEBRAS OF THE COHOMOLOGY OF LOCAL RINGS

BY

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ABSTRACT. A criterion is established, in terms of the Massey products structure carried by the homology of partial resolutions, for the Yoneda cohomology algebra $\text{Ext}_A(k, k)$ to be a free module over the universal envelope of a free graded Lie subalgebra. It is shown that several conjectures on the (co)homology of local rings, in particular on the asymptotic behaviour of the Betti numbers, follow from such a structure. For all rings with $\text{edim } A - \text{depth } A \leq 3$, and for Gorenstein rings with $\text{edim } A - \text{depth } A = 4$, the following dichotomy is proved: Either A is a complete intersection, or $\text{Ext}_A(k, k)$ contains a nonabelian free graded Lie subalgebra.

0. Ten conjectures on the homology of local rings (in place of Introduction). In this paper (A, \mathfrak{m}, k) will denote a noetherian local ring A with maximal ideal \mathfrak{m} and residue field k . A well-known homological invariant of A is its Poincaré series

$$P_A(t) = \sum_{i=0}^{\infty} b_i t^i \in \mathbb{Z}[[t]],$$

where $b_i (= b_i(A))$ is the i th Betti number of A , i.e. the rank of the i th free module in the unique, up to noncanonical isomorphism, A -free resolution X of k . Much of the work on the homology of local rings has been motivated by the following famous conjecture:

C_0 . $P_A(t)$ represents a rational function.

Recently this has been settled in the negative by D. Anick,¹ who constructed for each k a k -algebra A requiring 5 generators for \mathfrak{m} and having $\mathfrak{m}^3 = 0$, with transcendental Poincaré series. On the other hand, more than 20 years of research have produced different kinds of conditions, which ensure the rationality of $P_A(t)$. Before collecting for further reference several of them in a theorem, we introduce some notation, to be kept for the rest of the paper.

$\text{edim } A =$ embedding dimension of $A := \dim_k \mathfrak{m}/\mathfrak{m}^2 (= b_1(A))$;

$\dim A =$ Krull dimension of A ;

$\text{depth } A =$ maximal length of an A -regular sequence in \mathfrak{m} ;

$d(A) =$ complete intersection defect of $A := \dim_k \mathfrak{a}/\mathfrak{n}\mathfrak{a} - (\dim R - \dim A)$, where (R, \mathfrak{n}) is a regular local ring such that R/\mathfrak{a} is isomorphic to the \mathfrak{m} -adic

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¹ D. Anick, *Counter-examples to a conjecture of Serre*, Thesis, Massachusetts Institute of Technology, June 1980; cf. also D. Anick, *C. R. Acad. Sci. Paris Ser. A-B* **290** (1980), A729–A732.

completion of A ; note that $d(A) \geq 0$, the rings for which equality holds being called complete intersections.²

THEOREM 0.1. *Under each one of the following conditions $P_A(t)$ represents a rational function:*

- (1) A is a complete intersection (Tate, Zariski, 1957);
- (2) $\text{edim } A - \dim A \leq 1$ [Sh];
- (3) $\text{edim } A - \text{depth } A \leq 2$ [Sc];
- (4) $\text{edim } A - \text{depth } A = 3$ and A is Gorenstein [W];
- (5) $\text{edim } A - \text{depth } A = 3$ and $d(A) = 1$ [Go 2; Av 2];
- (6) $\text{edim } A = 3$ and $\mathfrak{m}^3 = 0$ (Backelin and Fröberg, 1978);
- (7) $\dim_k \mathfrak{m}^2/\mathfrak{m}^3 \leq 1$ (J. Sally, 1978; Backelin, 1978).

The striking disparity of these conditions leaves little hope for a satisfactory ring-theoretic characterization of the local rings with rational Poincaré series. The purpose of this section is to review several conjectures, formulated in terms of Betti numbers, deviations, and Yoneda products structure in cohomology, which can be organized around three main themes:

(i) the already acquired understanding that the complete intersections are remarkably disjoint from the rest of the local rings by their (co)homological behaviour (cf. $C_1, C_3, C_4(b)$, and C_9 below);

(ii) the asymptotic properties of the Betti numbers (cf. C_1, C_2, C_{10});

(iii) the fact that the homological invariants, and especially the deviations, give a certain hold on the singularity of A , and there are situations when this “should not get worse”: this is the content of C_4 – C_8 .

It should be emphasized that for some of these statements the status of open questions would have been more appropriate. However, for the sake of more precise formulations and unity of exposition, we have chosen, taking the usual risks, to have conjectures throughout.

Let r_A be the radius of convergence of $P_A(t)$. It is infinite if and only if A is regular, otherwise one has the double inequality $0 < r_A \leq 1$. The upper bound holds for all infinite series in $\mathbb{Z}[[t]]$, while the lower one is a consequence of the coefficient-wise inequality of power series (denoted \ll), due to Serre:

$$(0.2) \quad P_A(t) \ll \frac{(1+t)^{\text{edim } A}}{1 - \sum_{i=1}^{\text{edim } A - \text{depth } A} c_i t^{i+1}}$$

where $c_i = \dim_k H_i(K)$ (cf. §3A below).

C_1 (Golod, Gulliksen). If $r_A \geq 1$, then A is a complete intersection. (The converse is true by 0.5 below.)

From the product formula for the Poincaré series (cf. (0.4)) it follows that for $n > \text{edim } A$ the function $n \mapsto b_n(A)$ is nondecreasing (if one restricts to *nonregular* rings, then this is true for $n \geq 0$). The set of all almost everywhere nondecreasing

² L. L. Avramov, Math. Ann. **228** (1977), 27–37.

functions from the set \mathbf{N} of nonnegative integers to the nonnegative reals \mathbf{R}^+ has a partial preorder: $f \leq g$ if and only if there exist positive integers p and q such that for all $n \in \mathbf{N}$, $f(n) \leq pg(qn)$. The equivalence class of f for the relation $f \sim g \Leftrightarrow (f \leq g \text{ and } g \leq f)$ is called the growth of f and denoted $[f]$ (for details cf. [BK]). We write \leq for the induced partial order on the factor set.

EXAMPLE. For any real numbers $a > 1, b > 1, c > d > 0$, one has the inequalities $[a^n] = [b^n] \geq [n^c] \geq [n^d]$. The growth of f is said to be polynomial (resp. exponential) if there exists a $c \in \mathbf{N}$ for which $[f] = [n^c]$ (resp. if $[f] = [2^n]$). Note $[f] = [2^n]$ implies the existence of $b > 1, m \in \mathbf{N}$, such that $f(n) \geq b^n$ for all $n \geq m$.

Define $b_A: \mathbf{N} \rightarrow \mathbf{R}^+$ by setting $b_A(n) = b_n(A)$. From (0.2) and (0.4) one has the double inequality

$$[n^{\text{edim } A - \text{dim } A - 1}] \leq [b_A] \leq [2^n]$$

(in the regular case, i.e. when $\text{edim } A = \text{dim } A$, the left-hand term is defined to be $[0]$). By a theorem of Gulliksen [Gu], complete intersections are characterized by the fact that the growth of b_A is at most polynomial.

C_2 [Av 4]. Either A is a complete intersection, or b_A has exponential growth.

In the study of the Betti numbers, the main tools are the very rich algebraic structures carried by $X \otimes k = \text{Tor}^A(k, k) = T^A$, and by its graded k -space dual $\text{Hom}_A(X, k) = \text{Ext}_A(k, k) = E_A$. In the sequel we shall assume E_A is equipped with the product opposite (in the graded sense) to Yoneda's. It becomes a Hopf algebra when given the diagonal obtained by dualizing Cartan and Eilenberg's product \cap in T^A (for details cf. [GL]). We denote by $I(\)$ the kernel of the natural augmentation of $(\)$ onto k , and by $(\)_+$ the subset of homogeneous elements of positive even degree. The algebra T^A has divided powers γ_n , compatible with its diagonal (cf. [GL]). Denote by L_A the subspace of E_A , orthogonal to $T_0^A \oplus I^{(2)}(T^A)$, where $I^{(2)}(T^A) = I^2(T^A) + \sum_{n \geq 2} k \cdot \gamma_n(T^A)_+$, and set $Q(T^A) = I(T^A)/I^{(2)}(T^A)$. Assuming that in characteristic 2 all Lie algebras are "adjusted" in the sense of Sjödín [Sj], we have the following fundamental structure theorem due to Milnor and Moore, André, and Sjödín.

