THE RANK 3 PERMUTATION REPRESENTATIONS
OF THE FINITE CLASSICAL GROUPS

BY

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ABSTRACT. The permutation representations in the title are all determined, and no surprises are found to occur.

1. Introduction. A group G has rank 3 in its permutation representation on the cosets of a subgroup K if there are exactly 3 (K, K)-double-cosets; that is, if K has exactly 3 orbits on the set G/K of K-cosets. Such permutation representations have been studied a great deal during the past 15 years; classical groups have been intensively studied for more than a century. The purpose of this paper is to relate these two areas, by proving the following results.

THEOREM 1.1. *Let M be one of the groups*

$$Sp(2m - 2, q), \quad \Omega^{\pm}(2m, q), \quad \Omega(2m - 1, q) \quad or \quad SU(m, q)$$

for $m \geqslant 3$ and q a prime power. Let $M \trianglelefteq G$ with $G/Z(M) \leqslant \mathrm{Aut}(M/Z(M))$. Assume that G acts as a primitive rank 3 permutation group on the set X of cosets of a subgroup K of G. Then (at least) one of the following holds up to conjugacy under $\mathrm{Aut}(M/Z(M))$.

(i) *X is an M-orbit of singular (or isotropic) points.*

(ii) *X is an M-orbit of maximal totally singular (or isotropic) subspaces and $M = Sp(4, q), SU(4, q), SU(5, q), \Omega^{-}(6, q), \Omega^{+}(8, q)$ or $\Omega^{+}(10, q)$.*

(iii) *X is any M-orbit of nonsingular points and $M = SU(m, 2), \Omega^{\pm}(2m, 2), \Omega^{\pm}(2m, 3)$ or $\Omega(2m - 1, 3)$.*

(iv) *X is either orbit of nonsingular hyperplanes of $M = \Omega(2m - 1, 4)$ or $\Omega(2m - 1, 8)$ (where $G = \Omega(2m - 1, 8) \cdot 3$ in the latter case).*

(v) *$M = SU(3, 3), K \cap M = PSL(3, 2)$ (Mitchell [40], Suzuki [56]).*

(vi) *$M = SU(3, 5), K \cap M = 3 \cdot A_7$ (Mitchell [40], Higman [24]).*

(vii) *$M = SU(4, 3), K \cap M = 4 \cdot PSL(3, 4)$ (Hartley [21], McLaughlin [38]).*

(viii) *$M = Sp(6, 2), K = G_2(2)$ (Edge [15], Frame [19]).*

(ix) *$M = \Omega(7, 3), K \cap M = G_2(3)$.*

(x) *$M = SU(6, 2), K \cap M = 3 \cdot PSU(4, 3) \cdot 2$ (Fischer [17]).*

THEOREM 1.2. *Let $M = PSL(n, q) \leqslant G \leqslant$ Aut M. Assume that G acts as a primitive rank 3 permutation group on the set X of cosets of a subgroup K of G. Then (at least) one of the following occurs up to conjugacy under* Aut M.

    (i) *X is the set of lines for M, $n \geqslant 4$.*

    (ii) *$M = PSL(2, 4) \cong PSL(2, 5)$, $|X| = \binom{5}{2}$,*

        $M = PSL(2, 9) \cong A_6$, $|X| = \binom{6}{2}$,

        $M = PSL(4, 2) \cong A_8$, $|X| = \binom{8}{2}$, or

        $G = P\Gamma L(2, 8)$, $|X| = \binom{9}{2}$.

    (iii) *$M = PSL(3, 4)$, $M \cap K = A_6$.*

    (iv) *$M = PSL(4, 3)$, $M \cap K = PSp(4, 3)$.*

The reader is cautioned that isomorphisms between, and automorphisms of, the various groups allow many different ways of viewing some of these cases. Specifically, the isomorphisms $PSp(4, q) \cong \Omega(5, q)$, $PSU(4, q) \cong P\Omega^-(6, q)$, $PSL(4, q) \cong P\Omega^+(6, q)$ and $PSU(4, 2) \cong PSp(4, 3)$ lead to numerous representations under (1.1i). Similarly, a triality automorphism of $P\Omega^+(8, q)$ can be applied to (1.1iii) and a polarity automorphism can be applied to (1.2i).

The 2-transitive representations for G have been determined by Curtis, Kantor and Seitz [9]; from this result, the imprimitive rank 3 representations of G can be obtained (see §11). Our approach is similar to theirs, especially their note "Added in proof". Both arguments rely heavily on the degrees of the nonprincipal irreducible constituents of 1_B^G, where B is a Borel subgroup (and G is semilinear on the natural module V for M). All such degrees are divisible by p, with only 3 exceptions for $G = Sp(2n, 2)$. This fact, which follows from Hoefsmit [26], was not available in [9]; on the other hand, 2-transitivity permitted counting arguments in [9] which have no parallels here. (Only if $G = Sp(6, 2)$ or $Sp(8, 2)$ are we able to use the standard rationality conditions for the parameters of a rank 3 group.) When combined with a result of Seitz [50], this divisibility implies that 1_K^G and 1_B^G have exactly one nonprincipal constituent χ in common. In almost all cases, this fact by itself is strong enough to determine K. (A more general classification based upon this idea is found in (11.1).) Computations within the Weyl group yield strong transitivity properties for the action of K on V. Now a method of Perin [46] applies and K can be determined. The analogues of (1.1) and (1.2) for the remaining Chevalley groups seem blocked by the problem of finding all subgroups transitive (or almost transitive) on one or more classes of parabolic subgroups.

The only other results of this type are the beautiful theorem of Seitz [51] which proves the same result for q large relative to m, and the determination by Bannai [2] (resp., Cohen [5]) of all representations of rank at most 5 of A_n and S_n (resp., of rank 3 of all complex reflection groups). It seems quite difficult to extend our arguments to higher rank representations.

The paper is organized as follows. In §2 we present some notation and numerous preliminary results. Characters of the Weyl groups of type B_n and D_n are discussed in §3, and applied to G in §4; this is where some of Hoefsmit's results are described.

Highly transitive subgroups of G are crucial to our approach. These are studied for small dimensions in §5, and for large dimensions in §6 (using Perin's method).

We obtain a number of extensions of Seitz's flag-transitive theorem. For example, subgroups transitive on points, lines and planes (if planes exist) are all determined; mere transitivity on lines suffices for unitary geometries of dimension at least 8. On the other hand, §§5 and 6 contain many technical results with hypotheses tailored to the proof of (1.1). The proofs tend to be involved and unpleasant, due in part to several interesting (and annoying) exceptions. These exceptions, which arise in §5, are carried over to §6, where reductions are made to small dimensional situations.

The proof of (1.1) occupies §§7–9. The basic ideas in the proof of (1.1) can be clearly seen in §§7–8. In particular, the precision with which the Pigeon-Hole Principle applies is evident in (8.1). In §9 we deal with two exceptional and tedious cases: $Sp(2n, 2)$, where 1_B^G contains two nonprincipal characters of odd degree; and the possibility of graph automorphisms arising in the case of $P\Omega^+(8, q)$.

The proof of (1.2) appears in §10. This result is much simpler than (1.1), and its proof is relatively self-contained. The reader may wish to start the paper with that section.

One of the features of our proofs of (1.1) and (1.2) is that the rank 3 hypothesis is not fully used. Consequently, other subgroups of classical groups are able to occur which, from our point of view, behave very much like stabilizers in rank 3 representations. This is discussed in §11, where more general technical results are indicated (a generalization of much of (1.2) having already been proved in (10.1)). In §11 we have also listed the known examples of rank 3 representations of other Chevalley groups.

Finally, §12 represents a change of topic, though not of method. In [25], Higman and McLaughlin studied subgroups K of $\Gamma Sp(V)$ or $\Gamma U(V)$ which are rank 3 on the set of points of V. A complete determination was made by Perin [46], except in the case $Sp(2n, 2)$. (For a stronger unitary result, see our (6.1).) The orthogonal case was examined by Stark [54] for odd characteristic and small dimensions. By imitating Perin [46] and some portions of §6, we will prove the following result.

THEOREM 1.3. *Suppose that* $K \leqslant \Gamma O^\pm(d, q)$, $d \geqslant 5$, *and that* K *has rank* 3 *on points. If* d *is odd, assume that* q *is odd. Then* $K \geqslant \Omega^\pm(d, q)$.

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2. Preliminaries.

A. *Notation.* Let V be a finite vector space of characteristic p equipped with a symplectic, unitary, or orthogonal geometry. The corresponding Chevalley (or twisted) group Chev(V) is $Sp(V)$, $SU(V)$, or $\Omega^\pm(V)$ (or just $\Omega(V)$ if dim V is odd). Note that Chev(V) may have a nontrivial center.

The group of semisimilarities of V will be denoted by $\Gamma(V) = \Gamma Sp(V)$, $\Gamma U(V)$ or $\Gamma O^\pm(V)$. This is the group of semilinear transformations g of V such that $f(u^g, v^g) = cf(u, v)^\sigma$ (or $f(u^g) = cf(u)^\sigma$) for some scalar c, some field automorphism σ, and all $u, v \in V$; here, f is the sesquilinear (or quadratic) form on V. (See

Dieudonné [**13**, I.10, I.16].) Modulo scalars, $\Gamma(V)$ is the automorphism group of Chev(V) in almost all cases. The only exceptions occur for $Sp(4, 2^e)$ and $\Omega^+(8, q)$ (Carter [**3**, Chapter 12]).

Chev(V) is generated by "long root elements": nontrivial transvections in the symplectic and unitary cases; in the orthogonal cases (with dim $V > 4$), elements of order p having $L^\perp$ as an eigenspace, for some totally singular line L. For more information, see [**13**], [**3**] or [**33**].

There is a natural isomorphism $Sp(2n, q) \cong \Omega(2n + 1, q)$ for q even. We will therefore only have to consider whichever one of these groups is most convenient in a given situation, usually without even mentioning the other group. We will frequently make use of the isomorphisms $\Omega(5, q) \cong PSp(4, q)$, $P\Omega^+(6, q) \cong PSL(4, q)$ and $P\Omega^-(6, q) \cong PSU(4, q)$.

Totally singular (or isotropic) i-spaces will simply be called "i-spaces". All other subspaces will be modified by suitable adjectives. Points, lines and planes are, of course, just 1-spaces, 2-spaces and 3-spaces. The letters x, y, z will be reserved for points, L for lines, and E for planes. By abuse of language, we will write $x \in S$ in place of $x \subset S$.

The maximum dimension of an i-space will be denoted by n.

An (i, j)-*flag* consists of an i-space and a j-space containing it; also, in the $\Omega^+(2n, q)$ case, there are (n, n)-flags, consisting of two n-spaces meeting in an $n - 1$-space. A *flag* for $Sp(V)$, $SU(V)$ and $O^\pm(V)$ is a nested sequence of i-spaces, one for each $i = 1, \dots, n$; a flag for $\Omega^+(2n, q)$ is a sequence $(V_1, \dots, V_{n-2}, V_n, V_n')$ consisting of nested i-spaces for $i = 1, \dots, n - 2$, along with two n-spaces $V_n, V_n' \supset V_{n-2}$ meeting in an $n - 1$-space.

Throughout §§2–9, G will denote a group lying between Chev(V) *and* $\Gamma(V)$, *while K will be a subgroup of G.*

If $W \subseteq V$, then K_W and $C_K(W)$ are the set-wise and vector-wise stabilizers of W in K; the *semilinear* group induced by K_W on W is denoted $K_W^W \cong K_W/C_K(W)$. Also, W^K is the orbit of W under K; $[V, K] = [K, V] = \{v^k - v \mid k \in K\}$; K' is the commutator subgroup of K, $K^{(\infty)}$ is the last term of the commutator series for K, $Z(K)$ is the center of K, and $O_p(K)$ is the largest normal p-subgroup of K. If H is a group, $n \cdot H$ is an extension of H by a group of order n, while $H \cdot n$ is H with an automorphism of order n adjoined.

B. *Some subgroups of classical groups.* We will require a number of properties of classical groups related to generation and transitivity. We will also make very frequent use of the structure of the most obvious parabolic subgroups [**9**, §3]:

LEMMA 2.1. *Let x be a point, and set $Q = O_p(G_x)$, so that Q centralizes $x^\perp/x$. Let Q_1 be the group of transvections in Q.*

(i) *The representations of $C_G(x)$ (resp., G_x) on Q/Q_1 and $x^\perp/x$ are isomorphic (resp., projectively isomorphic), via $gQ_1 \to [V, gQ_1]/x$.*

(ii) *In the orthogonal case, singular vectors in $x^\perp/x$ correspond to long root elements in Q; nonsingular vectors correspond to nontrivial elements $g \in Q$ having* rad$[V, g] \neq [V, g]$.

(iii) *If V is unitary, or symplectic of odd characteristic, then $Q' = Q_1 = \Phi(Q)$, and the commutator function on $Q \times Q$ induces a nondegenerate alternating bilinear form on Q/Q_1.*

LEMMA 2.2. *Suppose that K is a subgroup of $\Gamma(V)$ containing a Sylow p-subgroup of $\mathrm{Chev}(V)$. Then either $K \geqslant \mathrm{Chev}(V)$, or K stabilizes some totally singular subspace.*

PROOF. This is due to Tits (see Seitz [**51**, (4.6)]).

The preceding lemma is, in effect, concerned with subgroups of Chevalley groups generated by root groups, even in the infinite case. We will require much more detailed information concerning groups generated by long root groups.

LEMMA 2.3. *Let K be a subgroup of $\Gamma(V)$ which is primitive (as a linear group) on $V/\mathrm{rad}\,V$, where V does not have type $\Omega^+(4, q)$. Assume that the subgroup K^* generated by all long root elements of K is nontrivial. Then either* (i) $K^* = \langle t^{K^*} \rangle$ *for some long root element t, or* (ii) *V has type $\Omega(5, q)$ with q even, and there is a nonsingular K-invariant hyperplane on which K^* induces $\Omega^+(4, q)$.*

PROOF. Clearly $O_p(K^*) = 1$. By Cooperstein [**7**, (3.8)], there are subgroups $K_1, \ldots, K_r$ of K^* such that $K^* = K_1 \cdots K_r$, $[K_i, K_j] = 1$ for $i \neq j$, $K_i = \langle t_i^{K_i} \rangle$ for some long root element t_i, and $\cup K_i$ contains all long root elements of K. We must show that $r = 1$ unless (ii) holds.

Suppose that $r > 1$, and consider $A_i = [V, t_i]$.

If V is symplectic or unitary, then A_1 and A_2 are points which are perpendicular since $[t_1, t_2] = 1$. Thus, the subspaces $[V, K_i]$ are pairwise orthogonal. Since V is spanned by these subspaces, while K permutes these subspaces, K cannot be primitive.

If V is orthogonal, then A_1 and A_2 are lines. This time, $[t_1, t_2] = 1$ implies that either $A_2 \subseteq A_1^\perp$ or $A_1 \cap A_2 \neq 0$. Consider the latter possibility. Since $O_p(K_i) = 1$, $W = \langle A_1, A_1^g \rangle$ is a nonsingular 4-space for some $g \in K_1$. Since $A_1^\perp \cap A_1^g = 0$, it follows that, if $h \in G$, then $W^h = W$ or $W^h \subset W^\perp$. Then $V = \langle W, \mathrm{rad}\,V \rangle$ by the primitivity of K.

Consequently, if we exclude case (ii), then the subspaces $[K_i, V]$ are again pairwise orthogonal and we obtain the same contradiction as before.

THEOREM 2.4. *Suppose that $n \geqslant 2$, V is not of type $\Omega^+(4, q)$, and K is a subgroup of $\Gamma(V)$ which is point-transitive and contains a long root element. Then one of the following holds*:

(i) $K \geqslant \mathrm{Chev}(V)$;

(ii) $K \trianglerighteq G_2(q)$ *and* $\Gamma(V) = \Gamma O(7, q)$;

(iii) $K \trianglerighteq N$ *with* $-1 \in Z(N)$ *and* $N/\langle -1 \rangle$ *conjugate in* $\mathrm{Aut}\, P\Omega^+(8, q)$ *to one of the usual subgroups* $P\Omega(7, q)$;

(iv) $K \trianglerighteq SU(m, q)$ *and* $\Gamma(V) = \Gamma O^\varepsilon(2m, q)$ *(with $\varepsilon = +$ iff m is even); or*

(v) $O_3(K) \nleqslant Z(K)$ *and K has an extraspecial subgroup of order 3^3, where* $K < \Gamma O^-(6, 2)$.

PROOF. This follows from (2.3) and a check of the lists in [33].

REMARKS. That the groups in (iii) are point-transitive follows from the fact that the usual subgroup $\Omega(7, q)$ is transitive on each class of 4-spaces of V. Aut $P\Omega^+(8, q)$ will enter into our arguments in a number of other situations.

There will also be other occasions when [33] will be quoted. Unfortunately, the lists in that paper are too long to reproduce here. Another use of [33] is the following lemma (compare Stark [53]).

LEMMA 2.5. *Let K be a primitive subgroup of $\Gamma O^+(m, q)$, where q is odd and $m \geqslant 5$. Let x be a point, and $b_1, b_2 \in x^\perp$ with $\dim\langle x, b_1, b_2 \rangle = 3$. Assume that $|C_K(\langle x, b_i \rangle^\perp)| = q$ for $i = 1, 2$. Then either $K \geqslant \Omega^+(m, q)$ or $K \trianglerighteq 2 \cdot \Omega(7, q)$, inside $\Gamma O^+(8, q)$.*

PROOF. Write $A(x) = C_K(x) \cap C_K(x^\perp/x)$, $V(x) = [V, A(x)]$ and $\Omega = x^K$. Note that $V = \langle \Omega \rangle$.

If K contains long root elements, then (2.3) and [33] provide a short list of possibilities. Since $|C_K(\langle x, b_1 \rangle^\perp)| = q$, the only ones that occur are those stated in the lemma.

From now on, we will assume that K contains no long root elements. By (2.1), $V(x)/x$ is an anisotropic 2-space and $|A(x)| = q^2$.

Let $y \in \Omega - x^\perp$. If $0 \neq b \in V(x) \cap y^\perp$, there is a nonzero vector $d \in V(y) \cap x^\perp$ such that exactly one of (b, b), (d, d) is a square in $GF(q)$. Then $\langle b, d \rangle$ is nonsingular, and hence so is $W = \langle x, y \rangle \oplus \langle b, d \rangle$. Write

$$K_1 = \left\langle C_K(\langle x, b \rangle^\perp), C_K(\langle y, d \rangle^\perp) \right\rangle.$$

Then K_1^W is generated by nonconjugate groups of order q, and hence is $\Omega(W)$. Also, K_1 centralizes $W^\perp$. Since K has no long root elements, W must have type $\Omega^-(4, q)$. Moreover, $V(x') \subset W$ and $\dim V(x) \cap V(x') \geqslant 2$ for each point x' of W (since a Sylow p-subgroup of K_1 has order q^2).

By primitivity, there is a point $z \in \Omega - (W \cup W^\perp)$. We may assume that $z \notin x^\perp$. As above, $\dim V(x) \cap V(z) = 2$. Thus, $Z = \langle W, V(z) \rangle$ has dimension 5. Clearly, $H = \langle A(x), A(y), A(z) \rangle$ acts on Z, while centralizing $Z^\perp$.

By Mitchell [41], Z cannot be nonsingular. (This also follows readily from the fact that K_1 would be transitive on the lines of Z.) Set $r = \text{rad } Z$ and $T = O_p(C_H(Z/r))$. Both K_1 and $\langle A(x), A(z) \rangle$ induce $\Omega^-(4, q)$ on Z/r, and they fix different hyperplanes of Z. Thus, $T \neq 1$. Here, T consists of transvections of Z with direction r. Since K_1 acts on T, we must have $|T| = q^4$. But now some element of T fixes x while moving $V(x)$. This contradiction proves the lemma.

THEOREM 2.6. *Suppose that $n \geqslant 2$ and that V is not of type $\Omega^+(4, q)$. Let K be a flag-transitive subgroup of $\Gamma(V)$ which does not contain $\text{Chev}(V)$. Then one of the following holds:*

(i) *K is A_7 or S_7, inside $O^+(6, 2)$;*

(ii) *K is A_6, inside $Sp(4, 2)$;*

(iii) *K is a semidirect product of an extraspecial group of order 32 with A_5, S_5 or a Frobenius group of order 20, inside $\Gamma Sp(4, 3)$;*

(iv) *K has a normal subgroup which is a semidirect product of an elementary abelian group of order 16 with A_5, S_5 or a Frobenius group of order 20, inside $\Gamma O(5, 3)$;*

(v) $K \trianglerighteq 2 \cdot PSL(3, 4) \cdot 2$, *inside* $\Gamma O^-(6, 3)$; *or*

(vi) $K \trianglerighteq 4 \cdot PSL(3, 4) \cdot 2$, *inside* $\Gamma U(4, 3)$.

PROOF. Seitz [50]. (Note that (iii) and (iv) are related via the isomorphism $PSp(4, 3) \cong \Omega(5, 3)$, while (v) and (vi) are related via the isomorphism $P\Omega^-(6, 3) \cong PSU(4, 3)$.)

LEMMA 2.7. *Let* $K \leqslant P\Gamma L(3, q)$.

(i) *If K is point-transitive on $PG(2, q)$, then $K \geqslant PSL(3, q)$ or K has a normal irreducible cyclic subgroup of order dividing $q^2 + q + 1$.*

(ii) *If K has 2 point-orbits and 3 flag-orbits, then K fixes a point or line, or else K is A_6 or S_6 stabilizing a hyperoval of $PG(2, 4)$.*

(iii) *If K has 3 point-orbits and at most 6 flag orbits, then K has 6 flag-orbits and fixes a point or a line, or an oval or its dual.*

(iv) *If L, L' are lines and both K_L^L and $K_{L'}^{L'}$ are transitive, then $L' \in L^K$.*

PROOF. Mitchell [40] and Hartley [22].

LEMMA 2.8. *Let* $K \leqslant P\Gamma U(3, q)$.

(a) *If K is transitive on isotropic points, then one of the following holds*:

(i) $K \geqslant PSU(3, q)$;

(ii) $K \leqslant P\Gamma U(3, 2)$;

(iii) K' *is $PSL(3, 2)$ inside $P\Gamma U(3, 3)$; or*

(iv) K' *is A_7 inside $P\Gamma U(3, 5)$.*

(b) *If K has 2 orbits on isotropic points, then one of the following holds*:

(i) *K fixes a line*;

(ii) $K \leqslant P\Gamma O(3, q)$;

(iii) *$K \cap PSU(3, q)$ is inside one of the groups appearing in (a iii) or (a iv); or*

(iv) *K lies in the normalizer of a Sylow 7-subgroup of $P\Gamma U(3, 5)$.*

PROOF. Mitchell [40] and Hartley [22].

C. *Some elementary geometric lemmas.*

LEMMA 2.9. *Let Ω be a nonempty set of points of $PG(d, q)$, $d \geqslant 2$, such that each line meets Ω in 0 or k points, where $1 < k < q + 1$. Then*

(i) $k \mid q$; *and*

(ii) *If $d > 2$, then Ω is the complement of a hyperplane.*

PROOF. If W is an i-space, then $|W \cap \Omega|$ is 0 or

$$n_i = 1 + (k - 1)(q^i - 1)/(q - 1).$$

In order to prove (i), assume that $d = 2$, pick a point $x \notin \Omega$, and note that there are $|\Omega|/k$ lines on x meeting Ω.

Now suppose that $d > 2$. If every hyperplane met Ω, then Ω and the hyperplanes would form a symmetric design with more blocks than points. Thus, some $d - 2$-space W misses Ω. There are then n_d/n_{d-1} hyperplanes containing W and meeting Ω. But

n_d/n_{d-1} is an integer only if $k = q$, in which case n_d is the number of points outside of a hyperplane missing Ω.

LEMMA 2.10. *Let V be a symplectic, unitary or orthogonal space with $n \geqslant 3$. Let Ω be a nonempty set of points such that every line meets Ω in 0 or q points. Then Ω is the set of all points not in some hyperplane.*

PROOF. If Ω' denotes the set of points not in Ω, then every line meeting Ω' twice must have all its points in Ω'. Such a line exists: a plane E meeting Ω has $|E \cap \Omega| = 1 + (q + 1)(q - 1) = q^2$, so that $E \cap \Omega$ is the complement of a line.

If $\langle \Omega' \rangle \neq V$, then Ω contains all points not in $\langle \Omega' \rangle$, and our assertion follows readily. If $\langle \Omega' \rangle = V$, then $\Omega = \varnothing$ by Tits [58, (8.10)], which is absurd. (N.B.–Hypotheses (A) and (B) of [58, (8.10)] assert that the sesquilinear form on V is trace-valued, which is always the case for finite fields.)

Combining (2.9) and (2.10) produces the following curious consequence.

COROLLARY 2.11. *Let Ω be a nonempty set of points of a finite symplectic, unitary or orthogonal space V. Assume that V contains totally isotropic (or totally singular) 4-spaces. If every line meets Ω in 0 or k points, for some constant $k > 1$, then Ω consists either of all points, or of all points off of some hyperplane.*

A group $K \leqslant G$ is called (i, j)-*transitive* if it is transitive on the set of (i, j)-flags.

LEMMA 2.12. *Suppose that $K \leqslant \Gamma(V)$ is transitive on the sets $\mathbb{S}_i$ and $\mathbb{S}_j$ of i-spaces and j-spaces. If $W \in \mathbb{S}_j$, then all orbits of K_W on $\mathbb{S}_i$ have length divisible by $|\mathbb{S}_i|/(|\mathbb{S}_i|, |\mathbb{S}_j|)$. (In particular, if $i < j$ and $|\mathbb{S}_i|/(|\mathbb{S}_i|, |\mathbb{S}_j|)$ is the number of i-spaces in a j-space, then K is (i, j)-transitive.)*

PROOF. Let U be any i-space. Then $\{(U^k, W^k) \mid k \in K\}$ has size $|\mathbb{S}_i||W^{K_U}| = |\mathbb{S}_j||U^{K_W}|$.

LEMMA 2.13. *Let Ω be a set of points, and $\mathbb{S}_k$, $\mathbb{S}_{k-1}$ collections of k-spaces (resp., $k - 1$-spaces) contained in Ω. Assume that every k-space on a member of $\mathbb{S}_{k-1}$ is in $\mathbb{S}_k$. Then Ω consists of all points. (If V has type $\Omega^+(2k, q)$, modify the hypothesis so that all k-spaces considered have a fixed type.)*

PROOF. The elementary connectedness argument is omitted.

D. *Cohomology.* We will also require some results concerning the first cohomology groups of classical groups. These are collected in the next theorem.

THEOREM 2.14. (i) *For a classical group $K = \mathrm{Chev}(V)$, set $d(K) = \dim H^1(K, V)$. Then $d(K) = 0$, except in the following cases: $d(K) = 1$ for $K = \Omega(3, 5)$, $\Omega(5, 3)$, $SU(4, 2)$, $\Omega^+(6, 2)$, $SU(2, 2^i)$, $Sp(2n, 2^i)$, and $\Omega(2n + 1, 2^i)$, except that $d(SU(2, 2)) = d(Sp(2, 2)) = 0$ and $d(\Omega(5, 2)) = 2$; and $d(\Omega^-(4, 3)) = 2$.*

(ii) *$d(SL(V)) = \dim H^1(SL(V), V)$ is 0 for $\dim V > 2$, except that $d(SL(3, 2)) = 1$.*

(iii) *$d(K) = 0$ for each of (2.6 i, iii–vi) and (2.8 ii–iv), while $d(K) = 1$ for (2.6 ii).*

PROOF. Parts (i) and (ii) are a summary of results of Higman [23], Pollatsek [47]–[49], Cline, Parshall and Scott [4], Jones [29], McLaughlin [39], Fischer [17,

(16.1.10)], and Finkelstein [**16**, p. 82] (cf. Jones and Parshall [**30**]). The cases $SU(m, q)$ and $\Omega^-(2m, q)$ with $q \leqslant 3$ (due to McLaughlin and to Pollatsek) are not in print, and will be dealt with here.

Our approach is very elementary, cocycles remaining invisible throughout. It is based primarily upon the following two elementary observations, which are simple consequences of the interpretation of first cohomology in terms of conjugacy classes of complements.

(α) *Let $D \leqslant GL(V)$, and assume that D has a normal r-subgroup $R \neq 1$ (where $r \neq p$). If $W = C_V(R)$ is zero, or if $C_D(W) = O^p(C_D(W))$ and $p \nmid |D^W|$, then $H^1(D, V) = 0$.*

(β) *If $K = \langle B, C \rangle$ with $H^1(B \cap C, V) = H^1(B, V) = H^1(C, V) = 0$ and $C_V(B \cap C) = 0$, then $H^1(K, V) = 0$.*

PROOF OF (α). If $DV = D_1V \geqslant RV$ and $D_1 \cap V = 1$, we may assume that $D \cap D_1 \geqslant R$. Then $DW = N_{DV}(R) = D_1W$, and $D = D_1$ if $|W| = 1$. If $|W| > 1$ then $C_D(W) = O^p(C_{DW}(W)) = C_{D_1}(W)$, so that $(D/C_D(W))W = (D_1/C_D(W))W$. Thus, $D^W = D/C_D(W)$ and $D_1/C_D(W)$ are conjugate under W.

PROOF OF (β). Let V be a hyperplane of a space U, and embed K into $GL(U)$ so as to fix a vector $u \notin V$. Set $T = O_p(GL(U)_V)$. Then D acts on T as it does on V. Let $KT = K_1T$ with $K_1 \cap T = 1$. Let $B_1, C_1 \leqslant K_1$ correspond to B, C. By hypothesis, there is a unique 1-space $\langle u_1 \rangle$ of U centralized by $B_1 \cap C_1$. Using T, we may assume that $u_1 = u$. There is also a unique 1-space centralized by B_1 (resp., C_1). Thus, $\langle B_1, C_1 \rangle = K_1$ centralizes u, so $K_1 = C_{KT}(u) = K$.

REMARKS. Assertion (β) is due to Alperin and Gorenstein [**1**], where an entirely different proof is given. (Their more general result is also immediate, using the above argument.) Yet another elementary proof can be obtained by translating the above one into statements concerning cocycles. Note that U and T were introduced in order to avoid using the elementary fact that $H^1(K, V) \cong \text{Ext}_{GF(q)K}(GF(q), V)$.

We will use (α) and (β) in several examples, primarily those not already in print. We may assume that (α) does not apply with $D = K$ (and in particular that $Z(K) = 1$). This handles (2.6 iii–vi) and (2.8 ii, iv). Of course, in situations in which $d(K) > 0$, ad hoc methods will be needed. Note that $d(K)$ can be proved to be nonzero by exhibiting an indecomposable K-module containing the K-submodule V as a hyperplane.

EXAMPLE 1. $PSL(2, 7) < SU(3, 3)$. Use $B \cong C \cong S_4$ with $B \cap C = D_8$ in (β).

EXAMPLE 2. $K = SU(m, q)$, $m \geqslant 3$, excluding $SU(4, 2)$ (McLaughlin [**39**]). Let b and c be perpendicular 1-spaces, and set $B = K_b$ and $C = K_c$. Note that $1 \neq Z(B) < C$ and $C_V(Z(B)Z(C)) = 0$, so $H^1(B \cap C, V) = 0$ by (α). Since $B \neq SU(3, 2)$, either $C_V(Z(B)) = 0$ or $C_B(b) = O^p(C_B(b))$. Thus, $H^1(B, V) = 0$ by (α), and (β) applies.

Now consider the case $K = SU(4, 2) \cong \Omega(5, 3)$. Let B and C be maximal parabolic subgroups of $\Omega(5, 3)$ such that $K = \langle B, C \rangle$, $B \cap C$ contains a Sylow 3-subgroup of K, and $B/O_3(B) \cong SL(2, 3)$. Then C acts monomially on V, so that $H^1(C, V) = 0$ by (α). Suppose that $KV = K_1V$ with $K_1 \cap V = 1$. We may assume that $K \cap K_1 \geqslant C$. If $R = O_3(B)$, then $K_1 = \langle C, N_{K_1}(R) \rangle$. But $N_{KV}(R)/R$ has at most 4 subgroups

$\cong SL(2, 3)$, since $B = N_K(R)$ is the stabilizer of a nonsingular 1-space of V. Thus, $d(K) \leqslant 1$. That $d(K) = 1$ is due to Fischer [17, (16.1.10)], and is discussed in [33, §2] and Case 4 of the proof of (2.16).

EXAMPLE 3. $K = \Omega(3, q)$. If $q > 5$ and q is odd, use $B \cong A_4$ and C dihedral of order $q \pm 1$, with $|B \cap C| = 4$. If $q = 3$, (α) applies. Higman [23] and Pollatsek [47] proved that $d(\Omega(3, q)) = 1$ for q even.

Consider $K = \Omega(3, 5)$. First use the standard permutation module for A_5 over $GF(5)$ in order to find that $d(K) \neq 0$. Then suppose that $KV = K_1V$ with $K_1 \cap V = 1$. By (α), we may assume that $K \cap K_1 \geqslant B \cong A_4$. Let $T \in Syl_3(B)$, and note that $K_1 = \langle B, N_{K_1}(T) \rangle$. Since $|C_V(T)| = 5$, $N_{KV}(T)$ has exactly 5 subgroups S_3, so there are at most 5 choices for $N_{K_1}(T)$. Thus, $d(K) \leqslant 1$.

EXAMPLE 4. $K = \Omega^-(4, q)$. Set $B = K_b$ for a nonsingular 1-space b. If q is odd and $q \geqslant 5$, let C be a dihedral group of order $q^2 - 1$, chosen so that $|B \cap C| = 2(q \pm 1) \equiv 0 \pmod 8$. Then $K = \langle B, C \rangle$ (since $q > 5$), and $H^1(B, V) = 0$ by Example 3, while $H^1(B \cap C, V) = 0$ by (α). Also, $B \cap C$ is not inside $C_K(b) = \Omega(3, q)$, and hence $C_V(B \cap C) = 0$. Hence, (β) applies.

If q is even and $q > 2$, then $\dim H^1(B, V/b) = 1$ (Example 3), while B acts indecomposably on V. Thus, $H^1(B, V) = 0$. Let $KV = K_1V$ with $K_1 \cap V = 1$. We may assume that $K \cap K_1 \geqslant B$. Let A be a dihedral subgroup of B of order $2(q - 1)$. Then $K_1 = \langle B, A, C_{K_1}(A') \rangle$, where $\langle A, C_K(A') \rangle$ is dihedral of order $2(q^2 - 1)$. Here, $\langle A, C_{KV}(A') \rangle = b \times \langle A, C_K(A') \rangle$ has exactly q such dihedral subgroups, and these are conjugate under b. Since b centralizes B, it follows that K and K_1 are also conjugate under b. Thus, $d(K) = 0$ (Jones [29]).

