SYMMETRIC SKEW BALANCED STARTERS
AND COMPLETE BALANCED HOWELL ROTATIONS

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Abstract. Symmetric skew balanced starters on n elements have been previously constructed for n = 4k + 3 a prime power and 8k + 5 a prime power. In this paper we give an approach for the general case n = 2^m k + 1 a prime power with k odd. In particular we show how this approach works for m = 2 and 3. Furthermore, we prove that for n of the general form and k > 9 · 2^m, then a symmetric skew balanced starter always exists. It is known that a symmetric skew balanced starter on n elements, n odd, can be used to construct complete balanced Howell rotations (balanced Room squares) for n players and 2(n + 1) players, and in the case that n is congruent to 3 modulo 4, also for n + 1 players.

1. Introduction. Let S₁, S₂,..., S_m be a family of subsets of the elements in GF(n) where n is an odd prime power. Let D_i = \{x - x' for all x and x' in S_i, x \neq x'\} denote the set of symmetric differences generated by S_i. Then S₁, S₂,..., S_m are called supplementary difference sets (mod n) if D₁, D₂,..., D_m together contain each nonzero element of GF(n) an equal number of times.

A set of m = (n - 1)/2 pairs (x₁, y₁), (x₂, y₂),..., (x_m, y_m) is called a starter if
(i) the m pairs contain each nonzero element of GF(n) exactly once and
(ii) the m pairs are supplementary difference sets (mod n).

A starter is strong if
(iii) x₁ + y₁, x₂ + y₂,..., x_m + y_m are all distinct elements of GF(n).

It is skew if in addition
(iv) ±(x₁ + y₁), ±(x₂ + y₂),..., ±(x_m + y_m) are all distinct.

A starter is balanced if
(v) the two sets \{x₁, x₂,..., x_m\} and \{y₁, y₂,..., y_m\} are supplementary difference sets (mod n).

A starter is symmetric if
(vi) \{x₁, x₂,..., x_m\} = {-x₁, -x₂,..., -x_m}.

It is well known [7] that a Room square of side n can be constructed from a strong starter modulo n by assigning the pair (x_j + y_j) to cell (j, x_j + y_j + j) for j = 0, 1,..., n - 1, and the pair (∞, j) to cell (j, j) for j = 0, 1,..., n - 1. If the starter is balanced or skew, then the constructed Room square is also balanced or skew. It is also known [4] that a strong balanced starter modulo n can be used to construct a complete balanced Howell rotation for n players and, if n ≡ 3 mod 4, then also a complete balanced Howell rotation for n + 1 players [1] (which is
equivalent to a balanced Room square of side \( n \). Finally, it has been shown \([5, 7]\)
that a balanced Room square (or a complete balanced Howell rotation) for \( 2(n+1) \)
players can be constructed from a symmetric skew balanced starter modulo \( n \).

Symmetric skew balanced starters have been constructed for the case \( n = 4k + 3 > 3 \) a prime power \([1]\), and for the case \( n = 8k + 5 > 5 \) a prime power \([4, 5]\). In this
paper we give an approach for the general case \( n = 2^m k + 1 \) a prime power with \( k \)
odd (the two previous cases correspond to \( m = 1 \) and \( m = 2 \)). In particular we give a
construction for the case \( m = 3 \) and prove an asymptotic result for the general case.

2. The general approach. Let \( \text{GF}^*(n) \) denote the multiplicative group of \( \text{GF}(n) \).
We quote a result of Bose \([2]\).

**Bose Lemma.** Let \( n = 4k + 1 \) be a prime power and let \( x \) be a generator of \( \text{GF}^*(n) \).
Then the two sets \( \{ x^2, x^4, \ldots, x^{4k} \} \) and \( \{ x, x^3, \ldots, x^{4k-1} \} \) are supplementary
difference sets \((\text{mod} \ n)\).

From now on we will always assume that \( n \) is a prime power of the form \( 2^m k + 1 \)
where \( k \) is odd and \( m > 2 \). Let \( x \) be a generator of \( \text{GF}^*(n) \) and for any element
\( y \in \text{GF}^*(n) \), we write \( T(y) = z \) if \( y = x^z \).