THEOREM 0.3 [An], [Sj]. *The subspace L_A is a graded Lie subalgebra of E_A (endowed with the commutator structure), and, as a Hopf algebra, E_A is isomorphic to the universal enveloping algebra $U(L_A)$.*

Note that a local homomorphism $(A, \mathfrak{m}, k) \rightarrow (B, \mathfrak{n}, l)$ induces a homomorphism of graded Lie algebras $L_B \rightarrow L_A \otimes l$. Also, L_A is closely connected to the cohomology of the cotangent complex of the A -algebra k . There is a canonical homomorphism of graded Lie algebras $H^*(A, k, k) \rightarrow L_A$, which, by results of Quillen and André is an isomorphism in degrees $\leq \pi(A)$, where $\pi(A) = 2p$ when $\text{char } k = p > 0$, and $\pi(A) = \infty$ when $\text{char } k = 0$. A consequence of the Poincaré-Birkhoff-Witt theorem for L_A is the equality of formal power series

$$(0.4) \quad P_A(t) = \prod_{j=0}^{\infty} \frac{(1 + t^{2j+1})^{e_{2j+1}}}{(1 - t^{2j+2})^{e_{2j+2}}}$$

which uniquely determines the integers $e_i = \dim_k L_A^i$; e_i is classically called the $(i - 1)$ th deviation of A and denoted ϵ_{i-1} .

The next theorem illustrates the ubiquitous appearance of the complete intersections in the theory.

THEOREM 0.5. *Each one of the following conditions is necessary and sufficient for A to be a complete intersection:*

(1) $P_A(t) = (1 + t)^n / (1 - t^2)^r$; in this case $n = \text{edim } A$, and $n - r = \dim A$ (Tate, Zariski, 1957);

(2) $e_3 = 0$ (Assmus, 1959);

(3) $e_4 = 0$ [GL];

(4) $e_i = 0$ for $i \geq 3$;

(5) $e_i = 0$ for all sufficiently large i (Gulliksen, 1971);

(6) $e_i = 0$ for all sufficiently large even i [Gu];

(7) $b_n(A) \leq f(n)$ for some $f \in \mathbf{Z}[t]$ [Gu];

(8) L_A is nilpotent and finitely generated (Sjödin, 1976).

The rigidity of the deviations is an open question of long standing (cf. [GL, Remark on p. 154]).

C_3 . If $e_i = 0$ for some $i \geq 1$, then A is a complete intersection.

A new type of deviation has recently been introduced by André: being a series in $1 + t\mathbf{Z}[[t]]$, the Poincaré series of A can be written uniquely in the form

$$P(t) = \frac{(1+t)^{e_1}}{\prod_{i \geq 2} (1-t^i)^{f_i}}.$$

Note that $f_2 = e_2$, $f_i = e_i - e_{i/2}$ for $i \equiv 2 \pmod{4}$ and $i > 2$, $f_i = e_i$ for $i \not\equiv 2 \pmod{4}$. Considering the case of residual characteristic 2, André³ has shown that $f_i \geq 0$ for all i . Gulliksen [Gu] has formulated as an open question in all characteristics, and verified in several cases when an explicit formula for $P_A(t)$ is available, a strengthened version of both André's result and C_3 .

C'_3 . If A is not a complete intersection, then $f_i(A) > 0$ for all $i \geq 2$.

The next group of conjectures, mostly based on work of Gulliksen, André, and the author, reflects the fact that the deviations measure the singularity of A ; cf. Theorem 0.5 and the characterization of regular local rings by the vanishing of e_2 .

Set

$$\chi_r(A) = \sum_{j=1}^r (-1)^{j-1} e_j(A).$$

C_4 . (a) $\dim A - \chi_{2i}(A) \geq 0$ for all $i \geq 0$;

(b) if equality holds in (a), then A is a complete intersection (the converse is true by Theorem 0.5);

(c) for any prime $\mathfrak{p} \subset A$, there is inequality: $\dim A_{\mathfrak{p}} - \chi_{2i}(A_{\mathfrak{p}}) \leq \dim A - \chi_{2i}(A)$.

³ M. André, Manuscripta Math. **18** (1976), 83–108.

C₅. For any prime $\mathfrak{p} \subset A$ and for all $r > 0$, there is inequality:

$$(-1)^{r-1} \chi_r(A) \geq (-1)^{r-1} (\chi_r(A_{\mathfrak{p}}) + \dim A/\mathfrak{p}).$$

C₆. For any prime $\mathfrak{p} \subset A$, $e_i(A_{\mathfrak{p}}) \leq e_i(A)$ for $i \geq 1$.

C₇. Let $A \rightarrow B$ be a flat local homomorphism. Then $e_i(A) \leq e_i(B)$ for $i \geq 1$.

C₈. Let $(A, m, k) \rightarrow (B, n, l)$ be a flat local homomorphism. Then the induced map of Lie algebras $L_B \rightarrow L_A \otimes_k l$ is surjective in degrees > 1 .

PROPOSITION 0.6. *The following implications hold:*

$$C_4(c) \Leftarrow C_5 \Rightarrow C_6 \Leftarrow C_7 \Leftarrow C_8;$$

$$C_4(a) \text{ for artinian rings and } \left. \begin{array}{l} C_4(c) \text{ for complete rings} \end{array} \right\} \Rightarrow C_4(a) \text{ and } C_4(c).$$

The proof is given in §5.

We describe briefly the present status of these conjectures. In a classical paper [L], Lech established C₇ for $i = 1$, and also proved C₅ for all A and $r = 1$. Under the additional assumption that (i) A is a homomorphic image of a regular local ring, Gulliksen [Gu] proved C₅ for $r < \pi(A)$. The map of Lie algebras of C₈ has been shown by the author⁴ to be surjective in all even degrees, and this has been extended by André⁵ to all degrees $\neq 1$, assuming $\text{char } k = 2$. Much evidence is available for C₆. It is proved for all A if either $i = 1$ [L] or i is even⁶; furthermore, it is known to hold for all i under any one of the following conditions: (i) as above; (ii) A is universally japanese; (iii) \mathfrak{p} is analytically unramified; (iv) $2 \in \mathfrak{p}$ (the first three cases are treated by the author⁷, and the last one by André⁸). Conjecture C₄ is known for $i = 1$: noting that $\dim A - \chi_2(A) = d(A)$, (a) becomes Krull's altitude theorem, (b) is the Cohen-Macaulay theorem, and (c) is a conjecture of Kiehl and Kunz, proved by the author⁹.

We conclude with two statements on the Yoneda product structure in cohomology, which, in view of Theorem 0.3, is the finest homological invariant of A .

C₉ (Roos). If $\text{Ext}_A(k, k)$ is a noetherian k -algebra, then A is a complete intersection. (The converse is true by Theorem 0.5.)

C₁₀. If A is not a complete intersection, then L_A contains a nonabelian free graded Lie subalgebra L .

PROPOSITION 0.7. *The following implications hold:*

$$C_0 \Rightarrow C_1 \Leftarrow C_2 \Leftarrow C_{10} \Rightarrow C_9.$$

The proof is deferred to §5.

⁴ See footnote 2.

⁵ M. André, *Gulliksen-Avramov exact sequences in characteristic two*, Preprint, EFP Lausanne, 1979.

⁶ See footnote 2.

⁷ See footnote 2.

⁸ See footnote 5.

⁹ See footnote 2.

1. The main results. Our purpose in this paper is to establish a criterion, in terms of the Massey products structure of partial resolutions, for the validity of C_{10} , and to verify its conditions for a class of rings, for which the rationality of the Poincaré series is not known (compare Theorem 0.1). These results have been announced without proofs in [Av 4].

THEOREM 1.1. *Suppose A satisfies the inequality $\text{edim } A - \text{depth } A \leq 4$, with equality admitted only for Gorenstein rings. Then either A is a complete intersection, or L_A contains a nonabelian free graded Lie subalgebra. In the latter case, $[b_A] = [2^n]$.*

The proof is carried out in §3. After some further elaborations, worked out in §4, we obtain, for the class of rings considered here, strengthened versions of C_1 and C_2 , and also partial results on the validity of C_3 .

THEOREM 1.2. *Under the hypotheses of the previous theorem, the following conditions are equivalent:*

- (i) A is a complete intersection;
- (ii) the growth of b_A is not exponential;
- (iii) $r_A > \sqrt{2} / 2$;
- (iv) $e_{6i}(A) = 0$ for some $i > 0$;
- (v) $e_{2i}(A) = 0$ for some $i > 0$, and A is Cohen-Macaulay;
- (vi) E_A is a noetherian k -algebra.

For the rest of this paper we denote by X a fixed minimal A -free resolution of k , equipped with its standard structure of DG Γ -algebra (i.e algebra with a system of divided powers). For details we refer to [GL]. Consider a DG subalgebra $F \subset X$ satisfying the following conditions:

- (i) $F_i = X_i$ for $i = 0, 1$ (i.e. F contains the Koszul complex K of A);
- (ii) F is closed under the divided powers of X ;
- (iii) $X^\#$ is free as a $F^\#$ -module, where $()^\#$ denotes the forgetful functor assigning to a DG object its underlying graded object.