Next, consider the case $q = 3$. If $KV = K_1V$ with $K_1 \cap V = 1$, we may assume that $K \cap K_1 \geqslant D$ with $|D| = 10$. Let t be an involution of D. Then $K_1 = \langle D, g \rangle$ with $g^2 = t$. But $|t^V| = 9$, so that $C_{KV}(t)$ has only 9 subgroups of order 4. Thus, at most 9 classes of complements exist. Using generators and relations, Finkelstein [16, p. 82] showed that 9 classes actually occur. An alternative procedure is as follows. Let W be the 6-dimensional permutation module for S_6 over $GF(3)$, equipped with the usual inner product. If w is its S_6-invariant 1-space, then S_6 has a unique orbit of length 6 on W/w. Thus, Aut S_6 cannot act on W/w, but it certainly acts on $H^1(A_6, w^\perp/w)$. Consequently, $\dim H^1(\Omega^-(4, 3), w^\perp/w) > 1$.

Finally, consider $K = \Omega^-(4, 2)$. Let $KV = K_1V$ with $K_1 \cap V = 1$. We may assume that $K \cap K_1$ contains $\langle s, t \rangle \cong D_{10}$ with $s^2 = t^2 = 1$. Then $K = \langle r, s, t \rangle$ for a uniquely determined involution $r \in C_K(t)$ such that $|rs| = 3$. Thus, $K_1 = \langle r_1, s, t \rangle$ with $r_1^2 = 1$, $r_1 \in C_K(t)$, $|r_1 s| = 3$, and $rr_1 \in V$. Since $|rr_1| = 2$, $rr_1 \in C_V(\langle r, t \rangle)$. Suppose that $r \neq r_1$. Then $v = rr_1$ is uniquely determined (since $\dim C_V(\langle r, t \rangle) = 1$), and is singular. But $(sr)^3 = (srv)^3 = v + v^{sr} + v^{(sr)^2}$. Since $|sr| = 3$ and sr moves v, the singular vectors v, v^{sr} and $v^{(sr)^2}$ are linearly independent. Thus, $|sr_1| \neq 3$. This contradiction shows that $K_1 = K$, as desired.

EXAMPLE 5. $K = \Omega^+(4, q)$. If q is odd, then $-1 \in Z(K)$ and (α) applies. If $q = 2$, (α) also applies. If q is even and $q > 2$, the corresponding argument in Example 4 applies.

EXAMPLE 6. $K = \Omega^-(2m, q)$, $m \geqslant 3$. Let B and C be the stabilizers of two perpendicular anisotropic 2-spaces. By (α), Example 4, or induction, (β) applies to $\langle B, C \rangle$. Here, $\langle B, C \rangle = K$ except when $K = \Omega^-(6, 3)$ or $\Omega^-(6, 2)$. Note that $-1 \in \Omega^-(6, 3)$ (since $-1 \in \Omega^-(2, 3)$), so that (α) applies here.

Finally, if $K = \Omega^-(6, 2) \cong \Omega(5, 3)$, let B_0 and C_0 be maximal parabolic subgroups of $\Omega(5, 3)$ containing a Sylow 3-subgroup of $\Omega(5, 3)$. Then (β) applies to $\langle B_0, C_0 \rangle = K$, by (α).

EXAMPLE 7. K is A_6, inside $Sp(4, 2)$ or $\Omega(5, 2)$. Using the standard 6-dimensional permutation module for K over $GF(2)$, we find that $d(K) \neq 0$.

Let $KV = K_1 V$ with $K_1 \cap V = 1$. By Example 4, we may assume that $K \cap K_1$ contains $A_5 \cong \Omega^-(4, 2)$. Let $T \in Syl_3(B)$. Then $K_1 = \langle B, N_{K_1}(T) \rangle$. However, $N_{KV}(T)$ has exactly two Sylow 3-subgroups containing $N_B(T)$. Thus, $d(K) \leqslant 1$, as required.

EXAMPLE 8. K is A_7 or S_7, inside $O^+(V) = O^+(6, 2)$. Here, $d(K) = 0$ (proved for S_7 in Pollatsek [47]). For, let $KV = K_1 V$ with $K_1 \cap V = 1$. We may assume that $K \cap K_1$ contains a Frobenius group B of order 21. Let $T \in Syl_3 B$. Then $K_1 = \langle B, C_{K_1}(T) \rangle$. But $C_{KV}(T) = C_K(T) \times C_V(T)$. Thus, $C_{K_1}(T) = C_K(T)$ and $K_1 = K$.

REMARK. We have concentrated on low dimensional examples. The higher dimensional ones can be readily handled inductively, as in Examples 2 and 6.

PROPOSITION 2.15. *Let $H \leqslant \Gamma L(W) = \Gamma L(n, q)$, $n \geqslant 3$. Suppose that H is not flag-transitive, but that H_x is point-transitive on W/x for some point x. Then one of the following holds*:

(i) *H fixes a point or hyperplane; or*

(ii) *H is $3 \cdot A_6$ or $3 \cdot S_6$ inside $\Gamma L(3, 4)$, fixing a hyperoval.*

PROOF. Clearly, $\Omega = x^H$ does not consist of all points, and we may assume that $|\Omega| > 1$. If $n = 3$, use Mitchell [40] and Hartley [22]. If $n \geqslant 4$, then (2.9) yields (i).

The next result is similar, but more interesting.

THEOREM 2.16. *Let $K \leqslant \Gamma(V)$, where V has rank $n \geqslant 3$ but does not have type $\Omega^+(6, q)$. Let x be a point moved by K, and assume that K_x is flag-transitive on $x^\perp/x$. Then one of the following holds.*

(i) *K fixes a nondegenerate hyperplane or a nonsingular 2-space.*

(ii) *$K \trianglerighteq 3 \cdot PSU(4, 3)$, inside $\Gamma U(6, 2)$.*

(iii) *K is A_8 or S_8 inside $O^+(8, 2)$, and fixes a point other than x.*

(iv) *K is A_9 or S_9 inside $O^+(8, 2)$.*

(v) *$K \trianglerighteq \Omega^-(6, 2)$, inside $\Gamma O(7, 3)$, where K fixes a point other than x.*

(vi) *$K \trianglerighteq Sp(6, 2)$, inside $\Gamma O(7, 3)$.*

REMARK. Some interpretation is required for (i) when V is of type $Sp(2n, 2^i)$ or $\Omega(2n + 1, 2^i)$. Namely, fixing a nonsingular 2-space of the former must be regarded as essentially the same as fixing a nondegenerate 3-space of the latter, while fixing an

orthogonal subgeometry $O^+(2n, 2^i)$ of the former must be identified with fixing a nonsingular hyperplane of the latter.

PROOF. We may assume that (i) does not hold and that K contains no long root elements ((2.3) and [33]). Let $Q = O_p(\Gamma(V)_x)$ and let Q_1 be the group of transvections in Q (as in (2.1)). Then $K \cap Q_1 = 1$.

In fact, $K \cap Q = 1$. For otherwise, $Q = (K \cap Q)Q_1$ by (2.1) and the irreducibility of K_x on $x^\perp/x$. If $Q' \neq 1$, then $Q_1 = Q' = \Phi(Q)$ implies that $K_x \cap Q = Q > Q_1$. Thus, $Q' = 1$, and similarly, $Q_1 \neq 1$, so we are in the case $Sp(2m, q)$ with q even. But here, K_x acts indecomposably on Q (by (2.6)), so that again $K_x \cap Q = Q$.

Clearly, $K_x^{x^\perp/x}$ is as in (2.6) or contains $\mathrm{Chev}(x^\perp/x)$. In the latter case, K_x has a normal subgroup $D \cong \mathrm{Chev}(x^\perp/x)$. If $H^1(D, x^\perp/x) = 0$, then $H^1(D, Q/Q_1) = 0$ by (2.1), so DQ_1 is unique up to conjugacy in $\mathrm{Chev}(V)$, and hence so is $(DQ_1)' = D$; in this case, K_x has a subgroup which centralizes a nonsingular 2-space containing x and induces $\mathrm{Chev}(x^\perp/x)$ on $x^\perp/x$, so that K_x contains long root groups. Consequently, we are only left with the possibilities that either $H^1(D, x^\perp/x) \neq 0$ or that $K_x^{x^\perp/x}$ is one of the exceptional cases occurring in (2.6). By (2.14), V must have type $Sp(2m, q)$ with q even, $Sp(6, 3)$, $\Omega(7, 3)$, $\Omega^+(8, 2)$, $SU(6, 2)$, $\Omega^-(8, 3)$ or $SU(6, 3)$.

We will consider these possibilities separately. Set $\Omega = x^K$.

Each line meets Ω in 0 or k points, for some constant k. If E is a plane meeting Ω more than once, then (2.9) applies to K_E^E. Thus, by (2.10), if $k > 1$ then V has type $SU(6, 2)$ and $E \cap \Omega$ is a hyperoval of E (which is precisely what happens in (ii)).

Case 1. $K \leqslant Sp(2n, 2)$, $n \geqslant 3$, q even, excluding $Sp(6, 2)$. By (2.14), $\dim H^1(D, Q) = 1$. By [33, §2, Example RL1], each complement to Q in DQ contains long root elements.

Case 2. K_x is A_7, S_7, A_8 or S_8, inside $O^+(8, 2)$. Here, $k = 1$, so that $x^\perp \cap \Omega = \{x\}$. Let $K_x \geqslant A = A_7$. According to (2.14 iii), there is essentially just one possibility for A. Thus, there is a basis $\{e_i\}$ of pairwise nonperpendicular nonsingular vectors of V such that (identifying each point with the corresponding vector) $x = \Sigma e_i$ and A permutes the basis, fixing e_8. The point-orbits of A have lengths 1, 1, 7, 21, 35, 35, 35. Since $k = 1$ it follows easily that $|\Omega| \leqslant 9$. Then Ω is preserved by the transvection t with direction $e_1 + e_2$, and $\langle K, t \rangle$ must be S_8 or S_9. Thus, (iii) or (iv) holds.

Case 3. K_x is A_6 or S_6, inside $O(7, 2)$. Let $K_x \geqslant A = A_6$. Once again, $x^\perp \cap \Omega = \{x\}$. By (2.14 iii), there are two possibilities for the action of A on V.

Suppose first that $C_V(A)$ is a nondegenerate 3-space. Then the point-orbits of A have lengths 1, 1, 1, 15, 15, 15, 15 (as is most easily seen by regarding A as being inside $Sp(6, 2)$). Since each nontrivial orbit contains a pair of perpendicular points, $|\Omega| \leqslant 3$. Thus, (i) holds when V is replaced by $V/\mathrm{rad}\, V$.

Thus, $\dim C_V(A) = 2$ and V has a basis $\{e_i\}$ of pairwise nonperpendicular singular vectors which A permutes while fixing $x = e_7$. This time, $x^\perp \cap \Omega = \{x\}$ implies that $|\Omega| = 7$ and that K is A_7 or S_7. But K then fixes the nonsingular hyperplane $\{\Sigma \alpha_i e_i \mid \Sigma \alpha_i = 0\}$.

Case 4. $K < \Gamma U(6, 2)$. Here, $K_x \trianglerighteq H = SU(4, 2)$, where H does not fix a nonsingular 2-space. By Fischer [17, (16.1.10)], H is uniquely determined up to

conjugacy in $\Gamma U(6, 2)$ (compare (2.14 i)), and there is a transvection $t \in SU(6, 2)_x$ normalizing but not centralizing H. Set $z = [V, t]$ and $K^+ = \langle H, K_z, t \rangle \trianglerighteq \langle H, K_z \rangle$.

If K^+ is irreducible on V, then [33] applies. In this case, $K^+ \cap SU(6, 2) = 3 \cdot PSU(4, 3) \cdot 2$, so that $K \geqslant 3 \cdot PSU(4, 3)$. The latter group has only two orbits on points: one of length 126 containing z, and one of length 567 (Fischer [17, (16.1.16)]). Consequently $(K\langle t \rangle) \cap SU(V) = 3 \cdot PSU(4, 3) \cdot 2$, and (2.16 ii) holds.

If K^+ is reducible on V, then K^+ must fix x (since H fixes only two proper subspaces of V, namely, x and $x^\perp$). Thus, $K_z \leqslant K_x$. Let E be any plane containing $\langle x, z \rangle$. Then K_E^E contains A_6, so that K_z cannot fix x, and this situation cannot occur.

Case 5. $K < \Gamma O^-(8, 3)$. Here $K_x \trianglerighteq H = 2 \cdot PSL(3, 4)$, so that $|Z(K_x)| = 2$. Then K_x fixes a hyperbolic line containing x, and hence also a second point y. Note that H is transitive on each class of nonsingular points in $\langle x, y \rangle^\perp$. (For, $\Omega^-(6, 3)$ has rank 3 on each class of nonsingular points, as well as on the conjugates of H, and it is easy to check the degrees of the irreducible constituents of the permutation characters.) Thus, each H-orbit of points not in $\langle x, y \rangle \cup \langle x, y \rangle^\perp$ has length 56 or 126. It is now easy to check that $x^\perp \cap \Omega = \{x\}$ forces Ω to be $\{x, y\}$, so that (i) holds.

Case 6. $K < \Gamma U(6, 3)$. Here, $K_x \trianglerighteq H = 4 \cdot PSL(3, 4)$, and K_x fixes a hyperbolic line T containing x. Moreover, H centralizes T and is point-transitive on $T^\perp$. Since K moves T, Ω must contain a point of the form $\langle u + v \rangle$ with $u \in T$, $v \in T^\perp$ and $(u, u) + (v, v) = 0$. Let $\langle a \rangle$ be a point in $T^\perp \cap v^\perp$. Then a Sylow 3-subgroup P of H_a fixes $\langle a, v \rangle$. Also, P moves v (as otherwise, P would act on $T^\perp \cap v^\perp$, and hence would have an element inducing a transvection on V). Thus, Ω contains $\langle u + v + \alpha a \rangle$ for some scalar $\alpha \neq 0$. Since $(u + v, u + v + \alpha a) = 0$, this contradicts the fact that $k = 1$.

Case 7. $K < \Gamma Sp(6, 3)$. Here, $K_x > H$ with $O_2(H)$ extraspecial of order 2^5 and $|H/O_2(H)| = 5$. Once again, K_x fixes a nonsingular 2-space T on x. Then H centralizes T and is transitive on $T^\perp$. Thus, since $k = 1$, (i) holds.

Our last case is the most interesting one, since it leads to (2.16 v, vi).

Case 8. $K < \Gamma O(7, 3)$. Here, $K_x > H$ with $A = O_2(H)$ elementary abelian of order 2^4 and $|H/A| = 5$. Then H fixes a point $y \neq x$. There is an orthogonal basis $\{b_i\}$ of $\langle x, y \rangle^\perp$ such that H permutes the 1-spaces $\langle b_i \rangle$. We may assume that $(b_i, b_i) = 1$ for all i. Fix $d_j \in \langle x, y \rangle$ with $(d_j, d_j) = j$ for $j = \pm 1$.

Recall that $x^\perp \cap \Omega = \{x\}$. Using $\{b_i\}$, it is easy to check that H has only 3 point-orbits outside of $\langle x, y \rangle$ which might belong to Ω, namely

$$B = \{\langle d_{-1} \pm b_i \rangle \mid 1 \leqslant i \leqslant 5\} \quad \text{and} \quad F_\mu = \left\{\left\langle d_1 + \sum_1^5 \varepsilon_i b_i \right\rangle \Big| \prod_1^5 \varepsilon_i = \mu \right\} \text{ for } \mu = \pm 1.$$

Here, $|B| = 10$ while $|F_\mu| = 16$. Thus, $|\Omega|$ is 11, 12, 17, 18, 27 or 28 (since $(d_1 + \Sigma_1^5 b_i, d_1 + b_1 + b_2 - b_3 - b_4 - b_5) = 0$ and $x^\perp \cap \Omega = \{x\}$). The third of these fails to divide $|\Gamma O(7, 3)|$, while the first leads to a nonexistent group of degree 11.

Suppose that $|\Omega| = 12$. Then K_x cannot induce $\Omega(5,3)$ on $x^\perp/x$, so that $K_x \rhd A$. Consequently, K acts imprimitively on Ω, with 6 blocks of size 2 which K permutes 2-transitively. However, the 2-space spanned by a block is $\langle x, y \rangle$ or $\langle d_{-1} + b_i, d_{-1} - b_i \rangle$, and hence contains d_{-1}. Thus, K fixes $\langle d_{-1} \rangle$, and (i) holds. (N.B.–This case actually occurs.)

Assume that $|\Omega| = 18$. Once again $K_x \rhd A$ and K acts imprimitively on Ω. However, K_x acts primitively on F_μ, so that this situation cannot occur.

Consequently, $|\Omega|$ is 27 or 28. Note that, in either case, Ω is uniquely determined (as $\{x\} \cup B \cup F_1$ or $\{x, y\} \cup B \cup F_1$) up to $\Gamma O(7,3)$-equivalence. Thus, it suffices to show that $K^* = \Gamma O(7,3)_\Omega$ has the required transitivity properties, and then to study subgroups of K^* containing H. For this we will require a digression involving root systems mod 3.

Digression. In order to examine the embedding of the Weyl group $W = W(E_7)$ of type E_7 into $O(7,3)$, we will consider the embedding of the Weyl group $W(E_8)$ into $O^+(8,3)$. The latter embedding will also arise in later sections.

Let $\{e_i\}$ be an orthonormal basis for an $O^+(8,3)$-space. Form the vectors $\varepsilon_i e_i + \varepsilon_j e_j$ ($i \neq j$) and $\Sigma_1^8 \varepsilon_i e_i$ with $\Pi_1^8 \varepsilon_i = -1$, where each ε_i is ± 1. Then $W(E_8)$ is generated by the reflections in the hyperplanes perpendicular to the above "root" vectors (Carter [3, p. 48]).

Set $\rho = e_7 - e_8$ and $u = e_6 + e_7 + e_8 = -\rho + (e_6 - e_7)$, and write $W = W(E_8)_\rho$. Then $W_u = W(E_6)$, so $(W_u)' \cong \Omega(5,3)$. Moreover, $(W_u)'$ acts faithfully on $u^\perp/\langle u \rangle$ as $\Omega(5,3)$. In particular, W' behaves as in (vi), with $x = \langle u \rangle$ and $|\langle u \rangle^W| = |W : W_{\langle u \rangle}| = 28$.

Now consider W_u. This group acts on the set of 27 roots α such that $\alpha \in \rho^\perp$ and $\alpha + (e_6 - e_7)$ is a root. (These roots α are $e_7 + e_8$, $\pm e_i - e_6$ with $1 \leqslant i \leqslant 5$, and $\Sigma_1^5 \varepsilon_i e_i + e_6 - e_7 - e_8$ with $\Pi_1^5 \varepsilon_i = -1$.) Then W_u also acts on the corresponding 27 points $\langle \alpha + u \rangle$. One of these is $\langle u' \rangle$ with $u' = e_6 - e_7 - e_8$, and $W_{uu'} = W(D_5)$. Consequently, (2.16 v) holds when $K = (W_u)'$ and $x = \langle u' \rangle$. (N.B.–The correspondence with our previous notation is as follows: $x = \langle u' \rangle$, $y = \langle u \rangle$, $d_j = -ju' - u$, $b_i = e_i$.)

We now return to Case 8. Which subgroups K of $K^* = \Gamma O(7,3)_\Omega$ behave as required in (2.16)? We may assume that $|\Omega| = 28$. Then $W \leqslant K^* < \Gamma O(7,3)$. By Fischer [17, (15.3.16)], $K^* = \langle -1 \rangle \times W$. **First suppose that**

$$K \leqslant K_y^* \cong \langle -1 \rangle \times O^-(6,2).$$

Two suitable K_y^*-conjugates of $A = O_2(H)$ generate $\Omega^-(6,2)$, so that K is as in (2.16 v). Now suppose that K is transitive on Ω. If $K_x \geqslant \Omega(5,3)$, then (2.16 vi) certainly holds. Suppose that $A \rhd K_x$. Then K acts imprimitively on Ω, with 14 blocks of size 2. Since an element of K^* of order 5 fixes only 3 members of Ω, this is impossible.

This completes the proof of (2.16).

REMARK 2.17. We will need further information concerning the embedding of $W(E_8)$ into $O^+(8,3)$, whose study was begun in Case 8 of (2.16). The vectors e_i, ρ and u were defined there. Changing notation somewhat, we will now write $x = \langle u \rangle$ and $K = W(E_8)$.

Note that $\langle u, \rho \rangle$ contains all root vectors β such that $\beta - u$ is also a root. Thus, $K_x = N_K(\langle u, \rho \rangle)$ (since $x = \mathrm{rad}\langle u, \rho \rangle$), where $K_x = K_u \times \langle -1 \rangle$. Recall that $K_{u\rho}$ is $W(E_6)$. Moreover, $K_u = K_{u\rho}\langle r_\rho, r_{u+\rho} \rangle$ where r_ρ is the reflection with axis $\rho^\perp$. Note that $\langle r_\rho, r_{u+\rho} \rangle \cong S_3$, and it contains the 3-cycle $t = (6, 7, 8)$ acting on $\{e_i\}$. Here, t centralizes both $K_{u\rho}$ and $x^\perp / x$.

The points of our $O^+(8, 3)$ have the shape $(\pm 1^3 0^5)$ or $(\pm 1^6 0^2)$. It follows that K is transitive on them. Also, K_x induces $O(5, 3)$ on $x^\perp / x$, so that K has 2 orbits of $(1, 2)$-flags and is transitive on $(1, 4)$-flags. Then $K' = \Omega^+(8, 2)$ is transitive on the $(1, 4)$-flags of each type. Moreover, if F is a 4-space then $W_F \cong 3 \cdot Sp(4, 3)$; this is easily proved directly, but also follows from the fact that $\mathrm{Aut}\, \Omega^+(8, 2) <$ $\mathrm{Aut}\, P\Omega^+(8, 3)$.

E. *Primitive divisors.* If $q > 1$, a *primitive divisor* of $q^k - 1$ is a prime r dividing $q^k - 1$, but not dividing $q^i - 1$ for $1 \leqslant i < k$. (Note that $k \mid r - 1$.) For convenience, a primitive divisor of $q^k + 1$ is defined to be the same as a primitive divisor of $q^{2k} - 1$. The following number-theoretic fact will be crucial in §6.

LEMMA 2.18. $q^k - 1$ *has a primitive divisor, with the following exceptions*:
(i) $q = 2$, $k = 6$; *or*
(ii) q *is a Mersenne prime, and* $k = 2$.

PROOF. Zsigmondy [67].
The following simple result will be used often, frequently without reference.

LEMMA 2.19. *Let* r *be a primitive divisor of* $q^k - 1$, *and* R *a nontrivial* r-*subgroup of* $\Gamma(V)$. *Then*
(i) $V = C_V(R) \perp [V, R]$, *and*
(ii) *each irreducible constituent of* $[V, R]$ *has dimension* k.

3. Characters of the Weyl groups. Let $B_n = Z_2 \,\mathrm{wr}\, S_n$ be the n-dimensional hyperoctahedral group, viewed as the group of all "signed" permutations of $\Omega = \{\pm 1, \ldots, \pm n\}$. Then B_n is generated by $\{(i, i + 1)(-i, -i - 1);$ $(n, -n) \mid i = 1, \ldots, n - 1\} \subseteq S_\Omega$. The groups B_n and $D_n = B_n \cap A_\Omega$ are the Weyl groups of classical groups, and hence the proof of our main theorem requires explicit information about certain of their permutation characters.

The purpose of this section is to describe how the appropriate permutation characters decompose as sums of irreducible characters. The authors acknowledge the generous help of Jan Saxl in the development of Table II.

A *partition* λ *of the natural number* s, written $\lambda \vdash s$, is a nonincreasing sequence of nonnegative integers $\lambda_1, \ldots, \lambda_s$ such that $\Sigma \lambda_i = s$. The *length* k of such a partition is defined by $\lambda_{k+1} = 0 \neq \lambda_k$. We say a subset X of Ω is *admissible* when $x \in X$ if and only if $-x \in X$. In case $X \subseteq \Omega$ is admissible we define $X^+ = X \cap \{1, 2, \ldots, n\}$ and $X^- = X \cap \{-1, -2, \ldots, -n\}$, and consider two groups of permutations of Ω:

$$A_X = \left\langle (x, x')(-x, -x') \mid x, x' \in X^+ \right\rangle$$

and

$$B_X = \left\langle A_X, (x, -x) \mid x \in X^+ \right\rangle.$$

Observe that $|X| = 2k$ if X is admissible and that A_X is isomorphic to the symmetric group S_k while B_X is isomorphic to the hyperoctahedral group B_k.

Let $(\lambda; u)$ be a pair of partitions where $\lambda \vdash s$ has length l and $\mu \vdash n - s$ has length m. Partition Ω into admissible sets $L_1, \ldots, L_l, M_1, \ldots, M_m$ so that $|L_i| = 2\lambda_i$ and $|M_j| = 2\mu_j$ for $1 \leq i \leq l$, $1 \leq j \leq m$, and consider the *Weyl subgroup* (notation of Mayer [35])

$$W(\lambda; \mu) = \left\langle A_{L_i}, B_{M_j} \mid i = 1, \ldots, l; \; j = 1, \ldots, m \right\rangle \leq B_n.$$

It is easy to see that the conjugacy class of $W(\lambda; \mu)$ is uniquely determined by $(\lambda; \mu)$.

The Weyl subgroups of B_n play a very important role in the character theory of B_n that is quite analogous to the role played by the Young subgroups $W(\lambda; \varphi)$ in the character theory of the symmetric groups (in the notation of Kerber [34]). In spite of this we are most interested in the parabolic Weyl subgroups, i.e., those that correspond to parabolic subgroups of a group with a (B, N)-pair of type B_n. These are the Weyl subgroups $W(\lambda; \mu)$ where $l \leq 1$.

Let $1(\lambda; \mu)$ denote the character of B_n obtained by inducing the trivial character of $W(\lambda; \mu)$ to B_n, and let $\varepsilon(\lambda; \mu)$ denote the character of B_n obtained by inducing the alternating character ε of $W(\lambda; \mu)$ to B_n. (Recall that $\varepsilon(g) = -1$ if g is an odd permutation in S_Ω and $\varepsilon(g) = 1$ if g is an even permutation in S_Ω.)

Let $C(l, m)$ denote the $2l + m$ by $2l + m$ matrix having block diagonal form:

$$C(l, m) = \text{diag}\left\{ \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{l}, \; \underbrace{1, \ldots, 1}_{m} \right\},$$

and let $M(a; b \mid l; m)$ denote the set of all nonnegative integral $2a + b$ by $2l + m$ matrices $M = (M_{ij})$ such that

(i) $C(a, b) M C(l, m) = M$ and

(ii) M_{ij} is even whenever both $i > 2a$ and $j > 2l$.

Suppose $W(\alpha; \beta)$ and $W(\lambda; \mu)$ are Weyl subgroups, where the associated partitions have lengths a, b and l, m, respectively. Define $M(\alpha; \beta \mid \lambda; \mu)$ to be the set of all $M \in M(a; b \mid l; m)$ such that the row sums of M are

$$\alpha_1, \alpha_1, \ldots, \alpha_a, \alpha_a, 2\beta_1, \ldots, 2\beta_b$$

and the column sums of M are

$$\lambda_1, \lambda_1, \ldots, \lambda_l, \lambda_l, 2\mu_1, \ldots, 2\mu_m.$$

Finally, let $N(\alpha; \beta \mid \lambda; \mu)$ be the set of elements of $M(\alpha; \beta \mid \lambda; \mu)$ of the form

$$\left(\begin{array}{c|c} * & * \\ \hline * & O_{b, m} \end{array} \right).$$

THEOREM 3.1. *Let* $\langle \ , \ \rangle$ *denote the usual inner product of characters and suppose* $W(\alpha; \beta)$, $W(\lambda; \mu)$ *are Weyl subgroups in* B_n. *Then*

$$\langle 1(\alpha; \beta), 1(\lambda; \mu) \rangle = |M(\alpha; \beta \mid \lambda; \mu)| = \langle \varepsilon(\alpha; \beta), \varepsilon(\lambda; \mu) \rangle$$

and

$$\langle 1(\alpha; \beta), \varepsilon(\lambda; \mu) \rangle = |N(\alpha; \beta \mid \lambda; \mu)|.$$

REMARK. This should be compared to Coleman [**6**, Theorem 15] and Snapper [**52**].

PROOF. By the Mackey subgroup theorem,

$$\langle 1(\alpha; \beta), 1(\lambda; \mu) \rangle = \sum_{y \in \Delta} \langle 1_{W(\alpha; \beta)} |_Y, 1_{W(\lambda; \mu)} |_Y \rangle$$

where $Y = W(\alpha; \beta) \cap (W(\lambda; \mu))^y$ and Δ is a complete system of $(W(\alpha; \beta), W(\lambda; \mu))$ double coset representatives in B_n. Since each term in this sum is 1, we have

$$\langle 1(\alpha; \beta), 1(\lambda; \mu) \rangle = |\Delta| = \text{the number of } W(\lambda; \mu) \text{ orbits on } B_n / W(\alpha; \beta).$$

Let $A_1, \ldots, A_a$, $B_1, \ldots, B_b$ and $L_1, \ldots, L_l$, $M_1, \ldots, M_m$ be the admissible partitions of Ω associated with $W(\alpha; \beta)$ and $W(\lambda; \mu)$, respectively. View the coset $gW(\alpha; \beta)$, $g \in B_n$, as the admissible partition $gA_1, \ldots, gA_a$, $gB_1, \ldots, gB_b$ of "type" $(\alpha; \beta)$. We claim that $gW(\alpha; \beta)$ and $hW(\alpha; \beta)$ are in the same $W(\lambda; \mu)$-orbit if and only if

$$\begin{aligned}
\left| g(A_i^+) \cap L_j^+ \right| &= \left| h(A_i^+) \cap L_j^+ \right| \quad \text{for } 1 \le i \le a, \, 1 \le j \le l, \\
\left| g(A_i^-) \cap L_j^+ \right| &= \left| h(A_i^-) \cap L_j^+ \right| \quad \text{for } 1 \le i \le a, \, 1 \le j \le l, \\
\left| gB_i \cap L_j \right| &= \left| hB_i \cap L_j \right| \quad \text{for } 1 \le i \le b, \, 1 \le j \le l, \\
\left| gA_i \cap M_j \right| &= \left| hA_i \cap M_j \right| \quad \text{for } 1 \le i \le a, \, 1 \le j \le m, \\
\left| gB_i \cap M_j \right| &= \left| gB_i \cap M_j \right| \quad \text{for } 1 \le i \le b, \, 1 \le j \le m.
\end{aligned} \tag{3.2}$$

It is clear that these conditions are necessary, since for example

$$\left| x\left(g(A_i^+) \cap L_j^+ \right) \right| = \left| xg(A_i^+) \cap L_j^+ \right| \quad \text{for } x \in W(\lambda, \mu)$$

and all relevant i, j. To see that (3.2) is also sufficient, observe for example, that the group $A_{L_1} \le W(\lambda; \mu)$ acts as the full symmetric group on L_1^+ and acts trivially on $\Omega - L_1$. Consequently there is an $x \in A_{L_1}$ taking $g(A_1^{\pm}) \cap L_1^+$ to $h(A_1^{\pm}) \cap L_1^+$ and $gB_k \cap L_1^+$ to $hB_k \cap L_1^+$, since these two partitions of L_1^+ have the same type. A similar observation holds for each group A_{L_j}, $2 \le j \le l$, and each group B_{M_j}, $1 \le j \le m$.

This shows that the orbit parameters given in (3.2) completely determine a $W(\lambda; \mu)$-orbit on $B_n / W(\alpha; \beta)$. Thus, each element of Δ is associated with a $2a + b$

by $2l + m$ matrix of the following form:

$$
M = \begin{pmatrix}
\cdots & \begin{matrix} \vdots \\ |gA_i^+ \cap L_j^+|, & |gA_i^+ \cap L_j^-| \\ |gA_i^- \cap L_j^+|, & |gA_i^- \cap L_j^-| \\ \vdots \end{matrix} & \cdots & \Bigg| & \cdots & \begin{matrix} \vdots \\ |gA_i^+ \cap M_j| \\ |gA_i^- \cap M_j| \\ \vdots \end{matrix} & \cdots & \Bigg) & \begin{matrix} gA_1^+ \\ \vdots \\ gA_i^+ \\ gA_i^- \\ \vdots \\ gA_a^- \end{matrix} \\
\hline
\cdots & \begin{matrix} \vdots \\ |gB_i \cap L_j^+|, & |gB_i \cap L_j^-| \\ \vdots \end{matrix} & \cdots & & \cdots & \begin{matrix} \vdots \\ |gB_i \cap M_j| \\ \vdots \end{matrix} & \cdots & & \begin{matrix} gB_1 \\ \vdots \\ gB_i \\ \vdots \\ gB_b \end{matrix} \\
L_1^+ \cdots L_j^+ & L_j^- \cdots L_l^- & M_1 \cdots M_j \cdots M_m
\end{pmatrix}
\tag{3.3}
$$

The sum of the entries in one of the rows of this matrix is just the cardinality of the set labeling that row, and similarly for columns. Since $\{\{x, -x\} \mid x = 1, \ldots, n\}$ is a system of imprimitivity for the action of B_n on Ω, we have

$$\left| gA_i^+ \cap L_j^+ \right| = \left| gA_i^- \cap L_j^- \right|, \qquad \left| gA_i^- \cap L_j^+ \right| = \left| gA_i^+ \cap L_j^- \right|,$$

$$\left| gA_i^+ \cap M_i \right| = \left| gA_i^- \cap M_i \right|, \qquad \left| gB_i \cap L_j^+ \right| = \left| gB_i \cap L_j^- \right|,$$

and it follows that $M \in M(\alpha; \beta \mid \lambda; \mu)$.