**Theorem 1.** Suppose that there exists an element \( y \in \text{GF}^*(n) \) satisfying

(i) \( T(y) \equiv -1 \pmod{2^m} \),
(ii) \( T(y - 1) \equiv T(x - 1) \pmod{2} \),
(iii) \( T(y + 1) \equiv T(x + 1) \pmod{2} \).

Then the set of \((n-1)/2\) pairs

\[
\begin{align*}
(x^{2^m i + 2 j + 1}, x^{2^m i + 2 j + 2}), & \quad i = 0, 1, \ldots, k - 1, j = 0, 1, \ldots, 2^{m-2} - 1, \\
(x^{2^m i + 2^m - 1 + 2 j + 2}, x^{2^m i + 2^m - 1 + 2 j + 2}), & \quad i = 0, 1, \ldots, k - 1, j = 0, 1, \ldots, 2^{m-2} - 1,
\end{align*}
\]

is a symmetric skew balanced starter.

**Proof.** That the \( n-1 \) elements in the \((n-1)/2\) pairs are all distinct powers of \( x \)
follows from condition (i). That the \((n-1)/2\) pairs are supplementary difference
sets follows from condition (ii). Therefore the set of \((n-1)/2\) pairs is a starter. The
"skew" property comes from condition (iii). The "symmetric" and the "balanced"
properties come from the fact that \( y \) is an odd power of \( x \) and the Bose Lemma.

The next task is to prove the existence of an element \( y \) satisfying the three
conditions of Theorem 1. Let \( Y \) denote the set of elements satisfying conditions (i),
(ii), (iii). Let \( Y' \) denote the set of elements \( y \) satisfying conditions (i), (ii'), (iii') where

(ii') \( T(y - 1) \equiv T(x - 1) + 1 \pmod{2} \),
(iii') \( T(y + 1) \equiv T(x + 1) + 1 \pmod{2} \).

Then clearly, \( Y \cap Y' = \emptyset \). Finally, let \( Z \) denote the set of elements \( z \) satisfying
conditions (iv), (v) where

(iv) \( T(z) \equiv -2 \pmod{2^m} \),
(v) \( T(z - 1) \equiv T(x^2 - 1) \pmod{2} \).

Since there exists a 1-1 mapping between \( z \) satisfying condition (iv) and \( y \) satisfying
condition (i), while condition (v) implies that \( y \) must satisfy either conditions (ii) and
(iii) or conditions (ii') and (iii'), we have \(| Z | = | Y | + | Y' | \).
Let $U$ denote the set of elements satisfying conditions (i) and (ii). Let $V$ denote the set of elements satisfying conditions (i) and (iii'). Then $Y = U \setminus V$ and $Y' = V \setminus U$. Suppose $Y = \emptyset$, i.e., $U \subseteq V$. Then $|Y'| = |V| - |U|$. Therefore if we can show $|Z| > |V| - |U|$, then $Y \neq \emptyset$.

3. The cases $m = 2$ and $m = 3$. The existence of a symmetric skew balanced starter for the $m = 2$, i.e., $n = 8k + 5$, case has been shown in [5] for $n > 5$. Here we use the approach given in §2 for a different proof. By using the cyclotomic matrix and equations (see pp. 28 and 48 of [8], for example) with $n = 4k + 1$, $k$ odd, we obtain

$$
|U| = B + E \quad \text{if } T(x - 1) \text{ is odd,}
$$

$$
= D + E \quad \text{if even,}
$$

$$
|V| = D + E \quad \text{if } T(x + 1) \text{ is even,}
$$

$$
= B + E \quad \text{if odd,}
$$

$$
|Z| = B + D \quad \text{if } T(X^2 - 1) \text{ is odd,}
$$

$$
= A + C \quad \text{if even,}
$$

with

$$
16B = n + 1 + 2s - 8t, \quad 16D = n + 1 + 2s + 8t, \quad 8(A + C) = n - 3 - 2s,
$$