The set of all elements of $IH(F)$, decomposable in terms of matric Massey products, is denoted $MH(F)$. It is proved in [M] that this is a vector subspace (even an ideal) in $IH(F)$; we refer to May's paper for the definitions and basic properties of matric Massey products. The following convenient formalism has been introduced by Gulliksen.

DEFINITION. The possibly infinite set $S = \{f_i \in IH(F) \mid i \in I\}$ is said to admit a trivial Massey operation (t.M.o.), if there exists a function γ , defined on the set of finite sequences of elements of S (with repetitions) and taking values in IF , subject to the following conditions:

- (i) $\gamma(f_i) = z_i$, with $[z_i] = f_i$ for each $i \in I$;
- (ii)

$$d\gamma(f_{i_1}, \dots, f_{i_n}) = \sum_{j=1}^{n-1} \bar{\gamma}(f_{i_1}, \dots, f_{i_j}) \cdot \gamma(f_{i_{j+1}}, \dots, f_{i_n}).$$

(Here and in what follows, homology classes, cycles, etc. are always supposed homogeneous; square brackets denote the homology class of a cycle; if $a \in F_m$, then $\bar{a} = (-1)^{m+1}a$.)

THEOREM 1.4. *Let $S = \{f_i\}_{i \in I}$ be a set of homology classes of $IH(F)$, such that:*

- (1) *S admits a trivial Massey operation; and*
- (2) *the images of $\{f_i\}_{i \in I}$ in $IH(F)/MH(F)$ are linearly independent.*

Then L_A contains a free graded Lie subalgebra on a set of generators $\{e_i \in L_A^{\deg(f_i)+1}\}_{i \in I}$.

The proof of this result occupies §2.

COROLLARY 1.5. *Suppose $A = B/(m_1, \dots, m_s)$, where B is an arbitrary local ring and the m_i 's are monomials in the elements of a B -regular sequence x_1, \dots, x_r . If $\text{grade}(m_1, \dots, m_s) \neq s$, then L_A contains a free Lie subalgebra on two generators of degree 2.*

Note that the rationality of $P_A(t)$ is not known even for the local rings with "monomial relations", i.e. for A 's obtained as above from a regular B . The next corollary is a sharpening of the structure theorem for the cohomology of Golod rings proved in [Av 1, (5.1)] and is extendable to the case of Golod homomorphisms. Recall that A is said to be Golod if the right-hand side inequality in (0.2) becomes an equality or, equivalently, if some (resp. any) k -basis of $IH(K)$ admits a t.M.o. [Go 1].

COROLLARY 1.6. *The ring A is Golod if and only if the Lie subalgebra of L_A generated by its elements of degree ≥ 2 is isomorphic to the free Lie algebra on $sIH(K)^*$ (s denotes the degree shifting functor $(sV)_i = V_{i-1}$).*

(1.7) **REMARKS ON THEOREM 1.4.** (1) Conditions (1) and (2) should be viewed as the proper homological substitutes for the algebraic notions of zero-divisors and indecomposable elements respectively. In this connection, note that if the ring A admits as a subset of a minimal set of generators for m elements t_1, \dots, t_m such that $t_i t_j = 0$ for $1 \leq i, j \leq m$, then it is well known that L_A contains a free Lie subalgebra on m generators of degree 1. In fact, a slight modification of our proof can produce a variant of the theorem, including this result as a degenerate case, when $F = A$. However, the interest of our theorem lies in the fact that even for integral domains A , zero-divisors abound in the homology of the strictly skew-commutative algebra F .

(2) The first condition is satisfied in each of the following cases: (a) the cycles z_i representing f_i can be chosen in such a way that $z_i z_j = 0$ for $i, j \in I$; (b) $f_i f_j = 0$ for $i, j \in I$, and $H_q(F) = 0$ for $q > 3 \min\{\deg f_i \mid i \in I\}$.

(3) If $H_q(F) = 0$ for $0 < q < m$, then $MH(F) = I^2H(F)$ in dimensions $\leq 3m$.

(4) Our result can be viewed as an algebraic counterpart to a theorem of Chen [Ch], which gives conditions for the Pontriagin ring of a loop space to contain a free subalgebra.

2. Proof of Theorem 1.4.

A. Free k-models.

DEFINITION 2.1. A DG k -algebra V is called a k -model of F if there exists a finite sequence of skew-commutative DG algebras $F = U_1, U_2, \dots, U_t = V$ and a sequence of homomorphisms of DG rings u_1, u_2, \dots, u_{t-1} , where u_j maps U_j to U_{j+1} or U_{j+1} to U_j , such that $H(u_j)$ are isomorphisms for $j = 1, \dots, t - 1$. The k -model is said to be free if, as an algebra, V is isomorphic to a tensor product of symmetric algebras on even-dimensional generators and exterior algebras on odd-dimensional generators.

Note that each k -model defines an isomorphism $\alpha: H(F) \rightarrow H(V)$. The existence of free k -models will be deduced from the following very general statement.

LEMMA 2.2. *Let $u_0: R \rightarrow A$ be a surjective homomorphism of commutative rings and let F be a skew-commutative connected (i.e. $F_0 = A$) DG A -algebra. Then there exist a free skew-commutative DG R -algebra U and a surjective homomorphism $u: U \rightarrow F$ of DG rings, such that u coincides with u_0 in dimension zero and $H(u)$ is an isomorphism. If, moreover, F_i is a finitely generated A -module for each i , and R is noetherian, then U can be chosen to be locally (i.e. in each degree) finitely generated.*

PROOF. Setting $u(-1) = u_0$ and $U(-1) = R$, we can assume by induction that a homomorphism of DG rings $u(i): U(i) \rightarrow F$ has been constructed, which is surjective in degrees $\leq i$, and induces an epimorphism $Z_i(u(i)): Z_i(U(i)) \rightarrow Z_i(F)$ and isomorphisms $H_j(u(i))$ for $j < i$. By standard diagram chasing we can find free R -modules T'_{i+1}, T''_{i+1} , and T_{i+1} , and dotted maps, such that the diagrams of R -modules depicted below commute and satisfy the following conditions:

$$\text{Im}(p\delta') = \text{Ker } H_i(u(i)),$$

$$\text{Im}(u'_{i+1}) = Z_{i+1}(F),$$

$$\text{Im}(u_{i+2}) = F_{i+2}$$

(of course, all A -modules are viewed as R -modules via u_0).

$$\begin{array}{ccccc}
 & & & u_{i+2} & \\
 & & & \rightarrow & \\
 T_{i+2} & & & & F_{i+2} \\
 & & & & \\
 \delta \downarrow & & & & \downarrow d_{i+2} \\
 & & & u''_{i+1} & \\
 & & & \rightarrow & \\
 T''_{i+1} & & & & F_{i+1} \\
 & & & & \\
 & & & u'_{i+1} & \\
 & & & \rightarrow & \\
 T'_{i+1} & & & & F_{i+1} \\
 & & & & \\
 \delta' \downarrow & & & & \downarrow d_{i+1} \\
 & & & Z_i(u(i)) & \\
 & & & \rightarrow & \\
 Z_i(U(i)) & & & & Z_i(F) \rightarrow 0 \\
 & & & & \\
 p \downarrow & & & & \downarrow \\
 & & & H_i(u(i)) & \\
 & & & \rightarrow & \\
 H_i(U(i)) & & & & H_i(F) \rightarrow 0 \\
 & & & & \\
 \downarrow & & & & \downarrow \\
 0 & & & & 0
 \end{array}$$

Assigning to $T'_{i+1} \oplus T''_{i+1}$ degree $i + 1$, to T_{i+2} degree $i + 2$, and denoting by $V(\)$ the free skew-commutative algebra functor, we set

$$\begin{aligned}
 U(i + 1) &= U(i) \otimes V(T'_{i+1} \oplus T''_{i+1}) \otimes V(T_{i+2}), \\
 u(i + 1) | U(i) &= u(i), & d | U(i) &= d, \\
 u(i + 1) | T'_{i+1} &= u'_{i+1}, & d | T'_{i+1} &= \delta', \\
 u(i + 1) | T''_{i+1} &= u''_{i+1}, & d | T''_{i+1} &= 0, \\
 u(i + 1) | T_{i+2} &= u_{i+2}, & d | T_{i+2} &= \delta.
 \end{aligned}$$

These conditions completely define the DG R -algebra $U(i + 1)$ and the homomorphism of DG rings $u(i + 1)$. By construction, $u(i + 1)$ is surjective in degrees $\leq i + 1$, and induces an epimorphism $Z_{i+1}(u(i + 1))$ and isomorphisms $H_j(u(i + 1))$ for $j < i + 1$. Of course, with $U = \varinjlim U(i)$ and $u = \varinjlim u(i)$, we have established our claim. The finiteness condition is achieved by choosing finitely generated T_i 's.

