UNIVERSAL BUNDLES FOR DEFORMATIONS
OF ASYMMETRIC STRUCTURES

BY

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ABSTRACT. Riemannian parallel transport of frames and Cartan connections can be considered as special cases of a type of structure in which the connection form on the principal bundle can transform by an arbitrary extension of the adjoint representation of the fiber group; there always exist universal deformations for asymmetric objects in any of these geometries.

The purpose of this paper is to give an elementary proof of the existence of universal objects for deformations of infinitesimally asymmetric, globally asymmetric structures in a wide range of geometries.

The class of geometries we consider is a generalization of linear connections and Cartan connections in which the connection form may transform by a more general representation of the fiber group. Within these geometries, we always require the structures under study to be both infinitesimally and globally asymmetric. Starting with a fixed geometry and a fixed principal bundle $Y \to M$ over a compact base, we construct a Y-bundle $\mathcal{E}^\infty \to \mathfrak{M}^\infty$, with a connection on each fiber, such that any deformation of asymmetric structures of the given type can be uniquely described by mapping its parameter space into $\mathfrak{M}^\infty$.

Let H be a Lie group of dimension n and θ a representation of H on $\mathbf{R}^{m+n}$ which preserves $\{0\} \times \mathbf{R}^n$ and restricts so as to be isomorphic to the adjoint representation there. A θ-*structure* on an m-manifold M is then a principal H-bundle $Y \to M$ with a 1-form ω on Y with values in $\mathbf{R}^{m+n}$, such that each $\omega_y: T_y Y \to \mathbf{R}^{m+n}$ is an isomorphism and $(R_a)^* \omega = \theta(a^{-1}) \omega$ for $a \in H$, and such that $\{0\} \times \mathbf{R}^n$ always corresponds to the vertical subspace of $T_y Y$. For example (§1B), linear connections can be viewed as θ-structures where θ is the sum of the standard representation of $H \subset \mathrm{GL}(m)$ on $\mathbf{R}^m$ and the adjoint representation on $\mathbf{R}^n \cong \mathfrak{h}$.

A θ-structure (Y, ω) is called *infinitesimally asymmetric* if there is no point $y \in Y$ such that either (i) $f^* \omega - (R_a)^* \omega$ vanishes to infinite order at y for some $a \in H$ not the identity and some local diffeomorphism f fixing y, or (ii) there is a vector field Z in Y with $Z_y \neq 0$ such that $\mathcal{L}_Z \omega$ vanishes to infinite order at y. A different definition, given in §1C, is shown to be equivalent to this one in §1E. The θ-structure is *globally asymmetric* if there is no nontrivial equivariant diffeomorphism $Y \to Y$ preserving ω.

The main result can then be precisely stated as follows:

THEOREM. *Let M be a compact m-manifold and Y a principal H bundle over M. Then there is a Y-bundle $\mathcal{E}^\infty(Y, \theta) \to \mathfrak{M}^\infty(Y, \theta)$ with a smooth θ-structure on each fiber which is infinitesimally and globally asymmetric, and on which the connection forms and all their higher order invariants are continuous, such that any other such objects can be obtained uniquely as a pull-back from this universal one.*

Our methods are quite simple. We construct an abstract jet space for fields of frames, parametrized by their local invariants. Considering the connections as framings of T_*Y, we show (§2D) that infinitesimally asymmetric structures' invariants-mappings are immersions, and thus can be given normal bundles. The coordinate charts used in constructing the above bundle are then defined using the spaces of sections of these normal bundles (§3). Finally, these charts are glued together to form the global bundle (§4A), which is then shown to have the universal property (§4B).

1. Frame field descriptions of various types of connections. This section has two purposes. First, it combines Cartan connections, linear connections, and other geometric constructions into one natural family, described at the end of §1B. Secondly, it illustrates the extent to which the information in any of these geometries (for instance, the principal bundle structure itself) is contained in the framing of the total space Y which is provided by the connection. Writing the connection as a frame field of N vectors, we see that the geometry is specified by conditions on their Lie brackets, and we describe in terms of these vector fields the hypotheses which we place on the geometry ("geometry A") whose deformations are analyzed in this paper.

A. *Cartan connections.* To define a Cartan connection, one must first specify a symmetric space to use as a *model* geometry. Suppose then that $G \supset H$ are connected finite-dimensional Lie groups, G/H is a manifold of dimension m, and $\mathfrak{g}$, $\mathfrak{h}$ are the Lie algebras.

DEFINITION. A *Cartan connection modelled on G/H* is a $\mathfrak{g}$-valued 1-form ω on an H principal bundle Y over a smooth m-manifold M such that: (1) $\omega_x: T_xY \to \mathfrak{g}$ is an *isomorphism* of vector spaces for all $x \in Y$; (2) if $A \in \mathfrak{h}$ and A^* is the associated invariant vector field on Y, then $\omega(A^*) = A$; (3) $(R_a)^*\omega = \mathrm{ad}(a^{-1}) \cdot \omega$ for $a \in H$, where $(R_a: Y \to Y, a \in H)$ defines the action of H.

An *equivalence* of these structures is a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{h} & Y' \\ \downarrow & & \downarrow \\ M & \xrightarrow{\bar{h}} & M' \end{array}$$

where h, $\bar{h}$ are diffeomorphisms and $h^*(\omega') = \omega$. Here the vertical arrows are the projections.

Up to equivalence, the entire structure can be reconstructed from the pair (Y, ω). That is, the condition $h^*(\omega') = \omega$ implies that the vector fields A^*, and thus the action of H, are preserved by h. Up to equivalence we can thus *replace M* by the orbit space of H on Y, and we are free also to *forget* the H action itself since it can be reconstructed from ω.

Suppose now that we choose a vector space basis $e_1, \ldots, e_N$ for $\mathfrak{g}$, so that $e_{m+1}, \ldots, e_N$ span $\mathfrak{h} \subset \mathfrak{g}$, and define vector fields $X_1, \ldots, X_N$ on Y by

$$\langle \omega, X_i \rangle_x = e_i, \qquad i = 1, \ldots, N, \quad x \in Y.$$

These vector fields define a frame at each point of Y from which we can recover ω.

We have now simplified the situation substantially; our picture now consists of just a manifold with N vector fields satisfying some conditions. It is reasonable to ask what the local consequences of these conditions are.

In general, the lowest order local invariants of a frame field $X_1, \ldots, X_N$ are the functions γ^l_{jk} which are defined as the coefficients in

$$[X_j, X_k] = \gamma^l_{jk} X_l$$

(where the summation convention is used). One instance is the definition of the structure constants of $\mathfrak{g}$ by the formula

$$[e_j, e_k] = c^l_{jk} e_l.$$

The infinitesimal version of condition (3) of the above definition is simply

(3′) $\gamma^l_{jk} = c^l_{jk}$ for $j > m$, k and l arbitrary.