We leave to the reader the verification that any element of $M(\alpha; \beta \mid \lambda; \mu)$ may be viewed as a matrix of intersection numbers as in (3.3), and hence determines an element in Δ, from which it follows that

$$\langle 1(\alpha; \beta), 1(\lambda; \mu) \rangle = |M(\alpha; \beta \mid \lambda; \mu)|.$$

The Mackey subgroup theorem also implies that

$$\langle \varepsilon(\alpha; \beta), \varepsilon(\lambda; \mu) \rangle = \sum_{y \in \Delta} \langle \varepsilon|_Y, \varepsilon|_Y \rangle = |\Delta|,$$

where $Y = W(\alpha; \beta) \cap W(\lambda; \mu)^y$ and Δ is a complete system of $(W(\alpha; \beta), W(\lambda; \mu))$-double coset representatives as above. Therefore

$$\langle \varepsilon(\alpha; \beta), \varepsilon(\lambda; \mu) \rangle = |M(\alpha; \beta \mid \lambda; \mu)|.$$

In fact, the same argument establishes the remaining equality provided only that the elements of $N(\alpha; \beta \mid \lambda; \mu)$ correspond to those double coset representatives $y \in \Delta$ having the property:

$$\langle 1_Y, \varepsilon_Y \rangle = 1 \quad \text{where } Y = W(\alpha; \beta) \cap W(\lambda; \mu)^y.$$

This condition that $Y \leqslant \ker \varepsilon$ amounts to the condition that none of the various intersections considered in (3.3) contains both $\pm k$, $1 \leqslant k \leqslant n$. These are just the elements of $N(\alpha; \beta \mid \lambda; \mu)$, and the theorem follows.

COROLLARY 3.4. $\langle 1(\alpha; \beta), 1(\lambda; \mu)\rangle$ is the coefficient of

$$\prod_{i=1}^{a} (x_i x_{-i})^{\alpha_i} \prod_{i=1}^{b} (y_i y_{-i})^{\beta_i} \prod_{j=1}^{l} (z_j z_{-j})^{\lambda_j} \prod_{j=1}^{m} (w_j w_{-j})^{\mu_j}$$

in the expansion of the generating function

$$\prod \frac{\left(1 - x_i x_{-i} z_j z_{-j}\right)}{\left(1 - x_i z_j\right)\left(1 - x_{-i} z_{-j}\right)\left(1 - x_i x_{-i} w_j^2\right)\left(1 - y_i^2 z_j z_{-j}\right)\left(1 - y_i^2 w_i^2\right)}.$$

Theorem 3.1 can be used to express the irreducible characters of B_n and the permutation characters $1(\alpha; \beta)$ as linear combinations of one another, since the characters $1(\alpha; \beta)$ span the character ring of B_n (Mayer [35, 3.1]). We will, however, not require this much information.

The irreducible characters of B_n are also labeled by ordered pairs of partitions $(\alpha \mid \beta)$ where $\alpha \vdash s$, $\beta \vdash n - s$ and $(\alpha \mid \beta)$ occurs as a constituent of $1(\beta; \alpha)$ (Kerber [34, 2.22]). A routine, though tedious calculation yields Tables I and II. An example is given after (3.7). We note in passing that rows 1, 2, 4, 5, 8, 9 and 10 of Table I can be found on p. 154 of Murnaghan [42].

TABLE I. The irreducible constituents of selected permutation characters $1(\alpha; \beta)$ and their multiplicities.

	$(n\mid0)$	$(n-1,1\mid0)$	$(n-1\mid1)$	$(n-2,2\mid0)$	$(n-2,1,1\mid0)$	$(n-2,1\mid1)$	$(n-2\mid2)$	$(n-3,3\mid0)$	$(n-3,2,1\mid0)$	$(n-3,1^3\mid0)$	$(n-3,2\mid1)$	$(n-3,1^2\mid1)$	$(n-3,1\mid2)$	$(n-3\mid3)$
$1(0; n)$	1													
$1(0; n-1, 1)$	1	1												
$1(1; n-1)$	1	1	1											
$1(0; n-2, 2)$	1	1		1										
$1(0; n-2, 1, 1)$	1	2		1	1									
$1(1; n-2, 1)$	1	2	1	1	1	1								
$1(2; n-2)$	1	1	1	1		1	1							
$1(0; n-3, 3)$	1	1		1				1						
$1(0; n-3, 2, 1)$	1	2		2	1			1	1					
$1(0; n-3, 1^3)$	1	3		3	3			1	2	1				
$1(1; n-3, 2)$	1	2	1	2	1	1		1	1		1			
$1(1; n-3, 1^2)$	1	3	1	3	3	2		1	2	1	1	1		
$1(2; n-3, 1)$	1	2	1	2	1	2	1	1	1		1	1	1	
$1(3; n-3) \, n \neq 4$	1	1	1	1		1	1	1			1		1	1
$1(3; 1)$	1	1	1			1	1	—	—		—		1	1

(N.B.–Characters involving nondecreasing sequences are to be ignored.)

TABLE II. The multiplicities to which the irreducible constituents of $1(1; n-1)$, $1(2; n-2)$ and $1(3; n-3)$ occur in selected parabolic characters of B_n (also the multiplicities to which the irreducible constituents of θ_1, θ_2 and θ_3 occur in selected parabolic characters of a group with (B, N)-pair of type B_n or C_n—see §4).

Character degree	$(n\mid0)$	$(n-1, 1\mid0)$	$(n-1\mid1)$	$(n-2, 2\mid0), n\geqslant 4$	$(n-2, 1\mid1), n\geqslant 3$	$(n-2\mid2), n\geqslant 2$	$(n-3, 3\mid0), n\geqslant 6$	$(n-3, 2\mid1), n\geqslant 5$	$(n-3, 1\mid2), n\geqslant 4$	$(n-3\mid3), n\geqslant 3$
	1	$n-1$	n	$\dfrac{n(n-3)}{2}$	$n(n-2)$	$\binom{n}{2}$	$\binom{n}{2}\dfrac{n-5}{3}$	$\binom{n}{2}(n-4)$	$\binom{n}{2}(n-3)$	$\binom{n}{3}$
$1(1; n-1)\ n\geqslant 2$	1	1	1	0	0	0	0	0	0	0
$1(2; n-2)\ n\geqslant 3$	1	1	1	1	1	1	0	0	0	0
$1(3; n-3)\ n\geqslant 5$	1	1	1	1	1	1	1	1	1	1
$1(3; 1)$	1	1	1	0	1	1	–	–	1	1
$1(4; n-4)\ n\geqslant 7$	1	1	1	1	1	1	1	1	1	1
$1(4; 2)$	1	1	1	1	1	1	0	1	1	1
$1(4; 1)$	1	1	1	0	1	1	–	0	1	1
$1(n; 0)$	1	0	1	0	0	1	0	0	0	1
$1(1^2; n-2)\ n\geqslant 3$	1	2	2	1	2	1	0	0	0	0
$1(2, 1; n-3)\ n\geqslant 5$	1	2	2	2	3	2	1	2	2	1
$1(2, 1; 1)$	1	2	2	1	3	2	–	–	2	1
$1(1^3; n-3)\ n\geqslant 4$	1	3	3	3	6	3	1	3	3	1
$1(1^3; 0)$	1	2	3	–	3	3	–	–	–	1
$1(3, 1; n-4)\ n\geqslant 7$	1	2	2	2	3	2	2	3	3	2
$1(3, 1; 2)$	1	2	2	2	3	2	1	3	3	2
$1(3, 1; 1)$	1	2	2	1	3	2	–	1	3	2
$1(n-1, 1; 0)\ n\geqslant 4$	1	1	2	0	1	2	0	0	1	2
$1(2, 1; 0)$	1	1	2	–	1	2	–	–	–	1

Since this paper was first written, Geissinger and Kinch [20] has appeared. These tables may be calculated directly from the elegant Theorem III.5 of [20].

This completes the data required to deal with groups of type B_n. The characters of D_n are related to those of B_n in a particularly simple manner.

THEOREM 3.5 (KERBER [34, 2.25]). *An irreducible character $(\alpha \mid \beta)$ of B_n remains irreducible when restricted to D_n if and only if $\alpha \neq \beta$. The characters of the form $(\alpha \mid \alpha)$ decompose into a sum of two irreducible characters $(\alpha \mid \alpha)^{\pm}$ conjugate in B_n. In any case, $(\alpha \mid \beta)\vert_{D_n} = (\beta \mid \alpha)\vert_{D_n}$.*

COROLLARY 3.6. *Let $W(\lambda; \mu)$ be a Weyl subgroup of B_n and define*

$$W^*(\lambda; \mu) = W(\lambda; \mu) \cap D_n, \qquad 1^*(\lambda; \mu) = 1^{D_n}_{W^*(\lambda; \mu)} \quad \text{for } \mu \neq 0$$

and

$$W^*(\lambda; 0)^+ = W(\lambda; 0) \cap D_n, \qquad 1^*(\lambda; 0)^+ = 1^{D_n}_{W^*(\lambda; 0)^+},$$

$$W^*(\lambda; 0)^- = \left(W(\lambda; 0)^\sigma \cap D_n\right), \qquad 1(\lambda; 0)^- = 1^{D_n}_{W^*(\lambda; 0)^-},$$

where σ is the permutation $(1, -1)$ *in* S_Ω. *Also, define* $(\alpha \mid \beta)$ *to be the restriction of the* B_n-*character* $(\alpha \mid \beta)$ *to* D_n. *Finally, set* $1^*(\lambda; 0) = 1^*(\lambda; 0)^+ + 1^*(\lambda; 0)^-$. *Then*

$$\langle 1^*(\lambda; \mu), (\alpha \mid \beta) \rangle = \langle 1(\lambda; \mu) + \varepsilon(\lambda; \mu), (\alpha \mid \beta) \rangle$$

and

$$\langle 1^*(\alpha; \beta), 1^*(\lambda; \mu) \rangle = |M(\alpha; \beta \mid \lambda; \mu)| + |N(\alpha; \beta \mid \lambda; \mu)|.$$

PROOF. Since $D_n = \ker \varepsilon$,

$$1^*(\lambda; \mu)^{B_n} = 1^{B_n}_{W^*(\lambda; \mu)} = (1 + \varepsilon)^{B_n}_{W(\lambda; \mu)} = 1(\lambda; \mu) + \varepsilon(\lambda; \mu)$$

for $\mu \neq 0$. In case $\mu = 0$,

$$1^*(\lambda; 0)^{B_n} = \left(1_{W^*(\lambda; 0)^+} + 1_{W^*(\lambda; 0)^-}\right)^{B_n} = 2\left(1_{W^*(\lambda; 0)^+} + 1_{W^*(\lambda; 0)^+}\right)^{B_n},$$

since $W^*(\lambda; 0)^+$ and $W^*(\lambda; 0)^-$ are conjugate in B_n. However, $W^*(\lambda; 0)^+ = W(\lambda; 0)$ and $1_{W(\lambda; 0)} = \varepsilon_{W(\lambda; 0)}$, so

$$1^*(\lambda; 0)^{B_n} = \left(1_{W(\lambda; 0)} + \varepsilon_{W(\lambda; 0)}\right)^{B_n} = 1(\lambda; 0) + \varepsilon(\lambda; 0).$$

The first claim now follows from Frobenius reciprocity.

By the Mackey subgroup theorem, $1^*(\lambda; \mu) = 1(\lambda; \mu)_{D_n}$. Consequently, if $1(\lambda; \mu) = \Sigma c_{\gamma\delta}(\gamma \mid \delta)$ for integers $c_{\gamma\delta}$, then $1^*(\lambda; \mu) = \Sigma c_{\gamma\delta}(\gamma \mid \delta)$ also. Thus,

$$\langle 1^*(\alpha; \beta), 1^*(\lambda; \mu) \rangle = \left\langle 1^*(\alpha; \beta), \sum c_{\gamma\delta}(\gamma \mid \delta) \right\rangle$$

$$= \langle 1(\alpha; \beta) + \varepsilon(\alpha; \beta), 1(\lambda; \mu) \rangle = |M(\alpha; \beta \mid \lambda; \mu)| + |N(\alpha; \beta \mid \lambda; \mu)|$$

by the first part of this corollary and (3.1).

COROLLARY 3.7. (i)

$$\left\langle 1^*(\lambda; \mu), (\alpha \mid \alpha)^+ \right\rangle = \left\langle 1^*(\lambda; \mu), (\alpha \mid \alpha)^- \right\rangle \quad if \, \mu \neq 0.$$

(ii) *If* $\alpha \neq \beta$ *and if* $\langle 1^*(3; n - 3); (\alpha \mid \beta) \rangle \neq 0$, *then*

$$\left\langle 1^*(\lambda; \mu), (\alpha \mid \beta) \right\rangle = \left\langle 1(\lambda; \mu), (\alpha \mid \beta) \right\rangle$$

whenever $\mu \vdash t$ *with* $t > 3$, *while*

$$\left\langle 1^*(\lambda; 0)^+, (\alpha \mid \beta) \right\rangle = \left\langle 1^*(\lambda; 0)^-, (\alpha \mid \beta) \right\rangle = \left\langle 1(\lambda; 0), (\alpha \mid \beta) \right\rangle.$$

(iii)

$$\left\langle 1^*(\lambda; 0)^+, (\alpha \mid \alpha)^+ \right\rangle = \left\langle 1^*(\lambda; 0)^-, (\alpha \mid \alpha)^+ \right\rangle = \left\langle 1^*(\lambda; 0)^+, (\alpha \mid \alpha)^- \right\rangle$$

$$= \left\langle 1^*(\lambda; 0)^-, (\alpha \mid \alpha)^- \right\rangle.$$

PROOF. Statement (i) follows from (3.6) and the fact that $1^*(\lambda; \mu)$ is B_n-invariant. If $\mu \vdash t$ and $t > 3$, then $N(\lambda; \mu \mid 3; n - 3) = 0$, and hence (ii) holds in this case.

Clearly, $N(\lambda; 0 \mid \alpha; \beta) = M(\lambda; 0 \mid \alpha; \beta)$ for all partition pairs $(\alpha; \beta)$. Thus, $\langle 1^*(\lambda; 0), (\alpha \mid \beta) \rangle = 2\langle 1(\lambda; 0), (\alpha \mid \beta) \rangle$. Here, $(\alpha \mid \beta)$ is B_n-invariant while $1^*(\lambda; 0)^\pm$ is not, so that

$$\left\langle 1^*(\lambda; 0)^\pm, (\alpha \mid \beta) \right\rangle = \tfrac{1}{2} \left\langle 1^*(\lambda; 0), (\alpha \mid \beta) \right\rangle = \left\langle 1(\lambda; 0), (\alpha \mid \beta) \right\rangle.$$

This completes the proof of (ii).

Write

$$1^*(\lambda;0)^+ = \sum_{\alpha \neq \beta} c_{(\alpha|\beta)}(\alpha|\beta) + \sum c_\alpha(\alpha|\alpha)^+ + \sum d_\alpha(\alpha|\alpha)^-$$

for integers $c_{(\alpha|\beta)}$, c_α and d_α. Since $(W^*(\lambda;0)^+)^\sigma = W^-(\lambda;0)^-$, we can conjugate by σ (cf. (3.6)) in order to obtain

$$1^*(\lambda;0)^- = \sum c_{(\alpha|\beta)}(\alpha|\beta) + \sum c_\alpha(\alpha|\alpha)^- + \sum d_\alpha(\alpha|\alpha)^+ .$$

We claim that $0 = \langle 1^*(\lambda;0)^+, 1^*(\lambda;0)^+ - 1^*(\lambda;0)^- \rangle$. For, let $\{a_i \mid 1 \leq i \leq t\}$ be a complete set of $(W^*(\lambda;0)^+, W^*(\lambda;0)^+)$ double coset representatives in D_n. Then $\{a_i\sigma \mid 1 \leq i \leq t\}$ is a complete set of $(W^*(\lambda;0)^+, W^*(\lambda;0)^-)$ double coset representatives; for, $w^+ a\sigma w^- = b\sigma$ if and only if $w^+ a\sigma w^- \sigma = b$, while $\sigma w^- \sigma \in W^*(\lambda;0)^+$ whenever $w^- \in W^*(\lambda;0)^-$. This proves our claim.

Thus,

$$0 = \left\langle 1^*(\lambda;0), \sum (c_\alpha - d_\alpha)(\alpha|\alpha)^+ + \sum (d_\alpha - c_\alpha)(\alpha|\alpha)^- \right\rangle$$
$$= \sum (c_\alpha - d_\alpha)\left\langle 1^*(\lambda;0), (\alpha|\alpha)^+ - (\alpha|\alpha)^- \right\rangle = \sum (c_\alpha - d_\alpha)^2.$$

This proves (iii).

Further inner products, not handled in (3.7) and Table II, will be needed later. These are found in Table III, which is obtained by further straightforward calculations, as in the following example.

EXAMPLE. We illustrate the use of (3.1), Table I and (3.6) in a very simple situation. For $n \geq 3$ we show that

$$1(n-2,1;1) = (n|0) + 2(n-1,1|0) + 2(n-1|1) + \chi,$$
$$1^*(n-2,1;1) = (n|0) + 2(n-1,1|0) + 3(n-1|1) + \chi'$$

where $\langle \chi, 1(1;n-1) \rangle = 0 = \langle \chi', 1^*(1;n-1) \rangle$. We see trivially that the elements of $M(0;n|n-2,1;1)$ and $M(0;n-1,1|n-2,1;1)$ are $(n-2,n-2,1,1,2)$ and

$$\begin{pmatrix} n-2 & n-2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} n-2 & n-2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} n-3 & n-3 & 1 & 1 & 2 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix},$$

respectively. It is almost as easy to see that $M(1;n-1|n-2,1;1)$ consists of

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ n-3 & n-3 & 1 & 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ n-3 & n-3 & 1 & 1 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ n-2 & n-2 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ n-2 & n-2 & 0 & 0 & 2 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ n-2 & n-2 & 1 & 1 & 0 \end{pmatrix}.$$

TABLE III. The multiplicities of the irreducible constituents of $1*(3, n-3)$ in selected parabolic characters of D_n (also the multiplicities of the irreducible constituents of θ_1, θ_2 and θ_3 in selected parabolic characters of a group with a (B, N)-pair of type D_n–see §4).

	$(n \mid 0)$	$(n-1, 1 \mid 0)$	$(n-1 \mid 1)$	$(n-2, 2 \mid 0)$	$(n-2, 1 \mid 1)$	$(n-2 \mid 2)$	$(n-3, 3 \mid 0)$	$(n-3, 2 \mid 1)$	$(n-3, 1 \mid 2)$	$(n-3 \mid 3)$
$1*(1; 3)$	1	1	1	0	0	0 + 0	—	—	0	
$1*(2; 2)$	1	1	1	1	1	1 + 1	—	—	0	
$1*(3; 1)$	1	1	2	0	1	1 + 1	—	—	1	See Column 3
$1*(1^2; 2)$	1	2	2	1	2	1 + 1	—	—	0	
$1*(2, 1; 1)$	1	2	3	1	4	2 + 2	—	—	3	
$1*(1^3; 1)$	1	3	4	2	8	3 + 3	—	—	6	
$1*(2; 3)$	1	1	1	1	1	1	—	0	0	
$1*(1^2; 3)$	1	2	2	1	2	1	—	0	0	
$1*(3; 2)$	1	1	1	1	1	2	—	1	1	See Column 6
$1*(2, 1; 2)$	1	2	2	2	3	3	—	2	3	
$1*(1^3; 2)$	1	3	3	3	6	4	—	3	5	
$1*(3, 1; 1)$	1	2	3	1	4	4	—	1	4	
$1*(3; 3)$	1	1	1	1	1	1	1	1	1	1 + 1
$1*(2, 1; 3)$	1	2	2	2	3	2	1	2	2	1 + 1
$1*(3, 1; 2)$	1	2	2	2	3	3	1	3	1	2 + 2
$1*(3, 1; 3)$	1	2	2	2	3	2	2	3	3	3
$1*(4; 3)$	1	1	1	1	1	1	1	1	1	2
$1*(4, 1; 2)$	1	2	2	2	4	2	1	3	2	2

By (3.1) we have

$$\langle 1(0; n), 1(n-2, 1; 1) \rangle = 1,$$
$$\langle 1(0; n-1, 1), 1(n-2, 1; 1) \rangle = 3$$

and

$$\langle 1(1; n-1), 1(n-2, 1; 1) \rangle = 5.$$

The first three rows of Table I imply $\langle (0 \mid n), 1(n-2, 1; 1) \rangle = 1$, $\langle (n-1, 1 \mid 0),$ $1(n-2, 1; 1) \rangle = 3 - 1 = 2$ and $\langle (n-1 \mid 1), 1(n-2, 1; 1) \rangle = 5 - 3 = 2$. The first

claim follows. By (3.6) and the above enumeration,

$$\langle 1^*(0;n), 1^*(n-2,1;1)\rangle = 1 + 0 = 1,$$

$$\langle 1^*(0;n-1,1), 1^*(n-2,1;1)\rangle = 3 + 0 = 3$$

and

$$\langle 1^*(1;n-1), 1^*(n-2,1;1)\rangle = 5 + 1 = 6,$$

since the last matrix listed in $M(1, n-1 \mid n-2, 1; 1)$ above is in $N(1, n-1 \mid n-2, 1, 1)$. The second claim follows from (3.7) (each of the characters $(n \mid 0)$, $(n-1, 1 \mid 0)$, $(n-1 \mid 1)$ is an irreducible character of D_n) and from Table I, just as above.

The last Weyl group we must discuss is the symmetric group S_n. Since S_n is a homomorphic image of B_n, some of the irreducible characters of B_n are also characters of S_n. It turns out that these characters are precisely those of the form $(\alpha \mid \beta)$ where $\beta = 0$ (this follows, for example, from Mayer [35, 1.1]). In fact, one can obtain a table describing characters of the symmetric group from Table II by deleting columns 3, 5, 6, 8, 9, and 10, and replacing the semicolons in the row labels by commas and rearranging into partitions.

The information we require about characters of S_n is included in the following lemma.

LEMMA 3.8. *Let θ_i be the permutation character of S_n on i-sets (alias $1(n-i, i)$), θ_{ij} ($i < j$) the character on pairs consisting of an i-set and a j-set containing it (alias $1(n-j, j-i, i)$), and so on. Then the following hold for $1 \le i \le n/2$.*
  (i) $\theta_i - \theta_{i-1}$ *is irreducible, $\theta_0 = 1$.*
  (ii) $\langle \theta_i - \theta_{i-1}, \theta_{i-1}\rangle = 0.$
  (iii) $\theta_{ij} = \theta_{j-i,j}.$
  (iv) $\langle \theta_1 - 1, \theta_{12\cdots n-1}\rangle n - 1.$
  (v) $\langle \theta_3 - \theta_2, \theta_{34}\rangle = \langle \theta_4 - \theta_3, \theta_{45}\rangle = 2$ *if $n > 8$.*
  (vi) $\langle \theta_2 - \theta_1, \theta_{12}\rangle = 1$; $\langle \theta_2 - \theta_1, \theta_{13}\rangle$ *is* 1 *if* $n = 4$, *and is* 2 *if* $n > 4$; $\langle \theta_2 - \theta_1, \theta_{123}\rangle = 3$; *and if $n > 5$ then $\langle \theta_2 - \theta_1, \theta_{124}\rangle = 4$ and $\langle \theta_2 - \theta_1, \theta_{1234}\rangle = 6$.*

For these inner products, see Murnaghan [42, p. 154], where more standard notation is used; or see (10.1) below.

4. Parabolic characters. In this section we are concerned with characters of $\mathrm{Aut}(\mathrm{Chev}(V))$. After a technical lemma that is used to deal with graph automorphisms in §§8–10, we turn to characters of parabolic type. We recall the fundamental correspondence, due to Curtis, Iwahori and Kilmoyer [8, (7.2)], between these characters and those of the Weyl group, which provides the link with §3. Finally, in (4.7) we use the remarkable formulae of Hoefsmit [26] to completely determine the characters of parabolic type that fail to have degree divisible by p (compare Howlett [27]).

Lemma 4.1. *Let G be as in (1.1) or (1.2), and let $G^+ = G \cap G^\natural$, where $G^\natural$ is generated by* Chev(V) *and its diagonal and field automorphisms. Let $K \leqslant G$ satisfy $G = KG^+$, and set $K^+ = K \cap G^+$. Write $\varphi = 1_K^G = 1 + \chi + \zeta$ where χ is irreducible, and let $\varphi^+ = \varphi|_{G^+}$, $\chi^+ = \chi|_{G^+}$ and $\zeta^+ = \zeta|_{G^+}$. Finally, let B be a Borel subgroup of G^+ (cf. [9, (2.6)]). Then*

(a) $\varphi^+ = (1_{K^+})^{G^+}$;

(b) *Each irreducible constituent of χ^+ occurs to multiplicity one;*

(c) *The irreducible constituents of χ^+ are conjugate under G;*

(d) *Either $\chi^+ \subseteq 1_B^{G^+}$ or $\langle \chi^+, 1_B^{G^+} \rangle = 0$.*

Remarks. (1) $\Gamma(V) = G^\natural$ unless V has type $\Omega^+(2n, q)$ in which case $|\Gamma(V): G^\natural| = 2$.

(2) A slightly more technical, similar lemma that can be used to restrict to Chev(V) is described in §11.

Proof. Since $K^+ = K \cap G^+$, $g \in G^+$ fixes a coset Kh, $h \in G^+$, if and only if it fixes $K^+ h$. This proves (a). Since $|G: G^+|$ divides 6, Isaacs [28, (6.18)] (applied twice if necessary) implies (b), and (c) follows from Clifford's theorem [28, (6.5)]. Finally, (d) follows from (b) and (c) together with the fact that $1_B^{G^+}$ is G-invariant [9, (2.6)].

The (B, N) structure of Chev(V) is discussed in detail in Carter [3] (and also Curtis, Kantor and Seitz [9, §2]). We observe that each of the groups G in Theorem 1.1 or 1.2 has a (B, N)-pair with the associated Coxeter system (W, R), $R = \{s_1, \ldots, s_n\}$ of type B_n, C_n, D_n or A_n.

For each subset $I \subseteq \{1, \ldots, n\}$, $W_I = \langle s_j | j \in I \rangle$ and $G_I = BW_I B$ are *parabolic* subgroups. (This is consistent with §3, where $s_i = (i, i+1)(-i, -i-1)$ for $i < n$ while $s_n = (-n, n)$, and $W = W(B_n) \cong W(C_n)$.) The irreducible characters ζ of G that occur as constituents in $1_{G_I}^G$ are said to be of *parabolic type*.

It is elementary to see that the normalizer of a Sylow p-subgroup of G fixes a unique flag in V. The group B is just such a normalizer [3, §8.6], [9, §2], and so fixes a flag $(V_1, V_2, \ldots)$. The group G_{V_i} is maximal in G and contains B. Consequently, G_{V_i} is the parabolic subgroup $G_i = G_{\{i\}}$ for a suitable labeling of the elements of R (cf. [3, 8.3.3]). It follows that the parabolic subgroup G_I is the stabilizer of an $(i_1, i_2, \ldots)$-flag, where $I = \{i_1, i_2, \ldots\}$; and the character $\theta_I = \theta_{i_1 i_2} \cdots = 1_{G_I}^B$ is just the permutation character of G on the set of all $(i_1, i_2, \ldots)$-flags.

Theorem 7.2 of Curtis, Iwahori and Kilmoyer [8] provides a natural 1-1 correspondence $\zeta \to \zeta_0$ between the irreducible characters ζ of parabolic type and the characters ζ_0 of W such that

$$\langle \zeta, \theta_I \rangle = \langle \zeta, 1_{G_I}^G \rangle = \langle \zeta_0, 1_{W_I}^W \rangle = \langle \zeta_0, 1(\lambda; \mu) \rangle, \tag{4.2}$$

where $I = \{i_1, i_2, \ldots, i_t\}$, λ is obtained by rearranging the sequence $(i_1, i_2 - i_1, \ldots, i_t - i_{t-1})$ into a partition of i_t, and $\mu = (n - i_t)$, provided that $i_t < n$ when G has type D_n. (The flags of groups of type D_n have the more complex structure described at the beginning of §2, but λ is obtained in an analogous way.) The examples most relevant to (1.1) are $\theta_i = 1(i; n - i)$, $\theta_{1i} = 1(i - 1, i; n - i)$, and $\theta_{123} = 1(1^3; n - 3)$ (cf. Table II).

A consequence of the fact that the right-hand side of (4.2) depends only on the partition $(\lambda; \mu)$ and not on I is that

$$\theta_{r,s} = \theta_{s-r,s} \tag{4.3}$$

for appropriate values of r, s.

For simplicity of notation we will use $(\alpha \mid \beta)$ to denote the irreducible character of W as well as the corresponding parabolic character of G.

The importance of (4.2) cannot be over-emphasized. The proof of Theorem 1.1 essentially reduces to the situation where one of the characters in 1_K^G corresponds under (4.2) to a character labeling a column of Table II. In each case, the table forces K to have so much transitivity of one sort or another that it cannot escape recognition.

The following is a simple example of (4.2).

COROLLARY 4.4. *Let* $\mathrm{Chev}(V) \leqslant G^+ \leqslant G \leqslant \Gamma(V)$, *and assume that* G *and* G^+ *have the same Weyl group. Let* B *be a Borel subgroup of* G, *and let* $B^+ = B \cap G^+$. *If* χ *is an irreducible constituent of* 1_B^G, *then* $\chi|_{G^+}$ *is an irreducible constituent of* $1_{B^+}^{G^+}$.

PROOF. Apply (4.2) to both G and G^+.

A few entries of Tables II and III can be computed using the following special case of Curtis, Iwahori and Kilmoyer [**8**, (2.2) and (9.17)].

THEOREM 4.5. (i) *The multiplicity to which the character* ζ *in* (4.2) *occurs in* 1_B^G *is the degree of the associated character* ζ_0.

(ii) *If* ρ *is the character of* G *corresponding to the reflection character* $(n-1 \mid 1)$ *of* $W(B_n)$ *or* $W(D_n)$, *then*

$$n - |I| = \langle \rho, \theta_I \rangle = \left\langle (n-1 \mid 1), 1_{W_I}^W \right\rangle.$$

As an example of the use of (4.5 ii), we will give another proof of the fact that $\langle \theta_{1n-1}, (n-1, 1 \mid 0) \rangle = 2$ for groups of type D_n (cf. §3). First, compute $\langle \theta_{1n-1}, \theta_1 \rangle = 6$ by considering the action of the stabilizer of a $(1, n-1)$-flag on points. Then note that $\theta_{1n-1} = \theta_{1nn}$ is the character of triples (V_1, V_n, V_n') with $V_n \cap V_n'$ an $n-1$-space on V_1. By (4.5 ii), $\langle \theta_{1n-1}, (n-1 \mid 1) \rangle = 3$. Since $\theta_1 = 1 + (n-1 \mid 1) + (n-1, 1 \mid 0)$, the desired result follows.

In his thesis [**26**], Hoefsmit gives formulae for the generic degrees for all irreducible representations of the generic ring corresponding to a classical group. These formulae require numerous preliminary definitions.

The *index parameters* x and y of G are defined by

$$x = |B : B \cap B^{s_1}|, \qquad y = |B : B \cap B^{s_n}|.$$

The index parameters of a classical group associated with the field $GF(q)$ are powers of q and are given explicitly in Curtis, Kantor and Seitz [**9**, §5].

The crux of our discussion concerns groups of type B_n, since groups of type D_n correspond to the situation $y = 1$ and groups of type A_{n-1} correspond to the situation $y = 0$ (Hoefsmit [**26**, (3.4.12), (3.4.15)]). It is important to note that $x > 1$ in every case.

Let α be a partition of s and let α also denote the associated Young diagram (cf. Kerber [34]). The conjugate partition $\bar{\alpha}$ is associated with the diagram obtained from α by interchanging the rows and columns of α. The (i, j)-hook of α for a node (i, j) of α is $A_{ij}^\alpha \cup \{(i, j)\} \cup L_{ij}^\alpha$ where $A_{ij}^\alpha = \{$nodes (i, t) in $\alpha \mid t > j\}$ is the hook's *arm* and $L_{ij}^\alpha = \{$nodes (t, j) in $\alpha \mid t > j\}$ is the hook's *leg*. The (i, j) *hook length* of α is

$$h_{ij}^\alpha = \left| A_{ij}^\alpha \cup L_{ij}^\alpha \cup \{(i, j)\} \right| = (\alpha_i - j) + (\bar{\alpha}_j - i) + 1.$$

Hook lengths were first introduced by Nakayama [45] in connection with modular representations of the symmetric groups. They play an important role in the intricate recursion relations that arise in the study of the symmetric groups.

Let $(\alpha; \beta)$ be a partition pair as in §3, where $\alpha \vdash s$, $\beta \vdash (n - s)$. The (i, j)-*split hook lengths* are defined by $g_{ij}^\alpha = (\alpha_i - j) + (\bar{\beta}_j - i) + 1$ for each node (i, j) of α and $g_{ij}^\beta = (\beta_i - j) + (\bar{\alpha}_j - i) + 1$ for each node (i, j) in β. (*Caution*: A split hook length may be negative; for example, if $(\alpha; \beta) = (1^2; 0)$ then $g_{2,1}^\alpha = -1$.) Following Hoefsmit [26], we define the following rational functions of x and y:

$$H_{ij}^\alpha = x^{i - \bar{\alpha}_j} \left(x^{h_{ij}^\alpha} - 1 \right) / (x - 1),$$

$$G_{ij}^\alpha = 1 + yx^{g_{ij}^\alpha},$$

$$H_{ij}^\beta = x^{i - \bar{\beta}_j} \left(x^{h_{ij}^\beta} - 1 \right) / (x - 1),$$

$$G_{ij}^\beta = x^{-\bar{\alpha}_j} \left(y^{-1} x^{g_{ij}^\beta} + 1 \right),$$

and the Poincaré polynomial

$$P(x, y) = \prod_{i=0}^{n-1} \left\{ (1 + x^i y)(x^{i+1} - 1) / (x - 1) \right\}.$$

Hoefsmit gives the following formula for the degree $d(\alpha \mid \beta)$ of the irreducible character $(\alpha \mid \beta)$ of G associated with the Weyl group character $(\alpha \mid \beta)$ by (4.2).

$$d(\alpha \mid \beta) = P(x, y) \Big/ \left(\prod_{(i,j) \in \alpha} H_{ij}^\alpha G_{ij}^\alpha \prod_{(i,j) \in \beta} H_{ij}^\beta G_{ij}^\beta \right). \tag{4.6}$$

THEOREM 4.7. *Suppose $(\alpha \mid \beta)$ is not $1 = (n \mid 0)$ and G has type A_n, B_n, C_n, or D_n, $n \geqslant 2$. Then p divides $d(\alpha \mid \beta)$ unless G of type $Sp(2n, 2)$ and $(\alpha \mid \beta) = (n - 1, 1 \mid 0)$, $(n - 1 \mid 1)$ or $(0 \mid n)$. In these cases*

$$d(n - 1, 1 \mid 0) = (1 + 2^n)(2^{n-1} - 1),$$

$$d(n - 1 \mid 1) = (1 + 2^{n-1})(2^n - 1)$$

and

$$d(0 \mid n) = (1 + 2^n)(1 + 2^{n-1})/3.$$

REMARKS. (1). The first two exceptional cases of (4.7) are the nontrivial irreducible constituents of the 2-transitive permutation representations of $Sp(2n, 2)$. While they do not yield exceptional cases for (1.1), they do appear in the characters of the imprimitive rank 3 representations of $Sp(2n, 2) = \Omega(2n + 1, 2)$ on the cosets of $\Omega^{\pm}(2n, 2)$.