PROPOSITION 2.3. *Suppose F satisfies condition (1.3)(i). Then F has a free locally finitely generated k -model.*

If S is a set of elements of $IH(F)$, which admits a trivial Massey operation, then the set $\alpha(S) \subset IH(V)$ admits a trivial Massey operation.

PROOF. Set $F = F \otimes_A \hat{A}$, where \hat{A} denotes the m -adic completion of A . Choose, by Cohen's structure theorem for complete local rings, a regular local ring R with $\text{edim } R = \text{edim } \hat{A}$ and a surjective ring homomorphism $u_0: R \rightarrow \hat{A}$. Let $u: U \rightarrow F$ be the homomorphism constructed in the lemma, with U free skew-commutative and locally finitely generated. Let S_1, \dots, S_n be elements of U_1 whose images under $u_0 d_1$ form a minimal set of generators of m . Then the S_i 's form part of a basis of the free R -module U_1 . Denote by J the DG ideal of U generated by $S_1, \dots, S_n, dS_1, \dots, dS_n$. The argument used to prove [AG, Proposition 1] shows, with an easy induction on n , that $H_i(J) = 0$ for all integers i . Setting $V = U/J$, we see that the natural projection from U to V induces an isomorphism in homology. Since J contains the maximal ideal of R , V is a locally finitely generated free k -algebra and, hence, the required model.

For the second statement of the proposition, it is clearly sufficient to establish for a homomorphism of DG rings $w: Y \rightarrow Y'$, such that $H(w)$ is an isomorphism, that (a) if $S \subset IH(Y)$ admits a t.M.o. γ , then $S' = H(w)(S)$ admits a t.M.o. γ' ; and (b) the converse statement, assuming that w is surjective. Part (a) is straightforward, since $\gamma' = w\gamma$ is a t.M.o. For (b), writing W for $\text{Ker}(w)$, we have the exact sequence

$$0 \rightarrow B(W) \rightarrow Z(Y) \xrightarrow{Z(w)} Z(Y') \rightarrow 0.$$

Hence we can lift γ' to a γ defined on the elements (i.e. on sequences of length one) of $S = H(w)^{-1}S'$. By induction we can assume that γ has been defined on sequences of elements of S of length $< n$ in such a way that $w\gamma(f_1, \dots, f_m) = \gamma'(w_* f_1, \dots, w_* f_m)$

$(w_* = H(w))$. If y is any element of Y mapping onto $\gamma'(w_*f_1, \dots, w_*f_n)$ then

$$\begin{aligned} w(dy) &= d\gamma'(w_*f_1, \dots, w_*f_n) \\ &= \sum_{i=1}^{n-1} \bar{\gamma}'(w_*f_1, \dots, w_*f_i) \gamma'(w_*f_{i+1}, \dots, w_*f_n) \\ &= w \left(\sum_{i=1}^{n-1} \bar{\gamma}(f_1, \dots, f_i) \gamma(f_{i+1}, \dots, f_n) \right). \end{aligned}$$

It follows from the exact sequence that there exists an $x \in Y$ such that

$$\sum_{i=1}^{n-1} \bar{\gamma}(f_1, \dots, f_i) \gamma(f_{i+1}, \dots, f_n) - dy = dx;$$

hence we can take $\gamma(f_1, \dots, f_n) = x + y$.

In order to fix notations we recall

DEFINITION 2.4 (cf. [Cl, (3.1); M, (1.4)]). A s.h.m. (= strongly homotopy multiplicative) map $h: G \rightarrow V$ of DG k -algebras is a collection of k -linear maps $\{h_n: (IG)^{\otimes n} \rightarrow IV\}_{n \geq 1}$ such that h_n has degree $n - 1$ and for each n the following identity is satisfied:

$$\begin{aligned} dh_n(g_1, \dots, g_n) &= \sum_{j=1}^n h_n(\bar{g}_1, \dots, \bar{g}_{j-1}, dg_j, g_{j+1}, \dots, g_n) \\ &\quad - \sum_{j=1}^{n-1} h_{n-1}(\bar{g}_1, \dots, \bar{g}_{j-1}, \bar{g}_j g_{j+1}, g_{j+2}, \dots, g_n) \\ &\quad + \sum_{j=1}^{n-1} h_j(\bar{g}_1, \dots, \bar{g}_j) h_{n-j}(g_{j+1}, \dots, g_n). \end{aligned}$$

Observation 2.5. Let V be a DG algebra over k and let $\{v_i\}_{i \in I}$ be a set of elements of $IZ(V)$. Let G be the connected DG k -algebra defined up to (a noncanonical) isomorphism by the following conditions:

- (i) IG has a basis $\{g_i\}_{i \in I}$ such that $\deg g_i = \deg v_i$;
- (ii) $(IG)^2 = 0$;
- (iii) $dG = 0$.

Then $S = \{[v_i]\}_{i \in I}$ admits a t.M.o. if and only if there exists a s.h.m. map $h: G \rightarrow V$ such that $h_1(g_i) = v_i$ ($i \in I$).

Indeed, for a given h define $\gamma([v_1], \dots, [v_n]) = h_n(g_1, \dots, g_n)$. Conversely, given a t.M.o. γ , set $h_n(g_1, \dots, g_n) = \gamma([v_1], \dots, [v_n])$.

In part C of the proof we are going to apply this observation through the following result of Clark.

LEMMA 2.6 [Cl, (3.2)]. *If $h: G \rightarrow V$ is a s.h.m. map, it induces a homomorphism $\bar{B}(h): \bar{B}(G) \rightarrow \bar{B}(V)$ of DG coalgebras over k , where $\bar{B}(\)$ denotes the reduced bar-construction with its standard diagonal map.*

B. Divided powers for DG torsion functors. We shall construct such a structure only in the special case of algebras satisfying the conditions (1.3); their existence in

general for connected algebras is an interesting open question. For the general properties of the Eilenberg-Moore DG torsion functors we refer to [GM], and for the Hopf algebra structure they carry in the cases considered here, to [Av 1].

LEMMA 2.7. *Suppose F satisfies conditions (1.3). Then $T^F = \text{Tor}^F(k, k)$ has a unique structure of Hopf Γ -algebra for which the map $T^\eta = \text{Tor}^\eta(k, k)$ induced by the structure map $\eta: A \rightarrow F$ is a homomorphism of Hopf Γ -algebras. Moreover, T^η is surjective.*

PROOF. It is shown in [Av 1, (1.1)] that T^F has a natural structure of Hopf algebra. Furthermore, it is proved in [Av 2, (2.2)], that T^η is a surjective homomorphism of Hopf algebras with kernel equal to $I(k \otimes F) \cdot T^A$. This ideal is closed under the divided powers of T^A by our assumption (1.3)(iii), and the claim follows.

In particular, as for Theorem 0.3, one can apply the result of [An] and [Sj] to see that the graded dual E_F of T^F is as a Hopf algebra the universal enveloping algebra of its Lie subalgebra formed by the elements orthogonal to $T_0^F \oplus I^{(2)}T^F$. We shall denote this Lie algebra by L_F .

Next we recall a definition from [Av 2, (2.4)]. For $z \in IZ(F)$ choose $y \in X$ such that $dy = -z$. Then $[z] \mapsto 1 \otimes y$ defines a degree one k -linear map $\sigma_F: IH(F) \rightarrow T^F$ called the suspension. It appears in the statement of the following proposition, which has been proved also by M. Andr e.

PROPOSITION 2.8. *The kernel of the composite map*

$$IH(F) \xrightarrow{\sigma_F} T^F \xrightarrow{\pi_F} T^F / I^{(2)}T^F$$

is the set $MH(F)$ of all elements of $IH(F)$ decomposable as matrix Massey products.

PROOF. By an application of some results of Gugenheim and May, it was shown in [Av 2, (2.6)] that $\text{Ker } \sigma_F = MH(F)$. Hence we have to show that $\text{Im } \sigma_F \cap I^{(2)}T^F = 0$. Let $P(T^F)$ denote the subspace of primitive elements of the Hopf algebra T^F . By [An, Proposition 8], one knows that the restriction of π_F to $P(T^F)$ is injective. Hence it suffices to show that $\text{Im } \sigma_F \subset P(T^F)$. From the definition of σ_F , given in [GM], $\text{Im } \sigma_F \subset I_1$, where $0 \subset I_0 = T_0^F \subset I_1 \subset \dots$ is the increasing filtration of T^F defined by the Eilenberg-Moore spectral sequence with $E_{p,q}^2 = \text{Tor}_{p,q}^{H(F)}(k, k) \Rightarrow \text{Tor}_{p+q}^F(k, k)$. This is a sequence of Hopf algebras [Av 1], hence the term $E^\infty = \bigoplus_{j \geq 0} I_j / I_{j-1}$ is a bigraded Hopf algebra associated to T^F . But the elements of $E_{1,*}^\infty = I_1 / I_0$ are primitive for dimensional reasons. It follows that the positive dimensional elements of I_1 are primitive in T^F , which implies $\text{Im } \sigma_F \subset P(T^F)$.