The germs of actual connection forms, written in this notation, are the same as the germs of frame fields satisfying (3′). To verify this, one chooses a frame field (X_i) in a neighborhood of $0 \in \mathbf{R}^N$ satisfying (3′), and notes that H acts locally (in the pseudogroup sense) so that (3) holds. Using (3) and the group law, it is then possible to piece together an actual connection form on a neighborhood of $\{0\} \times H \subset \mathbf{R}^m \times H$ whose germ (regarded as a frame field) at $(0, 1)$ is isomorphic to that of (X_i) at 0. Note that from this point of view the geometry depends on the Lie algebras rather than the Lie groups.

Now we must discuss the possible auxiliary defining conditions for the geometries we will consider. First it is reasonable to restrict ourselves to inhomogeneous linear conditions on the γ^l_{jk}, and these conditions must certainly be satisfied by the model geometry, *i.e.* by the structure constants c^l_{jk}. Second, we note that H acts on a frame field $X_1, \ldots, X_N$ by its adjoint representation on the vectorspace (namely $\mathfrak{g}$) generated by the e_i; this action induces transformations of the γ^l_{jk} under which the conditions should be invariant.

The limitation to homogeneous linear conditions on the quantities $\gamma^l_{jk} - c^l_{jk}$ is motivated by the fact that these quantities are precisely the coefficients of the *curvature*

$$\Omega = d\omega - [\omega \wedge \omega]$$

of the Cartan connection.

B. *Linear connections and other equivariant geometries.* Clearly other geometrical entities besides Cartan connections can be described in terms of framings of principal bundles. We next discuss the case of linear connections.

Given a principal H bundle Y_0 of linear frames on M ($H \subset \mathrm{GL}(m)$) and a connection ω_0 on Y_0, the horizontal subspace at each $y \in Y_0$ is tautologically framed since y is a frame. If Y_0 is replaced by a diffeomorphic manifold Y and ω_0 by the corresponding $\tilde{\omega}$, we will need these frames on the horizontal subspaces in order to recover (Y_0, ω_0). A frame field on Y is given at $y \in Y$ by these horizontal vectors $(X_1, \ldots, X_m)$ and the $X_j = \tilde{\omega}_j^{-1}(e_j)$ for a basis $e_{m+1}, \ldots, e_N$ of $\mathfrak{h}$. As in the case of Cartan connections, the manifold M, bundle $Y_0 \to M$, and connection ω_0 can then be recovered up to equivalence from $(Y; X_1, \ldots, X_N)$. The only real difference in the two sorts of geometries lies in the conditions that the γ_{jk}^l and the action of H must satisfy.

The structure constants appearing in (3') are only defined here if j, k, and l are all more than m; the corresponding equations must be combined with $\gamma_{jk}^l = 0$ for $j, k > m$ and $l \leq m$ to maintain the bundle structure on Y. The action of H on the horizontal vectors must be that given by the inclusion $H \hookrightarrow \mathrm{GL}(m)$; this specifies the γ_{jk}^l for $j > m$, $k \leq m$, and $l \leq m$. Finally, parallel translation must preserve the group structures on the fibers, so the γ_{jk}^l must vanish for $j > m$, $k \leq m$, and $l > m$. In all, we have fixed the γ_{jk}^l with $j > m$, just as in the case of Cartan connections; for notational convenience we will write c_{jk}^l for the constants in these formulas.

These equations are again the infinitesimal form of an equation similar to condition (3) in §1A which describes the behavior of a Cartan connection under translation by an element of H.

Just as before, we can impose auxiliary conditions on the other γ_{jk}^l (here the torsion and curvature) to obtain other geometries, *e.g.* Riemannian metrics, vacuum general-relativistic space-times, and so on.

Now consider the properties held in common by the two cases of Cartan connections and linear connections.

In each the group H has a representation θ given on $\mathbf{R}^N = \mathbf{R}^m \times \mathbf{R}^{N-m}$ such that

(i) θ induces a representation of H on $\mathbf{R}^m$ by projection to the first factor, and

(ii) some vectorspace isomorphism $f: \{0\} \times \mathbf{R}^{N-m} \to \mathfrak{h}$ carries θ to the adjoint representation of H on its Lie algebra $\mathfrak{h}$.

The geometry is then given by a 1-form ω on Y with values in $\mathbf{R}^N$ such that

(iii) ω takes the tangent space of the fiber to $\{0\} \times \mathbf{R}^{N-m}$,

(iv) for each $A \in \mathfrak{h}$, the vectorfield A^* on Y given by differentiating the action of H is taken to A by $f \circ \omega$, and

(v) for $x \in Y$ and $a \in H$, we have $(R_a)^* \omega = \theta(a^{-1}) \cdot \omega$ where R_a is the map giving the action of a on Y.

The infinitesimal version of condition (v) then states that if we write vectorfields $X_1, \ldots, X_N$ on Y with $\langle \omega, X_j \rangle \equiv e_j, j = 1, \ldots, N$, then certain of the quantities γ_{jk}^l in $[X_j, X_k] = \gamma_{jk}^l X_l$ must equal the corresponding structure constants c_{jk}^l of the representation of $\mathfrak{h}$ on $\mathbf{R}^N$ associated to θ.

For the remainder of this paper, *geometry A* will be any geometry defined in terms of connections of the above sort with auxiliary conditions similar to those discussed in §1A.

Where there are *additional* conditions imposed on the geometry which are not of this type, these can be simply held to one side until the construction of deformations for the less-restricted geometry is completed.

C. *A remark about the kernel of θ.* If, in the above construction, the representation θ of H on $\mathbf{R}^N$ has nontrivial kernel, then for any a in the kernel, we have $(R_a)^*\omega = \omega$: R_a is always a global symmetry of the θ structure (Y, ω). Thus we include here some remarks concerning the extension of our deformation theory to this case.

Let $1 \to H_0 \to H \to H' \to 1$ be an exact sequence of Lie groups, where H_0 is perhaps the kernel of θ. Writing $\mathfrak{h}'$ for the Lie algebra of H', suppose that θ preserves the fibering of $\mathbf{R}^m \times \mathfrak{h}$ ($\cong \mathbf{R}^N$) over $\mathbf{R}^m \times \mathfrak{h}'$ ($\cong \mathbf{R}^{N'}$), and that the induced action of H on $\mathbf{R}^{N'}$ factors through H'. Let θ' be the representation of H' on $\mathbf{R}^{N'}$ so obtained. If we construct $Y' = Y/H_0$ and the 1-form ω' which makes the diagram

$$
\begin{array}{ccccc}
T_*(Y) & \xrightarrow{\omega} & \mathbf{R}^N & \cong & \mathbf{R}^m \times \mathfrak{h} \\
\downarrow & & & & \downarrow \\
T_*(Y') & \xrightarrow{\omega'} & \mathbf{R}^{N'} & \cong & \mathbf{R}^m \times \mathfrak{h}'
\end{array}
$$

commute, we obtain an H' principal bundle $Y' \to M$ with θ' structure ω'.