(2). The case of G of type A_n in (4.7) is treated in Curtis, Kantor and Seitz [9, (5.9)] by a more elementary argument.

(3). In (4.7), the case of untwisted G of type B_n, C_n with $q > 2$, or D_n is due to Howlett [27].

PROOF. Consider the product

$$\prod_{(i,j)\in\alpha} H_{ij}^\alpha = \prod_{(i,j)\in\alpha} x^{i-\bar{\alpha}_j}\left(\frac{x^{h_{ij}^\alpha}-1}{x-1}\right) = \prod_{(i,j)\in\alpha}\left(\frac{x^{h_{ij}^\alpha}-1}{x-1}\right)\Big/x^{\Sigma(\bar{\alpha}_j-i)}.$$

Here,

$$\sum_{(i,j)\in\alpha}(\bar{\alpha}_j-i) = \sum_j\left[\sum_{i=1}^{\bar{\alpha}_j}(\bar{\alpha}_j-i)\right] = \sum_j\left[\sum_{i=1}^{\bar{\alpha}_j}i\right] = \sum_j\binom{\bar{\alpha}_j}{2},$$

and each factor of the form $(x^h-1)/(x-1)$ is congruent to 1 modulo p. It follows from (4.6) that

$$(d(\alpha\,|\,\beta))_p = x^{\Sigma(\bar{\alpha}_2)+(\bar{\beta}_2)}\left(P(x,y)\Big/\left(\prod_{(i,j)\in\alpha}G_{ij}^\alpha\prod_{(i,j)\in\beta}G_{ij}^\beta\right)\right)_p,$$

where $(Q)_p$ denotes the highest power of p dividing Q. Next, observe that

$$\prod_{(i,j)\in\alpha}G_{ij}^\alpha\prod_{(i,j)\in\beta}G_{ij}^\beta = \prod_{(i,j)\in\alpha}\left(1+yx^{g_{ij}^\alpha}\right)\prod_{(i,j)\in\beta}\left(1+y^{-1}x^{g_{ij}^\beta}\right)\Big/x^\Sigma,$$

where

$$\Sigma = \sum_{(i,j)\in\beta}\bar{\alpha}_j = \sum_j\left(\sum_{i=1}^{\bar{\beta}_j}\bar{\alpha}_j\right) = \sum_j\bar{\alpha}_j\bar{\beta}_j.$$

Since

$$\binom{\bar{\alpha}_j}{2}+\binom{\bar{\beta}_j}{2}+\bar{\alpha}_j\bar{\beta}_j = \frac{1}{2}(\bar{\alpha}_j+\bar{\beta}_j)(\bar{\alpha}_j+\bar{\beta}_j-1) = \binom{\bar{\alpha}_j+\bar{\beta}_j}{2},$$

we now have

$$(d(\alpha\,|\,\beta))_p = x^{\Sigma(\bar{\alpha}_j+\bar{\beta}_j)}\left\{\frac{P(x,y)}{\prod_{(ij)\in\alpha}\left(1+yx^{g_{ij}^\alpha}\right)\prod_{(i,j)\in\beta}\left(1+y^{-1}x^{g_{ij}^\beta}\right)}\right\}_p. \qquad (4.8)$$

(i) Suppose $p \neq 2$. Then the only way a factor in the denominator can influence $(d(\alpha\,|\,\beta))_p$ is if it has the form

$$1+p^{-k} = p^{-k}(1+p^k).$$

In this case it contributes a *positive* power of p. Therefore,

$$(d(\alpha\,|\,\beta))_p \geq x^{\Sigma(\bar{\alpha}_j+\bar{\beta}_j)}.$$

Since $x > 1$ is a power of p, $(d(\alpha\,|\,\beta))_p = 1$ only if $\bar{\alpha}_j + \bar{\beta}_j \leq 1$ for all j, that is, $(\alpha\,|\,\beta) = (n\,|\,0)$ or $(0\,|\,n)$. But if $(\alpha\,|\,\beta) = (0\,|\,n)$, then $g_{ij}^\beta = 0$ and $G_{11}^\beta = 1 + y^{-1}$.

Therefore,

$$(d(\alpha\,|\,\beta))_p = \frac{y}{(1+y)_p}\left(\frac{P(x,y)}{\prod\limits_{i=2}^{n}\left(1+y^{-1}x^{i-1}\right)}\right)_p \geqslant y.$$

This shows that $d(0\,|\,n)$ is divisible by p also, unless perhaps $y = 1$. If $y = 1$, G is of type D_n and $(0\,|\,n) = (n\,|\,0)$, by (3.5) and (4.3).

(ii) Suppose $p = 2$. Then there is the possibility that a factor in the denominator has the form $1 + 1 = 2$. Observe that g_{ij}^{α} and g_{ij}^{β} are both strictly decreasing functions of i (as $\alpha_i \geqslant \alpha_{i+1}$, $\beta_i \geqslant \beta_{i+1}$ by the definition of a partition of integers). Therefore, factors of this troublesome type occur at most $\bar\alpha_1 + \bar\beta_1$ times in (4.8). Set $x = 2^m$ and $\gamma_j = \bar\alpha_j + \bar\beta_j$. Then (4.8) yields

$$\log_2\left[(d(\alpha\,|\,\beta))_2\right] \geqslant m\left(\sum\binom{\bar\alpha_j + \bar\beta_j}{2}\right) - \bar\alpha_1 - \bar\beta_1$$

$$= \frac{m}{2}\left[\gamma_1\left(\gamma_1 - \left(1 + \frac{2}{m}\right)\right)\right] + m\sum_{j=2}^{n}\binom{\gamma_j}{2}.$$

Now γ_j is an integer so each summand is nonnegative, and the first term is nonpositive only on the interval $0 \leqslant \gamma_1 \leqslant 1 + 2/m \leqslant 3$. It follows easily that $d(\alpha\,|\,\beta)$ is divisible by p, except perhaps if
   (a) $\gamma_1 = 3$, $\gamma_2 \leqslant 1$ and $m = 1$,
   (b) $\gamma_1 = 2$, $\gamma_2 = 2$, $\gamma_3 \leqslant 1$ and $m = 1$, or
   (c) $\gamma_1 = 2$, $\gamma_2 \leqslant 1$ and $m \leqslant 2$.
   (d) $\gamma_1 = 1$.
Since $\{\gamma_i\}$ is a nonincreasing sequence of natural numbers (it is the sum of two such sequences), the corresponding pairs of partitions are:
   (a) $\{\alpha, \beta\} = \{(n-2, 1, 1); \varnothing\}$, $\{(n-2, 1); (1)\}$, $\{(n-2); (1, 1)\}$,
   (b) $\{\alpha, \beta\} = \{(n-2, 2); \varnothing\}$, $\{(n-2); (2)\}$,
   (c) $\{\alpha, \beta\} = \{(n-1); (1)\}$.
   (d) $\{\alpha, \beta\} = \{(n); \varnothing\}$.
Direct computation of (4.8) shows that 2 divides $d(\alpha\,|\,\beta)$ in each case except (c) and (d) when $m = 1$ and $x = y = 2$. These index parameters are those of $Sp(2n, 2)$ (Curtis, Kantor and Seitz [9, Tables 2 and 5]). This completes the proof of (4.7) in case G has type B_n or C_n.

If G has type A_n, then (4.6) still holds with $y = 0$ and $\beta \vdash 0$ (Hoefsmit [26, 3.4.12]), and so the above calculation also establishes (4.7) in this case.

Suppose G has type D_n. Then $y = 1$ and $(\alpha\,|\,\beta)$ is irreducible only if $\alpha \neq \beta$ (Hoefsmit [26, 3.4.15]). Since $G \geqslant \Omega^+(2n, q)$, G is of index 2 in a subgroup G^* of $\Gamma O^+(2n, q)$ of type B_n. By Clifford's theorem, the irreducible characters of G that do not extend to characters of G^* are conjugate in G^*, and hence have degree a half of the expression (4.6). Therefore, the above argument proves (4.7) for G, except perhaps for characters of the form $(\alpha\,|\,\alpha)$ when $y = 1$, $p = 2$ and the right-hand side

of (4.8) is not divisible by 4. In this case, split hook lengths are hook lengths and hence are positive, and

$$4 \nmid \left(x^{\Sigma(2\bar{q}_j)} \frac{P(x,1)}{\left[\prod_{(i,j)\in\alpha} (1+x^{g_{ij}^\alpha}) \right]^2} \right)_2 = \frac{2x^{\Sigma(2\bar{q}_j)}}{\left[\prod_{(i,j)\in\alpha} (1+x^{h_{ij}^\alpha}) \right]^2_2} \geqslant 2x^n,$$

contrary to the hypothesis that $n \geqslant 2$. This completes the proof of (4.7).

5. Transitivity in small dimensions. In this long section, we will consider classical groups of low rank: those of rank 2 or 3, as well as $\Gamma O^+(8, q)$. The goals are a number of results concerning subgroups having specific transitivity properties. Most of these are closely related to (1.1) or (11.1). However, some are designed for use in §6, where arbitrary dimensional situations are reduced, in part, to small dimensional ones.

There are two approaches to the required transitivity results. One is geometric, ad hoc, tedious, and very long, but reasonably elementary. The other uses classification theorems [14], [43], [44], [61], [64]–[66] which depend upon deep results concerning the classification of finite simple groups. We have chosen the latter approach: it requires much less space, and at the same time yields somewhat stronger results than are essential for our main theorems.

This section is divided into four subsections: (A) invokes the aforementioned classification theorems in order to deal with subgroups of $\Gamma L(4, q)$ and $\Gamma L(5, q)$; (B) uses (A) in order to discuss subgroups of rank 2 classical groups; (C) deals with subgroups of rank 3 classical groups; and (D) concerns $\Gamma O^+(8, q)$.

Recall that $H^{(\infty)}$ denotes the last term of the derived series of H. Throughout this section $K^{(\infty)}$ (or $H^{(\infty)}$) *will be uniquely determined up to conjugacy in* $\Gamma(V)$, *unless otherwise stated.* Since $K^{(\infty)}$ is usually irreducible on V, $K/Z(K)K^{(\infty)}$ is a group of outer automorphisms of $K^{(\infty)}$, and hence can be determined at a glance.

(A) *Subgroups of* $\Gamma L(4, q)$ *and* $\Gamma L(5, q)$. Throughout this subsection, let $H \leqslant \Gamma L(W) = \Gamma L(4, q)$ or $\Gamma L(5, q)$. The possibilities for H have been more or less determined by Mwene [43], [44], Wagner [61], DiMartino and Wagner [14], Zalesskiĭ [64], [65] and Zalesskiĭ and Suprunenko [66].

THEOREM 5.1. *Let H be a primitive subgroup of $\Gamma L(4, q)$. Then one of the following holds.*

(a) $H \geqslant SL(4, q)$.

(b) $H \leqslant \Gamma L(4, q')$ *with $GF(q') \subset GF(q)$.*

(c) $H \leqslant \Gamma L(2, q^2)$, *with the latter group embedded naturally.*

(d) $H \leqslant Z(H)H_1$, *where H_1 is an extension of a special group of order 2^6 by S_5 or S_6. Here, q is odd, H_1 induces a monomial subgroup of $O^+(6, q)$, and H_1 is uniquely determined up to $\Gamma L(4, q)$-conjugacy.*

(e) $H^{(\infty)}$ *is $Sp(4, q)'$ or $SU(4, q^{1/2})$.*

(f) $H \leqslant \Gamma O^{\cdot}(4, q)$.

(g) $H^{(\infty)}$ *is $PSL(2, q)$ or $SL(2, q)$ (many classes).*

(h) $H^{(\infty)}$ is A_5. Here, H arises from the natural permutation representation of S_5 in $O(5, q)$, and $p \neq 2, 5$.

(i) $H^{(\infty)}$ is $2 \cdot A_5$, $2 \cdot A_6$ or $2 \cdot A_7$. These arise from the natural permutation representation of S_7 in $O(7, q)$.

(j) $H^{(\infty)} = A_7$, and $p = 2$.

(k) $H^{(\infty)} = Sp(4, 3)$, and $q \equiv 1 \pmod 3$. This arises from the natural representation of the Weyl group $W(E_6)$ in $O^+(6, q)$.

(l) $H^{(\infty)} = SL(2, 7)$, and $q^3 \equiv 1 \pmod 7$. Here, $H^{(\infty)}$ lies in the group $2 \cdot A_7$ occurring in (i).

(m) $H^{(\infty)} = 4 \cdot PSL(3, 4)$, and q is a power of 9.

(n) $H^{(\infty)} = Sz(q)$, and $p = 2$.

PROOF. If q is even, see [43], [65]. (It is necessary to sift through the various cases examined in these references in the course of proving their main theorems.)

If q is odd, we will use [44]. (Note that [66] assumes that $p > 5$.)

By primitivity, if H has a noncentral abelian normal subgroup then (c) holds. Suppose that (c) does not hold, but that H has a noncentral normal subgroup E of prime power order. Since $\dim V = 4$, E is a 2-group. We may assume that $E/Z(E)$ is elementary abelian, and then (d) follows readily.

Let $\overline{M} = M/Z(H)$ be a minimal normal subgroup of $\overline{H} = H/Z(H)$. We may assume that $\overline{M}$ is a direct product $\overline{S}_1 \times \cdots \times \overline{S}_k$ of isomorphic nonabelian simple groups $\overline{S}_i$, where $\overline{S}_i = S_i/Z(H)$.

If S_1 is reducible, then (by Clifford's theorem) M preserves a decomposition $W = T_1 \oplus T_2$, where T_1 and T_2 are S_1-isomorphic 2-spaces. Then S_1 fixes a regulus Σ of $q + 1$ lines, which are permuted by $N_H(S_1)$. Thus, $N_H(S_1) \leqslant \Gamma L(4, q)_\Sigma = \Gamma O^+(4, q)$. If $k = 1$, then (f) holds. If $k > 1$, then $S_1 \times S_2 \leqslant \Omega^+(4, q)Z(H)$, from which (f) again follows.

Finally, suppose that S_1 is irreducible. Then $M = S_1$, and $K^{(\infty)}$ is quasisimple. This is precisely the situation studied by Mwene [44]. His theorem, and his study of the modular representations of covering groups of S_1, complete the proof.

For further information concerning (5.1m), see Finkelstein [16], McLaughlin [38] and Mwene [44].

THEOREM 5.2. Let H be a primitive subgroup of $\Gamma L(5, q)$. Then one of the following holds.

(a) $H \geqslant SL(5, q)$.

(b) $H \leqslant \Gamma L(5, q')$ with $GF(q') \subset GF(q)$.

(c) $H \leqslant \Gamma L(1, q^5)$.

(d) $H \leqslant Z(H)H_1$, where H_1 is an extension of an extraspecial group of order 5^3 by $SL(2, 5)$. Here, H_1 is uniquely determined up to $\Gamma L(5, q)$-conjugacy, and $p \neq 5$.

(e) $H^{(\infty)} = \Omega(5, q)$ or $SU(5, q^{1/2})$.

(f) $H^{(\infty)}$ is $PSL(2, q)$, in the representation afforded by homogeneous polynomials of degree 4 in 2 variables.

(g) $H^{(\infty)} = \Omega(5, 3)$, and $q \equiv 1 \pmod 6$.

(h) $H^{(\infty)}$ is A_5, A_6 or A_7, and is contained in $\Omega(5, q)$. Here, $p = 7$ if $H^{(\infty)} = A_7$. These cases arise from the permutation module for $PGL(2, 5)$, S_6 or S_7.

(i) $H^{(\infty)} = PSL(2, 11)$, $p \neq 11$. This arises from the irreducible complex representation of $PSL(2, 11)$ of degree $\frac{1}{2}(11 - 1)$.

(j) $H^{(\infty)} = M_{11}$, $p = 3$ (two classes).

PROOF. See [61], [65] for q even and [14] for q odd. (Note that [66] assumes that $p > 5$.)

REMARK. The precise fields over which the groups listed in (5.1, 2) can be realized are discussed at length in [14], [44], [61].

Various transitivity and related results are simple consequences of (5.1, 2).

COROLLARY 5.3. *Let H be a subgroup of $\Gamma L(4, q)$ of order divisible by a primitive divisor r of $q^2 + 1$. Then one of the following holds.*

(i) $H \geqslant SL(4, q)$.

(ii) $H \leqslant \Gamma L(2, q^2)$.

(iii) $H^{(\infty)} = Sp(4, q)'$.

(iv) $H \leqslant \Gamma O^-(4, q)$.

(v) $H = A_7 < SL(4, 2)$.

(vi) $H^{(\infty)} = Sz(q)$.

(vii) $r = 5$, *and one of* (5.1 d, h, i, j, k, l, *or* m) *holds.*

PROOF. Clearly, H is primitive, so we only need to sift through the list in (5.1). Note that $r \geqslant 5$ and $r \neq 7$. If $r = 5$ then (vii) can occur. All remaining cases of (5.1) are easily checked.

REMARK 5.4. We will use primitive divisors together with (5.1, 2) in other situations. It will always be as straightforward to check the lists as it was in (5.3). Another example is provided by the next result.

COROLLARY 5.5. *If H is a subgroup of $\Gamma L(n, q)$, $n = 4$ or 5, and if H is transitive on points, then one of the following holds.*

(a) $H \geqslant SL(n, q)$.

(b) $H \leqslant \Gamma L(1, q^n)$.

(c) $H \leqslant \Gamma L(2, q^2) < \Gamma L(4, q)$.

(d) $H^{(\infty)} = Sp(4, q)' < \Gamma L(4, q)$.

(e) $H = A_7$, *inside* $SL(4, 2)$.

(f) H *has a normal extraspecial subgroup of order 2^5, and $5 \mid\mid H\mid$, where* $H < GL(4, 3)$.

PROOF. If $n = 5$, then $\mid H \mid$ is divisible by a primitive divisor r of $q^5 - 1$. Here, $r > 11$, and it is straightforward to use (5.2).

If $n = 4$, then (5.3) applies. The possibilities listed in (5.3 vi) are easily checked.

(B) *Subgroups of rank 2 classical groups.* The results in (A) can clearly be used to study $\Gamma Sp(4, q)$, $\Gamma U(4, q)$, $\Gamma U(5, q)$, $\Gamma O(5, q)$ and $\Gamma O^-(6, q)$. We only need to consider the first three of these. $\Gamma Sp(4, q)$ was handled long ago by Mitchell [41], when q is odd. (Partial results for q even were obtained by Flesner [18]; he essentially only dealt with the case of subgroups containing transvections.)

THEOREM 5.6. *Let* $K \leqslant \Gamma Sp(4, q)$. *Then one of the following holds.*

(i) $K^{(\infty)} = Sp(4, q')'$ *with* $GF(q') \subseteq GF(q)$.

(ii) *K fixes a point, a line, a pair of skew lines, or a pair of perpendicular hyperbolic lines.*

(iii) $K \cap Sp(4, q) \leqslant Sp(2, q^2) \cdot 2$.

(iv) $K^{(\infty)}$ *is* $SL(2, q')$, *in its representation afforded by homogeneous polynomials of degree* 3 *in* 2 *variables, where* $GF(q') \subseteq GF(q)$ *and* $p > 3$.

(v) $K \leqslant \Gamma O^{\pm}(4, q)$, *where* $p = 2$.

(vi) $K^{(\infty)} = Sz(q)$, *where* $q > p = 2$.

(vii) *K induces a subgroup of* $\Gamma O(5, q)$ *fixing an anisotropic 2-space.*

(viii) *K fixes a quintuple of pairwise orthogonal nonsingular points when viewed inside* $\Gamma O(5, q)$, *and q is odd.*

(ix) $K^{(\infty)}$ *is* $2 \cdot A_5$ *or* $2 \cdot A_6$, *in the representation obtained from the* mod *p permutation representation of* A_6 *in* $\Omega^-(6, q)$.

(x) $K^{(\infty)}$ *is* $2 \cdot A_7$, *in the representation obtained from the* mod 7 *permutation representation of* S_7 *in* $O(7, q)$.

HISTORICAL REMARK. The first $q > 2$ for which this was proved was $q = 3$. Dickson [11] listed all 70 factors of 25920, and then sifted through them, primarily using Sylow's theorem and lists of permutation groups of small degree.

THEOREM 5.7. *Let* $K \leqslant \Gamma U(4, q)$. *Then one of the following holds.*

(i) $K^{(\infty)} = SU(4, q')$ *with* $GF(q') \subseteq GF(q)$.

(ii) *K fixes a point or line, a nonsingular point or line, a pair of skew lines, a pair of perpendicular nonsingular lines, or a quadruple of pairwise perpendicular nonsingular points.*

(iii) *K induces a subgroup of* $P\Gamma O^-(6, q)$ *fixing a nonsingular point or line, a pair of perpendicular nonsingular planes, or (when q is odd) a sextuple of pairwise perpendicular nonsingular points.*

(iv) $K^{(\infty)}$ *is* $2 \cdot A_7$ *or* $SL(2, 7)$. *Here,* $K^{(\infty)}$ *arises from the permutation representation of* S_7 *in* $O(7, q)$.

(v) $K^{(\infty)} = Sp(4, 3)$, *and* $q^3 \equiv 1 \pmod 7$. *Here,* $K^{(\infty)}$ *arises from the natural representation of* $W(E_6)$.

(vi) $K^{(\infty)} = 4 \cdot PSL(3, 4)$, *and q is a power of* 3.

THEOREM 5.8. *Let* $K \leqslant \Gamma U(5, q)$. *Then one of the following holds.*

(i) $K^{(\infty)} = SU(5, q')$ *with* $GF(q') \subseteq GF(q)$.

(ii) *K fixes a point or line, a nonsingular point or line, or a quintuple of pairwise perpendicular nonsingular points.*

(iii) $K \leqslant Z(K)\Gamma O(5, q)$, *where the* $O(5, q)$ *is embedded naturally.*

(iv) $K \leqslant \Gamma U(1, q^5)$.

(v) $K \leqslant Z(K)K_1$, *where* K_1 *is an extension of an extraspcial group of order* 5^3 *by* $SL(2, 5)$, *and* $p \neq 5$.

(vi) $H^{(\infty)} = \Omega(5, 3)$, *and* $p > 3$.

(vii) $H^{(\infty)}$ *is* A_5, A_6, *or* A_7, *and is contained in* $\Omega(5, q)$. *Moreover,* $p = 7$ *if* $H^{(\infty)} = A_7$.

(viii) $H^{(\infty)} = PSL(2, 11)$, *where* $p \neq 11$.

Theorems 5.6–5.8 are almost immediate consequences of the following elementary result.

LEMMA 5.9. *Let H be an absolutely irreducible subgroup of* $GL(W) = GL(n, q)$. *Let* $(\, , \,)_1$ *and* $(\, , \,)_2$ *be two nonsingular hermitian forms on W* (*or two nonsingular alternating forms on W*) *preserved by H. Then there is a scalar c such that* $(\, , \,)_1 = c(\, , \,)_2$.

PROOF. View H as a group of matrices, and $(\, , \,)_i$ as arising from a matrix A_i. Consider the unitary case. Here, $A_i^{-1} h A_i = (\bar{h}^{-1})^t$ for all $h \in H$. Then $A_2 A_1^{-1}$ centralizes H, and hence is a scalar by Schur's Lemma.

PROOF OF (5.6)–(5.8). We may assume that K is primitive. Suppose first that $K_0 = K \cap SL(V)$ is absolutely irreducible. By (5.9), if a group occurring in (5.1) or (5.2) can arise in (5.6), (5.7) or (5.8), then all occurrences are conjugate. The groups listed in the latter theorems all occur, for suitable q [**44**], [**14**], [**60**]. Note that neither (5.1j) nor (5.2j) can occur: in each case, there is a 1-dimensional subspace x such that $K_x^{(\infty)}$ does not fix any hyperplane.

If K_0 is not absolutely irreducible, then one of (5.1 c, f) or (5.2 c) holds. Suppose that K lies in both $\Gamma Sp(4, q)$ and $\Gamma L(1, q^4)$. Then K_0 has an irreducible normal cyclic r-subgroup for some prime r (cf. (2.18)) By Sylow's theorem, $K \leqslant \Gamma Sp(2, q^2)$, with the latter group embedded naturally. The same argument applies if K is in $\Gamma U(5, q) \cap \Gamma L(1, q^{10})$, this time producing (5.8 iv). However, the case $\Gamma U(4, q) \cap \Gamma L(1, q^8)$ cannot occur (acting irreducibly), since $|SU(4, q)|$ is not divisible by the required prime r.

This leaves us with the possibility that $\dim V = 4$ and (5.1 f) holds (with q replaced by q^2 in the unitary case). Then K has a normal subgroup S acting irreducibly on some 2-dimensional subspace T. By primitivity, T has at least 3 images under K, all of which must be S-isomorphic.

If T is a line, then (5.6 vii) or (5.7 iii) holds. (For, S fixes at least 3 points of the corresponding 5- or 6-dimensional orthogonal space.)

If T is nonsingular, then $T^\perp$ is also S-invariant. Moreover, T and $T^\perp$ are S-isomorphic. This uniquely determines S up to conjugacy. Then S fixes a nonsingular plane of the corresponding $O(5, q)$ or $O^-(6, q)$ space. Consequently, (5.6 vii) or (5.7 iii) holds for K. This completes the proof of (5.6)–(5.8).

The remainder of this subsection parallels (5 A): primitive divisors will be used in order to deduce transitivity properties.

LEMMA 5.10. *Let* $K \leqslant \Gamma U(4, q)$, *and assume that* $|K|$ *is divisible by a primitive divisor r of* $q^2 + 1$. *Then one of the following holds.*

(i) *K has a normal Sylow r-subgroup, and fixes a hyperbolic line when viewed inside* $\Gamma O^-(6, q)$.

(ii) *K fixes a line.*

(iii) $K^{(\infty)} = SU(4, q)$.

(iv) $K^{(\infty)} = Sp(4, q)'$.

(v) $K^{(\infty)} = Sp(2, q^2)$.

(vi) $K^{(\infty)} = \Omega^-(4, q)$.

(vii) $K^{(\infty)} = Sz(q)$.

(viii) $K^{(\infty)} = 4 \cdot PSL(3,4)$, and q is a power of 3.

(ix) $r = 5$, and either $K^{(\infty)}$ is A_5 (with q odd), $2 \cdot A_5$, $2 \cdot A_6$, $2 \cdot A_7$ or $Sp(4,3)$, or $K \cap SU(4,q)$ induces a monomial group inside $O^-(6,q)$.

PROOF. Check the lists in (5.6) and (5.7) (using Dickson [**12**, Chapter 12] in order to deal with subgroups in the cases (5.6 iii, v)).

LEMMA 5.11. *If* $K \leqslant \Gamma U(5,q)$, *and if* $|K/K \cap Z(\Gamma U(5,q))|$ *is divisible by both* $q^2 + 1$ *and* $q^3 + 1$, *then either*

(i) $K \geqslant SU(5,q)$,

(ii) $K \trianglerighteq Z_3^4 \rtimes A_5$, *acting monomially inside* $\Gamma U(5,2)$,

(iii) $K^{(\infty)} = SU(4,q)$,

(iv) $K^{(\infty)}$ *is* A_5, A_6 *or* $E \rtimes A_5$, *with* E *extraspecial of order* 32, *where* $K < \Gamma U(5,2)$ *and* K *fixes a nonsingular point, or*

(v) $K^{(\infty)} = 4 \cdot PSL(3,4)$, *inside* $\Gamma U(5,3)$.

PROOF. Assume that (i) does not hold. If $q^3 + 1$ has a primitive divisor, then K fixes a nonsingular point by (5.8), and hence (5.10) applies. If no primitive divisor exists, then $q = 2$ by (2.18). Thus, $|K|$ is divisible by 45. If (ii) does not hold, then K fixes a nonsingular point by (5.8), and hence $K \leqslant (Z_3 \times GU(4,2)) \cdot 2$. The subgroups of $\Gamma U(4,2)$ of order divisible by 15 give rise to the examples in (iv).

COROLLARY 5.12. *Let* $K \leqslant \Gamma Sp(4,q)$, $\Gamma U(4,q)$ *or* $\Gamma U(5,q)$. *If* K *is transitive on points or lines, then one of the following holds.*

(i) $K^{(\infty)} = \mathrm{Chev}(V)'$.

(ii) $K^{(\infty)} = Sp(2,q^2)$, *inside* $\Gamma Sp(4,q)$.

(iii) $K^{(\infty)} = SL(2,5) < Sp(2,3^2)$, *inside* $\Gamma Sp(4,3)$.

(iv) $K^{(\infty)} = \Omega^-(4,q)$, *inside* $\Gamma Sp(4,q)$ *with* q *even.*

(v) $K^{(\infty)} = SU(3,q)$, *inside* $\Gamma U(4,q)$.

(vi) $K \trianglerighteq SU(3,2)''$, *inside* $\Gamma U(4,2)$ (*and* K *fixes a nonsingular point*).

(vii) K *has a normal monomial subgroup* $Z_3^3 \rtimes A_4$, *and* $K < \Gamma U(4,2)$.

(viii) $K^{(\infty)} = PSL(3,2)$, *inside* $\Gamma U(4,3)$ (*and* K *fixes a nonsingular point*).

(ix) $K^{(\infty)} = 3 \cdot A_7$, *inside* $\Gamma U(4,5)$ (*and* K *fixes a nonsingular point*).

(x) $K \trianglerighteq E \rtimes Z_5$, *where* E *is extraspecial of order* 2^5, $E \leqslant O_2(K)$, $K < \Gamma Sp(4,3)$, *and* K *acts monomially within* $\Gamma O(5,3)$.

(xi) $K^{(\infty)} = 4 \cdot PSL(3,4)$, *inside* $\Gamma U(4,3)$.

PROOF. If $K \leqslant \Gamma Sp(4,q)$, then $(q+1)(q^2+1) \mid |K|$. Thus, (5.10) applies by (2.18). All possibilities are easily checked (although some computation is required in the monomial case of (5.10 ix)).

Let $K \leqslant \Gamma U(4,q)$. If K is point-transitive, then $(q^2+1)(q^3+1) \mid |K|$, and (5.10) again applies. If K is line-transitive, then $(q+1)(q^3+1) \mid |K|$. As in the proof of (5.11), if $q > 2$ then a primitive divisor of $q^3 + 1$ can be used: if K is irreducible then (i) or (xi) holds. For any q, (2.8) can be applied in case K fixes a nonsingular point. Finally, if $q = 2$ then $PSU(4,2) \cong PSp(4,3)$ and $(q+1)(q^3+1) = 27$, so that (5.6) and our transitivity lead to (vi) or (vii).

If $K \leqslant \Gamma U(5, q)$, then $q^5 + 1 \mid |K|$. By (2.18) and (5.8), only (i) can occur.

DEFINITION. K is an (a, b, c)-*group* if it has a point-orbits, b line-orbits and c flag-orbits. The next two results contain complete determinations of all (a, b, c)-groups for some triples (a, b, c) of special importance later.

COROLLARY 5.13. *Let $K \leqslant \Gamma Sp(4, q)$, $\Gamma U(4, q)$ or $\Gamma U(5, q)$, and let (a, b, c) be one of the triples $(1, 2, 2)$, $(1, 2, 3)$, $(2, 1, 2)$ or $(2, 1, 3)$. If K is an (a, b, c)-group, then one of the following holds.*

$(1, 2, 2)_\mathrm{i}$ $K^{(\infty)} = Sp(2, q^2)$, *inside $\Gamma Sp(4, q)$.*

$(1, 2, 2)_\mathrm{ii}$ $K/Z(K) = Z_2^4 \rtimes D$, *where D is dihedral of order 10, $K < \Gamma Sp(4, 3)$, and K acts monomially within $\Gamma O(5, 3)$ with respect to an orthonormal basis.*

$(1, 2, 2)_\mathrm{iii}$ $K/\langle -1 \rangle = S_5$, $A_5 \times Z_2$ *or* $S_5 \times Z_2$, $K < \Gamma Sp(4, 3)$, *and K acts monomially within $\Gamma O(5, 3)$ with respect to an orthonormal basis.*

$(1, 2, 2)_\mathrm{iv}$ $K^{(\infty)} = 4 \cdot PSL(3, 4)$, *inside $\Gamma U(4, 3)$.*

$(1, 2, 3)$ *There are no $(1, 2, 3)$-groups.*

$(2, 1, 2)_\mathrm{i}$ $K^{(\infty)} = \Omega^-(4, q)$, *with q even, inside $\Gamma Sp(4, q)$.*

$(2, 1, 2)_\mathrm{ii}$ $K^{(\infty)} = SU(3, q)$, *inside $\Gamma U(4, q)$.*

$(2, 1, 2)_\mathrm{iii}$ $K < \Gamma U(4, 2)$, *and either $K \trianglerighteq Z_3^3 \rtimes A_4$ with the latter group monomial with respect to an orthonormal basis, or K fixes a nonsingular point and $K \trianglerighteq SU(3, 2)'' \rtimes Z_8$.*

$(2, 1, 3)$ *There are no $(2, 1, 3)$-groups.*

PROOF. This is a consequence of (5.12) and some straightforward computations.

LEMMA 5.14. *Let $K \leqslant \Gamma Sp(4, q)$, $\Gamma U(4, q)$ or $\Gamma U(5, q)$. If K is a $(2, 2, 3)$-group, then one of the following holds.*

(i) $K^{(\infty)} = Sp(4, q')$, *inside $\Gamma U(4, q)$.*

(ii) $K^{(\infty)} = SL(2, q) \times SL(2, q)$, K *fixes a pair of perpendicular hyperbolic lines, and $K < \Gamma Sp(4, q)$.*

(iii) $K \trianglerighteq \Omega^+(4, q)$, *inside $\Gamma Sp(4, q)$ with q even.*

(iv) $K < Sp(4, 2)$, K *behaves as in* (ii), *and $18 \mid |K|$.*

(v) $K < \Gamma Sp(4, 3)$, K *behaves as in* (ii), *and $K \trianglerighteq SL(2, 3)' \times SL(2, 3)'$.*

(vi) $K/Z(K) \trianglerighteq Z_2^4 \rtimes A_4$, *where $K < \Gamma Sp(4, 3)$ and K acts monomially inside $\Gamma O(5, 3)$ with respect to an orthonormal basis.*

(vii) $K/Z(K) \trianglerighteq Z_2^4 \rtimes PSL(2, 5)$ *or* $Z_2^4 \rtimes A_6$, *where $K < \Gamma U(4, 3)$ and K acts monomially inside $\Gamma O^-(6, 3)$ with respect to an orthonormal basis.*

(viii) $K^{(\infty)} = Z_3^4 \rtimes A_5$, *where the latter group acts monomially inside $\Gamma U(5, 2)$ with respect to an orthonormal basis.*

(ix) $K^{(\infty)} = SU(4, q)$, *inside $\Gamma U(5, q)$.*

PROOF. There is a point x and a line L such that both $K_x^{x^\perp / x}$ and K_L^L are transitive on points.