C. *End of the proof.* Let V be a k -model of F as in Proposition 2.3. Denote by α the isomorphism $IH(F) \simeq IH(V)$ realized by the chosen homotopy equivalence of F to V , and by β the corresponding homomorphism of DG torsion functors; note that β is an isomorphism of Hopf algebras. With the notations of Theorem 1.4, choose $v_i \in IZ(V)$ with $[v_i] = \alpha(f_i)$ ($i \in I$). By Proposition 2.3, $\{[v_i]\}_{i \in I}$ admit a t.M.o., hence as observed in (2.5) there is a s.h.m. map $h: G \rightarrow V$ with $h_1(g_i) = v_i$. Now

consider the diagram, all of whose maps have already been defined:

$$\begin{array}{ccccccc}
 IG & \xrightarrow{IH(h_1)} & IH(V) & \xleftarrow{\alpha} & IH(F) & & \\
 \sigma_G \downarrow & & \sigma_V \downarrow & & \sigma_F \downarrow & & \\
 \bar{B}(G) & \xrightarrow{H\bar{B}(h)} & \text{Tor}^V(k, k) & \xleftarrow{\beta} & T^F & \xleftarrow{T^\eta} & T^A \\
 & & & & \pi_F \downarrow & & \downarrow \pi_A \\
 & & & & Q(T^F) & \xleftarrow{Q(T^\eta)} & Q(T^A).
 \end{array}$$

The commutativity of the lower square follows from the fact that T^η is a map of Γ -algebras over k (Lemma 2.7), that of the two upper squares is a consequence of the naturality of the suspension [Av 2, (2.5)]. Set $h_* = H\bar{B}(h)$.

Claim. The composite map $\delta = \pi_F \beta^{-1} h_* \sigma_G$ is injective.

First note that by [M, (1.5)], $\alpha MH(F) = MH(V)$, hence condition (2) of Theorem 1.4 together with Gugenheim's and May's evaluation of the kernel of the suspension show that $\text{Im } IH(h_1) \cap \text{Ker } \sigma_V = 0$. This implies that h_* is injective on $\text{Im } \sigma_G \simeq IG$. But $\text{Im } \sigma_G$ is the set of primitive elements of $\bar{B}(G)$; h_* being a homomorphism of coalgebras (Lemma 2.6), which is injective on the primitives, is injective globally. Having proved that $\beta^{-1} h_* \sigma_G$ is injective, we see that $\text{Im}(\alpha^{-1} \circ IH(h_1))$ intersects $\text{Ker } \sigma_F = MH(F)$ trivially, and the claim follows from Proposition 2.8.

After dualization, the relevant information obtained so far can be summarized in the commutative diagram

$$\begin{array}{ccccc}
 T(sIG^*) & \xleftarrow{h^*} & E_F & \xrightarrow{E_\eta} & E_A \\
 \rho \downarrow & & \iota_F \uparrow & & \iota_A \uparrow \\
 sIG^* & \xleftarrow{\delta'} & L_F & \xrightarrow{L_\eta} & L_A
 \end{array}$$

in which:

- $T(\)$ denotes the tensor algebra functor;
- ρ is the natural injection of the tensors of length one;
- h^* is a surjective homomorphism of algebras;
- E_η is an injective homomorphism of Hopf algebras;
- L_η is an injective homomorphism of Lie algebras;
- ι_F and ι_A are the injections of the corresponding Lie algebras in their universal envelopes;

δ' is the degree zero k -linear map, obtained by composing δ^* with the degree one isomorphism of IG^* to $s(IG^*)$; δ' is surjective by the claim established above.

Let ϵ be a splitting of δ' . By the universal property of the tensor algebra, there is a unique algebra homomorphism $\nu: T(sIG^*) \rightarrow E_F$, whose restriction to the tensors of length one coincides with $\iota_F \circ \epsilon$. The map ν is injective, since it is a splitting of h^* , and it sends into L_F the free Lie algebra, generated by $\rho(sIG^*)$ in $T(sIG^*)$ under the commutation operation. The image of this Lie algebra under $L_\eta \circ \nu$ is the required Lie subalgebra of L_A .

D. *Proof of Corollary 1.5.* Let $R = k[[X_1, \dots, X_r]]$ be the power series ring, and let M_1, \dots, M_s denote the monomials in the indeterminates, corresponding to the monomials m_1, \dots, m_s in the x_i 's. By [Av 2, (6.2)] there is an isomorphism of Hopf algebras

$$\text{Tor}^A(k, k) \otimes_{\text{Tor}^B(k, k)} k \simeq \text{Tor}^S(k, k) \otimes_{\text{Tor}^R(k, k)} k$$

(where $S = R/(M_1, \dots, M_s)$), which can be shown, using Lemma 2.7, to commute with the divided powers. Since both homomorphisms $B \rightarrow A$ and $R \rightarrow S$ induce injective maps in homology (loc. cit.) it follows from the above isomorphism and [Av 2, (1.3)] that $\text{Ker}(L_A \rightarrow L_B) \simeq \text{Ker}(L_S \rightarrow L_R)$ as Lie algebras; hence it suffices to prove the statement for $A = S$. In this case if $\text{grade}(M_1, \dots, M_s) \neq s$, S is not a complete intersection, and two of the monomials are divisible by one of the indeterminates. The corresponding one-dimensional homology classes in the homology of the Koszul complex of S satisfy the conditions of Remark 1.7(2)(a), hence Theorem 1.4 applies.

3. Proof of Theorem 1.1.

A. *Preliminaries on the Koszul complex.* We shall denote by K^A or simply by K the Koszul complex on a minimal set of generators of \mathfrak{m} . It is well known that $H(K)$ is up to isomorphism independent of the choice of the generators. For ease of reference we collect several known results.

LEMMA 3.0. *For every local ring A ,*

$$e_2(A) = \dim_k H_1(K), \quad e_3(A) = \dim_k H_2(K)/H_1^2(K).$$

LEMMA 3.1 [AG]. *If $\text{depth } A = d$, there exists a minimal set t_1, \dots, t_n of generators of \mathfrak{m} such that t_1, \dots, t_d form a regular sequence, and the epimorphism from the Koszul complex on t_1, \dots, t_n to the Koszul complex on the images of t_{d+1}, \dots, t_n in $\bar{A} = A/(t_1, \dots, t_d)$ induces an isomorphism $H(K^A) \xrightarrow{\sim} H(K^{\bar{A}})$.*

LEMMA 3.2. *The following are equivalent:*

- (1) A is a complete intersection;
- (2) $H(K)$ is the exterior algebra on $H_1(K)$;
- (3) $H_2(K) = H_1^2(K)$;
- (4) $(H_1(K))^{\text{edim } A - \text{depth } A} \neq 0$.

((1) \Rightarrow (2) is due to Tate, (3) \Rightarrow (1) is due to Assmus; for proofs cf. [GL]; (4) \Rightarrow (1) is Wiebe's main result in [W].)

LEMMA 3.3 [AG]. *The following are equivalent:*

- (1) A is Gorenstein;
- (2) there exists an $n \geq 0$ such that the natural pairings (induced by the multiplication in K) $H_i(K) \times H_{n-i}(K) \rightarrow H_n(K)$ are perfect for all $i \geq 0$.

When these conditions are satisfied, one has $n = \text{edim } A - \text{depth } A$, and $H_n(K) \simeq k$.

LEMMA 3.4 [Av 3]. *Let A be a Cohen-Macaulay almost complete intersection (i.e. $d(A) = 1$) with $\text{edim } A - \text{depth } A = n$. Then $e_3(A) = \dim_k H_n(K)$.*

LEMMA 3.5. *Suppose that one of the following conditions is fulfilled:*

- (a) $\text{edim } A - \dim A \leq 1$; or
- (b) $\text{edim } A - \text{depth } A \leq 2$.

Then A is either a complete intersection or a Golod ring. Moreover, Golod rings satisfying (b) have $e_3(A) = e_2(A) - 1$.

Condition (a) is treated by Shamash [Sh], while (b) is settled by Scheja [Sc]; note that a direct proof of Scheja's result can be obtained along the lines of Lemma 3.8 below.