To carry out the deformation theory of this paper in the case where $\ker(\theta) \neq 1$, we might begin by "reducing" the θ structure (Y, ω) to the θ' structure (Y', ω'), where $Y' = Y/H_0$, ω', and θ' are as above with H_0 equal to the kernel of θ. An advance is made by this simplification, since as θ' is faithful, no $R_{a'}$: $Y' \to Y'$ for $a' \in H'$ will be a symmetry of ω'. On the other hand, the functor from $(Y \to M, \omega)$ to $(Y' \to M, \omega')$ is forgetful of the components of curvature lying in the Lie algebra of H_0, *i.e.* of the γ_{jk}^l for $j, k \leq m$ and $l > N'$. These extra invariants will of course be well defined on Y'. Thus to extend the deformation theory to nonfaithful θ, we restrict to the part of J_F^∞ (§2) on which $\xi_{N'+1}, \dots, \xi_N$ vanish and then search for immersions of Y' rather than of Y in the construction of §§2 and 3.

2. Invariants, jet spaces, and normal forms. In this section, we develop the basic jet space techniques in the context of abstract frame fields, and discuss the consequence of infinitesimal asymmetry. The invariants of arbitrary finite order for frame fields parametrize a jet space J_F^∞; this jet space has certain structures defined on it, and there is a subspace $J_A^\infty \subset J_F^\infty$ of formal geometry A structures. Infinitesimally asymmetric structures are discussed from the viewpoint of their invariant mappings, and it is seen that the latter can be easily distinguished among the mappings of Y into J_A^∞.

A. *Normal forms for abstract frame field jets.* We have seen in §1 that Cartan connections and linear connections are essentially just fields of frames in the

principal bundles. These fields can be written locally as ordered N-tuples $X_1, \ldots, X_N$ of vectorfields on (a coordinate neighborhood) $\mathbf{R}^N$ (in Y). We now investigate the invariants of finite order at the origin for such N-tuples of vectorfields under the equivalence relation given by diffeomorphisms of $(\mathbf{R}^N, 0)$.

These invariants are precisely the coefficients γ_{jk}^l in the relations $[X_j, X_k] = \gamma_{jk}^l X_l$ evaluated at the origin and their derivatives for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_N)$ given by the formula

$$\gamma_{jk;\alpha}^l = X_1^{\alpha_1} \cdots X_N^{\alpha_N} \gamma_{jk}^l$$

where the ordering of the differentiations is of significance. If two such frame fields have the same invariants at the origin, then there is a diffeomorphism of $(\mathbf{R}^N, 0)$ taking the Taylor series of one family of vectorfields to the Taylor series of the other, as we shall see below.

To establish this claim that the above form a complete family of invariants of the Taylor series (or infinite jets) of the frame fields at the origin, we provide a normal form for any germ of frame field. This is very easy to construct. We choose new coordinates $(x^1, \ldots, x^N)$ in a neighborhood of $0 \in \mathbf{R}^N$ by a simple procedure. First we parametrize the integral curve of X_N through the origin as the x^N axis so that X_N will equal $\partial/\partial x^N$ when $x^1 = \cdots = x^{N-1} = 0$. Next we use X_{N-1} to similarly extend the definition of coordinates to the (x^{N-1}, x^N) plane, so that (near the origin)

$$X_{N-1} = \partial/\partial x^{N-1}, \qquad x^1 = \cdots = x^{N-2} = 0.$$

By induction coordinates can be constructed on a full neighborhood of the origin so that

$$X_i = \partial/\partial x^i, \qquad x^1 = \cdots = x^{i-1} = 0, \quad \text{all } i.$$

The coordinates are thus uniquely defined by the frame field itself, and thus the frame field is in a normal form in the new coordinates.

If the normal form expression for the frame field (X_i) in terms of $x = (x^1, \ldots, x^N)$ is

$$X_j = \partial/\partial x^j + a_j^k(x) \partial/\partial x^k, \qquad 1 \leq j \leq N,$$

then the (normal form) conditions on the a_j^k are simply

$$(*) \qquad\qquad a_j^k(x) = 0 \quad \text{when } x^1 = \cdots = x^{j-1} = 0.$$

Writing (for each multi-index α)

$$A_{j,\alpha}^k := a_{j,\alpha}^k(0) = (\partial/\partial x)^\alpha a_j^k(0),$$

the infinitesimal form of $(*)$ is given as

$$(*') \qquad\qquad A_{j,\alpha}^k = 0 \quad \text{when } \alpha_i = 0 \text{ for all } i < j.$$

The $A_{j,\alpha}^k$ for which $\alpha_i \neq 0$ for some $i < j$ can thus be used as invariants in place of the $\gamma_{jk;\alpha}^l$. In fact, since for any α the operators $X_1^{\alpha_1} \cdots X_N^{\alpha_N}$ and $(\partial/\partial x)^\alpha$ agree *at* the origin by $(*)$, we easily compute

$$\gamma_{jk;\alpha}^l = A_{j,(k)+\alpha}^l - A_{k,(j)+\alpha}^l + \sum_{\beta < \alpha} \left\{ A_{k,\alpha-\beta}^m A_{j,(m)+\beta}^l - A_{j,\alpha-\beta}^m A_{k,(m)+\beta}^l \right\};$$

the conditions $(*')$ then permit us to solve recursively for the $A^k_{j,\alpha}$ in terms of the $\gamma^l_{jk;\alpha}$. Here $(j)_i = \delta^j_i$.

DEFINITION. Let $\tilde{J}^r_F$ be the vectorspace consisting of germs at $0 \in \mathbf{R}^N$ of frame fields $X^0_1, \ldots, X^0_N$ which are in normal form with polynomial coefficients of order no greater than $(r - 1)$. Let $\tilde{J}^\infty_F$ be the projective limit

$$\tilde{J}^\infty_F \to \cdots \to \tilde{J}^{r+1}_F \xrightarrow{\pi} \tilde{J}^r_F \to \cdots \to \tilde{J}^2_F,$$

where the map $\tilde{J}^{r+1}_F \to \tilde{J}^r_F$ is given by deleting the rth order terms in the coefficients.

To any frame field $X_1, \ldots, X_N$ on an N-manifold Y and any $x \in Y$, the above constructions associate invariants $A^k_{j,\alpha}$ and thus a point of $\tilde{J}^\infty_F$ (or of $\tilde{J}^r_F$). The equivalence classes under the relation of mapping to the same point of $\tilde{J}^\infty_F$ (resp. of $\tilde{J}^r_F$) are called *infinite jets* (resp. *r-jets*) of abstract frame fields and form a space J^∞_F (resp. J^r_F) called the *infinite jet space* (resp. the *r-jet space*) of abstract frame fields.