Suppose first that $K \leqslant \Gamma U(5, q)$, but that (viii) does not hold. Then (5.11) applies, so that K fixes a nonsingular point b. Clearly, K must be transitive on the points of $b^\perp$, as well as on the $q^3(q^4 - 1)$ points not in $b^\perp$. Thus, only (5.11 ii or iii) can occur.

Next, suppose that $K \leqslant \Gamma U(4, q)$. This time, $|K|$ is divisible by both $q + 1$ and $q^2 + 1$, and (5.10) applies. (5.10 i–vii) are easily handled. Consider (5.10 ix). The first four cases lie in $\Gamma Sp(4, q)$, but are not transitive on points there (cf. (5.12)). If $K^{(\infty)} = Sp(4, 3)$ then $q \geqslant 7$, and hence $q^2 + 1 \nmid |K|$.

Finally, suppose that $K \leqslant \Gamma Sp(4, q)$. Then $q + 1 \mid |K|$. Consider the possibilities listed in (5.6). We may assume that K is primitive. This takes care of (5.6 ii), and (5.6 iii) is easily eliminated. In (5.6 iv, vii and ix) there are more than 2 point-orbits, while in (5.6 vi) there are 4 flag-orbits.

If $K \trianglerighteq \Omega^+(4, q)$, and if K has an element interchanging the two normal subgroups $SL(2, q)$ of $\Omega^+(4, q)$, then K has the desired orbit structure. Similarly, this and (v) are the only possibilities arising from (5.6 v).

Finally, in (5.6 viii) it is again straightforward to check the orbit structure.

(C) *Rank $n = 3$.* We now turn to spaces V of rank $n = 3$. The only interesting result is (5.19), which characterizes both the groups $G_2(q)$ and (for $q = 2$ or 8) the groups $\Gamma L(2, q^3)$.

The case of V of type $\Omega^+(6, q)$ is especially easy, and will usually be ignored in view of the following result.

LEMMA 5.15. *A point-transitive subgroup of $\Gamma O^+(6, q)$ either contains $\Omega^+(6, q)$ or is A_7 or S_7 inside $O^+(6, 2)$.*

PROOF. (5.1) (but compare (10.2)).

REMARK. The next two results concern groups K which are transitive on points and planes. The numbers of points, lines and planes are given in §6, Table IV. By (2.12), if V is not unitary then K is $(1, 3)$-transitive.

LEMMA 5.16. *If $n = 3$, then $\Gamma(V)$ has no subgroup transitive on points and planes and having 2 orbits on lines and $(1, 3)$-flags, and 3 orbits on $(1, 2)$-flags.*

PROOF. Let K be such a subgroup. As just noted, V must be unitary. By hypothesis, there is a line L such that K_L^L is transitive. An element of K_L of order a primitive divisor of $q^2 + 1$ (cf. (2.18)) will fix each plane $E \supset L$. On the other hand, $q^2 + q + 1 \mid |K_E^E|$ by (2.12). By Mitchell [40] and Hartley [22], $K_E^E \geqslant SL(3, q^2)$, whereas K is not $(1, 3)$-transitive.

THEOREM 5.17. *If $n = 3$, and if K is a subgroup of $\Gamma(V)$ which is transitive on points, lines and planes, then $K \geqslant \operatorname{Chev}(V)$ (or K is A_7 or S_7 inside $O^+(6, 2)$).*

PROOF. By (2.6), we may assume that K is not flag-transitive. Also, we may assume that $K = O^{p'}(K)$. Let E be a plane and x a point of E. Set $K^* = K_x^{x^\perp/x}$. We will consider the various possibilities for V separately.

Case $K < \Gamma Sp(6, q)$ or $\Gamma O(7, q)$. Since K^* is line-transitive but not flag-transitive, (5.12) can be applied: $K < \Gamma O(7, q)$, and either K^* fixes a nonsingular 4-space F or $q = 3$ and $K^* \trianglerighteq Z_2^4 \rtimes Z_5$.

Also, K_E^E is point-transitive but not flag-transitive. By (2.7), if $|K_{xE}^{E/x}|$ is divisible by q then $q = 3$ and q^2 cannot divide this order. Thus, if F exists then $K^{*F} \ntrianglerighteq \Omega^-(4, q)$.

Consequently, $q = 3$, and either $K^{*(\infty)} = A_5$ or $K^* \trianglerighteq Z_2^4 \rtimes Z_5$. If $K^* \trianglerighteq Z_2^4 \rtimes Z_5$, there is an involution $t \in C_K(x)$ such that $[V, t]$ has type $\Omega^+(4, 3)$; then t fixes a plane and acts nontrivially on it, and this contradicts the known behavior of K_E^E. Thus, $K^{*(\infty)} = A_5$. Let $A < K_x$, where $|A^{x^\perp/x}| = 4$, $A < K^{(\infty)}$, and A is a 2-group. Then $3 \mid |N_K(A)_x|$, and A centralizes x. Let $T \in Syl_3 N_K(A)_x$. Then AT centralizes nonsingular 2-spaces of both V and $x^\perp/x$. Consequently, T centralizes $y^\perp/y$ for some point y. There is a plane E' on y not centralized by T. Then $T^{E'}$ contains a transvection, and we again obtain a contradiction.

Case $K < \Gamma O^-(8, q)$. Since K^* is line-transitive but not flag-transitive, it cannot be point-transitive by (5.12).

Let r be a primitive divisor of $q^2 + 1$ (cf. (2.18)). Let $R \in Syl_r K$, and set $W = C_V(R)$ and $N = N_K(R)$. Then W has type $\Omega^+(4, q)$, and N^W is point-transitive by Sylow's theorem. Thus, N_L^L is point-transitive for each line $L \subset W$. Now K is $(1, 2)$-transitive, and hence K^* is point-transitive.

Case $K < \Gamma U(6, q)$. By (2.12), each line-orbit of K_E^E has length divisible by $(q^6 - 1)/(q^2 - 1)d$ (where $d = (q + 1, 3)$), while each line-orbit of K^* has length divisible by $q^3 + 1$. Exclude for the moment the possibility that $q = 2$ and K^* is monomial but irreducible. By (5.7) and (2.8), K^* fixes a nonsingular point, and $K^* \trianglerighteq SU(3, q)$, $3 \cdot A_7$, $PSL(3, 2)$ or $SU(3, 2)''$. In the first two cases, q^2 or 5 divides $|K_{xE}^E|$, and we find that K_E^E is flag-transitive.

Assume that $K^{*(\infty)} = PSL(3, 2)$. By (2.14 iii), K_x has a subgroup $S = PSL(3, 2)$ such that $\dim C_V(S) = 3$. An element of S of order 3 fixes a point $y \in [V, S]$ and induces a transvection on $y^\perp/y$. This contradicts the known action of K^*.

This leaves us with the case $q = 2$. Note that $K = O^{2'}(K) \leqslant GU(6, 2)$. Let $R \in Syl_7 K$. Then R fixes exactly 2 planes. Since $|C_{GU(6,2)}(R)|$ is odd, there is an involution $j \in N_K(R)$ interchanging these planes. If v is a vector in one of these planes, then $(v + v^j, v + v^j) = 0$. Thus, $C_V(j)$ is a plane, which we may assume is E. Set $P = C_K(E)$. In view of the known action of K^*, $P^{x^\perp/x}$ lies inside of a dihedral group of order 8. On the other hand, a Sylow 7-subgroup R_1 of K_E acts fixed-point freely on P. If $|P| = 8$, then all nontrivial elements of P are R_1-conjugate to j, and hence P acts faithfully on $x^\perp/x$. Thus, we must have $|P| > 8$, and hence $|P| \geqslant 64$. If $x \neq y \in x^{R_1}$, it follows that $C_P(x^\perp/x) \cap C_P(y^\perp/y)$ contains an involution i. Thus, $\dim C_V(i) \geqslant 5$, which is clearly impossible.

Case $K < \Gamma U(7, q)$. By (2.12), each point-orbit of K^* has length divisible by $q^5 + 1$. By (5.8), $K^* \leqslant \Gamma U(1, q^5)$. The latter group has order $(q^5 + 1)10e$, where $q = p^e$, and it contains all scalars in $\Gamma U(5, q)$. Thus, $q^5 + 1$ must be a divisor of $(q^5 + 1)10e/(q + 1)$. This is only possible if q is 9 or 4. However, $\Gamma U(1, 9^5)$ has a point-orbit of length $(9^5 + 1)/2$. Thus, $q = 4$. Since $K = O^{2'}(K)$, $|K^*| = (4^5 + 1)5$. By (2.1), $|K_x|$ is odd. If $R \in Syl_7 K$, then R fixes 2 planes, and hence $|N_K(R)|$ is even; but then so is $|K_x|$. This contradiction completes the proof of (5.17).

LEMMA 5.18. *If* $n = 3$, *then* $\Gamma(V)$ *has no subgroup which is* $(1, 2)$-*transitive and has* 2 *orbits of planes,* $(1, 3)$-*flags and flags.*

PROOF. Let K be such a subgroup. Let E be a plane, let $x \in E$, and consider both K_E^E and $K^* = K_x^{x^\perp/x}$. Since K_E^E is flag-transitive, it contains $SL(E)$, except perhaps

if $q = 2$ or 8 (by (2.7)). On the other hand, K^* is a $(1, 2, 2)$-group, so that (5.13) applies. Use of $K_{xE}^{E/x}$ eliminates $(1, 2, 2)_{ii}$. In every other case in (5.13), there is a plane E such that $p \mid \mid (K_E^*)^{E/x} \mid$, and then $K_E^E \geqslant SL(E)$.

Thus, $K_{xE}^E \geqslant A \rtimes SL(2, q)$, where $\mid A \mid = q^2$. Since K^* is irreducible, (2.1) shows that K^* must also contain a section $A \rtimes SL(2, q)$. By (5.13), either $K^* \unrhd Sp(2, 2^2)$ or $K^* \unrhd Z_3^3 \rtimes A_4$, where $K < \Gamma Sp(6, 2)$ or $\Gamma O^-(8, 2)$, respectively. In either case, the existence of an S_4 section in K^* forces $\mid (K_E^*)^{E/x} \mid$ to be even for every plane E on x. But then K_E^* has an S_4 section for every E (as above), and this is not the case.

THEOREM 5.19. *Let $n = 3$, and let K be a point-transitive subgroup of $\Gamma(V)$ having 2 orbits of lines, planes and $(1, 2)$-flags, 3 of $(1, 3)$- and $(2, 3)$-flags, and 4 of flags. Then either*

(i) *$K \unrhd G_2(q)'$, embedded naturally in $\Gamma O(7, q)$ or (if q is even) in $\Gamma Sp(6, q)$; or*

(ii) *$K \unrhd \Gamma Sp(2, q^3)$, $q = 2$ or 8, embedded naturally in $\Gamma Sp(6, q)$ (or $\Gamma O(7, q)$).*

PROOF. We will proceed in several steps.

(I) There are planes E and F such that K_E^E is point-transitive while K_F^F has 2 point-orbits Ω, Ω' and 2 line-orbits. Each line meeting both Ω and Ω' produces 2 flag-orbits. Hence, K_F^F has 3 flag-orbits, K_E^E is flag-transitive.

By (2.7), $K_E^E \geqslant SL(3, q)$ except, perhaps, if $q = 2$ or 8 and K_E^E has order $(q^2 + q + 1)(q + 1)$.

Let x be a point. By hypothesis, $K^* = K_x^{x^\perp/x}$ is a $(2, 3, 4)$-group.

(II) For each line L, K_L^L is transitive. Suppose that V is unitary. Let r be a primitive divisor of $q^2 + 1$ (cf. (2.18)). An element of order r in K_L centralizes $L^\perp/L$. Thus, if $L \subset F$ then $r \mid \mid K_{LF}^L \mid$. Then K_F^F is line-transitive by Sylow's theorem. Thus, V is not unitary.

Suppose that $q \neq 2$, $K < \Gamma O(7, q)$ or $\Gamma O^-(8, q)$, and $K_E^E \geqslant SL(3, q)$. Then $C_K(E^\perp/E)^E \geqslant SL(3, q)$. By (2.14), K_E has a subgroup H inducing $SL(3, q)$ on $V/E^\perp$ and such that V/E has an H-complement W/E to $V/E^\perp$. Here, W has type $\Omega^+(6, q)$. Passing to $SL(4, q)$ and again applying (2.14), we find that H may even be assumed to fix a second plane of W. Now H contains long root groups, and (2.4) applies. Only (2.4 ii) can occur.

(III) Since F contains representatives from both line-orbits, there is a line L contained only in planes from F^K, and having K_L transitive on these planes. We may assume that $L \subset F$, and that $E \cap F = M$ is a line. Note that K_M is intransitive on $M^\perp/M$.

Suppose that $K < \Gamma O^-(8, q)$, where $q = 2$ or 8. Since $q^2 + 1 \mid \mid K_L \mid$, if $D \in Syl_5 K$ then $D \neq 1$ and $C_V(D)$ has type $\Omega^+(4, q)$. Then $N_K(D)$ is point-transitive on $C_V(D)$, and hence $(q + 1)^2 \mid \mid N_K(D) \mid$. Since $\mid D \mid = 5$, $(q + 1)^2 \mid \mid C_K(D) \mid$. It follows that $C_K(D)$ has an element r of order 3 such that $C_V(r)$ has type $\Omega^+(6, q)$.

Let R be a 3-group maximal with respect to centralizing a line. Set $W = C_V(R)$, $N = N_K(R)$ and $C = C_N(W^\perp)$. Then $N_L^{L'}$ is transitive for each line $L' \subset W$ (by the Frattini argument). Also, W has type $\Omega^+(6, q)$. (If W had type $\Omega^+(4, q)$ or $\Omega^-(6, q)$ then $R^{W^\perp}$ would fix a point.) By (5.1), $N^W \geqslant \Omega^+(6, q)$ or A_7. Thus, $C^W \geqslant \Omega^+(6, q)$ or A_7. By (2.4), $C^W \geqslant A_7$ and $q = 2$.

There is a nonsingular vector $w \in W$ such that $C_W \geqslant A_6$. Let $0 \neq v \in W^\perp$, and set $y = \langle v + w \rangle$. Then $K_y^{y^\perp/y}$ contains a subgroup A_6 fixing a nonsingular vector. Since K^* is a $(2, 3, 4)$-group, this is impossible.

(IV) From now on, we will assume that $\Gamma < \Gamma Sp(6, q)$, where q is odd, 2 or 8.

Suppose that q is odd. There is an involution $t \in K_{ME}$ inducing -1 on M. We may assume that $-1 \in K$. Let $R \in Syl_2 C_K(M)$, and define W, N and C as above. Then $\dim W = 4$, so R cannot be properly contained in a larger 2-group centralizing a line. Note that $N_M^M \geqslant GL(2, q)$, while $N_{L'}^{L'}$ is transitive for any line $L' \subset W$ and N^W has at most 2 line-orbits. By (5.12), C^W is transitive on points. This contradicts the intransitivity of K^*.

(V) Thus, $q = 2$ or 8.

Next, suppose that K^* has an element of order 3 centralizing a point. Let R be a 3-group maximal with respect to centralizing a line, define W, N and C as above, and again find that C^W is point-transitive. Thus, K^* has no such element.

However, K^* is a $(2, 3, 4)$-group. By (5.6), K^* fixes a line. Thus, K_x fixes a unique plane $F(x)$ on x.

We must distinguish between the cases $F(x) \in E^K$ and $F(x) \in F^K$.

(VI) If $F(x) \in E^K$, note that $F(x^g) = F(x)^g = F(x)$ whenever $g \in K_{F(x)}$. Thus, $|E^K| = q^3 + 1$. The cosets of the members of E^K form an affine translation plane $\mathfrak{A}$ (Dembowski [10, p. 133]).

Since K^* is transitive on the q^3 lines opposite $F(x)/x$, $\mathfrak{A}$ is desarguesian and $K \leqslant \Gamma L(2, q^3)$ [10, pp. 130–131]. Since $|K|$ is divisible by

$$q^3(q + 1) \cdot (q^6 - 1)/(q - 1),$$

it follows that $K \geqslant \Gamma L(2, q^3)$.

(VII) Now suppose that $F(x) \in F^K$. Note that K_x is transitive on the lines of $F(x)$ through x. If L' is such a line, then $|L'^K| = |x^K|(q + 1)/(q + 1) = (q^6 - 1)/(q - 1)$. The points of V, together with L'^K, form a generalized hexagon with parameters q, q. (For, K^* has just 2 point-orbits, so all lines in L'^K through x lie in $F(x)$. If $x_1, \ldots, x_5$ are distinct points such that $\langle x_i, x_{i+1} \rangle \in L'^K$ for all i (mod 5), then all x_i are perpendicular since, for example, $\langle x_1, x_2 \rangle$, $\langle x_1, x_3 \rangle \subset F(x_1)$; thus, all $F(x_i)$ coincide, which is absurd. Thus, there are no triangles, quadrangles or pentagons, and a count shows that we have a generalized hexagon.) By Yanushka [63, (1.1)], this is the $G_2(q)$ hexagon.

Since K has just 2 plane-orbits, $|E^K| = q^3(q^3 + 1)$. Then $|K| = q^3(q^3 + 1)|K_E|$. If $q = 2$ then K contains a Sylow 3-subgroup of $G_2(2)' \cong PSU(3, 3)$, and hence $K \unrhd G_2(2)'$.

Finally, if $q = 8$ then we may assume that K does not contain any nontrivial scalars, and hence that $|K| = 8^3(8^3 + 1)(8^2 + 8 + 1)(8 + 1)$. Set $H = K \cap G_2(8)$. Then $|H| = |K|/3$ (since $K_E^E \not\leqslant GL(3, 8)$).

$G_2(8)$ has a class of $8^3(8^3 - 1)$ subgroups $S \cong SU(3, 8)$. Choose S so that $S \cap H$ contains a Sylow 3-subgroup of H. Note that $|S \cap H| \geqslant |H|/8^3(8^3 - 1) = (8^3 + 1)3/7 > 2(8 + 1)^2$. Then $S \cap H = S$ by Hartley [22], and (2.4) yields the desired contradiction in this case. This completes the proof of (5.19).

REMARK 5.20. (i) In its natural embedding in $\Gamma Sp(6, q)$ with $q = 2$ or 8, the group $K = \Gamma L(2, q) = \Gamma Sp(2, q)$ actually satisfies the hypotheses of (5.19). Indeed, K produces a partition E^K as in (VII), consisting of planes since isotropic 1-spaces over $GF(q^3)$ have dimension 3 over $GF(q)$. Also, $|K_E| = q^3(q^3 - 1)(q + 1)$ and $|K_E^E| = (q^3 - 1)(q + 1)$, where K_E^E is flag-transitive. Let K^* be the group induced by K_x on $x^\perp/x$. Certainly K^* stabilizes E/x, and permutes its points transitively. Since $O_2(K_x)$ is transitive on $E^K - \{E\}$, K^* is transitive on the lines missing E/x. A subgroup of K^* of order $q + 1$ fixes a line missing E/x and acts transitively on that line. Thus, K^* has 2 orbits on points, 3 on lines and 4 on flags. Then K has just 2 line-orbits. Finally, if a plane meets E in a line, it meets each other plane in E^K in at most one point, so K has 2 plane-orbits.

However, the permutation representation of $\Gamma Sp(6, q)$ on the set X of cosets of $\Gamma L(2, q^3)$ does not have rank 3. For, if it did, the permutation character would be $1 + \chi + \zeta$ with $\chi = (1 \mid 2)$, by Table II of §3. However, $|X| - 1 - \chi(1)$ does not divide $|\Gamma Sp(6, q)|$.

(ii) The groups $K = G_2(q)'$ also satisfy the hypotheses of (5.19), since K_x behaves as indicated in (VII).

It is known that $Sp(6, 2)$ has rank 3 on its class of subgroups $G_2(2)$ (Frame [19], Edge [15]). In view of (1.1), we must consider the rank for other values of q as well.

The group $\Omega(7, q)$ has 2 classes of subgroups $G_2(q)$, which are all in a single class under $\Gamma O(7, q)$. These can be viewed as follows. Consider $\Omega^+(8, q)$, with underlying vector space V. There is a subgroup G of $\Omega^+(8, q)$ such that $-1 \in G$, $G/\langle -1 \rangle \cong \Omega(7, q)$, and $G^* = (G/\langle -1 \rangle)^\tau$ fixes a nonsingular point b, where τ is a triality automorphism of $P\Omega^+(8, q)$. Since G^* is transitive on each class of 4-spaces, G is point-transitive. Moreover, $G^* \cap G^{*\tau^{-1}} = C_{G^*}(\tau) = G_2(q)$. Note that b can be chosen to be in any $\Omega^+(8, q)$-orbit of nonsingular points of V. There are $(2, q - 1)$ such orbits, conjugate in $N_{\Gamma O^+(8, q)}(G)$. Checking orders, we find that G is transitive on each class X of nonsingular points of V.

If $b, c \in X$, $b \neq c$, then $\langle b, c \rangle$ may be anisotropic or contain a unique singular point; it can also contain 2 singular points if $q > 3$. Thus, if $q > 3$ then G cannot have rank 3 on X.

Suppose that $q = 3$. If $\langle b, c \rangle$ is anisotropic, then $G_{bc} = SU(3, 3) \cdot 2$. If $\langle b, c \rangle$ contains a singular point r, then G_{br} is a parabolic subgroup of $G_b = G_2(3)$, has an element inverting r, and hence acts nontrivially on $\langle b, c \rangle$. Since G_b is transitive on the points of $b^\perp$, it is transitive on $\{c \in X \mid \langle b, c \rangle$ is an anisotropic 2-space$\}$. Consequently, G has rank 3 on X. (Of course, the case $q = 2$ can be handled similarly.)

(D) *Subgroups of* $\Gamma O^+(8, q)$. The case of $\Gamma O^+(8, q)$ is somewhat similar to the rank 3 cases already considered. The only interesting result is (5.22), which deals with $W(E_8)'$. We begin with an analogue of (5.15).

LEMMA 5.21. *If K is a subgroup of $\Gamma O^+(8, q)$ which is transitive on both points and lines, then $K \geqslant \Omega^+(8, q)$.*

PROOF. By (2.12), $K_x^{x^\perp/x}$ is either point-transitive or has 2 point-orbits of length $\frac{1}{2}(q^2 + 1)(q^2 + q + 1)$. By (5.1), K_x is flag-transitive, and hence (2.6) applies.

PROPOSITION 5.22. *Let $K < \Gamma O^+(8, q)$, and assume that K stabilizes each class of 4-spaces, is transitive on each class of $(1, 4)$-flags, has 2 orbits of lines, $(1, 2)$-flags, and each class of $(2, 4)$-, $(1, 2, 4)$- and $(1, 3)$-flags, and has 3 orbits of flags. Then K is $W(E_8)'$, embedded naturally in $\Omega^+(8, 3)$.*

PROOF. Set $K^* = K_x^{x^\perp/x}$, $A(x) = C_K(x) \cap C_K(x^\perp/x)$ and $V(x) = [V, A(x)]$. By hypothesis, K^* is transitive on each class of planes, and has 2 orbits of points, lines and $(1, 3)$-flags and 3 orbits of flags. By (5.1), $K^{*(\infty)} = \Omega^-(4, q)$ or $\Omega(5, q)'$, embedded naturally in $\Omega^+(6, q)$.

Case $K^{*(\infty)} = \Omega^-(4, q)$. Let F be a 4-space. By (5.1), $(K_F^F)^{(\infty)} = Sp(4, q)'$ or $SL(2, q^2)$. Computing $|K|$ using points or a class of 4-spaces, we find that $(K_F^F)^{(\infty)} = SL(2, q^2)$.

Suppose that $A(x) \neq 1$. By (2.1, 4), $V(x)/x$ is anisotropic. If $x \in F$ then $F \not\subset V(x)^\perp$. Thus, $A(x)^F$ is nontrivial and contains transvections, which is not the case.

Consequently, $A(x) = 1$ and $K_x^{(\infty)} \cong \Omega^-(4, q)$. If $q \neq 3$, then $K_x^{(\infty)}$ centralizes a 4-dimensional space, by (2.14); but then $A(y) \neq 1$ for some point y.

Thus, $q = 3$ and $\dim W(x) = 3$, where $W(x) = C_V(K_x^{(\infty)})$. Here, $W(x)/x$ is anisotropic.

Let $f \in K_x$ have order 5. Then $\dim C_V(f) = 4$, and f centralizes $W(y)$ for each $y \in C_V(f)$. Thus, $T = W(x) \cap W(y)$ is an anisotropic 2-space if $y \in C_V(f)$ but $y \neq x$. Then $H = \langle K_x^{(\infty)}, K_y^{(\infty)} \rangle$ acts on $T^\perp$. By (5.7), H is $2 \cdot PSL(3, 4)$. Set $\langle j \rangle = Z(H)$.

Let T_1 be a second anisotropic 2-space in $W(x)$, and let y_1 denote the point other than x in $T_1^\perp \cap C_V(f)$. This produces a second group $2 \cdot PSL(3, 4)$, and a second involution j_1. Both j and j_1 invert $W(x)^\perp$, and both centralize $W(x)/x$. Hence $jj_1 \in A(x) = 1$, and $K^{*(\infty)}$ cannot be $\Omega^-(4, q)$.

Case $K^{*(\infty)} = \Omega(5, q)'$. If $q > 3$, then (2.14) applies: K_x has a subgroup $\Omega(5, q)$ centralizing a 3-dimensional subspace. Then (2.4) yields a contradiction.

Suppose that $q = 3$. Then $K_x^{(\infty)}$ has a subgroup $J = Z_2^4 \rtimes A_5$, where $\dim C_V(O_2(J)) = 3$. Thus, $\dim C_V(J) = 3$, and J acts monomially on $[V, J]$. In particular, $A(y) \neq 1$ for some y, so that $|A(y)| = 3$ and $\dim V(y) = 2$ by (2.1, 4) and the action of K^*. Moreover, there are points $y, z \in [V, J]$ such that $V(y) \cap V(z)$ is a nonsingular point b.

Since $K_y^{(\infty)}$ centralizes $V(y)/y$, it centralizes $V(y)$. Then $K_b \geq \langle K_y^{(\infty)}, K_z^{(\infty)} \rangle$, where $K_y^{(\infty)} \neq K_z^{(\infty)}$ (as otherwise $K_y^{(\infty)}$ would act on the nonsingular 3-space $\langle b, y, z \rangle$, in which case $|A(y)|$ would be at least 3^3). By (2.16), $K_b^{b^\perp} \geq W(E_7)'$.

Note that $A(y)$ moves b to another nonsingular point c in $V(y)$. Then $K_c^{c^\perp} \geq W(E_7)'$. Moreover, $(K_y^{(\infty)})^{b^\perp}$ lies in a unique subgroup $W(E_7)'$ (see the proof of (2.16), Case 8); a similar statement holds for $(K_y^{(\infty)})^{c^\perp}$. Thus, $\langle K_b^{(\infty)}, K_c^{(\infty)} \rangle$ is contained in a group $W(E_8)'$, and hence is $W(E_8)'$. Also, b^K contains all "roots" for the latter group. Consequently, $K = W(E_8)'$, as desired.

Finally, if $q = 2$ then $K^* > \Omega^-(4, 2)$. By (2.14), K_x has a subgroup $\Omega^-(4, 2)$ centralizing a 4-dimensional subspace. This time, $|A(x)| \geqslant 4$ in view of the action of this $\Omega^-(4, 2)$, and (2.1, 4) yield a contradiction.

LEMMA 5.23. $\Gamma O^+(8, q)$ *has no subgroup stabilizing each class of* 4-*spaces, transitive on points and each class of* 4-*spaces, and having* 2 *orbits on lines, planes and each class of* $(1, 4)$-*flags,* 3 *orbits on* $(1, 2)$-*flags and each class of* $(2, 4)$-*flags,* 5 *orbits on* $(1, 3)$- *and* $(2, 3)$-*flags and each class of* $(1, 2, 4)$-*flags, and* 9 *orbits of flags.*

PROOF. Let K be such a group. Let F be a 4-space, and set $H = K_F^F$. Then H has 2 point-orbits, 3 line-orbits, 5 orbits of $(1, 2)$-flags and 9 flag-orbits. Corresponding statements hold for $K^* = K_x^{x^\perp/x}$. (In fact, our hypotheses are invariant under triality.)

First consider the possibility that $H^{(\infty)} = \Omega^-(4, q) \cong K^{*(\infty)}$. If q is odd, then $K^{*(\infty)}$ is $\Omega(3, q^2)$, in its natural embedding into $\Omega^+(6, q)$. By (2.1), (2.4) and (2.14), $K_x^{(\infty)}$ centralizes a hyperbolic line. If $P \in Syl_p(K_x \cap \Omega^+(8, q))$, then $\dim C_V(P) = 4$, $\dim \operatorname{rad} C_V(P) = 2$, and $N_K(P)$ cannot be transitive on the fixed points of P.

If q is even, then $K^{*(\infty)}$ fixes a nonsingular 1-space b, and induces at least $SL(2, q^2)$ on $b^\perp/b$. By (2.1) and (2.4), $A(x) = C_K(x) \cap C_K(x^\perp/x)$ has order at most q. Let $x \neq y \in C_V(A(x))$. Then $A(x)^{y^\perp/y}$ lies inside an $SL(2, q^2)$, and hence has order 1 or at least q^2. Consequently, $A(x) = 1$. By (2.14), $H^1(K^{*(\infty)}, x^\perp/x) = 0$. Thus, $K_x^{(\infty)}$ again centralizes a hyperbolic line, and we obtain the same contradiction as before.

It remains for us to show that there are no possibilities for $H^{(\infty)}$ other than $\Omega^-(4, q)$. This involves checking all the possibilities in (5.1). Most are eliminated by the following facts.

H has 2 point-orbits.

There is a line $L \subset F$ such that H_L^L is transitive.

There is a plane E such that H_E^E fixes a point, a line, or (if $q = 4$) a hyperoval; moreover, H_E^E contains either an elementary abelian group of order q^2 or A_6. (For, let E^H and E'^H be the plane-orbits of H, where H_E^E has fewer point-orbits than $H_{E'}^{E'}$. If H_E^E has 2 point-orbits, then it has at least 3 flag-orbits; by (2.7 iii), $H_{E'}^{E'}$ has at least 6 flag-orbits, and hence H_E^E behaves as desired. If H_E^E is transitive let Ω_1, Ω_2, Ω_3, Ω_4 be the point-orbits of $H_{E'}^{E'}$. If no line of E' meets all orbits Ω_i, then joining the Ω_i in pairs accounts for at least 9 flag-orbits, and leaves H_E^E with none. Thus, some line meets Ω_i, and accounts for 4 flag-orbits. This leaves only 4 flag-orbits for the remaining 3 line-orbits of $K_{E'}^{E'}$, and (2.7 iv) yields a contradiction.)

Applying (5.1) we find only one situation requiring further comment: that in which H fixes a line L. Here, H is transitive on L, on the $q^2(q + 1)$ points $y \notin L$, on the $q(q + 1)^2$ lines meeting L in a point, and on the q^4 lines skew to L. As in (2.12), H_y has just 2 orbits of lines on y. The same holds for H_x, $x \in L$. Then H has only 4 orbits of $(1, 2)$-flags, and this situation cannot occur.

LEMMA 5.24. $\Gamma O^+(8, q)$ *has no subgroup stabilizing each class of* 4-*spaces and having* 2 *orbits on points and each class of* 4-*spaces,* 4 *orbits on planes and each type of* $(1, 4)$-*flags,* 5 *orbits on* $(1, 2)$-*flags and each type of* $(2, 4)$-*flags,* 7 *orbits on* $(1, 3)$-*flags and each type of* $(1, 2, 4)$-*flags, and* 10 *orbits of flags.*

PROOF. Let K be such a subgroup. As in the proof of (5.23), if E is a plane such that K_E^E has i orbits of points and lines, then K_E^E has at least $2i - 1$ flag-orbits. The only partitions of 7 with 4 parts are

$$7 = 4 + 1 + 1 + 1 = 3 + 2 + 1 + 1 = 2 + 2 + 2 + 1.$$

Since there are only 10 flag-orbits, it follows that K_E^E is flag-transitive for some plane E.

Let F be a 4-space containing E. By (2.15), K_F^F contains $SL(4, q)$ or A_7, or else fixes a point or a plane.

If K_F^F contains $SL(4, q)$ or A_7, let τ denote a triality automorphism of $P\Omega^+(8, q)$. Then (2.16) applies to $(K/Z(K))^\tau$; but then K either fixes F or contains $\Omega^+(8, q)$ or A_9, and hence cannot satisfy the required orbit conditions.

Thus, K_F^F fixes a point or a plane. If K_F^F fixes both a point and a plane, then it has at least 10 flag-orbits, which is impossible. Thus, K_F^F has 2 orbits of points and lines, and 3 orbits of $(1, 2)$-flags.

Let $F' \in F^{\Omega^+(8,q)} - F^K$. Then $K_{F'}^{F'}$ has 2 orbits of points, lines and $(1, 2)$-flags. Then $K_{L,F'}^{L}$ is transitive for each line L of F', and hence $K_{F'}^{F'}$ has only one point-orbit, which is not the case.

LEMMA 5.25. $\Gamma O^+(8, q)$ *has no point-transitive subgroup having* 2 *plane-orbits and* 3 *orbits of* $(1, 3)$*- and* $(1, 4)$*-flags.*

PROOF. Let K be such a subgroup. There is a plane E such that K_E^E is transitive. By (2.7), if F is a 4-space containing E then K_{EF}^E is still transitive, as is K_{xE}^E for some point x of F. By (2.15), $K_x^{x^\perp/x}$ fixes a plane or a pair of planes. In either case $K_x^{x^\perp/x}$ has more than 3 plane-orbits, contrary to hypothesis.

6. Subspace transitivity. For rank $n \geq 4$, our study of highly transitive subgroups of $\Gamma(V)$ will employ the elegant method of Perin [46]. This method was, in fact, already used several times in §5C. As in §5, we will require numerous technical results. The only general results are (6.1, 6.4, 6.5).