where $n = s^2 + 4t^2$ with $s \equiv 1 \mod 4$. As the parity of $T(x^2 - 1)$ is determined by the parities of $T(x - 1)$ and $T(x + 1)$, therefore there are only four choices for $|U|$, $|V|$ and $|Z|$. The possible value of $|Z| - |V| + |U|$ in these four possible choices are $2B, 2D, \text{ and } A + C$. Therefore it suffices to prove

$$
\min\{n + 1 + 2s + 8|t|, n - 3 - 2s\} > 0.
$$

Note that $n + 1 + 2s - 8|t| = (s + 1)^2 + 4(|t| - 1)^2 - 4$ and $n - 3 + 2s = (s + 1)^2 + 4t^2 - 4$. Using the property that $s \equiv 1 \mod 4$, the minimum of the two equations can be $\leq 0$ only for the following set of pairs: $s = 1, |t| \leq 1; s = -3, |t| \leq 1$. The values of $n$ corresponding to these pairs are $1, 5, 9, 13$, of which only $13$ is of the form $n = 8k + 5 > 5$. But $2$ is a generator of $GF(13)$ and it is straightforward to check that $y = 2^3$ satisfies conditions (i)-(iii) of Theorem 1. Therefore the $m = 2$ case is settled. Next we deal with the case $m = 3$.

**Theorem 2.** There exists a symmetric skew balanced starter (mod $n = 16k + 9$) for every $k \geq 1$.

**Proof.** Using the cyclotomic matrix and equations (see pp. 29 and 79 of [8]) with $n = 8k + 1$, $k$ odd, we obtain

$$
|U| \text{ is either } H + K + J + 0 = \frac{1}{16}(n - 1 - 4y + 4b) \text{ if } T(x - 1) \text{ is even, or}
$$

$$
M + B + 0 + I = \frac{1}{16}(n - 1 + 4y - 4b) \text{ if } T(x - 1) \text{ is odd,}
$$

$$
|V| \text{ is either } J + L + D + M = \frac{1}{16}(n - 1 - 4y - 4b) \text{ if } T(x + 1) \text{ is odd, or}
$$

$$
K + F + L + I = \frac{1}{16}(n - 1 + 4y + 4b) \text{ if } T(x + 1) \text{ is even,}
$$

$$
|Z| \text{ is either } G + C + N + N = \frac{1}{16}(n - 3 + 2x) \text{ if } T(x^2 - 1) \text{ is even, or}
$$

$$
L + K + 0 + M = \frac{1}{16}(n + 1 - 2x) \text{ if } T(x - 1) \text{ is odd,}
$$

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(even though the values of the upper case variables depend on whether 2 is a fourth power of GF(n), the above sums remain unchanged,) with

(i) \( n = x^2 + 4y^2, x \equiv 1 \pmod{4} \) is the unique proper representation of \( n = p^n \) if \( p \equiv 1 \pmod{4} \); otherwise, \( x = \pm p^{n/2}, y = 0 \).

(ii) \( n = a^2 + 2b^2, a \equiv 1 \pmod{4} \) is the unique proper representation of \( n = p^n \) if \( p \equiv 1 \) or 3 \pmod{8} \); otherwise, \( a = \pm p^{n/2}, b = 0 \).

Consider the four possible choices of \(|U|, |V|\) and \(|Z| = |Z| - |V| + |U|\). It suffices to prove

\[
\begin{align*}
n - 3 &> 2 |x| + 8 |y|, \\
n + 1 &> 2 |x| + 8 |b|.
\end{align*}
\]

The first inequality is of the same type as we encountered in the \( m = 2 \) case. The only values of \( n \) not satisfying the inequalities are \( n = 1, 5, 9, 13, 17 \) of which none is of the form \( n = 16k + 9, k \geq 1 \). To prove the second inequality, note that \( x \leq \sqrt{n/2} \) and \( b \leq \sqrt{n/2} \). Therefore it suffices to prove \( n + 1 > (2 + 4\sqrt{2})\sqrt{n} \) which is equivalent to requiring that

\[
\sqrt{n} > \frac{2 + 4\sqrt{2} + \left((2 + 4\sqrt{2})^2 - 4\right)^{1/2}}{2} = 1 + 2\sqrt{2} + 2(2 + \sqrt{2})^{1/2}.
\]