B. *Jets for geometry A*. Let "geometry A" be one of the geometries described in §1. The conditions defining the geometry (and other conditions which can be formally derived from these) will be used to define for each r a subspace J^r_A of J^r_F describing candidates for jets of frame fields that describe objects in geometry A.

For each condition $\varepsilon^{jk}_l \gamma^l_{jk} = c$ in the specifications of geometry A there are conditions on the frame field jets of the form

$$\varepsilon^{jk}_l \left(A^l_{j,(k)} - A^l_{k,(j)} \right) = c,$$

where (j) is the multi-index with $(j)_i = \delta^j_i$, and the derived conditions

$$\varepsilon^{jk}_l \left[A^l_{j,(k)+\alpha} - A^l_{k,(j)+\alpha} \right] = \varepsilon^{jk}_l \left[\sum_{\beta < \alpha} \left\{ A^m_{k,\alpha-\beta} A^l_{j,(m)+\beta} - A^m_{j,\alpha-\beta} A^l_{k,(m)+\beta} \right\} \right].$$

These conditions will be called *frame field jet conditions* for the geometry A.

DEFINITION. The infinite jets (resp. r-jets) of abstract frame fields satisfying the frame field jet conditions for geometry A will be called the *infinite jets* (resp. *r-jets*) *of abstract connections* and form a subspace of J^∞_F (resp. of J^r_F) which will be denoted J^∞_A (resp. J^r_A).

From the representation of H on $\mathbf{R}^N$, which is called θ in §1B, we obtain consistent actions of H on all the J^r_F by replacing X_i by $(\theta(a^{-1}))^j_i X_j$ in the equations which define the invariants. These actions restrict to actions of H on the J^r_A for $r = 2, 3, \ldots, \infty$.

We should now stress that the above definition is quite formal, and does not mention existence of an actual connection with the given jet. It would perhaps be better to have the actual spaces of r-jets of smooth or analytic connections. However, we have no need to compute these spaces for general geometries in order to make our constructions. The spaces

$$J^r_{A,2} = \left\{ j \in J^r_F \colon \text{the 2-jet of } j \text{ is in } J^2_A \right\}$$

will do as well in the constructions of §3, since any actual frame field r-jet invariant mapping with image in $J^r_{A,2}$ will have image in J^r_A.

C. *The canonical vectorfields on the jet space. Infinitesimal asymmetry.* There is a structure on the jet space J_A^∞ analogous to that structure on the space of infinite jets of maps $\mathbf{R}_x \to \mathbf{R}_y$ which is given by the Pfaffian forms $d(y) - (y')\, d(x)$, $d(y') - (y'')\, d(x), \ldots$, or as a better analogy, by the "vectorfield" $\partial/\partial x + (y')(\partial/\partial y) + (y'')(\partial/\partial(y')) + \ldots$ which generates the nullspace of this infinite Pfaffian system. The structure on J_A^∞ is the restriction of a similar one on J_F^∞ which we will examine first.

For $j^r \in J_F^r$, let $(X_1^0, \ldots, X_N^0)$ be the corresponding polynomial frame field in $\tilde{J}_F^r$ defined in a neighborhood U of 0 in $\mathbf{R}^N$. There is a smooth map $i_0: U \to J_F^{r-1}$ giving the invariants of (X_k^0) of order up to $(r-1)$ at points of U. Define vectors $\xi_1^{r-1}, \ldots, \xi_N^{r-1}$ in the tangent space to J_F^{r-1} at $i_0(0)$ by

$$\xi_k^{r-1} = i_{0*}(\partial/\partial x^k)\big|_{x=0}.$$

The coordinates of these vectors are invariants of (X_k^0) at 0 of order no more than r, so the same vectors would be obtained in this way from any representative of j^r; it is also evident that the vectors depend smoothly on j^r.

If we write $j \in J_F^\infty$ as a projective limit $j = \lim(j^r)$, we have for each $r < \infty$ and $k = 1, \ldots, N$ the vectors $\xi_k^r \in T_{j^r}(J_F^r) \cong J_F^r$ and thus the projective limit vector

$$\xi_k = \lim_{\leftarrow}(\xi_k^r) \in T_j(J_F^\infty).$$

We may thus consider the ξ_k as "smooth" vectorfields defined on J_F^∞.

If $i: Y \to J_F^\infty$ is the natural mapping giving the invariants of a frame field $X_1, \ldots, X_N$ on Y, then $i_*(X_k) = \xi_k$ for $k = 1, \ldots, N$.

The higher-order conditions among the frame field jet conditions for the geometry are exactly what is needed to assure that the ξ_k will be tangent to J_A^∞; thus we can restrict the ξ_k to obtain a similar structure on J_A^∞.

DEFINITION. An *infinitesimally asymmetric* jet in J_A^∞ is a jet which (a) is a point of trivial isotropy for the action of H on J_A^∞ and (b) is not a point at which the vectors ξ_k are linearly dependent. An *infinitesimally asymmetric* connection in the geometry A is one for which the image of the invariant mapping contains only infinitesimally asymmetric jets.

D. *Immersions into J_F^r.* We wish to simplify the recognition of invariant mappings of infinitesimally asymmetric frame fields by some construction based on the vectors ξ_k. A point $x \in J_F^r$ determines the vectors ξ_k^{r-1} at $\pi(x) \in J_F^{r-1}$, and thus also their linear dependent or independence, but at one stage of the discussion we will need two derivatives, so we introduce the following open set.

DEFINITION. For $r \geq 4$, let $W^r \subset J_F^r$ be the set of points $x \in J_F^r$ such that

(1) the vectors ξ_k^{r-2} at $\pi^2(x) \in J_F^{r-2}$ determined by $\pi(x) \in J_F^{r-1}$ are linearly independent, and

(2) the isotropy of H at $\pi(x) \in J_F^{r-1}$ is trivial.

On W^r there is a Pfaffian system $\psi_1, \ldots, \psi_K$ of rank $(\dim(J_F^{r-1}) - N)$: the null space $\{\psi_i = 0\}$ at $x \in W^r$ consists of all vectors $v \in T_x(J_F^r)$ with $\pi_*(v)$ a linear combination of the vectors ξ_k^{r-1} at $\pi(x)$ determined by the point x.