The following notation will be used throughout this section, as well as in §12.

$K \leq \Gamma L(V)$, with K point-transitive.

x is a point.

r is a prime and R is an r-subgroup of K_x.

$W = C_V(R)$, $N = N_K(R)$ and $C = C_N(W^\perp)$.

Usually, $R \in Syl_r K_x$. Then, by Sylow's theorem, N^W is point-transitive. Perin's method involves comparing N^W and $N^{W^\perp}$ in order to show that C^W is relatively large. We begin with a simple example.

THEOREM 6.1. *Let K be a point-transitive subgroup of $\Gamma U(d, q)$, where $d \geq 4$. Assume that K is transitive on lines; if $d = 6$ or 7, assume further that K is also $(1, 2)$-transitive. Then $K \geq SU(d, q)$.*

PROOF. By (5.12), we may assume that $d \geq 6$. Let r be a primitive divisor of $q^{d-3} + (-1)^d$ (cf. (2.18)); we will exclude the cases $\Gamma U(6, 2)$ and $\Gamma U(9, 2)$ until the end of the proof. Let $R \in Syl_r K_x$. By Table IV, $R \neq 1$. Since r divides the number of nonzero vectors of $W^\perp$, from Table IV it also follows that W is a nonsingular 3-space.

TABLE IV

The numbers of points, lines, planes and n-spaces (of each type in case V has type $\Omega^+(2n, q)$)

	points	lines	planes	n-spaces
$Sp(2n, q)$ $\Omega(2n+1, q)$	$\dfrac{q^{2n}-1}{q-1}$	$\dfrac{(q^{2n}-1)(q^{2n-2}-1)}{(q-1)(q^2-1)}$	$\dfrac{(q^{2n}-1)(q^{2n-2}-1)(q^{2n-4}-1)}{(q-1)(q^2-1)(q^3-1)}$	$\displaystyle\prod_{i=1}^{n}(q^i+1)$
$SU(2n, q)$	$\dfrac{(q^{2n}-1)(q^{2n-1}+1)}{q^2-1}$	$\dfrac{(q^{2n}-1)(q^{2n-1}+1)(q^{2n-2}-1)(q^{2n-3}+1)}{(q^2-1)(q^4-1)}$	$\dfrac{(q^{2n}-1)(q^{2n-1}+1)(q^{2n-2}-1)(q^{2n-3}+1)(q^{2n-4}-1)(q^{2n-5}+1)}{(q^2-1)(q^4-1)(q^6-1)}$	$\displaystyle\prod_{i=1}^{n}(q^{2i-1}+1)$
$SU(2n+1, q)$	$\dfrac{(q^{2n+1}+1)(q^{2n}-1)}{q^2-1}$	$\dfrac{(q^{2n+1}+1)(q^{2n}-1)(q^{2n-1}+1)(q^{2n-2}-1)}{(q^2-1)(q^4-1)}$	$\dfrac{(q^{2n+1}+1)(q^{2n}-1)(q^{2n-1}+1)(q^{2n-2}-1)(q^{2n-3}+1)(q^{2n-4}-1)}{(q^2-1)(q^4-1)(q^6-1)}$	$\displaystyle\prod_{i=2}^{n}(q^{2i+1}+1)$
$\Omega^+(2n, q)$	$\dfrac{(q^n-1)(q^{n-1}+1)}{q-1}$	$\dfrac{(q^n-1)(q^{n-1}-1)(q^{n-2}+1)}{(q-1)(q^2-1)}$	$\dfrac{(q^n-1)(q^{n-1}+1)(q^{n-1}-1)(q^{n-2}-1)(q^{n-3}+1)}{(q-1)(q^2-1)(q^3-1)}$	$\displaystyle\prod_{i=1}^{n-1}(q^i+1)$
$\Omega^-(2m, q)$, $m=n+1$	$\dfrac{(q^m+1)(q^{m-1}-1)}{q-1}$	$\dfrac{(q^m+1)(q^{m-1}-1)(q^{m-1}+1)(q^{m-2}-1)}{(q-1)(q^2-1)}$	$\dfrac{(q^m+1)(q^{m-1}-1)(q^{m-1}+1)(q^{m-2}-1)(q^{m-2}+1)(q^{m-3}-1)}{(q-1)(q^2-1)(q^3-1)}$	$\displaystyle\prod_{i=2}^{m}(q^i+1)$

By (2.12), $N^W \trianglerighteq SU(3, q)$, $PSL(3, 2)$, $3 \cdot A_7$, or an extraspecial group of order 27. Also, $N^{W^\perp}$ is a subgroup of $\Gamma L(2, q^{d-3})$.

If N^W contains $SU(3, q)$, $PSL(3, 2)$ or $3 \cdot A_7$, then so does C^W. If $C^W \geqslant SU(3, q)$, then (2.4) applies.

We claim that K_x acts irreducibly on $x^\perp/x$. This is clear if $d = 6$ or 7. If $d \geqslant 8$, then $|K_x|$ is divisible by primitive divisors of $q^{d-2} - (-1)^d$ and $q^{d-3} + (-1)^d$, from which irreducibility follows immediately. (If $K < \Gamma U(8, 2)$, this argument fails. However, in this case K is $(1, 2)$-transitive by (2.12), and then the irreducibility of K_x is obvious.)

In particular, by (2.1) and (2.4), we may assume that $C \cap Q = 1$, where $Q = O_p(SU(V)_x)$. But in both of the exceptional cases $q = 3$ and 5, $C^W \cap Q^W \neq 1$. Thus, we are left with the case $q = 2$.

Now all we know is that N^W has a normal extraspecial subgroup of order 27 and exponent 3. Since $N^{W^\perp} \leqslant \Gamma L(2, 2^{d-3})$, it follows that C^W contains the center $\langle t \rangle$ of $SU(3, 2)$. Note that t centralizes the hyperplane $(W^\perp + x)/x$ of $x^\perp/x$. Thus, Wagner [60] applies to the group H induced by K_x on $x^\perp/x$. Using r, we find that $H \geqslant SU(d - 2, 2)$. Consequently (2.6) applies.

Finally, suppose that $K \leqslant \Gamma U(6, 2)$ or $\Gamma U(9, 2)$. In the former case, $K_x^{x^\perp/x} \geqslant SU(4, 2)$ by (5.12). Consider the case $\Gamma U(9, 2)$. By (2.12), K is $(1, 2)$-transitive. Let $R \in Syl_7 K_x$, and define W, N and C as before. Then N^W is transitive on points, and $\dim W = 3$ or 6. If $\dim W = 3$, we can proceed as before. If $\dim W = 6$, then N^W is $(1, 2)$-transitive, so that $N^W \geqslant SU(6, 2)$. Then $C^W \geqslant SU(6, 2)$, and (2.4) completes the proof of (6.1).

REMARK. By Table II and (4.2), a subgroup of $\Gamma U(m, q)$, $m \geqslant 6$, is point-transitive if it is line-transitive.

LEMMA 6.2. *Suppose that K is a point-transitive subgroup of $\Gamma(V)$ which is either transitive on lines or has 2 line-orbits and at most 3 orbits of $(1, 2)$-flags. (Moreover, if there are 2 line-orbits and V has type $\Gamma O(2m + 1, 3)$ or $\Gamma O^-(2m, 2)$, assume that K has 2 plane-orbits and at most 4 orbits of $(2, 3)$-flags.) Define s by the following table.*

Type of V	$Sp(2m, q)$	$\Omega(2m + 1, q)$	$\Omega^\pm(2m, q)$	$SU(2m, q)$	$SU(2m + 1, q)$
s	$q^{m-2} + 1$, $m \geqslant 4$	$q^{m-2} + 1$, $m \geqslant 4$	$q^{m-3} \pm 1$ (either sign), $m \geqslant 6$ $q^2 + 1, m = 5$	$q^{2m-5} + 1$, $m \geqslant 4$	$q^{2m-3} + 1$, $m \geqslant 4$

Assume that, for some line L, $|K_L|$ is divisible by a primitive divisor r of s. Then $K \geqslant \mathrm{Chev}(V)$.

REMARK 6.3. (i) The parenthetical hypothesis in (6.2) is only used in cases (h) and (j) of the proof.

(ii) Several cases are automatically excluded in (6.2): those in which s has no primitive divisor. For future reference, these excluded cases are listed in the following continuation of the preceding table

Type of V	$Sp(10, 2)$	$\Omega(11, 2)$	$\Omega^\pm(12, 2)$, $\Omega^\pm(18, 2)$	$SU(8, 2)$	none

Proof of 6.2. The argument falls into several steps.

(I) Let $R \in Syl_r K_L$. In each case, $W = C_V(R)$ is a nonsingular subspace; its type is given in the following continuation of the previous tables.

Type of W	$Sp(4, q)$	$\Omega(5, q)$	$\Omega^\pm(6, q)$	$SU(5, q)$	$SU(4, q)$.

Clearly, $R \in Syl_r K_{xL}$ for $x \in L$. Also, $r \nmid |\Gamma(W)_x|$ except when W has type $\Omega^-(6, q)$ and V has type $\Omega^-(10, q)$. Thus, if $R \notin Syl_r K_x$ then W has type $\Omega^-(6, q)$, and $r \mid |N_y^W|$ for each point y of W, in which case (5.10) applies. On the other hand, if $R \in Syl_r K_x$ then N^W is point-transitive. In any event, N^W has at most 2 line-orbits and at most 3 orbits of $(1, 2)$-flags, by Sylow's theorem. Note that if N^W has 2 line-orbits then so does K.

In view of (5.12, 5.13, 5.15), one of the following holds (and is handled in the step indicated):

(a) $N^W \geqslant \text{Chev}(W)$ (II);

(b) $N^W \trianglerighteq SU(3, q)$ for W of type $\Omega^-(6, q)$ (II);

(c) $N^W \trianglerighteq SL(2, q^2)$ for W of type $Sp(4, q)$ (V);

(c') $N^W \trianglerighteq SL(2, 5)$ for W of type $Sp(4, 3)$ (V);

(d) N^W is A_6, for W of type $Sp(4, 2)$ (VI);

(e) N^W is A_7 or S_7, for W of type $\Omega^+(6, 2)$ (VI);

(f) $N^W \trianglerighteq 4 \cdot PSL(3, 4)$ for W of type $SU(4, 3)$ (VII);

(g) $N^W \trianglerighteq 2 \cdot PSL(3, 4)$ for W of type $\Omega^-(6, 3)$ (VIII);

(h) $O_2(N^W)$ is elementary abelian of order 16 or 32, and $20 \mid |N^W/O_2(N^W)|$, for W of type $\Omega(5, 3)$ (IX);

(i) $O_2(N^W)$ is extraspecial of order 32, and $5 \mid |N^W|$, for W of type $Sp(4, 3)$ (X); or

(j) $|O_3(N^W)| \geqslant 27$, for W of type $\Omega^-(6, 2)$ (XI).

(II) By (2.4), we may assume that K contains no long root elements. (In (2.4 iv), there are 3 line-orbits, 2 of which arise from lines of the unitary geometry.)

If $s = q^i \pm 1$, then the normalizer of $R^{W^\perp}$ in $\Gamma(W^\perp)$ is contained in $\Gamma L(2, q^i)$. Consequently, this statement holds for $N^{W^\perp}$. However, N^W is not contained in any $\Gamma L(2, q^i)$.

If (a) or (b) holds (excluding $Sp(4, 2)$ and $SU(3, 2)$), then $(N^W)''$ has no homomorphic image in any $\Gamma L(2, q^i)$, so that $(C^W)'' = (N^W)''$ and C contains long root elements.

If $N^W \trianglerighteq SU(3, 2)$, then C^W contains a transvection (since $\Gamma L(2, 2^i)$ never has a quaternion subgroup). Once again, C contains long root elements.

Finally, if N^W is $Sp(4, 2)$, or if one of (c)–(g) holds for N^W, then one of (c)–(g) also holds for C^W in place of N^W. We will have to consider these possibilities for C^W separately. Note that, if (h) or (i) holds for N^W, it might not hold for C^W: only $|O_2(C^W)| = 16$ or 32 is clear at this point. Similarly, if (j) holds for N^W it need not hold for C^W.

(III) Set $Q = O_p(\Gamma(V)_x)$, $A(x) = K \cap Q$ and $V(x) = [V, A(x)]$. Clearly, K_x acts on both $A(x)$ and $V(x)$. Recall that $A(x)$ is assumed to contain no long root

elements. However, in several of our cases, $C \cap A(x)$ is nontrivial:

(c) $|C \cap A(x)| = q$ and $V(x)$ is a line (since a Sylow p-subgroup of C_x has order q^2 and acts on the $q + 1$ lines through x, centralizing one of these lines: $V(x)$);

(c') $|C \cap A(x)| = q$ and $V(x)$ is a line;

(d) $|C \cap A(x)| = 4$ and $V(x)/x$ is an anisotropic 2-space when K is regarded as lying in $O(2n + 1, 2)$ (since W is an obvious section of the standard mod 2 permutation module for A_6);

(e) $|C \cap A(x)| = 4$ and $V(x)/x$ is again an anisotropic 2-space;

(f) $|C \cap A(x)| = 9$ and $V(x) = W \cap x^{\perp}$;

(h) If $3 \mid |N^W|$ then $|C \cap A(x)| = 3$ and $V(x)/x$ is a nonsingular 1-space;

(j) $|C \cap A(x)| \leqslant 2$.

In cases (g) and (i), $C \cap A(x) = 1$. In all cases, we will use (2.1) in order to show that $|A(x)|$ is small.

(IV) We claim that $A(x) \leqslant C$. For, suppose not. Then $A(x) \not\leqslant C_K(R)$. In particular, $|A(x)| > q^2$. By (2.1), V is neither symplectic of characteristic 2 nor orthogonal.

If V is of type $Sp(2m, q)$, then $r \mid q^{m-2} + 1$, so $|A(x)| \geqslant q^{2m-4}$. By (2.1), $A(x)$ is abelian and $A(x)Q'/Q'$ contains q^{2m-4} mutually orthogonal vectors in a $2m - 2$-dimensional symplectic space. This is clearly impossible.

Thus, V is of type $SU(d, 3)$. By hypothesis, $s = 3^{d-4} + 1$ or $3^{d-5} + 1$. Thus, $|A(x)| \geqslant 3^{2d-10}$. But $|Q/Q'| = 3^{2(d-2)}$ and $2d - 10 > d - 2$ if $d > 8$. Consequently, (2.1) produces the same contradiction as above if $d > 8$. This leaves us with the case of V of type $SU(8, 3)$. Here, $|A(x)| = 3^6$ by the above argument. However, $|C \cap A(x)| = 3^2$ in case (f), whereas our group R of order 7 cannot act fixed-point-freely on a group of order 3^4.

This contradiction proves our claim. In particular, $A(x) = 1$ in cases (g) and (i).

(V) In (c) and (c'), use of N^W shows that $V(x) = V(x')$ whenever $x' \in V(x)$. Let L be a line on x such that $L \subset V(x)^{\perp}$ but $L \neq V(x)$. If $y \in L$ and $y \neq x$, then $y \in V(x)^{\perp}$ implies that $A(x)$ acts on $V(y)$. Since $V(x) \cap V(y) = 0$ and $V(x) = [V, A(x)]$, it follows that $V(y) \subset V(x)^{\perp} \subset x^{\perp}$. Thus, $L = \langle x, y \rangle \subset V(y)^{\perp}$.

Consequently, there are at least 3 types of lines: those of the form $V(x)$, those lines L contained in $V(x)^{\perp}$ for all $x \in L$ (but not of the form $V(x)$), and all other lines. Since K has at most 2 line-orbits, this is impossible.

(VI) We treat cases (d) and (e) together. Recall that $|A(x)| = 4$ and $V(x)/x$ is an anisotropic 2-space. We first show that, if $y \in V(x)^{\perp}$, then $V(y) \subseteq V(x)^{\perp}$. For, let $B(x)$ be the subgroup of C such that $A(x) \not\leqslant B(x) \cong A_4$ and $C_V(B(x)) = C_V(A(x)) = V(x)^{\perp}$. Then $B(x)$ acts on $A(y)$, and hence $A(x)$ centralizes $A(y)$. In particular, $A(x)$ acts on $V(y)$, fixing each subspace through y. If $v \in V(y)$, $a \in A$ and $v^a \neq v$, then $v^a - v$ must be the third nonzero vector in $\langle v, v^a \rangle$, and hence is in $y \cap V(x)$. Thus, if $y \neq x$ then $A(x)$ centralizes $V(y)$, and hence $V(y) \subset V(x)^{\perp}$.

Let M be a line on x such that $M \not\subseteq V(x)^{\perp}$. Then $A(x)^M \neq 1$. If $x \neq y \in M$ then $y \notin V(x)^{\perp}$, so $x \notin V(y)^{\perp}$ by the preceding paragraph, and hence $A(y)^M \neq 1$. Thus, K_M^M is transitive.

Since K has at most 2 line-orbits, it has 2 such orbits: those lines M as above, and all remaining lines. In particular, K is transitive on the $(1, 2)$-flags of the form $(x; M)$ with $M \not\subseteq V(x)^\perp$. Then K_x is transitive on the points on $x^\perp/x$ not in $V(x)^\perp/x$. Consequently, K_x is transitive on the nonsingular vectors in $V(x)^\perp/x$. In particular, rad $V = 0$.

Fix x, let $x \neq y \in V(x)^\perp$, and set $V(\langle x, y \rangle) = \langle V(y), V(y') \rangle$, where y' is the third point of $\langle x, y \rangle$. Since $V(y)$ and $V(y')$ are perpendicular, $V(\langle x, y \rangle)$ is a 6-dimensional subspace of $V(x)^\perp$, and $V(\langle x, y \rangle)/\langle x, y \rangle$ has type $\Omega^+(4, 2)$.

Consider the pairs $(\langle x, y \rangle, \langle x, d \rangle)$ with $y \in V(x)^\perp$, $d \in V(\langle x, y \rangle)$, and d nonsingular. Each point of $V(x)^\perp/x$ arises as a first component; $V(\langle x, y \rangle)$ determines $6 \cdot 2$ second components; and K_x acts transitively on the set of second components. Thus, the number of nonsingular points of $V(x)^\perp/x$ divides twelve times the number of singular points, which is not the case (since rad $V = 0$).

(VII) In (f), $V(x)/x$ is a nonsingular 2-space. There are thus 4 types of lines on x: those in $V(x)$, those in $V(x)^\perp$, those perpendicular to exactly one line of $V(x)$ through x, and all others. (If M is one of the latter lines, then $(M^\perp \cap V(x))/x$ is nonsingular.) Thus, K has more than 3 orbits of $(1, 2)$-flags.

(VIII) In (g), C_x acts irreducibly on $(x^\perp \cap W)/x$: it induces at least $\Omega^-(4, 3)$ there. We claim that K_x acts irreducibly on $x^\perp/x$. For, let M be a proper K_x-invariant subspace of $x^\perp/x$ of least dimension. If dim $M \leqslant 4$, then R centralizes M, so that $M = (x^\perp \cap W)/x$ and K_x has at least 4 point-orbits on $x^\perp/x$ (namely, those in M, those in $M^\perp$, those on lines meeting M and $M^\perp$, and those on no such line). Thus, dim $M > 4$. Also, dim $M \leqslant \frac{1}{2} \dim(x^\perp/x) = m - 1$ (by minimality). Since R acts on M, we cannot have $s = 3^{m-3} + 1$. Thus, $s = 3^{m-3} - 1$. Using C_x again, we see that dim $M \neq m - 2, m - 1$. Consequently, M is a totally singular $m - 3$-space. The lines on x must fall into K_x-orbits of lengths $\frac{1}{2}(3^{m-3} - 1)$, $10 \cdot 3^{m-3}$ and $\frac{1}{2}3^m(3^{m-3} - 1)$. Let L' be any line such that $K_{L'}^{L'}$ is intransitive. Counting suitable $(1, 2)$-flags in four ways, we find that $|L'^K| e_i = |x^K| l_i$ with l_1 and l_2 two of the above numbers and e_1, e_2 integers $\leqslant 3$. However, the equation $e_1 l_2 = e_2 l_1$ is easily seen to have no solution.

Thus, K_x acts irreducibly. Set $E = C_K(L) \cap C_K(L^\perp/L)$, so that $|E \cap C| = 9$. If $e \in E$ then dim$[x^\perp/x, e] \leqslant 2$ by (2.1).

We next show that K_x acts primitively on $x^\perp/x$. For, suppose that K_x preserves a nontrivial decomposition $x^\perp/x = W_1 \oplus \cdots \oplus W_k$. The above property of $E \cap C$ shows that $E \cap C$ cannot act faithfully on $\{W_1, \ldots, W_k\}$. Since $(C_x)' \cong A_6$, we may assume that $(x^\perp \cap W)/x \subseteq W_1$. Then R fixes W_1. Since $r \mid 3^{m-2} \pm 1$, it follows that $x^\perp \cap W/x = W_1$. But K_x has at most 3 point-orbits on $x^\perp/x$. Thus, $k = 2$, and V has type $\Omega^+(10, 3)$. Now the 3 point-orbits have lengths $2 \cdot 10$, $10 \cdot 10 \cdot 2$ and $30 \cdot 15 \cdot 2$. This leads to an impossible equation $e_1 l_2 = e_2 l_1$ as before.

Now K_x is primitive, $|E| \geqslant 9$, and E acts faithfully on $x^\perp/x$ (since $A(x) = 1$ by (IV)). By (2.5), K_x induces at least $\Omega(x^\perp/x)$ or $2 \cdot \Omega(7, 3)$ on $x^\perp/x$. The first case is eliminated by (2.6) or (2.14). By (α) in (2D), the second case leads to a subgroup $2 \cdot \Omega(7, 3)$ of K_x containing long root elements. Thus, (g) cannot occur.

(IX) In (h), $|O_2(C)| \geqslant 2^4$. There is an involution $j \in O_2(C)$ inverting L such that $F = [V, j]$ has type $\Omega^+(4, 3)$. Then j centralizes $L^\perp / L$. Since

$$K_L \trianglerighteq \langle j \rangle (C_K(L) \cap C_K(L^\perp / L)),$$

by the Frattini argument we have $K_L^{L^\perp / L} = H_L^{L^\perp / L}$, where $H = C_K(j)_L$. Note that $L^\perp / L$ and $F^\perp$ are H-isomorphic.

If L' is any line on x such that $L' \cap F^\perp \neq 0$, then j acts nontrivially on L'. However, if L'' is any line of W then $N_{L''}^{L''}$ acts regularly on points since N^W is monomial; moreover $N_{L''}^{L''} = K_{L''}^{L''}$ by the Frattini argument. Then $L' \notin L^K$, and hence $L'' \in L^K$ since K has at most 2 line-orbits. In particular, N^W is line-transitive.

Note that $K_{L'}^{L'^\perp / L'}$ cannot be point-transitive. (For, if it were, then we would have $r \mid |K_{L'}|$, and then L' would belong to L^K by Sylow's theorem.) Thus, by the parenthetical hypotheses of (6.2), $H^{F^\perp}$ has at most 2 point-orbits. There are two subcases to consider: (α) H fixes W, and (β) H moves W.

(α) Here, H is transitive on the $(3^{m-2} + 1)(3^{m-3} - 1)/2$ points of $F^\perp$. Suppose that $m \geqslant 6$. Let r_1 be a primitive divisor of $3^{m-3} - 1$ (cf. (2.18)), let $R_1 \in Syl_{r_1} H$, and set $Z = C_V(R_1)$, $N_1 = N_K(R_1)$ and $C_1 = C_{N_1}(Z^\perp)$. Then N_1^Z is point-transitive, $W \subset Z$, and $N_{1W}^W = K_W^W = N^W$ by the Frattini argument. Moreover, $\dim Z = 7$. Set $N_1^* = (N_1^Z)^{x^\perp / x}$. Then $5 \mid |N_1^*|$. Also, N_1^* contains an elementary abelian group of order 8 none of whose involutions induces -1 on $Z \cap x^\perp / x$. (Namely, $(N_{1W})^Z$ has a subgroup Z_2^3 centralizing $W^\perp \cap Z$ and fixing a member of the distinguished orthogonal basis of W.) Since Z contains a line L' meeting both F and $F^\perp \cap Z$, N_1^* has 2 or 3 point-orbits, and it follows from (5.6) that $N_1^* \geqslant Z_2^4 \rtimes Z_5$. The latter group is line-transitive, so N_1^Z is $(1, 3)$-transitive. But then N_1^Z is also $(2, 3)$-transitive, whereas it is not even line-transitive (since $L, L' \subset Z$). Thus, $m \leqslant 5$.

If $m = 5$, then $H^{F^\perp}$ is a point-transitive subgroup of $\Gamma O^-(6, 3)$. By (5.12), $C_K(F) \trianglerighteq \Omega^-(6, 3)$ or $2 \cdot PSL(3, 4)$. In particular, $5 \mid |C_K(L)|$. Setting $r_1 = 5$ and proceeding as before, we again obtain a contradiction.

If $m = 4$ then $H^{W^\perp}$ is transitive on the points of $W^\perp$, and on those of $F^\perp$ not in $W^\perp$. Thus, $C_K(F) \trianglerighteq \Omega^-(4, 3)$ or A_5. By (IV), $|A(x)| = 3$ and $C_K(F) \trianglerighteq A_5$. Since $A(x) < C_K(R)$, we have $C \trianglerighteq Z_2^4 \rtimes A_5$. If y is a point of $W^\perp$, then C acts on $y^\perp / y$. Also, $V(x)^\perp / x$ has type $\Omega^+(6, 3)$. (For, $V(x)^\perp \cap W/x$ and $W^\perp$ have respective types $\Omega^-(2, 3)$ and $\Omega^-(4, 3)$.) Thus, K_y induces a subgroup of $\mathrm{Aut}\, P\Gamma L(4, 3)$ having at most 2 line-orbits and containing the point-transitive group induced by C. By (5.12), $K_y^{y^\perp / y} \trianglerighteq \Omega^+(6, 3)$, $\Omega(5, 3)$ or $Z_2^4 \rtimes A_5$. The latter 2 cases require that $K_y^{y^\perp / y}$ fixes more than one nonsingular point, and hence that it has more than 3 point-orbits. In the first case, (2.14) produces long root groups.

(β) Let $h \in H$ with $W^h \neq W$. Then $F \subset W^h$. Set $Z = \langle W, W^h \rangle$ and $D = \langle C, C^h \rangle$. Then $\dim Z = 6$, D acts on Z, and D centralizes $Z^\perp$.

Since $C_W(j)$ and $C_W(j)^h$ are isometric, $C_Z(j)$ is either anisotropic or has a nontrivial radical.

Assume that $z = \mathrm{rad}\, Z \neq 0$. Since C and C^h agree on Z/z, while centralizing $Z^\perp$, they are conjugate under $O_3(D)$. Thus $|O_3(D)| \neq 1$. If $1 \neq d \in O_3(D)$, then $\dim C_Z(d) = 5$; since d centralizes $Z^\perp$, we have $d \in A(z)$. Now $C \trianglerighteq Z_2^4 \rtimes A_5$ once again. Thus, $|A(z)| > 3$ (since C acts on $A(z)$), which contradicts (IV).

Consequently, $C_Z(j)$ is an anisotropic 2-space. Since $C_W(j)$ and $C_W(j)^h$ are isometric, it follows that $Z = F \perp C_W(j) \perp C_W(j)^h$. However, $O_2(C)^F = O_2(C^h)^F$. Thus, $O_2(C)$ and $O_2(C^h)$ commute.

Now $N = \langle O_2(C)^H \rangle$ is an elementary abelian normal subgroup of H, and acts nontrivially on $F^\perp$. Then H acts monomially on $[F^\perp, N]$, while having at most 2 point-orbits on $F^\perp$. Thus, $2m - 3 = 5$ and $H^{F^\perp} \geqslant Z_2^4 \rtimes Z_5$. Then $C_K(F)^{F^\perp} \geqslant Z_2^4 \rtimes Z_5$.

Let $\{e_i\}$ be an orthogonal basis of $\langle F^\perp, x \rangle / x$ with respect to which $C_K(F)$ is monomial. We may assume that $(e_i, e_i) = 1$. Let $\{e, f\}$ be a basis of $x^\perp \cap F / x$ with $(e, e) = 1 = -(f, f)$. We may assume that $R^{F^\perp}$ is generated by $g = (1, 2, 3, 4, 5)$. Then $C_{\overline{V}}(g) = \langle e, f, \Sigma_i e_i \rangle$, where $\overline{V} = x^\perp / x$. Consequently, N_x acts transitively on $\{\langle e + f \rangle, \langle e - f \rangle, \langle e + \Sigma_i e_i \rangle, \langle e - \Sigma_i e_i \rangle\}$. Set $K^* = K_x^{x^\perp / x}$ and $\Omega = \langle e + f \rangle^{K^*}$. Then $C_K(F)$ acts on Ω, and hence Ω contains all the points $\langle e + \Sigma_i \varepsilon_i e_i \rangle$ with each $\varepsilon_i = \pm 1$. Then Ω contains two perpendicular points, such as $\langle e + \Sigma_i e_i \rangle$ and $\langle e + e_1 + e_2 - e_3 - e_4 - e_5 \rangle$. Also, $C_K(F)$ is transitive on the lines of $\overline{V}$ through $\langle e + f \rangle$. By (2.10), K^* fixes a hyperplane. The only one available is $f^\perp$, which has type $\Omega^+(6, 3)$. This produces the same contradiction as in the last part of (α).

(X) In (i), C^W has a normal extraspecial subgroup of order 2^5. Let j be an involution in C such that $[V, j]$ is a hyperbolic line, and let $x \in [V, j]$. Then j inverts x and centralizes $x^\perp / x$.

By (IV), $Q \cap K = 1$. Thus, $\langle j \rangle = C_K(x^\perp / x) \leqslant Z(K_x)$. Write $j_x = j$ and $U(x) = [V, j_x]$. If $y \in U(x)$ then $U(y) = U(x)$. If $y \in U(x)^\perp$, then j_x centralizes j_y and $U(x) \cap U(y) = 0$, so that $U(y) \subseteq U(x)^\perp$.

Let $y \in U(x)^\perp$ and $z \in U(x)^\perp \cap U(y)^\perp$, and consider $S = \langle U(x), U(y), U(z) \rangle$. If $s \in S^\perp$ then $U(s) \subseteq S^\perp$. Thus, $S^\perp = \langle U(s) \mid s \in S^\perp \rangle$. Similarly, $S = \langle U(s) \mid s \in S \rangle$. There are $(3^6 - 1)/(3^2 - 1) = 91$ hyperbolic lines $U(s)$ in S, of which $(3^4 - 1)/(3^2 - 1) = 10$ lie in $U(s)^\perp \cap S$. Call each such $U(s)$ a Point, and each of these sets of 10 Points of S a Line. Then two distinct Points are in at most one Line. A standard counting argument shows that we have a projective plane of order 9. Since $U(s) \to U(s)^\perp \cap S$ induces a polarity without absolute Points, this contradicts a result of Baer [**10**, p. 152].

(XI) In (j), there are two possibilities for N^W, which are dual to examples $(2.1, 2)_{\text{iii}}$ of (5.13). In both cases, $|O_3(C)| \geqslant 3^3$. Let $\langle i \rangle$ be a subgroup of order 3 in $O_3(C)$ such that $\langle i \rangle^W$ is central in a Sylow 3-subgroup of $\Omega^-(6, 2)$. Then $[V, i] = W$, and there are exactly 9 lines of V fixed but not centralized by i, all of which lie in W.

There is also an element j of $O_3(C)$ such that $F = [V, j]$ has type $\Omega^+(4, 2)$ and for which there are exactly 3 lines of V fixed but not centralized by j, all of which lie in F. We may assume that $x \in L \subset F$, and that L is the fixed line of j through x. Note that $\langle j \rangle$ is transitive on L, while $\langle i \rangle$ is transitive on the second line L' of F through x. Also, $L^K \neq L'^K$ since $L^N \neq L'^N$.

If E is any plane containing L, then j fixes E and $K_{L, E}^L$ is transitive. If E' is any plane containing L' and fixed by i, then $K_{L', E'}^{L'}$ is transitive. Here, $E' \notin E^K$. (For otherwise, some conjugate i' of i acts nontrivially on E. By Sylow's theorem, $[E, i']$

and $[E, j]$ are in the same line-orbit of $K_{E'}^{E'}$, and hence of K, which is not the case.) Thus, E^K and E'^K are the two plane-orbits of K (cf. (6.3 i)).

In particular, all planes containing L lie in E^K. Moreover, K_E^E is transitive on the members of L^K lying in E (again by Sylow's theorem). By considering $(L, E)^K$, it follows that K_L is point-transitive on $L^\perp/L$.

Since $\langle j \rangle (C_K(L) \cap C_K(L^\perp/L)) \trianglelefteq K_L$, we have $K_L^{L^\perp/L} = H^{L^\perp/L}$ by the Frattini argument, where $H = N_K(\langle j \rangle)_L$. Then H is point-transitive on $L^\perp/L$, and hence also on $F^\perp$. Since $\langle j \rangle$ is transitive on L, it follows that H_x is also point-transitive on $F^\perp$. Here, H_x fixes both L and L'. Consequently, $K_{L'F}$ is transitive on $F^\perp$, and hence on $L'^\perp/L'$.

Some plane contains lines from both of the orbits L^K and L'^K (cf. (2.13)). In view of the transitivities we have found, K must be transitive on planes, which is not the case.

This completes the proof of (6.2).

THEOREM 6.4. *Let K be a subgroup of $\Gamma(V)$, where V has rank $n \geq 3$. If K is transitive on points, lines and planes, then either $K \geq \mathrm{Chev}(V)$ or K is A_7 or S_7 inside $O^+(6, 2)$.*

PROOF. By (5.15, 5.17, 5.21), we may assume that $n \geq 4$ and that V does not have type $\Omega^+(8, q)$. By (6.1), we may also assume that V is not unitary.

By Table IV, a suitable prime divisor r of the number of planes can be used in (6.2), except for cases listed in (6.3), and except when a candidate for r already divides the number of lines. A straightforward check shows that we must only consider the cases $\Omega^+(10, q)$, $\Omega^\pm(12, q)$, $Sp(8, q)$, $\Omega(9, q)$, $\Omega^-(14, q)$ and $\Omega^-(18, q)$.

For $m = 5$ and 6, the number of planes of an $\Omega^+(2m, q)$ space is divisible by $(q^{m-3} + 1)^2$. Thus, (6.2) applies, except in the case of $\Omega^+(12, 2)$.