It is easily seen that if \( n \geq 64 \), the above inequality is satisfied. There are two values of \( n, 9 < n < 64 \), of the form \( n = 16k + 9 \) (\( n \) a prime power), i.e., \( n = 25, 41 \). We deal with these two cases separately.

\( n = 25 \). Then \( x = -3, y = \pm 2, a = 5, b = 0 \).

\[
n - 3 = 22 > 2 |-3| + 8 |0| = 6.
\]

\( n = 41 \). Then \( x = 5, y = \pm 2, a = (-3), b = \pm 4 \).

\[
n - 3 = 38 > 2 |5| + 8 |4| = 42.
\]

But \( x = 13 \) is a generator of GF(41) while \( y = 13^{15} = 14 \) satisfies conditions (i)–(iii) of Theorem 1.

**Corollary 1.** There exists a complete balanced Howell rotation for \( n = 16k + 9 \) players, \( k \geq 1 \).

**Corollary 2.** There exists a complete balanced Howell rotation, and also a balanced Room square, for \( n = 32k + 20 \) players for every \( k \geq 0 \) (since such a rotation exists for \( n = 20 \) using the method of [1], no exception is needed).

The complete balanced Howell rotations constructed by using Corollaries 1 and 2 are all new, except for those \( n \) for which \( n - 1 \equiv 3 \pmod{4} \) and \( n - 1 \) is a prime power.

**4. An asymptotic result.** The cyclotomic numbers for the \( m = 4 \) case are known [3, 9]. However, there are too many equations and parameters which determine the cyclotomic numbers to go through and there are too many cases of \(|Z| + |U| - |V|\) to check. Therefore we change direction from proving complete results for a single \( m \) to proving asymptotic results for all \( m \).
THEOREM 3. For each fixed \( m \), let \( n = 2^m k + 1 \) be a prime power where \( k \) is odd. Then a symmetric skew balanced starter, hence complete balanced Howell rotations for \( n \) and \( 2n \) players, always exists for \( k > 9 \cdot 2^3m \).

PROOF. Let \( q = ef + 1 \) be a prime power. Then it is well known [8] that any cyclotomic number \((i, j)\) with order \( e \) satisfies
\[
(i, j)_e = \frac{1}{e^2} \sum_{u=0}^{e-1} \sum_{v=0}^{e-1} (-1)^{u} \beta^{-iu-jv} J(x^u, x^v),
\]
where \( \beta = \exp(2\pi i / e) \) and \( J(x^u, x^v) \) is the Jacobi sum
\[
\sum_{\alpha \in GF(q)} \chi^u(\alpha) \chi^v(1 - \alpha) = J(x^u, x^v)
\]
for a character \( \chi \) on \( GF(q) \) of order \( e \). Furthermore, it is well known (see Chapter 8.3 of [6], Theorem 1 and Corollary) that \( J(x^0, x^0) = q - 2 \) and all other \( J(x^u, x^v) \) have absolute value \( \sqrt{q} \) or 1. Thus for \( i, j, e \) fixed,
\[
| (i, j)_e - q/e^2 | < \sqrt{q}.
\]
To prove Theorem 3, let \( q = n \) and \( e = 2^m \). Then each of \( Z, U \) and \( V \) is a sum over \( 2^{m-1} \) cyclotomic numbers. Therefore
\[
| Y | = | Z | + | U | - | V | > n/2^{m+1} - 3 \cdot 2^{m-1} \sqrt{n} > 0
\]
if \( \sqrt{n} > 3 \cdot 2^m \), or equivalently, if \( k > 9 \cdot 2^3m \). Theorem 3 is now an immediate consequence of Theorem 1.

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