LEMMA. (1) *For some neighborhood of any given point in an infinitesimally asymmetric frame field in geometry A, there is $r < \infty$ for which the r-jet invariant mapping has image in W^r.*

(2) *Given a smooth map f of an N-manifold U into W^r, f is the invariant mapping of a frame field if and only if $\pi \circ f$ is an immersion and f satisfies the Pfaffian system in the sense that $\langle \psi_i, f_* X \rangle = 0$ for all $X \in T_* U$ and $i = 1, \ldots, K$.*

PROOF. Statement (1) is trivial, as is the "only if" part of (2). Thus let f satisfy the Pfaffian system and let $\pi \circ f$ be an immersion. By dimension-counting, $D(\pi \circ f)(u)$ must have image spanned by the ξ_k^{r-1} for each $u \in U$. There is then certainly a frame field $X_1, \ldots, X_N$ on U with $(\pi \circ f)_* X_k = \xi_k^{r-1}$; we must check that f is in fact the r-jet invariant mapping for this frame field. Applying the identities which define the vectorfields ξ_k, we see that it suffices to check only the γ_{jk}^l at each point of U. This is no more difficult: we know the second derivatives $I_{:jk}^s$ and $I_{:kj}^s$ of N quantities I^s (invariants of order $\leq r - 2$) for which the $I_{:1}^s$ form an invertible matrix.

E. *The geometrical characterization of infinitesimal asymmetry.*

PROPOSITION. *A frame field $(X_1, \ldots, X_N)$ on $\mathbf{R}^N$ is infinitesimally asymmetric at 0 if and only if there is no nonvanishing smooth vectorfield Z on $\mathbf{R}^N$ with the Lie derivatives $\mathcal{L}_Z(X_i)$ all vanishing to infinite order at 0. For fields in geometry A, infinitesimal asymmetry at 0 is equivalent to this condition plus the nonexistence of a diffeomorphism $h: (\mathbf{R}^N, 0) \to (\mathbf{R}^N, 0)$ for which $Dh(0) = (h_i^j)$ is a transformation in the representation θ of H and $h_i^j X_j - h_*(X_i)$ vanishes to infinite order at 0 for all i.*

PROOF. The second part of the statement is clear given the first, since in the case of infinitesimal symmetry arising from H isotropy the diffeomorphism H can be constructed as the transformation from normal form coordinates for (X_i) to those for $(h_i^j X_j)$.

Now we introduce an auxiliary variable t and lift the X_i, $i = 1, \ldots, N$, "horizontally" to vectorfields $\tilde{X}_i = X_i + 0(\partial / \partial t)$ in $\mathbf{R}_x^N \times \mathbf{R}_t$. Suppose Z is a vectorfield on $\mathbf{R}_x^N$ as in the proposition, and introduce $\tilde{X}_{N+1} = Z + (\partial / \partial t)$ on $\mathbf{R}_x^N \times \mathbf{R}_t$ to complete the $(N + 1)$-frame field $(\tilde{X}_1, \ldots, \tilde{X}_{N+1})$; note that the $[\tilde{X}_i, \tilde{X}_{N+1}]$ vanish to infinite order at $(0, 0)$. Putting this field in normal form using a diffeomorphism h of $(\mathbf{R}^N \times \mathbf{R}, 0)$, we note that h preserves the t variable and that for each value of t, a frame field in normal form is induced on $\mathbf{R}^N \times \{t\}$ by restricting the $h_*(\tilde{X}_i)$. The invariants of these induced frames are equal to those of (X_i) along the flow line of Z through the origin. The t derivatives of these invariants at 0 are invariants of $(\tilde{X}_1, \ldots, \tilde{X}_{N+1})$ at $(0, 0)$ which involve t, and thus vanish by the Lie derivative condition. Thus $(X_1, \ldots, X_N)$ is infinitesimally symmetric.

The converse is similar; we suppose that Z_0 is a vector which is annihilated by the invariant mapping of $(X_1, \ldots, X_N)$ at the origin and write $\tilde{X}_{N+1}^0$ for the constant vectorfield $Z_0 + \partial / \partial t$ on $\mathbf{R}^{N+1}$. To construct the vectorfield Z, we put $(\tilde{X}_1, \ldots, \tilde{X}_N, \tilde{X}_{N+1}^0)$ in normal form at the origin by a diffeomorphism $h: (\mathbf{R}^{N+1}, 0) \to (\mathbf{R}^{N+1}, 0)$ and set

$$Z_x = \pi_*(h^{-1})_* \{ (\partial / \partial t) \mid_{(x,0)} \}$$

where $\pi(x, t) = x$ for $(x, t) \in \mathbf{R}^N \times \mathbf{R}$. After writing $\tilde{X}_{N+1} = Z + \partial/\partial t$ and transforming to normal form for $(\tilde{X}_1, \ldots, \tilde{X}_{N+1})$, it is easy to compute the required brackets $[Z, X_i]$ and their derivatives.

3. Deformations of an infinitesimally asymmetric, globally asymmetric geometry A structure on a compact manifold. The purpose of this section is to provide the constructions which are summarized in the following theorem. Here the concrete objects $(Y \to M, \omega)$ in geometry A are first given the topology of $\mathcal{C}^\infty$ convergence of connection forms on compact subsets of Y.

THEOREM 1. *Given an infinitesimally asymmetric, globally asymmetric object* $(Y \to M, \omega)$ *in gemoetry A, with M compact, there are*
  (1) *a finite-dimensional vector bundle $E \to M$,*
  (2) *a system of differential equations on E, and*
  (3) *a subset $E_A \subset E$*
such that those solutions of the system (2) *which are in a $\mathcal{C}^\infty$ neighborhood of the zero section and have values in E_A are in one-to-one correspondence with the equivalence classes of those objects in geometry A which are $\mathcal{C}^\infty$ perturbations of the given one.*

Choosing $(Y \to M, \omega)$ as in the theorem, we begin by writing the structure as a family of N vectorfields $X_1, \ldots, X_N$ on Y. There is then as in §2A a mapping i_ω: $Y \to J_F^\infty$ describing all the finite-order local invariants of this N-frame field on Y: this map has image in the subset J_A^∞ described in §2B. By infinitesimal asymmetry and compactness of M there must be $r < \infty$ such that the r-jet invariant mapping i_ω^r: $Y \to J_F^r$ has image in the open set W^r of §2D.

Define V^r as the space of orbits of H on W^r, and let $V_A^r = (W^r \cap J_A^r)/H \subset V^r$. There is a mapping of principal H-bundles

$$
\begin{array}{ccc}
Y & \overset{i_\omega^r}{\to} & W^r \\[1em]
\downarrow & & \downarrow \\[1em]
M & \overset{i_0}{\to} & V^r
\end{array}
$$

(this diagram defines i_0), and $i_0(M) \subset V_A^r$. This map i_0 clearly determines up to equivalence the bundle $Y \to M$ and the connection ω.