In order to prove Theorem 1.1 we shall use a specialization of Theorem 1.4, which we now state explicitly.

PROPOSITION 3.6. (a) *Let $\text{edim } A - \text{depth } A \leq 3$. Suppose $L \subset IH(K)$ is a graded subspace such that:*

- (1) $L^2 = 0$; and
- (2) $L \cap I^2H(K) = 0$.

Then L_A contains a Lie subalgebra isomorphic to the free graded Lie algebra on sL^ .*

(b) *Let A be Gorenstein with $\text{edim } A - \dim A = 4$. Suppose there exist linearly independent $f_1, f_2 \in H_1(K)$ such that $f_1 \cdot f_2 = 0$. Then L_A contains a free graded Lie subalgebra on two generators of degree 2.*

PROOF. Clearly $F = K$ satisfies conditions (1.3). Choose \bar{A} as in Lemma 3.1. Suppose we have proved that a t.M.o. can be defined on the image in $H(K^{\bar{A}})$ of a basis of L (resp. of f_1 and f_2). Then by Lemma 3.1 and by (the proof of) Proposition 2.3, a t.M.o. can be defined on a basis of L (resp. on $\{f_1, f_2\}$), hence condition (1) of Theorem 1.4 is satisfied by this set. Since the second condition of the theorem is satisfied in view of our assumption (2) and Remark 1.7(3), we get our result as a special case of the theorem.

It remains to prove that a t.M.o. can be defined on a basis f_1, \dots, f_m of L (resp. on $\{f_1, f_2\}$), assuming $\text{depth } A = 0$. In case (a) choose a set z_1, \dots, z_m of cycles such that $[z_i] = f_i$. According to our condition (1) there exist $y_{ij} \in IK$ such that $dy_{ij} = z_i z_j$. Set $\gamma(f_i) = z_i$, $\gamma(f_i, f_j) = y_{ij}$, and $\gamma(f_{i_1}, \dots, f_{i_l}) = 0$ for $l > 2$. For dimensional reasons it is clear that γ is a t.M.o. on $\{f_1, \dots, f_m\}$ since $K_i = 0$ for $i > 3$.

In case (b) choose $y \in K_3$ such that $dy = z_1 z_2$ ($[z_i] = f_i$). Since A is artinian of embedding dimension 4, $Z_4(K) = H_4(K)$ is one dimensional over k generated say by e . By Poincaré duality (Lemma 3.3), one can choose $u_1, u_2 \in Z_3(K)$ such that $f_i \cdot [u_j] = \delta_{ij} e$ ($i, j \in \{1, 2\}$); in particular, this means that $z_1 u_2 = 0 = z_2 u_1$. Let $z_i y = a_i e$ with $a_i \in k$ ($i = 1, 2$). Set $\gamma(f_i) = z_i$, $\gamma(f_1, f_2) = -\gamma(f_2, f_1) = y - a_1 u_1 - a_2 u_2$, and $\gamma = 0$ on all other sequences of f_1, f_2 . Since $z_i \gamma(f_1, f_2) = 0$, it is clear that γ is a t.M.o. on $\{f_1, f_2\}$.

B. $\text{edim } A - \text{depth } A \leq 2$. Assume that A is not a complete intersection. Then $e_2(A) > 1$, and by Lemma 3.5(b) A is a Golod ring. By Corollary 1.6 or by Proposition 3.6(a) L_A contains a free Lie subalgebra on $sIH(K)^*$ which proves the theorem for this class.

C. $\text{edim } A - \text{depth } A = 3$. In order to apply Proposition 3.6 in this case we have first to establish some results on the structure of $H(K)$.

LEMMA 3.7. *Suppose A satisfies condition C and that $\text{edim } A - \dim A = 2$. Then $e_3(A) \geq e_2(A) - 2$.*

PROOF. By Lemmas 3.1 and 3.0 we can assume $e_1(A) = \text{edim } A = 3$, $\dim A = 1$, $\text{depth } A = 0$. Let \mathfrak{p} be a nonmaximal prime of \hat{A} . In the notation of the introduction, $\chi_3(A) = \chi_3(\hat{A})$. On the other hand, by Gulliksen's theorem [Gu, (3.3)] (i.e., Conjecture C_5 for $r = 3$ and complete rings), we have

$$\chi_3(\hat{A}) \geq \chi_3(\hat{A}_{\mathfrak{p}}) + \dim \hat{A}/\mathfrak{p},$$

hence

$$e_3(A) \geq e_2(A) - 2 + \chi_3(\hat{A}_{\mathfrak{p}}).$$

By Lech's theorem [L, Theorem 3] (i.e. Conjecture C_5 for $r = 1$, which is easily checked directly for complete rings), we see that $e_1(\hat{A}_{\mathfrak{p}}) \leq e_1(\hat{A}) - \dim \hat{A}/\mathfrak{p} \leq 2$. It follows from Lemma 3.5(b) that the artinian ring $\hat{A}_{\mathfrak{p}}$ is either a complete intersection, in which case $\chi_3(\hat{A}_{\mathfrak{p}}) = 0$, or is a Golod ring, with $\chi_3(\hat{A}_{\mathfrak{p}}) = 2 - e_2(\hat{A}_{\mathfrak{p}}) + (e_2(\hat{A}_{\mathfrak{p}}) - 1) = 1$. This proves our claim.

LEMMA 3.8. *Suppose A is a Cohen-Macaulay ring satisfying condition C. Then either A is a complete intersection, or $H_1(K)$ contains linearly independent elements f_1, f_2 , such that $f_1 \cdot f_2 = 0$.*

PROOF. By Lemma 3.1 we can assume A is artinian, hence $A = R/\mathfrak{a}$ with (R, \mathfrak{n}) regular of dimension three and \mathfrak{a} \mathfrak{n} -primary with $\mathfrak{a} \subset \mathfrak{n}^2$. Choose a minimal set of generators x_1, \dots, x_r of \mathfrak{a} such that x_1, x_2, x_3 form an R -regular sequence. If A is not a complete intersection, the ring homomorphism $R/(x_1, \dots, x_4) \rightarrow A$ induces a homomorphism of Koszul homology algebras which is injective in degree one by the minimality of the chosen set of generators. Hence it is sufficient (and necessary) to prove the lemma for almost complete intersections. Setting $t = \dim_k H_3(K)$, we have $\dim_k H_2(K) = t + 4 - 1 = t + 3$. From Lemma 3.4 we have $\dim_k H_1^2(K) = 3$. Our claim now follows from the next general statement.

LEMMA 3.9. *Let A be an arbitrary local ring, with $\dim_k H_1(K) = r$. If $\dim_k H_1^2(K) \leq 2(r - 2)$, then there exist linearly independent $f_1, f_2 \in H_1(K)$ such that $f_1 \cdot f_2 = 0$.*

PROOF. Consider the kernel M of the canonical epimorphism $\Lambda^2 H_1(K) \rightarrow H_1^2(K)$. Our assumption means that the linear variety defined by M in $\mathbf{P} = \mathbf{P}_k(\Lambda^2 H_1(K))$ is of codimension $\leq 2(r - 2)$. Hence it has a nonempty intersection with the $2(r - 2)$ -dimensional Grassmannian of projective lines in $\mathbf{P}_k(H_1(K))$ embedded in \mathbf{P} by the Plücker embedding. This implies the assertion of the lemma.

REMARK. In fact, the algebra $H(K)$ for Cohen-Macaulay almost complete intersections of embedding codimension 3 has been explicitly computed in [Go 2] and [Av 3].

We now turn to the proof of Theorem 1.1 assuming condition C. Note that it implies the inequality $\text{edim } A - \dim A \leq 3$. When this difference does not exceed 1, we are done by applying Lemma 3.5(a) and Corollary 1.6.

Assume $\text{edim } A - \dim A = 2$, and let L' be the vector space generated by some lifting to $H_2(K)$ of a k -basis of $H_2(K)/H_1^2(K)$. Clearly $L = L'$ satisfies the conditions of Proposition 3.6(a), hence our claim follows from Lemma 3.7 when $e_2(A) > 3$. The algebras $H(K)$ for noncomplete intersections with $e_2(A) = 3$ have been computed in [Av 2, (7.2)]. In particular, it is shown there that either $H_1H_2 = 0$, or $A = A'/(x)$, where (A', m') is a Golod ring with $\text{edim } A' - \text{depth } A' = 2$, and x is a nonzero divisor contained in m'^2 . In the second case $H(K) \simeq H(K^{A'}) \otimes_k k[f]$, $\deg f = 1, f^2 = 0$, hence $L = \text{Im}(IH(K^{A'}) \rightarrow IH(K))$ satisfies, according to Lemma 3.5(b), the conditions of Proposition 3.6(a). In the first case we take $L = L' \oplus H_3(K)$, and note that $H_3(K) \neq 0$ by the assumption of depth zero, while $L' \neq 0$ by Lemma 3.2.