Similarly, in the cases $Sp(8, q)$, $\Omega(9, q)$, and $\Omega^-(14, q)$, $(q^2 + 1)^2 \,|\, |K|$; while for $\Omega^-(18, q)$, $(q^3 + 1)^2 \,|\, |K|$. Thus, (6.2) applies, except in the case $\Omega^-(18, 2)$.

For $\Omega^-(12, q)$, the number of planes is divisible by $(q^2 + 1)^2$. Let r be a primitive divisor of $q^2 + 1$, and let $R \in Syl_r K_L$. Define W and C as usual. Since $R \neq 1$, W has type $\Omega^-(6, q)$ or $\Omega^+(8, q)$. The first possibility is dealt with exactly as in (6.2). If W has type $\Omega^+(8, q)$, let r^* be a primitive divisor of $q^3 - 1$. Then N contains an element of order r^*. Since $r^* \,|\, |N^{W^\perp}|$, we can apply (6.2) with r^* in place of r.

For $\Omega^-(10, q)$, the number of lines is divisible by $(q^3 - 1)/(q - 1)$. Let d be a primitive divisor of $q^3 - 1$. If $D \in Syl_d K$, then $C_V(D)$ has type $\Omega^-(4, q)$. Let r be a primitive divisor of $q^2 + 1$. Since $N_K(D)$ is point-transitive on $C_V(D)$, its order is divisible by r. An element of order r in $N_K(D)$ must centralize $C_V(D)^\perp = [V, D]$ (since $r \,|\, |N_K(D)^{[V,D]}|$). Consequently, (6.2) applies.

Similarly, in the case $\Omega^+(12, 2)$ we find that $|K|$ is divisible by $d = 2^4 + 1$. If $D \in Syl_d K$, then $C_V(D)$ has type $\Omega^-(4, 2)$, and an element of order $r = 5$ in $N_K(D)$ centralizes $C_V(D)^\perp$. If E is a plane of $C_V(D)^\perp$, then $r \,|\, |C_K(E)|$. Let $R \in Syl_r C_K(E)$, and define W, N and C as usual. Then W has type $\Omega^-(8, 2)$. By (2.12), K is $(1, 3)$-transitive. Thus, N^W is $(1, 3)$-transitive. By (5.17) and Table II, $N^W \geq \Omega^-(8, 2)$. Then also $C^W \geq \Omega^-(8, 2)$, and (2.4) applies.

In the case of $\Omega^-(18,2)$ set $d = 2^7 - 1$ and $r = 2^2 + 1$. If $D \in Syl_d K$, then $C_V(D)$ has type $\Omega^-(4,2)$. Once again, some element of order r centralizes $C_V(D)^\perp$. Let $R \in Syl_r K_L$. Then W has type $\Omega^+(6,2)$, $\Omega^-(10,2)$ or $\Omega^+(14,2)$, and N^W is transitive on lines and planes (by Table IV). In the second and third cases, N^W is also transitive on points (since $\theta_2 \supset \theta_1$ by Table II); then $N^W \geqslant \Omega(W)$, and we can proceed as usual. Suppose that $\dim W = 6$. Then $N^W \leqslant O^+(6,2) \cong S_8$. It follows that $C^W \geqslant A_7$ or A_8, and we can proceed as in steps (II) and (VI) of (6.2) in order to obtain a contradiction.

Finally, in the case of $Sp(10,2)$, K is $(1,2)$-transitive by (2.12). Let $R \in Syl_3 C_K(L)$. Then R has index 3^2 in a Sylow 3-subgroup of K, and $R \neq 1$ since 3^3 divides the number of planes. Note that $\dim W$ is 4, 6 or 8. If $\dim W > 4$ then N^W is transitive on points, lines and planes; but then $N^W \geqslant Sp(W)$, and we obtain the usual contradiction. Thus, $\dim W = 4$ and N^W is flag-transitive. It follows that $C^W \geqslant A_6$, and we can again proceed as in steps (II) and (VI) of (6.2) in order to complete the proof of (6.4).

REMARK. By Tables II and III, if $n \geqslant 5$ then plane-transitivity implies both point- and line-transitivity.

THEOREM 6.5. *Let V have rank $n \geqslant 3$. If K is a subgroup of $\Gamma(V)$ which is transitive on points, lines and n-spaces (of at least one type, when V has type $\Omega^+(2n,q)$), then either $K \geqslant \mathrm{Chev}(V)$ or K is A_7 or S_7 inside $O^+(6,2)$.*

LEMMA 6.5′. *Let V have rank $n \geqslant 4$, but not have type $\Omega^+(8,q)$. Then $\Gamma(V)$ has no subgroup K having the following properties:*

(i) *K is transitive on points and n-spaces (of each type, when V has type $\Omega^+(2n,q)$);*

(ii) *K has 2 orbits of lines, at most 3 orbits of $(1,2)$-flags, and at most 2 orbits of planes and $(1,n)$-flags (for each type on n-space, when V has type $\Omega^+(2n,q)$); and*

(iii) *if K has more than 3 orbits of $(2,3)$-flags, then it has 4 such orbits, as well as 2 orbits of planes and $(1,n)$-flags (of each type); if, in addition, $n = 4$, then there are 7 orbits of $(1,2,3)$-flags and 9 orbits of flags.*

PROOF OF (6.5) AND (6.5′). The proof parallels that of (6.4). By (5.15, 5.17, 5.21), we may assume that $n \geqslant 4$ and that V does not have type $\Omega^+(8,q)$. By (2.18) and Table IV, we can use the number of n-spaces in order to find a prime r required in (6.2), except for cases occurring in (6.3), or cases in which a candidate for r already divides the number of lines. Thus, this time we must consider the cases $\Omega^\pm(10,q)$, $\Omega^+(12,q)$, $SU(8,q)$, $SU(10,q)$, $Sp(10,2)$, $\Omega^-(12,2)$, $Sp(8,q)$ and $\Omega(9,q)$.

For $\Omega^+(10,q)$, $q \neq 2$, use primitive divisors d of $q^3 + 1$ and r of $q^2 + 1$. If $D \in Syl_d K$ then $C_V(D)$ has type $\Omega^-(4,q)$, and hence $N_K(D)$ has an element of order r. That element cannot act nontrivially on $C_V(D)^\perp$. Thus, (6.2) applies.

The case $SU(8,q)$, $q \neq 2$, is handled in the same manner, using primitive divisors d of $q^5 + 1$ and r of $q^3 + 1$. Also, for $SU(10,q)$ we find that $(q^3 + 1)^2$ divides $|K|$ (Table IV), so that (6.2) applies when $q \neq 2$.

For $SU(2n,2)$, $n = 4$ or 5, let M be an n-space. By (2.12), K_M^M is point-transitive if $n = 4$, and has order divisible by $2^5 - 1$ if $n = 5$. By hypothesis, K_M^M has at most 2

point-orbits. Now (5.1, 5.2) imply that K_M^M is transitive unless $n = 5$ and $K_M^M \geqslant SL(5, 2)$. If $n = 5$ and $7 \mid \mid K \mid$, then (6.2) applies. Thus, we may assume that K is $(1, n)$-transitive. By (6.1) and (6.5' ii, iii), there is a line L such that K_L is point-transitive on $L^\perp / L$. If $n = 4$, then $2^2 + 1 \mid \mid C_K(L) \mid$ by (5.12); if $n = 5$ then $2^5 + 1 \mid \mid K_L \mid$. In either case, (6.2) yields a contradiction.

For $\Omega^-(10, q)$, assume first that K is line-transitive. Then primitive divisors of $q^3 - 1$ and $q^2 + 1$ can be used as before. Now assume that we are in the situation of (6.5'). Then K has 2 orbits of $(1, 4)$-flags, and hence also 2 of $(3, 4)$-flags and at most 2 of planes. If K is plane-transitive then it is also line-transitive (by Table II), and (6.4) applies. If K has 2 plane-orbits, then $K_E^{E^\perp / E}$ is point-transitive for each plane E; thus, $q^2 + 1$ divides $\mid K_E \mid$, and (6.2) applies.

In the cases $\Omega^+(10, 2)$, $\Omega^-(12, 2)$ and $Sp(10, 2)$, K is $(1, 5)$-transitive (by Table IV and (2.12)). Let M be a 5-space. If $\mid K_M^M \mid$ is even, then $K_M^M = SL(5, 2)$ by (5.5). Thus, $\mid K_M^M \mid$ is odd. However, $\mid K \mid$ is even. (A Sylow 31-subgroup S fixes exactly two 5-spaces, which $N_K(S)$ permutes transitively.) Let t be an involution in K. Then we may assume that t fixes M, and hence centralizes M. But now it is easy to find another fixed 5-space of t not in $C_V(t)$.

For $\Omega^+(12, q)$, let d and r be primitive divisors of $q^4 + 1$ and $q^2 + 1$, respectively. Using a Sylow d-subgroup of K, we find (as earlier in the present proof) an element of order r centralizing a subspace W of type $\Omega^-(8, q)$. Let $R \in Syl_r C_K(W)$. If $R \in Syl_r K$, then N^W is transitive on the $(q^4 + 1)(q^3 - 1)/(q - 1)$ points of W, and (6.2) applies with $s = q^3 - 1$. Thus, we may assume that R is not Sylow in K (and that N^W is not point-transitive).

If K is line-transitive, then $r \mid \mid N_L^W \mid$ for every line L of W. Then also $r \mid \mid C_N(L)^W \mid$. Fix $x \in W$, and set $N^* = N^{x^\perp \cap W / x}$. Each point of $(x^\perp \cap W)/x$ is fixed by an element of N^* of order r. By (5.10), $N^{*(\infty)} = \Omega^-(6, q)$ or $2 \cdot PSL(3, 4)$. In the former case, (2.14) produces long root elements in K; while in the latter case, (6.2) applies with $s = q^3 + 1$.

Thus, we must now consider (6.5') for $\Omega^+(12, q)$. Here, N^W has at least 2 point-orbits and at most 4 orbits of $(1, 3)$-flags. As in the preceding paragraph, we find that there are 2 point-orbits and 4 orbits of $(1, 3)$-flags, both for N^W and for K. In particular, each plane-orbit of K has a member in W. Since $q^2 + q + 1$ divides the number of points of V but not the number of planes, there is a plane E such that $\mid K_E^E \mid$ is divisible by a primitive divisor r^* of $q^2 + q + 1$. We may assume that $E \subset W$. By the Frattini argument, $K_E^E = N_E^E$. Thus, $r^* \mid \mid N_E^W \mid$. An element of order r^* in N_E centralizes $W^\perp$, and hence also centralizes a line of V. Once again, (6.2) applies with $s = q^3 - 1$, and finishes the case $\Omega^+(12, q)$.

The cases $Sp(8, q)$ and $\Omega(9, q)$ are somewhat different from the preceding ones, in that a suitable prime is harder to find. Suppose first that we are in (6.5'). If L is a line, we may assume that $q^2 + 1 \mid \mid K_L \mid$, as otherwise (6.2) applies. In particular, $K_L^{L^\perp / L}$ is intransitive on both points and lines. Thus, we must be in (6.5' iii). Let (x_i, L_i), $i = 1, 2, 3$, be representatives of the orbits of $(1, 2)$-flags; let K_i be the corresponding group induced on $L_i^\perp / L_i$. Since K has 7 orbits of $(1, 2, 3)$-flags, we may assume that K_1 and K_2 have 2 point-orbits, while K_3 has 3 point-orbits. Since

K_1 has at least 2 line-orbits, it has at least 3 flag-orbits; and so does K_2. Thus, K_3 has at most 3 flag-orbits. Then $(K_3)_y$ is transitive on $y^\perp/y$ for each point y of $L_3^\perp/L_3$. Consequently, K_3 is line-transitive, which is not the case.

Thus, we must be in the situation of (6.5): K is transitive on the $(q^4 + 1)(q^3 + 1)(q^2 + 1)(q^2 + q + 1)$ lines of V. By (2.12), if F is a 4-space then $q^2 + q + 1$ divides the length of each line-orbit of K_F^F. Since K is not flag-transitive, K_F^F fixes a point or a plane (by (5.1)). In either case, there is a plane $E \subset F$ such that K_{EF}^E is transitive. By Sylow's theorem, K_F^F is transitive on the set of such planes. Hence, $K_E^{E^\perp/E}$ is also transitive.

Consequently, there is an orbit of $|L^K|\alpha(q + 1) = |F^K|\epsilon(q^2 + q + 1)$ triples (L, E, F), where α and ϵ are the numbers of planes E containing L and contained in F, respectively. Here, ϵ is 1 or q^3 (cf. (2.7 i) and (2.14)), and hence so is α (by Table IV). Then $\alpha = 1$. (For, if $\alpha = q^3$, a Sylow p-subgroup of K_L acts transitively on the corresponding q^3 points of $L^\perp/L$. Then these are the points not in some line of $L^\perp/L$. Thus, K_L fixes a 4-space containing L, and this contradicts (2.12).)

Note that $K_E^E \ngeq SL(3, q)$ if $q > 2$. For, suppose that $K_E^E \geqslant SL(3, q)$ and $q \neq 2$. Then $C_K(E^\perp/E)^E \geqslant SL(3, q)$. By (2.14), K_E has a subgroup $S \cong SL(3, q)$ fixing two planes. Then S_x has an element of order p centralizing $x^\perp/x$. This is impossible by (2.1) and (2.4).

Now suppose that $q \neq 2, 8$. Recall that $K_L = K_{LF}$, and that $K_{LE}^{E^\perp/E}$ is transitive. There is a prime $r \neq 3$ dividing $q + 1$ but not dividing $\log_p q$. Let $R \in Syl_r K_L$. Then $R^E = 1$ by the preceding paragraph and (2.7 i), so that $\dim W = 6$ or 7, N^W is a line-transitive subgroup of $\Gamma Sp(6, q)$, $\Gamma O^+(6, q)$ or $\Gamma O(7, q)$, $q^2 + 1 \mid |N^W|$, and hence (6.2) applies.

Thus, q is 2 or 8. Let d be a primitive divisor of $q^3 - 1$, and let $D \in Syl_d K_E$. Then $N_K(D)$ is transitive on the $2(q + 1)$ fixed 4-spaces of D. Thus, $|K|$ is even, and hence so is $|K_L^L| = |K_{LE}^L|$. Then $K_E^E \geqslant SL(3, q)$. A Sylow 3-subgroup of K_{EF} centralizes a line. If $R \in Syl_3 C_K(L)$, we obtain the same contradiction as before.

This completes the proof of (6.5) and (6.5').

The remainder of this section degenerates into those types of technical results which dominated §5 (and resemble (6.5″)). They appear here only because they will be needed in (8.3): there, (6.6) and (6.8) are used to deal with $(n - 2 \mid 2)$, while (6.7) and (6.9) are used for $(n - 3 \mid 3)$.

LEMMA 6.6. *Suppose that $n \geqslant 4$ but V does not have type $\Omega^+(8, q)$. Then $\Gamma(V)$ has no subgroup K with the following properties*:

(i) *K is point-transitive;*

(ii) *K has 2 orbits of lines, planes and $(1, 2)$-flags;*

(iii) *K has 3 orbits of $(1, 3)$- and $(2, 3)$-flags; and*

(iv) *if $n = 8$ then K has 2 plane-orbits and 3 orbits of $(3, 4)$-flags.*

PROOF. By (ii) and (iii), there is a line L such that K_L is transitive on $L^\perp/L$. By Table IV and (2.18), $|K_L|$ is divisible by a prime required in (6.2), unless V has type $Sp(10, 2)$, $\Omega^+(12, 2)$ or $\Omega^-(18, 2)$ (cf. (6.3)). Thus, by (6.2) we only need to consider these three cases.

If V has type $Sp(10,2)$, then $7 \,|\, |K_L^{L^\perp/L}|$, and the proof of (6.2) can be repeated in order to obtain a contradiction.

If V has type $\Omega^+(12,2)$, set $d = 2^4 - 1$ and $r = 3$. Since $d \,|\, |K_L|$, a Sylow d-subgroup of K centralizes a subspace W_0 of type $\Omega^+(4,2)$. There is then a group of order 9 centralizing $W_0^\perp$. Let E be a plane of $W_0^\perp$; if possible, choose E so that K_E^E is transitive (such planes exist in V, by (ii) and (iii)). Let $R \in Syl_3 C_K(E)$. Then $W = C_V(R) = W_0^\perp$. If 7 divides $|N^W|$, then an element of order 7 in N centralizes $W^\perp$, and (6.2) applies. Since $K_E^E = N_E^E$ by the Frattini argument, K_E^E cannot be transitive. Thus (by (ii)), all planes of W lie in the same W-orbit. Consequently, N^W is transitive on planes; but then 7 again divides $|N^W|$.

If V has type $\Omega^-(18,2)$, then (iv) provides us with a plane E such that K_E is point-transitive on $E^\perp/E$. Then $2^6 + 1 \,|\, |K_E|$, and (6.2) yields a contradiction.

LEMMA 6.7. *Suppose that $n \geqslant 4$ but V does not have type $\Omega^+(8,q)$. Then $\Gamma(V)$ has no subgroup K with the following properties*:

(i) K is $(1,2)$-transitive;

(ii) K has 2 orbits of planes, $(1,2,3)$-flags and 4-spaces; and

(iii) K has 3 orbits of $(1,4)$- and $(3,4)$-flags.

PROOF. By (6.1), V is not unitary. Set $K^* = K_x^{x^\perp/x}$. Then K^* is transitive on points and has 2 orbits of lines and of $(1,2)$-flags. Clearly, (6.2) applies to neither K nor K^*. However, by (ii) and (iii) there is a plane E such that K_E is point-transitive on $E^\perp/E$. By Table IV and (2.18), the failure of (6.2) implies that V has type $Sp(12,2)$, $\Omega^-(12,2)$, $\Omega^+(14,2)$, $Sp(8,q)$, $\Omega(9,q)$, $\Omega^+(10,q)$, $\Omega(2n+1,3)$ or $\Omega^-(2n+2,2)$.

If V has type $\Omega^+(14,2)$, then $r = 2^2 + 1 \,|\, |K_E|$. Let $R \in Syl_r K_L$. Then N^W is a $(1,2)$-transitive subgroup of $\Gamma O^-(10,2)$, $\Gamma O^+(8,2)$ or $\Gamma O^+(6,2)$. In the first case, $2^4 + 1$ divides the number of lines of W, and (6.2) applies to K. In the remaining cases, N^W contains $\Omega^+(W)$ or A_7 (by (5.15), (5.21)), and we can proceed exactly as in steps (II) and (VI) of the proof of (6.2).

If V has type $Sp(12,2)$, a Sylow 3-subgroup of K_E has all its point-orbits on $E^\perp/E$ of length at least 9. Thus, there is a 3-group centralizing E but fixing no point of $E^\perp/E$. Let $R \in Syl_3 K_{xL}$ for $x \in L$. Then N^W is a $(1,2)$-transitive subgroup of $Sp(4,2)$ or $Sp(6,2)$. Either N^W or $(N_x)^{(x^\perp \cap W)/x}$ contains $Sp(2,4)$, and we can again proceed exactly as in (6.2).

If V has type $\Omega(2n+1,3)$, $n \geqslant 5$, let r be a primitive divisor of $3^{n-3} + 1$, and note that $r \,|\, |C_K(E)|$. Let $R \in Syl_r C_K(L)$. Then N^W is a $(1,2)$-transitive subgroup of $\Gamma O(7,3)$. By (6.3 i), it is even flag-transitive, and we obtain a contradiction from (2.6) just as in step (II) of (6.2).

If V has type $\Omega^-(2n+2,2)$, $n \neq 5$, let r be a primitive divisor of $2^{n-2} + 1$, and let $R \in Syl_r C_K(L)$. Then N^W is a $(1,2)$-transitive subgroup of $\Gamma O^-(8,2)$. By (6.3 i) and (2.1), $|O_3(N_x^W)| \geqslant 27$. Thus, N_x^W fixes a second point y, and is point-transitive on $\langle x, y \rangle^\perp$. Then y is uniquely determined, and $\{x, y\}$ is an imprimitivity block for the action of N on the points of W. Since W has an odd number of points, this is impossible.

Consider the remaining cases. If E is any plane then K_E^E is flag-transitive. Thus, by (2.7 i), $C_K(E^\perp/E)^E$ is transitive on E. Let F be any 4-space. If $E \subset F$ then K_{EF}^E is transitive. Thus, K_F^F is transitive. This contradicts (ii) and (iii).

LEMMA 6.8. $\Gamma O^+(10, q)$ *has no subgroup transitive on points and having* 2 *orbits of each type of* 5-*spaces,* 3 *orbits of planes and of each type of* $(1,5)$-*flags, and* 4 *orbits of* $(1, 3)$-*flags.*

PROOF. Let K be such a subgroup. Let r be a primitive divisor of $q^3 - 1$. There are two plane-orbits E_i^K, $i = 1, 2$, such that K_{E_i} is transitive on E_i. In particular, $r \mid |K_{E_i}|$. Let $R \in Syl_r K$; note that a Sylow r-subgroup of $\Gamma O^+(10, q)$ is cyclic, and fixes exactly 2 planes.

We may assume that R fixes E_1 and E_2. Clearly, R centralizes $E_i^\perp/E_i$. Let M be a 5-space containing E_1. Since R fixes only one plane of M, Sylow's theorem shows that M contains no member of E_2^K. But there are only 2 orbits of 5-spaces having the same type as M. Thus, we can choose E_1 so that K_M^M is transitive (recall that there are 3 orbits of each type of $(1, 5)$-flags). Now (5.5) implies that $K_M^M \geqslant SL(5, q)$.

Set $Q = O_p(\Gamma O^+(V)_M)$. Then Q is elementary abelian of order q^{10}, and K_M^M acts on Q as it does on skew-symmetric 2-tensors. Hence K_M acts irreducibly. Since Q contains long root groups, we have $Q \cap K = 1$ by (2.4). Since $H^1(SL(5, q), Q) = 0$ (Jones and Parshall [30]), K_M must fix a second 5-space. But then K_M contains long root groups, which is not the case.

LEMMA 6.9. *If* K *is a* $(1, 2)$-*transitive subgroup of* $\Gamma O^+(14, q)$, *then* $K \geqslant \Omega^+(14, q)$.

PROOF. Let d and r be primitive divisors of $q^5 + 1$ and $q^2 + 1$, respectively. Note that d divides the number of lines. If $D \in Syl_d K$, then $C_V(D)$ has type $\Omega^-(4, q)$. By Sylow's theorem, $r \mid |N_K(D)|$. An element of order r in $N_K(D)$ cannot act nontrivially on $[V, D]$. Thus, $r \mid |K_L|$ for some line L.

Let $R \in Syl_r K_L$, and define W and N as usual. Then W has type $\Omega^+(6, q)$ or $\Omega^-(10, q)$, and N^W is $(1, 2)$-transitive. The first possibility for N^W is handled as in (6.2) (cf. (5.15)). If W has type $\Omega^-(10, q)$ then $(q^3 - 1)/(q - 1)$ divides the number of lines of W, so that (6.2) applies with $s = q^3 - 1$.

REMARK. A similar argument deals with $(1, 2)$-transitive subgroups of $\Omega^+(2m, q)$ whenever m is relatively small.

7. Theorem 1.1: Reductions. Let G and K be as in (1.1), but assume that the pair G, K is *not* listed in that theorem. We will assume, until §9, that G actually acts on the vector space V as a group of semilinear maps. We may also assume that rad $V = 0$. The case of V of type $\Omega^+(6, q)$ will be postponed until §10.

Note that K is a maximal subgroup of G.

Write $\varphi = 1_K^G = 1 + \chi + \zeta$ with χ and ζ irreducible characters.

Let B be a Borel subgroup of G. To each irreducible character $(\alpha \mid \beta)$ or $(\alpha \mid \alpha)^+$ of the Weyl group of G there naturally corresponds an irreducible constituent of 1_B^G, also called $(\alpha \mid \beta)$ or $(\alpha \mid \alpha)^+$, as explained in §4.

Throughout §§7–9, frequent reference will be made to Tables II and III of §3. Recall from §4 the fact that, for example, the permutation character $1(1^2; n - 2)$ appearing in Table II is just the permutation character θ_{12} of G on the set of $(1, 2)$-flags of V. The proof of (1.1) will involve computing inner products of φ with several such permutation characters. Some such characters not explicitly listed in a table are implicitly in the table, because of (4.4); for example, $\theta_{23} = \theta_{13}$. Most relevant inner products for D_n are in Table II, not Table III, because of (3.5, 6, 7); consequently, we will be able to deal with $\Omega^+(2n, q)$ and $O^+(2n, q)$ simultaneously in most situations.

LEMMA 7.1. *Every noncentral normal subgroup of K is irreducible on V.*

PROOF. Let N be a noncentral reducible normal subgroup, and let W be a minimal proper N-subspace. Then rad $W = 0$ or W. If $N = K$, it is straightforward to check whether G has rank 3 on W^G; only instances on our list can occur (namely, (1.1 i, ii, iii, iv) arise in the present context). Thus, K is irreducible and (by Clifford's theorem) $V = W_1 \oplus \cdots \oplus W_r$ with $W_1 = W$ and each W_i an N-subspace conjugate to W_1 under K. The maximality of K forces K to be the stabilizer of the set $\{W_1, \ldots, W_r\}$. Once again it is easy to check whether G has rank 3 on W^G.

LEMMA 7.2. *If $n \geq 2$ and K is transitive on points then K contains no long root element.*

PROOF. Deny! Then the subgroup N of K generated by long root elements is irreducible by (7.1), and $N = \langle t^N \rangle$ for some long root element t by (2.3). Thus, (2.4) applies. (This time, (1.1 viii, ix, x) arise.)

LEMMA 7.3. *We may assume that $\chi \in 1_B^G$.*

PROOF. If $\langle \varphi, 1_B^G \rangle = 1$, then Seitz's result (2.6) applies if $n \geq 2$. If $n = 1$, then K is transitive on points, so that (2.7), (2.8) and Dickson [12, Chapter 12] can be used to find K. In all cases, it is straightforward to check whether or not K is maximal and G has rank 3 on G/K. (The cases (1.1 v, vi, vii) occur here.)

LEMMA 7.4. $\zeta \notin 1_B^G$, *except perhaps if G is $Sp(2n, 2)$ and ζ is $(n - 1, 1 \mid 0)$, $(n - 1 \mid 1)$ or $(0 \mid n)$.*

PROOF. Suppose $\varphi \subset 1_B^G$. Then, excluding the stated possibilities for ζ in the case $Sp(2n, 2)$, we have $|G : K| = \varphi(1) \equiv 1 \pmod{p}$ by (4.7). By (2.2), K is then reducible on V. (The cases (1.1 i, ii) appear here.)

Convention. Throughout §§7, 8, we will assume that $\zeta \notin 1_B^G$. The inevitable $Sp(2n, 2)$ case will be dealt with in §9.

LEMMA 7.5. $n > 2$.

PROOF. By (7.3, 4), K has 2 point-orbits, so that (2.8b) applies when $n = 1$. If $n = 2$, then $\chi \in \theta_1$ or θ_2, or $\chi = 1_B^G - \theta_1 - \theta_2 + 1_G$ (by 4.5(i) and Table II). Then using Table II in order to compute $\langle \varphi, 1_B^G \rangle$, we find that (5.13) or (5.14) applies. The rank on G/K is straightforward but tedious to check geometrically. (The cases (1.1 v, vi, vii) occur here.)

8. Transitivity. The most enjoyable part of the proof of (1.1) is the following crucial fact.

THEOREM 8.1. *K is transitive on points*.

PROOF. Assume that K is not point-transitive. Then $\langle \chi, \theta_1 \rangle = 1$, so that $\chi = (n-1, e \mid 1-e)$ for $e = 0$ or 1, and K has just $\langle \varphi, \theta_1 \rangle = 2$ point-orbits Ω, Ω'. By (4.5) and the argument following it, we have $\langle \varphi, \theta_{n-e} \rangle = 2$ and $\langle \varphi, \theta_{1,n-e} \rangle = 3$. Thus, K has 2 orbits on $(1, n-e)$-flags (of each type if G has Weyl group $W(D_n)$). There must then be an $n - e$-space whose stabilizer is transitive on its points; we may assume that Ω contains this $n - e$-space. By (2.13) there is an $n - e$-space W not in Ω which contains an $n - e - 1$-space of Ω.

Since flags behave differently for types B_n and D_n, we will consider these cases separately.

Type B_n. Each flag has the form $(V_1, V_2, \dots, V_n)$, and there is an i with $V_i \subseteq \Omega$, $V_{i+1} \not\subseteq \Omega$. Flags having different values of i cannot be in the same K-orbit. Every value of i between 0 and $\dim W = n - e$ can occur (using an i-space inside $W \cap \Omega$ if $i < n - e$). However, by (4.5) we have $\langle \varphi, 1_B^G \rangle = 1 + n - e$, so any two flags with the same i are in the same K-orbit. Set $i = 0$, and apply (2.16) with $x = V_1$ in Ω', in order to see that K is listed in (1.1).

Type D_n. Each flag has the form $(V_1, \dots, V_{n-2}, V_n, V_n')$, where V_n and V_n' are n-spaces on an $n - 1$-space $V_{n-1} \supset V_{n-2}$, and are in different G-orbits.

Once again, K has $\langle \varphi, 1_B^G \rangle = 1 + n - e$ flag-orbits. As above, at least $n - 2$ of these are accounted for by flags having $V_{n-2} \not\subseteq \Omega$. We will exhibit $3 - e$ orbits of flags having $V_{n-2} \subseteq \Omega$.

By (4.5), there are $\langle \varphi, \theta_2 \rangle = 2$ orbits of lines: those contained in Ω and those meeting both Ω and Ω' (the latter type can be seen in W). In particular, Ω' contains no $n - 1$-space. By (4.5), there are $\langle \varphi, \theta_{n-1} \rangle = 3 - e$ orbits of $n - 1$-spaces and $\langle \varphi, \theta_{1,n-1} \rangle = 4 - e$ orbits of $(1, n-1)$-flags. There are thus $2 - e$ orbits of $n - 1$-spaces contained in Ω, and one further orbit of $n - 1$-spaces. One of the latter $n - 1$-spaces contains an $n - 2$-space in $W \cap \Omega$. Thus, each of the $3 - e$ orbits of $n - 1$-spaces has a member containing a V_{n-2} contained in Ω. This produces at least $3 - e$ further flag-orbits.

As before, (2.16) applies, and completes the proof of (8.1).

REMARK. Cases (iii)–(v) of (2.16) clearly cannot occur in (1.1). Also, $\Omega(7, 3)$ has rank 4 on the cosets of $W(E_7)'$, with 4 distinct subdegrees (Fischer [17, (15.3.16)]).

LEMMA 8.2. $\chi \in \theta_2$ or θ_3.

PROOF. If $\chi \notin \theta_2, \theta_3$ then (by Tables II and III) K is transitive on points, lines and planes, so that (6.4) applies. (Recall that the case of $\Omega^+(6, q)$ has been postponed until §10.)

LEMMA 8.3. *Either $n \le 3$ or V has type $\Omega^+(8, q)$.*

PROOF. Assume that $n \ge 4$ and V does not have type $\Omega^+(8, q)$. Suppose first that $\chi \ne (n - 2 \mid 2), (n - 3 \mid 3)$. By Tables II and III, $\chi \notin \theta_n$, so that K is transitive on

n-spaces (of each type, if G has type D_n). Again by those tables, K has at most 2 orbits of lines and 3 orbits of $(1, 2)$-flags. Then by (6.5), K has 2 line-orbits. Thus, χ is $(n - 2, 2 \mid 0)$ or $(n - 2, 1 \mid 1)$. By Tables II and III, K has at most 2 orbits of planes and $(1, n)$-flags (for each type of n-spaces in the case of $\Omega^+(2n, q)$). If $\chi = (n - 2, 2 \mid 0)$ then K has 3 orbits of $(2, 3)$-flags. If $\chi = (n - 2, 1 \mid 1)$ then K has 2 orbits of planes and $(1, n)$-flags (for each type of n-spaces), 4 orbits of $(2, 3)$-flags and 7 orbits of $(1, 2, 3)$-flags; moreover, if $n = 4$, there are 9 orbits of flags by Table II and (4.5 i). Thus, $(6.5')$ eliminates this possibility for K.

Now suppose that $\chi = (n - 2 \mid 2)$. Again by Tables II and III, K is point-transitive, has 2 orbits of lines, planes and $(1, 2)$-flags, and 3 orbits of $(2, 3)$-flags (except when G has Weyl group $W(D_5)$, in which case there are 4 such orbits); and if $n = 8$, then K has 2 orbits of planes and 3 of $(3, 4)$-flags. By (6.6), G must have Weyl group $W(D_5)$. But here, Table III implies that K has 2 orbits of each type of 5-spaces, 3 orbits of planes and each type of $(1, 5)$-flags, and 4 orbits of $(1, 3)$-flags. Thus, (6.8) eliminates this case.

Finally, suppose that $\chi = (n - 3 \mid 3)$. We may assume that G does not have Weyl group $W(D_5)$ (as otherwise we are back in the case $(3 \mid 2)$ by Table III). This time, K is $(1, 2)$-transitive, has 2 orbits of planes, $(1, 2, 3)$-flags and 4-spaces, and 3 orbits of $(3, 4)$-flags (except that there are 4 such orbits if G has Weyl group $W(D_7)$). Now (6.7) and (6.9) eliminate this case.

LEMMA 8.4. $n \neq 3$.

PROOF. Suppose that $n = 3$. By (8.1), (8.2) and Table II, $\chi = (1, 1 \mid 1)$, $(1 \mid 2)$ or $(0 \mid 3)$.

If $\chi = (1, 1 \mid 1)$, then (by Table II) K is transitive on points and planes, has 2 orbits of lines and $(1, 3)$-flags and 3 orbits of $(1, 2)$-flags. This is impossible by (5.16).

If $\chi = (1 \mid 2)$, then K is transitive on points, has 2 orbits of lines, planes and $(1, 2)$-flags, 3 orbits of $(1, 3)$-flags and 4 orbits of flags. Then (5.19) describes all possible groups K, and (5.20) discusses which produce rank 3 permutation representations.

Finally, if $\chi = (0 \mid 3)$, then K is $(1, 2)$-transitive and has 2 orbits of planes, $(1, 3)$-flags and flags. This is impossible by (5.18).

LEMMA 8.5. V does not have type $\Omega^+(8, q)$.

PROOF. Suppose that V has type $\Omega^+(8, q)$. Define $G^+ = \Omega^+(8, q)$, K^+, φ^+ and χ^+ as in (4.1). Then $\langle \varphi^+ - 1 - \chi^+, 1_B^G \rangle = 0$.

If $\langle \chi^+, \theta_2 \rangle = 0$, then K^+ is transitive on points and lines. This contradicts (5.21).