The Pfaffian system $\psi_1, \ldots, \psi_K$ on W^r which was described in §2D is easily seen to push down to a Pfaffian system $\eta_1, \ldots, \eta_K$ on V^r which has the same rank, since the ψ_i annihilate tangents to the H-orbits and the system (ψ_i) behaves apropriately under the action of H. Any immersion i': $M \to V^r$ which satisfies the Pfaffian system (η_i) and is homotopic to i_0 clearly lifts to an H-equivariant immersion f: $Y \to W^r$ which satisfies (ψ_i).

Remembering the immersion condition on $(\pi \circ f)$ in Lemma (2) of §2D, we put additional structure on V^r. Since the action of H preserves the fibering of J_F^r over J_F^{r-1}, there is a well-defined *submersion* $\tilde{\pi}$: $V^r \to \pi(W^r)/H$, and $\pi \circ f$ is an immersion if and only if $\tilde{\pi} \circ i'$ is an immersion. Then by §2D we have

LEMMA 1. *A smooth map $i': M \to V^r$ which is homotopic to i_0 lifts to the invariant mapping of a connection on Y in geometry A if and only if*
(1) $\langle \psi_i, i'_* X \rangle = 0$ *for all* $X \in T_* M$ *and* $i = 1, \ldots, K$,
(2) $\tilde{\pi} \circ i'$ *is an immersion, and*
(3) $i'(M) \subset V_A$.

Now we will make use of the global asymmetry to choose a particular immersion i' of this type to represent *each* equivalence class of structures which approximates the given one. To make this choice we use a "tubular neighborhood of the immersed submanifold" to pull back ψ_i to a system on the normal bundle. Our procedure will automatically eliminate compositions of allowable immersions with diffeomorphisms near the identity. However, new discrete global symmetries can appear in the deformed structures unless we take some care in limiting the size of the tubular neighborhood. Thus we need the following estimate:

LEMMA 2. *Let $V = V^r$ and M be given Riemannian metrics so that i_0 is an isometry. Then there is $\delta_0 > 0$ such that each $\{y: d(x, y) \leq 2\delta_0\}$ is a disk embedded by i_0, and such that if $0 < \delta < \delta_0$ and $i', i'': M \to V$ are maps with $d(i_0(x), i'(x)) + d(i_0(x), i''(x)) < \delta/3$ for all $x \in M$, then any diffeomorphism $h: M \to M$ with $i' \circ h = i''$ must satisfy $d(x, h(x)) < \delta$, all $x \in M$.*

PROOF (BY CONTRADICTION). Suppose $0 < \delta_1$ with $d(i_0 x, i_0 y) > \frac{2}{3} d(x, y)$ whenever $d(x, y) < \delta_1$, and suppose also that the first condition on δ_0 holds for all δ with $0 < \delta \leq 5\delta_1$. Choose $\delta_2 > 0$ so that for each $x \in M$, each path component of $\{y \in M: d(i_0 x, i_0 y) < \delta_2\}$ is contained in a ball of radius $\delta_1/3$, and let δ_3 satisfy $0 < 6\delta_3 < \delta_2$. Choose a finite set $A \subset M$ such that each point of M is within a distance of δ_3 of some point of A. Now for each $k = 4, 5, \ldots$, choose maps i'_k, i''_k and diffeomorphisms h_k which contradict the condition in the conclusion for values δ_k of δ with $\delta_k \downarrow 0$; by taking a subsequence we may assume that the sequence $h_k(a)$ converge for $a \in A$.

Since $i'_k \circ h_k = i''_k$ we know for each $x \in M$ that

$$d(i_0(x), i_0 h_k(x)) \leq d(i_0(x), i''_k(x)) + d(i'_k h_k(x), i_0 h_k(x)) < \tfrac{2}{3}\delta_k.$$

Thus if $d(x, h_k(x)) < \delta_1$ for all $x \in M$, then the first condition on δ_1 implies that $d(x, h_k(x)) < \delta_k$, contradicting the hypothesis on h_k. It follows that $h_k(x)$ cannot be a Cauchy sequence for all $x \in M$: in fact, the limit would then define a covering map $h_\infty \neq \text{id}$ with $i_0 h_\infty = i_0$, and h_∞ would be one-to-one since only finitely many sheets of $i_0(M)$ pass through a given point of V. In particular, we cannot have large k_0 such that for $k, k' > k_0$, $d(h_k(x), h_{k'}(x)) < \delta_1$ for all x, for then the above estimate and the first condition on δ_1 would imply that $d(h_k(x), h_{k'}(x)) < \delta_k + \delta_{k'}$, for all x.

For $a \in A$, where the h_k converge, we have $d(h_k(a), h_{k'}(a)) < \delta_1/3$ when $3(\delta_k + \delta_{k'}) < \delta_1$. For arbitrary $x \in M$, choose a path x_t, $0 \leq t \leq 1$, with $x_0 = a \in A$, $x_1 = x$, and $d(a, x_t) < \delta_3$ for all t. We can show that the path $h_k(x_t)$ is contained in the ball of radius $\delta_1/3$ about $h_k(a)$, so that the absurd estimate $d(h_k(x), h_{k'}(x)) < \delta_1$

holds when $3(\delta_k + \delta_{k'}) < \delta_1$. In fact, if we note that

$$d(i_0(a), i_0 h_k(x_t)) \leqslant d(i_0(a), i_0(x_t)) + d(i_0(x_t), i_k''(x_t))$$
$$+ d(i_k' h_k(x_t), i_0 h_k(x_t))$$
$$< \delta_3 + \delta_k + \delta_k,$$

so that $d(i_0 h_k(x_0), i_0 h_k(x_t)) < 6\delta_3 < \delta_2$ for all t, $0 \leqslant t \leqslant 1$, then we can use the condition on components of sets $\{y: d(i_0(y), i_0(x)) < \delta_2\}$. This proves the lemma.

The *normal bundle* of an immersion $i_0: M \to V$ is the vector subbundle E of $i_0^*(T^*V)$ given by

$$E_x = \left\{ i_0^*(\theta): \theta \in T^*_{i_0(x)}(V), \langle \theta, i_{0*}(v) \rangle = 0 \text{ for all } v \in T_x(M) \right\}.$$

We write p_E for the projection of E and σ_0 for the zero section. Without specifying a metric on V, we can define a *tube* of i_0 as an immersion $\varphi: U \to V$ of a neighborhood of $m(\sigma_0) \subset E$ into V such that φ extends smoothly to $\overline{U}$ so as to be injective on each $E_x \cap \overline{U}$ and such that $\langle \theta, (\partial/\partial t)\varphi(i_0^*(t\theta))_{t=0} \rangle$ is positive for all $\theta \in E$. If $d(i_0 p_E(y), \varphi(y)) < \delta$ for all $y \in U$, in the metric used in Lemma 2, then we will call φ a δ-*tube*. As a straightforward consequence of Lemma 2 we then have

LEMMA 3. *If φ is a δ-tube of i_0 for $0 < 10\delta < \delta_0$ where δ_0 is as in Lemma 2, and if σ', σ'' are distinct sections of E with image in U, then there is no diffeomorphism h: $M \to M$ (i.e. distinct from the identity) for which $\varphi \circ \sigma' \circ h = \varphi \circ \sigma''$.*

To complete the proof of Theorem 1, we now observe that condition (2) of Lemma 1 does not need to be reflected explicitly in the final formulation:

LEMMA 4. *A tube φ of i_0 can be chosen so that $\tilde{\pi} \circ \varphi \circ \sigma$ is an immersion for each section $\sigma: M \to U$.*

PROOF. Construct a tube first for $\tilde{\pi} \circ i_0$ and use the fact that $\tilde{\pi}$ is a submersion.