Finally let $\text{edim } A - \dim A = 3$, i.e. let A be a Cohen-Macaulay ring. Then the elements f_1, f_2 of Lemma 3.8 generate a two-dimensional subspace L satisfying the conditions of 3.6(b).

D. $\text{edim } A - \dim A = 4$, A Gorenstein. The following lemma, shown to us by J. Backelin, and independently proved by A. Kustin and M. Miller, gives a shortcut to the original treatment of this case.

LEMMA 3.10. *Suppose A satisfies condition D. Then either A is a complete intersection (i.e. $e_3(A) = 0$), or $e_3(A) \geq e_2(A) - 1$.*

PROOF. Assume $0 < e_3(A) < e_2(A) - 1 = r - 1$. Since $H(K)$ is a Poincaré algebra (Lemma 3.3), and $H_i(K) = 0$ for $i > 4$, $\dim_k H_2(K) = 2(r - 1)$, hence $\dim_k H_1^2(K) = \dim_k H_2(K) - e_3(A) \geq r$. Denoting by $^\perp$ orthogonal complements with respect to the nondegenerate bilinear form $H_2(K) \times H_2(K) \rightarrow H_4(K) \simeq k$, which is symmetric if $\text{char } k \neq 2$ and alternating otherwise (cf. [Av 2, (9.4)]), we get $\dim_k (H_1^2(K))^\perp \leq r - 2$. It follows that $H_1^2(K) \not\subset (H_1^2(K))^\perp$, hence $H_1^4(K) \neq 0$, which contradicts Wiebe's result; cf. Lemma 3.2.

Now let A be a noncomplete intersection satisfying condition D. By the lemma $\dim_k H_1^2(K) = \dim_k H_2(K) - e_3(A) \leq r - 1$, and this integer is $\leq 2(r - 2)$ since $r > 4$. Lemma 3.9 shows that the conditions of Proposition 3.6(b) are satisfied, and this finishes the proof of the theorem.

4. Proof of Theorem 1.2. Suppose that A satisfies the conditions of the theorem, and is not a complete intersection. The equivalence of (i), (ii), and (vi) is a consequence of Theorem 1.1 and Proposition 0.7. If, moreover, A is Cohen-Macaulay, it has been shown in Lemmas 3.5(b), 3.8 and 3.10 that $H_1(K)$ contains two linearly independent elements admitting a t.M.o., hence L_A contains a free Lie subalgebra generated by two elements of degree 2. The same conclusion holds also under the assumption $\text{edim } A - \dim A = 1$ by Lemma 3.5(a). It follows that in these cases $e_{2i}(A) \neq 0$ for all $i > 0$, and also that $r_A \leq \sqrt{2}/2$, the last inequality being a consequence of the relation $P_A(t) \gg (1 - 2t^2)^{-1}$.

In order to finish the proof we have still to treat the rings satisfying the conditions

$$(4.1) \quad \text{edim } A - \text{depth } A = 3, \quad \text{edim } A - \dim A = 2.$$

We want to prove for them the relations:

- (iii)^o $r_A \leq \sqrt{2}/2$; and
- (iv)^o $e_{6i} \neq 0$ for $i > 0$.

From Lemma 3.7 and Proposition 3.6(a) L_A contains a free graded Lie subalgebra generated by $e_2(A) - 2$ elements of degree 3. As in the proof of the implication $C_{10} \Rightarrow C_2$ of Proposition 0.7 (cf. §5), it follows that it also contains a free graded Lie subalgebra on $e_2(A) - 2$ generators of degree 6. Hence for $e_2(A) \geq 4$, (iv)^o is established, while in the remaining case there is the more precise

LEMMA 4.2. *Suppose A satisfies (4.1) and has $e_2 = 3$. Then $f_i(A) > 0$ for $i \geq 2$ (cf. Conjecture C_3' in the introduction).*

PROOF. We use the technique and notation developed by Gulliksen [Gu, §4]: φ denotes the well-known isomorphism from the multiplicative group $1 + t\mathbb{Z}[[t]]$ to the additive group $t\mathbb{Z}[[t]]$, which in particular sends $1/G(t) = P_A(t)/(1+t)^{e_1(A)}$ to $-\sum_{i \geq 0} f_i(A)t^i$. According to [Av 2, (7.2)], the following expressions for $G(t)$ can occur:

$$(4.3) \quad \begin{aligned} & (1 - t^2)(1 - 2t^2 - t^3), \\ & 1 - 3t^2 - (m - s + 2)t^3 - (m - s)t^4, \\ & (m, s) = (1, 0), (1, 1), \text{ or } m > 1, 0 \leq s \leq 2, \\ & 1 - 3t^2 - (m - 1)t^3 - (m - 3)t^4 - t^5 - t^6, \quad m > 1, \end{aligned}$$

where $m = \dim_k H_3(K)$, $s = \dim_k H_1^2(K)$. In order to establish the inequality $\varphi(G(t)) \gg t^2(1 - t)^{-1}$, we use the inequality $\varphi(1 - 2t^2 - t^3) \gg t^2(1 - t)^{-1}$ obtained in the proof of [Gu, (4.1)(i)]. φ being a group homomorphism, it settles the first case. In all other cases, except for $m = 2$ in the last formula, $G(t)$ is of the form $(1 - F(t))$, with $F(t) \gg 2t^2 + t^3$, hence by [Gu, (4.2)(ii)],

$$\varphi(G(t)) \gg \varphi(1 - 2t^2 - t^3).$$

In the last case $G(t)$ can be written $(1 - t^2)(1 - t^2 - t^2(1 + t^2 + t^3))(1 - t)^{-1}$, which by the same result of Gulliksen implies

$$\varphi(G(t)) \gg t^2 + t^2(1 + t^2 + t^3)(1 - t)^{-1}.$$

Finally we prove (iii)^o under condition (4.1). When $e_2(A) = 3$, the inequality $r_A \leq \sqrt{2}/2$ is seen by direct inspection of the power series (4.3). When $e_2(A) > 4$, L_A contains a free Lie subalgebra on at least 3 generators of degree 3, hence $P_A(t) \gg (1 - 3t^3)^{-1}$, which implies $r_A \leq 3^{-1/3} < 2^{-1/2}$. When $e_2(A) = 4$ and $\dim_k H_1^2(K) \leq 4$, Lemma 3.9 and Proposition 3.6(b) imply that L_A contains a free Lie subalgebra on two generators of degree 2, hence $r_A \leq \sqrt{2}/2$ as above.

In the last case to remain undecided, $e_2(A) = 4$ and no two linearly independent elements of $V = H_1(K)$ have zero product. Since under our condition (4.1) Lemma

3.2 implies $V^3 = 0$, it is easy to see that for a suitably chosen basis v_1, v_2, v_3, v_4 of V , the subalgebra W of $H(K)$, generated by V , is isomorphic to one of the following k -algebras:

$$W' = \Lambda(V) / (\Lambda^3 V), \text{ or } W'' = \Lambda(V) / (v_1 v_2 + v_3 v_4).$$

I do not know shorter ways to finish the proof of the theorem than to compute the Poincaré series $P_W(t) = \sum_{n \geq 0} \sum_{p+q=n} \dim_k \text{Tor}_{p,q}^W(k, k) t^n$ (here p denotes the homological and q the internal degrees of the components of the graded Tor functor).

Consider the divided powers algebra $U = U'$ (resp. $U = U''$) obtained by adjoining to U' (resp. U'') variables S_1, S_2, S_3, S_4 of bidegree $(1, 1)$ with differential mapping S_i to v_i (cf. [GL]). The bigraded homology of U is computed by counting Euler characteristics in the complexes of vector spaces

$$0 \leftarrow U_{q-2,q} \xleftarrow{d_{q-1,q}} U_{q-1,q} \xleftarrow{d_{q,q}} U_{q,q} \leftarrow 0.$$

In bidegrees $\neq (0, 0)$, the result is

$$\dim_k H_{p,q}(U') = \begin{cases} 0 & \text{if } p \neq q - 2, q \geq 1, \\ (q + 1)(q - 1)(q - 2)/2 & \text{if } p = q - 2 \geq 1; \end{cases}$$

$$\dim_k H_{p,q}(U'') = \begin{cases} 0 & \text{if } p \neq q - 2, q \geq 1, (p, q) \neq (1, 2), \\ 1 & \text{if } (p, q) = (1, 2), \\ (q + 1)(q - 1)(q - 3)/3 & \text{if } p = q - 2 \geq 2. \end{cases}$$

For dimensional reasons, it is easy to see that any basis of $IH(U')$ (resp. of $IH(U'')$) admits a trivial Massey operation. Applying the (graded analogue of) Golod's theorem [Go 1], cf. also Corollary 1.6, we obtain after a computation with formal power series the following expressions:

$$P_{W'}(t) = ((1 - t^2)^4 + (t^2 - 4)t^5)^{-1},$$

$$P_{W''}(t) = ((1 - t^2)^4(1 - t^3 - t^4) + (t^4 - 4t^2 + 1)t^3)^{-1}.$$

It remains to observe that, since $I^3 W = 0$, W is an algebra retract of $H(K)$, hence $P_W(t) \ll P_{H(K)}(t)$. Together with the equality $P_A(t) = (1 + t)^3 P_{H(K)}(t)$ (e.g. [Av 1, (5.11)]), this gives $r_A \leq r_W < \sqrt{2}/2$, and the proof of Theorem 1.2 is now complete.