Thus, $\chi^+ \subset \theta_2$. If $\chi^+ \in \theta_1$, the proof of (8.1) applies with K^+ in place of K. Consequently, $\chi^+ \subseteq \theta_2 - \theta_1$.

Suppose that $\chi^+ = (2 \mid 2)^+$. Let τ denote a triality automorphism of $P\Omega^+(8, q)$, and consider $(K^+ \langle -1 \rangle / \langle -1 \rangle)^\tau$. This group behaves as in the preceding paragraph, if $\chi^{+\tau} = (3, 1 \mid 0)$.

Thus, $\chi^+ = (2, 2 \mid 0)$, $(2, 1 \mid 1)$ or $(2 \mid 2)$.

If $\chi^+ = (2, 2 \mid 0)$ then (by Table III) K^+ is transitive on each class of $(1, 4)$-flags, and has 2 orbits of lines, $(1, 2)$-, $(1, 3)$- and $(2, 3)$-flags and 3 of flags. Since χ^+ is τ-invariant, K^+ also has 2 orbits of each class of $(2, 4)$- and $(1, 2, 4)$-flags. By (5.22), K^+ is $W(E_8)'$, embedded in $\Omega^+(8, 3)$. Then G has rank at least 4 on the cosets of K (Fischer [**17**, (18.3.12)]).

If $\chi^+ = (2, 1 \mid 1)$ then χ^+ is again τ-invariant. Now Table III and (5.23) yield a contradiction.

Finally, if $\chi^+ = (2 \mid 2)$ then G has type B_4, and $\chi = (2 \mid 2)$. This time, Table II and (5.25) yield a contradiction.

9. Proof of Theorem 1.1: Conclusion. In view of §§7, 8, the proof of (1.1) is complete except for two situations: (A) $G = Sp(2n, 2)$ and $\varphi \subseteq 1_B^G$; and (B) G has elements not arising from semilinear transformations of V. (Recall however, that the case of V of type $\Omega^+(6, q)$ has been postponed until §10.)

(A) $G = Sp(2n, 2)$, $n \geqslant 3$. (Of course, $Sp(4, 2) \cong S_6$.) By (2.2), $\mid G : K \mid$ is even. Thus, by (4.7), $\varphi = 1 + \chi + \zeta$ with $\zeta = (n - 1, 1 \mid 0)$, $(n - 1 \mid 1)$ or $(0 \mid n)$.

LEMMA 9.1. $2^{2n-3} \mid \varphi(1)$. *In particular,*
(i) 2^{n-1} *is the highest power of 2 dividing* $\chi(1)$ *if* ζ *is* $(n - 1, 1 \mid 0)$ *or* $(n - 1 \mid 1)$, *and*
(ii) 4 *is the highest power of 2 dividing* $\chi(1)$ *if* $\zeta = (0 \mid n)$.

PROOF. Set $Q = C_G(x^\perp / x)$, and consider its action on the $\varphi(1)$ cosets of K. The orbit of K has length $\mid Q : K \cap Q \mid$. By (7.1), (2.3) and [**33**], $K \cap Q$ does not contain long root elements of $\Omega(2n + 1, 2)$. By (2.1), $\mid K \cap Q \mid \leqslant 4$. Thus, $\mid Q \mid / 4 = 2^{2n-3}$ divides $\varphi(1) = 1 + \chi(1) + \zeta(1)$. By (4.7), $\varphi(1) = 2^{n-1}(2^n \pm 1) + \chi(1)$ or $4(2^{2n-3} + 2^{n-2} + 2^{n-3} + 1)/3 + \chi(1)$. Assertions (i) and (ii) are now immediate.

LEMMA 9.2. *If* $n \geqslant 4$ *then* $\chi \in \theta_{23}$.

PROOF. Suppose that $\chi \notin \theta_{23}$. If $\zeta = (0 \mid n)$ then (6.4) applies (by Table II). Thus, $\zeta = (n - 1, 1 \mid 0)$ or $(n - 1 \mid 1)$. By Table II, K has 2 orbits of points, lines and planes, and 3 orbits of $(1, 2)$-, $(1, 3)$- and $(2, 3)$-flags. Call the point-orbits Ω and Ω'. We may assume that Ω contains a plane. Then each line meets Ω' in 0 or k points, for some $k \geqslant 1$. By (2.11), $k = 1$.

Let $x \in \Omega'$. Then K_x is transitive on the planes through x. If r is a primitive divisor of $2^{n-2} + 1$ (use $r = 2^3 - 1$ if $n = 5$), then $r \mid \mid K_x \mid$. Let $R \in Syl_r K_x$, and set $W = C_V(R)$, $N = N_K(R)$ and $C = C_N(W^\perp)$ as in §6. Then N^W is a subgroup of $Sp(4, 2)$ having at most 2 orbits of points and lines. Since $k = 1$, it follows that $\mid W \cap \Omega \mid = 5$ and $N^W \trianglerighteq \Omega^-(4, 2)$. Then $C^W \trianglerighteq \Omega^-(4, 2)$ as well.

Set $Q = C_G(x^\perp / x)$. Then $\mid C \cap Q \mid = 4$. Since $K_x^{x^\perp / x}$ is point-transitive, (2.1) implies that $K \geqslant Q$, which is ridiculous.

LEMMA 9.3. $n \leqslant 4$.

PROOF. Suppose that $n \geqslant 5$. Using Mayer [**35**, (1.1)], we find that θ_{13} has 15 irreducible constituents, 12 of which have even degree. Of these, only the character

$(n - 3 \mid 3)$, $n \geqslant 6$, has degree $\equiv 4 \pmod 8$, by (4.8). Suppose that $\zeta = (0 \mid n)$. By (9.1), $\chi = (n - 3 \mid 3)$. By (4.8),

$$|G : K| = 4(2^{2n-3} + 2^{n-2} + 2^{n-3} + 1)/3 + 4(2^{2n} - 1)(2^{2n-2} - 1)(2^{n-2} + 1)/315$$

is not divisible by 2^{2n-3}. Thus, $\zeta \neq (0 \mid n)$ by (9.1).

By (9.2), $\chi \in \theta_{23}$, and by (9.1), $\chi(1) \equiv 2^{n-1} \pmod{2^n}$. By (4.8), χ must be $(3, 1 \mid 1^2)$, $(3, 1^2 \mid 1)$ or $(2 \mid 2, 1)$. However, $1 + \zeta(1) + \chi(1) \nmid |G|$ in each of these cases.

PROPOSITION 9.4. *There is no counterexample to* (1.1) *when* $G = Sp(2n, 2)$.

PROOF. We must eliminate the cases $n = 3, 4$. Let $n = 4$. As in (9.3), we find that φ is $(4 \mid 0) + (3, 1 \mid 0) + (2, 1^2 \mid 0)$ or $(4 \mid 0) + (3 \mid 1) + (2, 1^2 \mid 0)$. In neither case is there a divisor d of $|G|$ such that $\varphi(1)d(\varphi(1) - d - 1)/\chi(1)\zeta(1)$ is a square, contrary to Frame's theorem [62, (30.1)].

Similarly, if $n = 3$, a routine calculation shows that there are only 3 characters $\varphi \subseteq 1_B^G$ such that $\langle \varphi, 1 \rangle = 1$, $\langle \varphi, \varphi \rangle = 3$ and $\varphi(1) \mid \mid G \mid$. Once again, none of these satisfies Frame's theorem. This proves (9.4).

(B) *Graph automorphisms.* There are two cases to consider: $Sp(4, 2^e)$ and $P\Omega^+(8, q)$. (The graph automorphisms of $\Omega^+(2n, q)$, $n > 4$, of order 2 were already considered when we discussed $O^+(2n, q)$.)

As usual, we have $\varphi = 1 + \chi + \zeta$. Define K^+, φ^+, χ^+, ζ^+ and B as in (4.1). By (2.6), we may assume that $\langle \varphi^+, 1_B^{G^+} \rangle > 1$. By (4.1d), we may assume that $\chi^+ \subseteq 1_B^{G^+}$. If $\varphi^+ \subseteq 1_B^{G^+}$ then (2.2) applies by (4.7), except in the easy case $Sp(4, 2)$. We may thus also assume that $\langle \zeta^+, 1_B^{G^+} \rangle = 0$.

Suppose that $Sp(4, 2^e) < G \leqslant \operatorname{Aut} Sp(4, 2^e)$. If χ^+ is irreducible, use (5.6). If χ^+ is reducible then $\chi^+ = (1, 1 \mid 0) + (0 \mid 2)$. Here, K^+ has 3 point-, line- and flag-orbits. No such group exists. (If L is any line then $(K^+)_L^L$ would be transitive, and hence K^+ would be point-transitive.)

Next suppose that $P\Omega^+(8, q) < G \leqslant \operatorname{Aut} P\Omega^+(8, q)$. If χ^+ is irreducible, the arguments in §§7, 8 apply with K^+ replacing K. We may thus assume that χ^+ is reducible. By Table III, θ_2 is G-invariant, so $\chi^+ \subseteq \theta_2$ by (5.22). Again by Table III, χ^+ must be a sum of 2 or 3 of the characters $(3, 1 \mid 0)$, $(2 \mid 2)^+$ and $(2 \mid 2)^-$.

By (5.25), the case $\chi^+ = (3, 1 \mid 0) + (2 \mid 2)^+ + (2 \mid 2)^-$ is impossible. If χ^+ has just 2 irreducible constituents, we can use $\operatorname{Aut} P\Omega^+(8, q)$ in order to assume that $\chi^+ = (2 \mid 2)^+ + (2 \mid 2)^-$. In this case, $G \leqslant P\Gamma O^+(8, q)$, and G has type B_4. Thus, this case was handled in §§7, 8.

This completes the proof of (1.1).

10. Proof of Theorem 1.2. The more complicated results appearing in §§2–5 are not required for (1.2), as the following remarks indicate.

REMARK 10.1. We will require (3.8) when θ_i is regarded as the permutation character on *i*-spaces of a group lying between $SL(n, q)$ and $\Gamma L(n, q)$, and when θ_{ij}, etc., are viewed similarly. The assertions in (3.8) can then be proved directly and easily as follows. For (i), consider $\langle \theta_{i-1}, \theta_{i-1} \rangle$, $\langle \theta_i, \theta_{i-1} \rangle$ and $\langle \theta_i, \theta_i \rangle$. Statements (ii) and (iii) are especially easy. For (iv)–(vi), proceed as in the following example. In order to compute $\langle \theta_3 - \theta_2, \theta_{34} \rangle$, calculate the number of orbits of the stabilizer of a $(3, 4)$-flag in its actions on both 3-spaces and 2-spaces, and subtract.

REMARK 10.2. From §2, we will require the relatively elementary results (2.7, 2.15). All transitivity results in §5 will be avoided. Instead, we will refer to [23], [58], [46], [31], [32] for transitivity statements. It should be noted that the results quoted from [32] are merely simple cases of Perin's method, and should be straightforward exercises for anyone who has read [46] or examined (6.1, 2).

We now begin the proof of (1.2). We are given $PSL(n, q) \leqslant G \leqslant \text{Aut } PSL(n, q)$ and $K \leqslant G$. Define $G^+ = G \cap P\Gamma L(n, q)$, K^+, φ, φ^+, χ, χ^+, ζ, ζ^+ and B as in (4.1).

If $\langle \varphi^+, 1_B^{G^+} \rangle = 1$ then K^+ is flag-transitive. Thus, Higman [23] can be applied if $n > 2$, while Dickson [12, Chapter 12] provides a short list to check if $n = 2$.

Similarly, if $n = 2$ and $\langle \varphi, 1_B^G \rangle \neq 1$ then $\langle \varphi, 1_B^G \rangle = 2$ and K has exactly 2 point-orbits. This time a longer list of possibilities for $K \cap SL(2, q)$ can be obtained from Dickson [12, Chapter 12]. In most cases there are elementary geometric arguments showing that the rank of G on G/K is greater than 3. The remaining ones are dealt with by arithmetic arguments. (For example, $|G : K| - 1 - q$ is divisible by the degree of an irreducible character of $PSL(2, q)$, namely, $q \pm 1$ or $\frac{1}{2}(q \pm 1)$.)

If $\varphi^+ \subseteq 1_B^{G^+}$, then $|G^+ : K^+| \equiv 1 \pmod{p}$ by (4.7). (This simple case of (4.7) is implicit in Steinberg [55]; cf. [9, (5.9)].) Thus, (2.2) applies.

In view of (4.1), we may assume that $\chi^+ \subseteq 1_B^{G^+}$ and $\langle \zeta^+, 1_B^{G^+} \rangle = 0$. Note that χ^+ is irreducible. For, by (4.1 c), χ^+ is a sum of characters conjugate in G, while each irreducible constituents of $1_B^{G^+}$ is G-invariant. (In order to see this, note that the permutation character of G^+ on $(i_1, \ldots, i_t)$- and $(n - i_t, \ldots, n - i_1)$-flags coincide, as in (4.3). Since every constituent of $1_B^{G^+}$ is a linear combination of such permutation characters by Steinberg [55] or (4.2), this implies the desired invariance.)

At this stage, we no longer require information of a rank 3 nature. All that is needed is the following more general result (in which our notation has been altered somewhat).

THEOREM 10.3. *Let* $SL(n, q) \leqslant G \leqslant \Gamma L(n, q)$, *where* $n > 2$. *Let* B *be a Borel subgroup of* G, *and let* $K < G$. *Assume that* 1_B^G *and* 1_K^G *have a common nonprincipal irreducible constituent* χ *such that* $\langle 1_K^G - 1 - \chi, 1_B^G \rangle = 0$. *Then one of the following holds* (*with the obvious embedding*):

(i) *K fixes a point or hyperplane*;
(ii) $K \trianglerighteq 3 \cdot A_6$, *inside* $\Gamma L(3, 4)$;
(iii) $K \trianglerighteq Sp(4, q)$, *inside* $\Gamma L(4, q)$;
(iv) $K = A_6$, *inside* $\Gamma L(4, 2)$; *or*
(v) $K \triangleright SL(2, q^2)$, *inside* $\Gamma L(4, q)$.

REMARK. In the context of (1.2), only (ii), $Sp(4, 2)$ and $Sp(4, 3)$ can arise, corresponding to (1.2 iv, v, vi). For, (i) and (iv) clearly cannot occur, by primitivity. Examples (iii) and (v) are best viewed by passing to $\Gamma O^+(6, q)$, where they correspond to subgroups of the stabilizer of an anisotropic 1- or 2-space; from this perspective, it is easy to check for the rank 3 property.

PROOF OF (10.3). We will proceed in several steps. Let V denote the underlying vector space.

(I) Suppose that $\chi = \theta_1 - 1$ is the reflection character. By (3.8), K then has 2 orbits of points and hyperplanes, and 3 orbits of $(1, n - 1)$-flags. Thus, K has a point-orbit Ω containing a hyperplane. By (4.5) (or a simple argument as in (10.2)), there are $1 + (n - 1)$ flag-orbits. Each flag has the form $(V_1, \ldots, V_{n-1})$ with $V_i \subset \Omega$ and $V_{i+1} \not\subset \Omega$, for some i. This accounts for n possibilities, one for each i between 0 and $n - 1$. Thus, if $x \notin \Omega$ then K_x is flag-transitive on V/x. By (2.15), (i) or (ii) holds.

(II) From now on, we may assume that K is transitive on points. By (2.7 i), $n \geqslant 4$.

Assume that $\chi \neq \theta_2 - \theta_1$. Then K is line-transitive. By Kantor [31], [32], K is 2-transitive on points and $n \geqslant 10$. (The only possible counterexample given in [31] is a Sylow 31-normalizer in $SL(5, 2)$, which has 7 orbits of $(1, 3)$-flags.) By Perin [46], K is intransitive on 4-spaces, and even on planes if $q > 2$. Thus, χ is $\theta_3 - \theta_2$ or $\theta_4 - \theta_3$.

Suppose $\chi = \theta_3 - \theta_2$. By (3.8), K has 2 plane-orbits and 3 orbits of $(3, 4)$-flags. Thus, there is a plane E such that K_E is transitive on the $(q^{n-3} - 1)/(q - 1)$ points of V/E. By (2.18), $q^{n-3} - 1$ has a primitive divisor. Thus, $K \geqslant SL(V)$ by [32, (5.2)].

Similarly, if $\chi = \theta_4 - \theta_3$ and $q = 2$ then $2^{n-4} - 1 \, | \, |K|$, and [32] again shows that $K \geqslant SL(V)$.

(III) Thus, $\chi = \theta_2 - \theta_1$.

Suppose $n = 4$. By (3.8) and (5.5), (iii), (iv) or (v) holds.

This completes the proof of (1.1) *when* $K \leqslant \Gamma O^+(6, q)$.

REMARK 10.4. In view of remarks (10.1, 2), it seems desirable to outline a more direct argument when $n = 4$, in order to avoid using (5.5). Note that K is transitive on points and planes, has 2 orbits of lines, $(1, 2)$- and $(1, 3)$-flags, and 3 orbits of flags. Let E be a plane. By (2.7), K_E^E fixes a point, line or (if $q = 4$) a hyperoval. If K_E^E fixes a hyperoval, then $|Z(K_E^E)| = 3$, and Wagner [60] applies.

Suppose that K_E^E fixes a point x. Then $K_E = K_x$. Here, $x \leftrightarrow E$ defines a symplectic polarity. (For, K_x is transitive on the $q + 1$ lines of E through x, and on the remaining q^2 lines through x. Let $\langle x, y \rangle$ be one of the latter lines. Then $K_{x, \langle x, y \rangle}$ is still transitive on the aforementioned $q + 1$ lines, by (2.12). Thus, K_{xy} cannot fix any plane on both x and y.) Since K_x^E has just 2 line-orbits, it contains all transvections with direction x. Thus, if $x \in L \subset E$ then K_x^L is transitive on $L - x$. Now $K_L^L \geqslant SL(2, q)$. Also, $q \, | \, |C_K(L)|$. Applying (2.1) to $\Gamma O(5, q)$, we find that $C_K(L) \geqslant O_p(G_L)$ if $q > 2$. Thus, (iii) or (iv) holds.

Next suppose that K_E^E fixes a line L. Then K_E^E is transitive on the remaining lines of E, so that L is the only member of L^K lying in E. Consequently, $|L^K| = q^2 + 1$, and L^K partitions the points of V. Note that $O_p(C_K(L)_E)$ is transitive on $E - L$, and hence on $L^K - \{L\}$. It follows that L^K is uniquely determined up to $GL(4, q)$-conjugacy [10, pp. 130–131]. Hence, (v) holds.

(IV) From now on we will assume that $n \geqslant 5$ in order to eventually obtain a contradiction.

Since $\chi = \theta_2 - \theta_1$, (3.8) implies that there are 2 orbits of lines, planes and $(1, 2)$-flags, 3 orbits of $(2, 3)$-flags and 4 orbits of $(1, 2, 3)$-flags. Thus, there is a plane E such that K_E^E is transitive, and a plane E' such that $K_{E'}^{E'}$ has 2 point-orbits. Then $K_{E'}^{E'}$ has at least 3 flag-orbits, and hence K_E^E is flag-transitive.

There is a line L contained in E' but contained in no member of E^K. Then K_L is transitive on the points and hyperplanes of V/L. Consequently, if H is a hyperplane, then K_H is transitive on those members of L^K lying in H.

Similarly, there are 2 orbits of $(1, 2, 3)$-flags on the form (x, L, E'), and hence also 2 of the form (x, L, H). Thus, K_{LH}^l has 2 point-orbits.

By (3.8), K_H^H has 2 point-orbits (called Ω and Ω'), 3 line-orbits and 4 orbits of $(1, 2)$-flags. Thus, $L \cap \Omega$ and $L \cap \Omega'$ are the orbits of K_{LH}^L, and only lines in L^K can meet both Ω and Ω'.

Again by (3.8), K has 3 orbits of $(1, 3)$-flags and at most 5 orbits of $(1, 3, 4)$-flags. Thus, K_{xT} is point-transitive on V/T for some $(1, 3)$-flag (x, T).

(V) Assume that K_x fixes a proper subspace $U(x) \supset x$. Since K_L is transitive on V/L, $U(x)$ is either a line or a hyperplane.

If $U(x)$ is a line, then $U(x) = L$ and no two members of $U(x)^K$ meet nontrivially. Then T meets q^2 or $q^2 + q + 1$ members of L^K in just one point, and K_{xT} has a point-orbit on V/T of length at most q^2. Since K_{xT} is transitive on V/T, this is impossible.

Thus, $U(x)$ is a hyperplane. As in (10.4), $x \leftrightarrow U(x)$ defines a symplectic polarity. However, $K_T^{V/T}$ cannot be transitive in a symplectic geometry.

Thus, $K_x^{V/x}$ is irreducible. Dually, so is K_H^H.

(VI) That $n \neq 5$ follows from (5.5).

Alternatively, since K_E^E is flag-transitive, K_{EH}^E is transitive whenever $H \supset E$ (by (2.7 i)). Thus, (2.15) shows that K_H^H cannot be irreducible.

(VII) Thus, $n \geq 6$. By (3.8), K_H^H has 3 plane-orbits, 5 orbits of $(1, 3)$-flags and 7 orbits of $(1, 2, 3)$-flags. By (2.7 iii), two of these plane-orbits produce subgroups of $\Gamma L(3, q)$ having 2 point-orbits and 3 flag-orbits, while the third produces a flag-transitive subgroup.

We may assume that $L \subset E' \subset H$. By (2.7 ii), $K_{LE'H}^L$ has orbit lengths 1 and q, except perhaps if $q = 4$ and the lengths are 2 and 3.

(VIII) If $|L \cap \Omega| = 1$, then any line meeting Ω twice must be contained in Ω. (For, we have seen that K_H^H has a unique orbit of lines meeting both Ω and Ω'.) Thus, Ω is a subspace, and this contradicts (V).

Consequently, $q = 4$ and $K_{E'}^{K'} \trianglerighteq 3 \cdot A_6$. Also, $K_E^E \geqslant SL(3, 4)$. Let r be a primitive divisor of $4^{n-3} - 1$, and let $R \in Syl_r K_{xT}$. Then R centralizes T, and $N_K(R)^T = K_T^T$ by the Frattini argument. It follows that $K_C([V, R])^T$ contains transvections, and hence so does K, which is ridiculous.

This contradiction completes the proof of (10.3), and hence also of (1.2).

11. Remarks concerning Theorems 1.1 and 1.2.

(A) Just as in the case of (1.2) and (10.1), the proof of (1.1) did not make full use of the rank 3 hypothesis. The crux of the argument was the following companion to Seitz's flag-transitive theorem (2.6).

THEOREM 11.1. *Suppose that V has rank at least 2, but is not of type $\Omega^+(4, q)$ or $\Omega^+(6, q)$. Let $\text{Chev}(V) \leqslant G \leqslant \Gamma(V)$, and let $K < G$. Assume that there is an irreducible character χ common to 1_K^G and 1_B^G such that $\langle 1_K^G - \chi, 1_B^G \rangle = 1$, where B is a Borel*

subgroup of G. Then one of the following holds, with K embedded in $\Gamma(V)$ in the natural manner:

(i) $K \geqslant \mathrm{Chev}(H)'$ *for a nonsingular hyperplane H, where V is orthogonal or unitary and* $\chi = (n - 1 \mid 1)$ *or* $(n - 1, 1 \mid 0)$;

(ii) *V, K and* χ *are given in Table* V; *or*

(iii) *K arises from Table* V *via one of the isomorphisms* $PSp(4, q) \cong P\Omega(5, q)$, $PSU(4, q) \cong P\Omega^-(6, q)$, *or* $\Omega(2n + 1, 2^i) \cong Sp(2n, 2^i)$.

The proof proceeds exactly as in §§7, 8. However, more care is needed in checking the examples occurring in the course of the proof. This is accomplished as follows. K has $(1_K^G, \theta_1) \leqslant 2$ point-orbits; this eliminates most instances. In order to then verify that a prospective example behaves as desired, one computes the number of flag-orbits geometrically. If K is not point-transitive, (4.5 i) is used; for example, (11.1 i) is checked by reversing the proof of (8.1).

(B) *Imprimitive rank 3 groups.* In order to underscore the previous remark we observe that Theorems 1.1 and 1.2 can now be generalized to include imprimitive rank 3 groups. For example, if $G \leqslant \Gamma(V)$ as in (1.1) and G acts on G/K as an imprimitive rank 3 group, then one of (2.6), (11.1) or (2.2) applies depending upon the number of parabolic characters in the associated permutation character. Comparing the resulting list of groups with those in the main theorem of [9] (and using the fact that G acts 2-transitively on a system of imprimitivity) we have:

THEOREM 11.2. *Let G be as in* (11.1) *and let* $K \leqslant G$. *Assume that G acts as an imprimitive rank 3 group on* G/K. *Then* $\Omega^{\pm}(2n, 2) = K < Sp(2n, 2) = G$.

In order to see that these are rank 3 representations, view $Sp(2n, 2)$ as the stabilizer of a nonsingular point in $O^{\pm}(2n + 2, 2)$ and let K be the centralizer of a nonsingular 2-space on x not in $x^{\perp}$. In a similar manner we obtain:

THEOREM 11.3. *Let G be as in* (10.1) *and let* $K \leqslant G$. *Assume that G acts as an imprimitive rank 3 group on* G/K. *Then either* $K = A_7 < S_8 \cong \mathrm{Aut}\ PSL(4, 2)$ *or K has index 2 in the stabilizer of a point or hyperplane.*

(C) *Other Chevalley groups.* It would be worthwhile to extend the methods presented here to the other Chevalley groups. Clearly, transitivity questions would have to be handled somewhat differently, due to the seeming lack of a convenient module. The known examples of rank 3 representations are as follows: (i) $E_6(q)$ on either of two classes of parabolic subgroups; (ii) $G_2(2) = PSU(3, 3) \cdot 2$ as in (1.1 x); and (iii) $G_2(4)$ on a class of Hall-Janko subgroups (Suzuki [56]).

(D) Although we have dealt primarily with $\Gamma(V)$, it may be helpful to observe that our main theorems could have been proven by working entirely inside $\mathrm{Chev}(V)$. While we have chosen the present exposition hoping that some of our numerous intermediate technical results might find greater application, such "linearization" might be necessary to deal with the other Chevalley groups. This kind of restriction is facilitated by (4.1), (4.4) and the following lemma.

TABLE V

Chev(V)	A normal subgroup of K	χ	Reference
$\Omega^+(8,q)$	$\Omega(7,q)$ or $2\cdot\Omega(7,q)$ A_9, $q=2$	$(2\|2)^\pm$ or $(3,1\|0)$	(2.16) and triality
	$2\cdot\Omega^+(8,2)$, $q=3$	$(2,2\|0)$	(5.22)
$\Omega(7,q)$	$G_2(q)'$ $P\Gamma L(2,q^3)$, $q=2$ or 8	$(1\|2)$	(5.19)
	$Sp(6,2)$, $q=3$	$(2,1\|0)$	(2.16)
$SU(6,2)$	$3\cdot PSU(4,3)$	$(2\|1)$	(2.16)
$SU(5,2)$	$Z_3^4 \rtimes A_5$	$(1\|1)$	(5.14)
$SU(4,q)$	$Sp(4,q)'$ $4\cdot Z_2^4\cdot PSL(2,5)$ or $4\cdot Z_2^4\cdot A_6$, $q=3$	$(1\|1)$	(5.14)
	$4\cdot PSL(3,4)$, $q=3$	$(0\|2)$	(5.13)
	$Z_2^3 \rtimes A_4$, $q=2$ $SU(3,2)'' \rtimes Z_8$, $q=2$	$(1,1\|0)$	(5.13)
$Sp(4,q)$	$SL(2,q)\times SL(2,q)$ $\Omega^+(4,q)$, q even $SL(2,3)'\times SL(2,3)'$, $q=3$ $2\cdot Z_2^4\cdot A_4$, $q=3$ Z_3^2, $q=3$	$(1\|1)$	(5.14)
	$Sp(2,q^2)$ $2\cdot Z_2^4 \rtimes D_{10}$, $SL(2,5)$, $q=3$	$(0\|2)$	(5.13)
	$\Omega^-(4,q)$, q even	$(1,1\|0)$	(5.13)

LEMMA 11.4. *Suppose $G \trianglerighteq G^+$ and $K^+ = K \cap G^+$ for some $K \leqslant G$. Suppose χ is an irreducible character of G such that $\langle \chi, 1_K^G \rangle \neq 0$. If $\chi^+ = \chi|_{G^+}$ is irreducible then*

$$\langle 1_K^G, \chi \rangle = \langle 1_{K^+}^{G^+}, \chi^+ \rangle.$$

PROOF. By Frobenius reciprocity and the fact that $1_K^G = (1_K^{G^+ K})^G$, it suffices to consider the case $G = G^+ K$. In this case,

$$\langle 1_{K^+}^{G^+}, \chi^+ \rangle = \langle 1_K^{G^+}, \chi \rangle = \langle \sum \omega(1)\omega^G, \chi \rangle = \sum \omega(1)\langle \omega, \chi|_H \rangle$$

where the summations range over the irreducible characters ω of K with kernel containing K^+. The irreducible constituents of $(\chi^+)^G$ are of the form $\chi^+ \varphi$ where φ is an irreducible character of K with kernel containing K^+ (Isaacs [28, (6.17)]). Since χ is an extension of χ^+, Frobenius reciprocity implies $\chi = \chi^+ \varphi$ for some such φ. However, an easy calculation shows

$$\langle \chi|_K, \omega \rangle = \langle \chi|_{K^+}, 1_{K^+} \rangle \langle \varphi, \omega \rangle$$

for any ω as above, and $0 \neq \langle \chi, 1_K^G \rangle = \langle \chi|_K, 1_K \rangle \langle \theta, 1_K \rangle$ by hypothesis and the preceding equation. Therefore $\varphi = 1_K$, and the same equation shows $\langle \chi|_K, \omega \rangle = \langle \chi|_K, 1_K \rangle \delta_{\omega, 1_H}$. Thus

$$\langle 1_{K^+}^{G^+}, \chi^+ \rangle = \sum \omega(1)\langle \omega, \chi|_H \rangle = \langle 1_K^G, \chi \rangle,$$

as desired.

12. Proof of (1.3). Since K is $(1, 2)$-transitive, by (2.6) and (5.15) we may assume that $d \geqslant 7$. Set $Q = O_p(\Gamma O(V)_x)$; we may also assume that $K \cap Q = 1$, by (2.1) and (2.4).

Let r be a primitive divisor of $q^{m-2} \pm 1$, for V of type $\Omega^+(2m, q)$ or $\Omega(2m - 1, q)$; define $R \in Syl_r K_x$, W, N and C as in §6. By (2.18), r exists except in the cases $\Omega^+(10, 2)$, $\Omega^-(16, 2)$, and $\Omega^-(8, q)$ with q a Mersenne prime, all of which we temporarily exclude. By Table IV, $R \neq 1$.

Note that W contains no line, while N^W is transitive on ordered pairs of points. Thus, $N^W \trianglerighteq \Omega^-(4, q)$ or $\Omega(3, q)$, while $N^{W^\perp}$ is isomorphic to a subgroup of $\Gamma O^\cdot(2, q^{m-2})$. Consequently, unless $N^W \trianglerighteq \Omega(3, 3)$, we find that $C^W \geqslant \Omega^-(4, q)$, and hence that $C \cap Q \neq 1$.

Thus, $K \leqslant \Gamma O(2m - 1, 3)$. Here, $|O_2(C^W)| \geqslant 4$. Let b be a nonsingular point of W whose corresponding reflection t fixes 2 points of W. Set $H = \langle K_b, C \rangle \langle t \rangle$. Since $|b^{\Gamma O(V)} \cap \langle x_1, x_2 \rangle| = 1$ for any pair of nonperpendicular points x_1, x_2, $\langle t \rangle K_b$ is transitive on the set of points not in $b^\perp$. Since C moves b, it follows that $b^h \notin b^\perp$ for some $h \in H$. But then $tt^h \in K$, and tt^h is in the conjugate $O_3(\Gamma O(V)_{x'})$ of Q (where $x' = \text{rad}\langle b, b^h \rangle$), contrary to our assumption.

We now turn to the excluded cases. If $K \leqslant \Gamma O^-(16, 2)$, use $r = 7$ and proceed as before. Note that R fixes a 6-space of $W^\perp$, and hence that $N^{W^\perp}$ is contained in $\Gamma O^+(4, 2^3)$. However, $(N^W)^{(\infty)} = A_5$ is not contained in the latter group.

If $K \leqslant \Gamma O^+(10, 2)$, let $R_1 \in Syl_3 K_x$. Since $x^\perp/x$ has $(2^4 - 1)(2^3 + 1)$ points, $|R_1| \geqslant 3^3$. However, $\Gamma O^+(8, 2)$ has no element of order 3^3, so R_1 cannot be fixed-point-free on $x^\perp/x$. Let R be a 3-group maximal with respect to having $\dim W > 2$, where $W = C_V(R)$; define N and C as usual. If W has type $\Omega^-(4, 2)$, the usual approach works. Since an $\Omega^+(6, 2)$ space has $(2^3 - 1)(2^2 + 1)$ points, W cannot have type $\Omega^+(4, 2)$. If $\dim W > 4$, it follows similarly that W has type $\Omega^+(6, 2)$ or $\Omega^-(8, 2)$. By Sylow's theorem, N^W is then line-transitive, so that $7 \mid |N^W|$ and we can proceed as in the preceding paragraph.

Finally, suppose that $K \leqslant \Gamma O^-(8, q)$ with q Mersenne. If L is a line, clearly $K_L^L \geqslant SL(2, q)$. We may assume that $-1 \in K$. If $R \in Syl_2 C_K(L) \cap SL(8, q)$, then $R \neq 1$. Define W, N and C as usual. Then N^W is $(1, 2)$-transitive, and $N_L^L \geqslant SL(2, q)$ by the Frattini argument. Clearly, $\dim W = 4$, 5 or 6. If $\dim W = 6$, then $C \cap Q \neq 1$ by (2.6) and (5.15). If $\dim W = 5$ then $q = 3$ or $C \cap Q \neq 1$ by (2.6). If W has type $\Omega^+(4, q)$ then $N^W \geqslant O^+(4, q)$ while $N^{W^\perp}$ normalizes the 2-group $R^{W^\perp} \leqslant \Gamma O^-(4, q)$; consequently, $q = 3$ (as otherwise, $C^W \geqslant \Omega^+(4, q)$ and $C \cap Q \neq 1$). We can now introduce a reflection as before in order to complete the proof of (1.3).

REMARK. It should be noted that the linearity assumed by Perin was as superfluous for his arguments as it was for ours.

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