For later reference we also note that there is a principal H-bundle $\tilde{U} \to U$ to which the tube φ lifts in an equivariant fashion. In fact, the action of H on Y lifts naturally to an action on $p^*(E) \to Y$; thus $(p^*E) \to E$ is itself a principal H-bundle. A tube $\tilde{\varphi}$ of $i^r(Y)$ then can be easily constructed so that the diagram

$$
\begin{array}{ccccc}
(p^*E) & \supset & \tilde{U} & \stackrel{\tilde{\varphi}}{\to} & W^r \\
\downarrow & & \downarrow & & \downarrow \\
E & \supset & U & \stackrel{\varphi}{\to} & V^r
\end{array}
$$

commutes and $\tilde{\varphi}$ is H-equivariant. A specific equivariant immersion of Y is thus associated with each section σ of E with image in U by the correspondence $\sigma \mapsto \tilde{\varphi} \cdot (p^*\sigma)$. If $\varphi \circ \sigma$ satisfies the conditions of Lemma 1, this specific immersion $\tilde{\varphi} \cdot (p^*\sigma)$ is the invariant mapping of a connection in geometry A.

4. The universal bundle.

A. *Coordinate transformations*. In the preceding section, we described a coordinate chart in a moduli space and associated to each point a particular geometric structure on Y. By studying the coordinate transformations, we will now fit these together to

form a Y-bundle over the global moduli space (of asymmetric objects) with a structure on each fiber.

Choosing an H principal bundle Y over a compact manifold M of the right dimension, we construct the asymmetric part $\mathfrak{M}^\infty(Y, A)$ of the moduli space of geometry A structures on $Y \to M$. As a set, $\mathfrak{M}^\infty = \mathfrak{M}^\infty(Y, A)$ consists of the classes of infinitesimally asymmetric, globally asymmetric geometry A structures on $Y \to M$ under the relation of equivalence defined in §1. $\mathfrak{M}^\infty$ is filtered by subsets $\mathfrak{M}^4 \subset \mathfrak{M}^5 \subset \cdots \subset \mathfrak{M}^\infty$, where $\mathfrak{M}^r$ consists of the classes of those structures whose r-jet mappings have images in W^r.

To topologize $\mathfrak{M}^\infty$, we will define the basic open neighborhoods of its points in a noninvariant way and later justify our definition by showing its independence of choices. For $[\omega] \in \mathfrak{M}^r \subset \mathfrak{M}^\infty$, we choose a tube $\varphi\colon U \to V^r$ of an immersion $i_0\colon M \to V^r$ representing the equivalence class of A-objects $[\omega]$; we write $\{i_0\} := [\omega]$. For each $\mathcal{C}^\infty$ open neighborhood $\mathfrak{U}$ of the zero section in the space of sections of the normal bundle E (§3), a basic neighborhood of $[\omega]$ in $\mathfrak{M}^\infty$ is given as the set of all classes $\{\varphi \circ \sigma\}$ represented by immersions $\varphi \circ \sigma\colon M \to V^r$ with $\sigma \in \mathfrak{U}$ and $(\mathrm{im}\,\sigma) \subset E_A \cap U$.

Given that this is a consistent definition of the topology, it will be appropriate to call the maps $\varphi_*\colon \sigma \mapsto \{\varphi \circ \sigma\}$ *coordinate charts*. The required justification amounts to showing that the "coordinate transformations" are homeomorphisms. The same must be checked for the bundle coordinate transformations, so we first describe the bundle.

There is a coordinate chart of the bundle for each of the lifted tubes $\tilde\varphi$ described in the last remark of §3. We make the bundle space out of these charts by gluing, so we must specify the identifications. For each coordinate chart $\varphi_*\colon \mathfrak{U} \to \mathfrak{V}$ in $\mathfrak{M}^\infty$ and lift $\tilde\varphi\colon \tilde U \to W^r$ of φ as in the remark above, we obtain a map of $\mathfrak{U} \times Y$ to W^r given by

$$(\sigma, y) \mapsto \{\tilde\varphi \circ p^*(\sigma)\}(y).$$

If $r' \geqslant r$ and another chart $\varphi'_*\colon \mathfrak{U}' \to \mathfrak{V}'$ is given with $\mathfrak{V} \cap \mathfrak{V}' \neq \varnothing$, where $\varphi'\colon U' \to V^{r'}$, then let $\varphi(\mathfrak{U}_0) = \mathfrak{V} \cap \mathfrak{V}' = \varphi'(\mathfrak{U}'_0)$. The points $(\sigma, y) \in (\mathfrak{U}_0 \times Y)_{\tilde\varphi}$ and $(\sigma', y') \in (\mathfrak{U}'_0 \times Y)_{\tilde\varphi'}$ will then be identified if and only if:

(i) σ and σ' correspond by $\varphi_*(\sigma) = \varphi'_*(\sigma')$;

(ii) $\{\tilde\varphi \circ p^*(\sigma)\}(y) = p_{r,r'}\{\tilde\varphi' \circ p^*(\sigma')\}(y')$ where $p_{r,r'}\colon J_F^{r'} \to J_F^r$ is the projection.

Once we know that the maps $(\mathfrak{U}_0 \times Y)_{\tilde\varphi} \to (\mathfrak{U}'_0 \times Y)_{\tilde\varphi'}$ are homeomorphisms, we can define the total space $\mathcal{E}^\infty(Y, A)$ of our bundle as the space obtained from the disjoint union of the $(\mathfrak{U} \times Y)_{\tilde\varphi}$ by identifying as above. The coordinate charts will then be the natural identification maps $\tilde\varphi_*\colon \mathfrak{U} \times Y \to \mathcal{E}^\infty(Y, A)$, and the group of the bundle will be $\mathrm{Diff}(Y)^H$. That is, to complete the proof of the following theorem we now need only check certain continuity requirements.