5. Some relations between the conjectures.

(5.1) PROOF OF PROPOSITION 0.6. Assume C_5 holds. Then

$$\chi_{2i}(A) \leq \chi_{2i}(A_{\mathfrak{p}}) + \dim A/\mathfrak{p} \leq \chi_{2i}(A_{\mathfrak{p}}) + \dim A - \dim A_{\mathfrak{p}},$$

which is the assertion of $C_4(c)$. Furthermore, the identity

$$e_r(A) - e_r(A_{\mathfrak{p}}) = (-1)^{r-1}(\chi_r(A) - \chi_r(A_{\mathfrak{p}})) + (-1)^r(\chi_{r-1}(A) - \chi_{r-1}(A_{\mathfrak{p}}))$$

shows that C_5 implies C_6 .

For $i > 1$, C_7 obviously follows from C_8 , while the inequality $e_1(A) \leq e_1(B)$ is known from [L].

The fact that C_6 is a consequence of C_7 has been established in the proof of Theorem (2.6).¹⁰

Assume $C_4(a)$ holds for artinian rings and $C_4(c)$ holds for complete rings. Let \mathfrak{p} be a prime ideal of A and pick a prime ideal \mathfrak{p}' of \hat{A} minimal over \mathfrak{p} . We have a local flat extension $A_{\mathfrak{p}} \rightarrow A_{\mathfrak{p}'}$ with artinian fibre $C = A_{\mathfrak{p}'}/\mathfrak{p}A_{\mathfrak{p}'}$. The following chain of inequalities establishes $C_4(c)$:

$$\begin{aligned} \dim A_{\mathfrak{p}} - \chi_{2i}(A_{\mathfrak{p}}) &\leq \dim A_{\mathfrak{p}} - \chi_{2i}(A_{\mathfrak{p}}) + \dim C - \chi_{2i}(C) && \text{(by the artinian } C_4(a)) \\ &= \dim \hat{A}_{\mathfrak{p}'} - \chi_{2i}(\hat{A}_{\mathfrak{p}'}) && \text{(by Theorem (1.1))}^{11} \\ &\leq \dim \hat{A} - \chi_{2i}(\hat{A}) && \text{(by the complete } C_4(c)) \\ &= \dim A - \chi_{2i}(A). \end{aligned}$$

Furthermore, taking a minimal prime \mathfrak{p} of A , the already established $C_4(c)$ and the artinian case of $C_4(a)$ imply $C_4(a)$ in general.

REMARK 5.2. Recall that Gulliksen [Gu, (3.3)] has proved C_5 for homomorphic images of regular local rings and $r < \pi(A)$, hence $C_4(c)$ is known for complete rings and all i , when $\text{char } k = 0$, and for $i < \text{char } k$ otherwise.

(5.3) PROOF OF PROPOSITION 0.7. The implication $C_0 \Rightarrow C_1$ is [Gu, (2.5)].

The assertion of C_2 can be written in the form $(1 - 2t^q)^{-1} \ll p \cdot P_A(t)$ for some positive integers p and q . This implies the inequality $r_A \leq 2^{-1/q}$, hence C_1 .

Assuming C_{10} , let l_1 and l_2 be free generators of L , $l_1 \neq l_2$. If $\text{deg } l_i$ is odd, we can replace l_i by $\kappa(l_i)$, where κ is the adjusting operator of L (cf. [Sj]); note that if $\text{char } k \neq 2$, then $\kappa(l_i) = 1/2[l_i, l_i]$. The graded Lie subalgebra of L generated by (the new) l_1, l_2 is free as a graded subalgebra of a free Lie algebra,¹² and since it is situated in even degrees, is a Lie algebra in the usual sense (i.e. $[a, b] = -[b, a]$ for any pair of elements). Then $[l_1, [l_2, [l_2, [l_2, l_1]]]]$ and $[[l_2, l_1], [l_2, [l_2, l_1]]]$ are distinct elements from its Hall basis,¹³ having equal degrees, say q . Applying Lemaire¹⁴ once more, we see they generate a free subalgebra. Passing to the universal envelopes, we have, for the Poincaré series, $(1 - 2t^q)^{-1} \ll P_A(t)$, hence C_2 follows.

Finally, the implication $C_{10} \Rightarrow C_9$ is simply faithfully flat descent, by remarking that E_A is free over the universal envelope $U(L)$ of L , which by assumption is the tensor algebra on a (graded) vector space of dimension > 1 , hence is not noetherian.

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¹⁰ See footnote 2.

¹¹ See footnote 2.

¹² J.-M. Lemaire, *Algèbres connexes et homologie des espaces de lacets*, Lecture Notes in Math., no. 422, Springer-Verlag, Berlin, 1974 (Theorem A.1.10).

¹³ N. Bourbaki, *Groupes et algèbres de Lie*. Chap. II: *Algèbres de Lie libres*, Hermann, Paris, 1972.

¹⁴ See footnote 12.

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REFERENCES

- [An] M. André, *Hopf algebras with divided powers*, J. Algebra **18** (1971), 19–50.
- [Av 1] L. L. Avramov, *On the Hopf algebra of a local ring*, Izv. Akad. Nauk SSSR Ser. Mat. **38** (1974), 253–277 = Math. USSR-Izv. **8** (1974), 259–284.
- [Av 2] _____, *Small homomorphisms of local rings*, J. Algebra **50** (1978), 400–453.
- [Av 3] _____, *Poincaré series of almost complete intersections of embedding dimension three*, PLISKA Stud. Math. Bulgar. **2** (1981), 167–172.
- [Av 4] _____, *Sur la croissance des nombres de Betti d'un anneau local*, C. R. Acad. Sci. Paris, Ser. A-B **289** (1979), A369–A372.
- [AG] L. L. Avramov and E. S. Golod, *On the homology of the Koszul complex of a local Gorenstein ring*, Mat. Zametki **9** (1971), 53–58 = Math. Notes **9** (1971), 30–32.
- [BK] W. Borho and H. Kraft, *Über die Gelfand-Kirillov dimension*, Math. Ann. **220** (1976), 1–24.
- [Ch] K.-T. Chen, *Free subalgebras of loop space homology and Massey products*, Topology **11** (1972), 237–243.
- [Cl] A. Clark, *Homotopy-commutativity and the Moore spectral sequence*, Pacific J. Math. **15** (1965), 65–74.
- [Go 1] E. S. Golod, *On the homology of certain local rings*, Dokl. Akad. Nauk SSSR **144** (1962), 479–482 = Soviet Math. Dokl. **3** (1962), 745–748.
- [Go 2] _____, *On the homology of certain local rings*, Uspehi Mat. Nauk **33** (1978), no. 5, 177–178.
- [GM] V. K. A. M. Gugenheim and J. P. May, *On the theory and applications of differential torsion products*, Mem. Amer. Math. Soc. **142** (1974).
- [Gu] T. H. Gulliksen, *On the deviations of a local ring*, Math. Scand. **47** (1980), 5–20.
- [GL] T. H. Gulliksen and G. Levin, *Homology of local rings*, Queen's Papers in Pure and Appl. Math., No. 20, Queen's Univ., Kingston, Ont., 1969.
- [L] C. Lech, *Inequalities related to certain couples of local rings*, Acta Math. **112** (1964), 69–89.
- [M] J. P. May, *Matric Massey products*, J. Algebra **12** (1969), 533–568.
- [Sc] G. Scheja, *Über die Bettizahlen lokaler Ringe*, Math. Ann. **155** (1964), 155–172.
- [Sj] G. Sjödin, *Hopf algebras and derivations*, J. Algebra **64** (1980), 218–229.
- [Sh] J. Shamash, *The Poincaré series of a local ring*, J. Algebra **12** (1969), 453–470.
- [W] H. Wiebe, *Über homologische Invarianten lokaler Ringe*, Math. Ann. **179** (1969), 257–274.

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