THEOREM 2. *Let $Y \to M$ be an H principal bundle over a compact manifold of the appropriate dimension. There exists a topological space $\mathfrak{M}^\infty(Y, A)$ whose points are the equivalence classes of infinitesimally and globally asymmetric geometry A structures on $Y \to M$; convergence in this space means $\mathcal{C}^\infty$ convergence on compact sets for some representative connection forms on Y.*

Over $\mathfrak{M}^\infty(Y, A)$ *there is a bundle with fiber* Y, *group* $\mathrm{Diff}(Y)^H$, *and projection* Π: $\mathcal{E}^\infty(Y, A) \to \mathfrak{M}^\infty(Y, A)$. *This bundle has the additional structure of a continuous mapping* inv: $\mathcal{E}^\infty(Y, A) \to J_A^\infty$ *such that for each class* $[\omega] \in \mathfrak{M}^\infty(Y, A)$, *the restriction* $(\mathrm{inv} \mid \Pi^{-1}\{[\omega]\})$ *is the invariant mapping for a geometry* A *structure on* $\Pi^{-1}\{[\omega]\} \cong Y$ *which represents* $[\omega]$.

The structure on each fiber is of course well defined due to the asymmetry.

We have reduced the theorem to a few easy estimates which we now make explicit. By continuity of addition of sections of E, $p^*(E)$, etc., we can restrict our attention to continuity at the zero section for those coordinate transformations which fix the zero section. Moreover, we need only consider the case of the total space transformations, since then the result about the base will follow. Each such transformation is a composition of two types: those with $r = r'$, and those where $r = r' - 1$ and the tube $\tilde\varphi'$ is obtained from $\tilde\varphi$ in a trivial fashion (or vice versa).

When $r = r'$, the map $h = \tilde\varphi'^{-1} \circ \tilde\varphi$ is defined in a neighborhood (say U) of the zero section of E as an H-equivariant but nonlinear diffeomorphism which fixes the zero section. Write $\| \cdot \|_{r, K}$ for the pointwise norm on the partials of order $\leqslant r$ on compact K, where the differentiations are with respect to the vectorfields $X_1, \ldots, X_N$ on Y given by the structure for the zero section. We then need only the obvious estimate that $\| h \circ \sigma \|_{r, K} \leqslant C \| \sigma \|_{r, K}$ for small σ.

Now let $r = r' - 1$ and define $\tilde\varphi'$ from $\tilde\varphi$ by using the product structure on $J_F^{r'} \cong J_F^r \times \mathbf{R}^t$, first to lift $\tilde\varphi$ to $J_F^{r'}$ so that $\tilde\varphi(p^{-1}\{y\} \cap U)$ is horizontal for $y \in Y$, p: $E \to Y$, and then to extend this lift vertically by means of an isomorphism $E' \cong E \times \mathbf{R}^t$. In one direction the coordinate transformation is then norm-decreasing. In the reverse direction, a section σ of E is mapped to a section $\sigma \times f$ of $E \times \mathbf{R}^t$. The function f is defined by differentiating the function $(\tilde\varphi \circ \sigma - \tilde\varphi \circ \sigma_0)$ with respect to the vectorfields $X_k(\sigma)$ on Y given by the structure associated with $\tilde\varphi \circ \sigma$; σ_0 is here the zero section. As the $X_k(\sigma)$ are bounded relative to the $X_j = X_j(\sigma_0)$, there is again no difficulty with $\mathcal{C}^\infty$ continuity, and the theorem is proved.

Note that the connection forms on the fibers combine to form a continuous function from a $T_* Y$ bundle over $\mathfrak{M}^\infty$ to $\mathbf{R}^N$.

B. *Universality.* We now describe the characteristic property of the object we have just constructed in Theorem 2.

DEFINITION. A *deformation* of infinitesimally and globally asymmetric geometry A structures on an H principal bundle $Y \to M$ is a bundle with fiber Y and with an infinitesimally and globally asymmetric structure on each fiber, such that each of the invariants of these structures defines a continuous function on the total space.

THEOREM 3. *For any deformation* $D \to X$ *as above, there is a unique map of bundles*

$$
\begin{array}{ccc}
D & \overset{h}{\to} & \mathcal{E}^\infty(Y, A) \\
\downarrow & & \downarrow \\
X & \overset{\bar h}{\to} & \mathfrak{M}^\infty(Y, A)
\end{array}
$$

for which h pulls back the canonical connection on the fibers of $\mathcal{E}^\infty$ *to the given one on the fibers of* D.

PROOF. It is clear that the map $\bar{h}$ is uniquely defined as a function, and that there exists a unique function h so that the diagram commutes and (inv) $\circ$ h restricts to the invariant map on each fiber.

Given $r < \infty$, let $D^r \to X^r$ be the restriction to the largest $X^r \subset X$ mapping to $\mathfrak{M}^r$, and let $j^r: D^r \to W^r$ be the map $z \mapsto [r\text{-jet of inv}(h(x))]$. We know, for every $r < \infty$ and $z \in D^r$ and every open neighborhood Q^r of $j^r(z)$ in W^r, that z has an open neighborhood in D^r which maps into Q^r by j^r.

Continuity of h will follow quickly from continuity of $\bar{h}$ since the topology on a bundle chart is the product topology, and since the charts in $\mathfrak{S}^\infty$ are given by tubes in W^r. Continuity of $\bar{h}$ can be expressed in terms of h with fiberwise-saturated open subsets of $\mathfrak{S}^\infty$ and D.

If Y_x is the fiber over $x \in X$, let $\tilde{\varphi}_*: \mathfrak{U} \to \mathfrak{S}^\infty$ be a chart centered along the fiber $h(Y_x)$, where φ is a tube in W^r. The basic open fiberwise saturated neighborhoods of $h(Y_x)$ are given, using the equations of Lemma 1 of §3, by $\mathcal{C}^s$ neighborhoods of the zero section of the source bundle of φ, for varying s. The solutions of these equations satisfying a $\mathcal{C}^s$ estimate are, however, the same as the solutions satisfying the condition that their natural lifts to W^{r+s} satisfy a $\mathcal{C}^0$ estimate. The fiberwise saturated neighborhoods of $h(Y_x)$ are thus determined by H-invariant open neighborhoods $\tilde{Q}^r$ of $j^r(Y_x)$ where $r \geqslant r_0$ is allowed to vary.

Suppose such an open set $\tilde{Q}^r$ is given, and let $K \subset Y_x$ be a compact subset which meets each H-orbit in Y_*. Each point of K has an open neighborhood mapping into $\tilde{Q}^r$; we may take a finite subcover $V_1, \ldots, V_q$ of K. The intersection of the images of the V_j in X is an open neighborhood of x which is mapped by $\bar{h}$ into the appropriate open neighborhood of $\bar{h}(x)$. That is, if $p: D \to X$ is the projection, then h maps $p^{-1}(\bigcap_j p(V_j))$ into $\tilde{Q}^r$.

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