ABSTRACT. The non-Euclidean counterparts of Hardy-Littlewood's theorems on Lipschitz and mean Lipschitz functions are considered. Let $1 < p < \infty$ and $0 < \alpha \leq 1$. For $f$ holomorphic and bounded, $|f| < 1$, in $|z| < 1$, the condition that
\[
f'(z) = O\left(\left(1 - |z|\right)^{-\alpha - 1}\right)
\]
is necessary and sufficient for $f$ to be continuous on $|z| \leq 1$ with the boundary function $f(e^{it}) \in \sigma \Lambda_\alpha$, the hyperbolic Lipschitz class. Furthermore, the condition that the $p$th mean of $f^*$ on the circle $|z| = r < 1$ is $O(1 - r)^{-1}$ is necessary and sufficient for $f$ to be of the hyperbolic Hardy class $H_p^r$ and for the radial limits to be of the hyperbolic mean Lipschitz class $\sigma \Lambda_\alpha^p$.

1. Introduction. We shall prove the non-Euclidean counterparts of the following Theorems A and B due to G. H. Hardy and J. E. Littlewood [2, Theorem 4, p. 627 and Theorem 3, p. 625] (see [1, Theorem 5.1, p. 74 and Theorem 5.4, p. 78]).

Let $\Phi$ be the family of complex-valued functions $\varphi$ defined on the real axis such that $\varphi$ is periodic with period $2\pi$. We say that $\varphi \in \Phi$ is of Lipschitz class $\Lambda_\alpha$ $(0 < \alpha \leq 1)$ if
\[
\sup_{|t-s| \leq \tau} |\varphi(t) - \varphi(s)| = O(\tau^\alpha) \quad \text{as} \quad \tau \to +0.
\]
Let $D = \{|z| < 1\}$ and let $D^* = \{|z| \leq 1\}$ in the plane.

**Theorem A.** Let $f$ be a function holomorphic in $D$ and let $0 < \alpha \leq 1$. Then $f$ is continuous on $D^*$ and the function $f(e^{it})$ is of class $\Lambda_\alpha$ if and only if
\[
f'(z) = O\left(\left(1 - |z|\right)^{-\alpha - 1}\right) \quad \text{as} \quad |z| \to 1 - 0.
\]

We say that $\varphi \in \Phi$ is of mean Lipschitz class $\Lambda_\alpha^p$ $(1 \leq p < \infty, 0 < \alpha \leq 1)$ if the restriction of $\varphi$ to $[0, 2\pi]$ is of $L^p[0, 2\pi]$ and if
\[
\left(1^{|\varphi(t + h) - \varphi(t)|^p} dt\right)^{1/p} = O(\tau^\alpha)
\]
as $\tau \to 0$. For $0 \leq r < 1$, $0 < p < \infty$, and for $v$ nonnegative and subharmonic in $D$, we set
\[
\mu_p(r, v) = \left[\frac{1}{2\pi} \int_0^{2\pi} v(r e^{it})^p dt\right]^{1/p}.
\]

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this is an increasing function of $r$. The Hardy class $H^p (0 < p < \infty)$ consists of $f$ holomorphic in $D$ such that $\mu_p (r, |f|) = O(1)$ as $r \to 1$, or equivalently, the subharmonic function $|f|^p$ has a harmonic majorant in $D$. By the boundary value of a complex-valued function $g$ in $D$ at the point $e^{it}$ of the unit circle we mean the radial limit $g(e^{it}) = \lim_{r \to 1} g(re^{it})$. Each function $f \in H^p (0 < p < \infty)$ admits the boundary value $f(e^{it})$ at a.e. point $e^{it}$, and $f(e^{it}) \in L^p [0, 2\pi]$.

**Theorem B.** Let $f$ be a function holomorphic in $D$, and let $1 \leq p < \infty, 0 < \alpha \leq 1$. Then $f \in H^p$ and the function $f(e^{it})$ is of class $\Lambda^p_\alpha$ if and only if

\[(1.2) \quad \mu_p (r, |f'|) = O\left(\left(1 - r\right)^{\alpha - 1}\right) \quad \text{as} \quad r \to 1.\]

In the case $\alpha = 1$, (1.2) says that $f' \in H^p$.

The non-Euclidean hyperbolic distance between $z$ and $w$ in $D$ is defined by

\[\sigma(z, w) = \frac{1}{2} \log \frac{|1 - \overline{z}w| + |z - w|}{|1 - \overline{z}w| - |z - w|}.\]

We set $\sigma(z) \equiv \sigma(z, 0)$, the hyperbolic counterpart of $|z|, z \in D$. We say that $\varphi \in \Phi$ is of class $\sigma\Lambda_\alpha (0 < \alpha \leq 1)$ if $\varphi$ is bounded, $|\varphi| < 1$, and if

\[\sup_{|r| < 1} \sigma(\varphi(t), \varphi(s)) = O(\tau^\alpha) \quad \text{as} \quad \tau \to +0.\]

Let $B$ be the family of functions $f$ holomorphic and bounded, $|f| < 1$, in $D$. Then, apparently, $f(e^{it})$ exists a.e. For $f \in B$, the Schwarz-Pick lemma reads

\[f^*(z) \equiv |f'(z)| / \left(1 - |f(z)|^2\right) \leq \left(1 - |z|^2\right)^{-1}, \quad z \in D.\]

We note that $\log f^*$ is subharmonic in $D$, so that $f^{*p} = \exp(p \log f^*) (0 < p < \infty)$ is subharmonic in $D$. The hyperbolic analogue of Theorem A is

**Theorem 1.** Let $f \in B$, and let $0 < \alpha \leq 1$. Then $f$ is continuous on $D^\#$ and the function $f(e^{it})$ is of class $\sigma\Lambda^p_\alpha$ if and only if

\[(1.3) \quad f^*(z) = O\left((1 - |z|)^{\alpha - 1}\right) \quad \text{as} \quad |z| \to 1 - 0.\]

We say that $\varphi \in \Phi$ is of class $\sigma\Lambda^p_\alpha$ ($1 \leq p < \infty, 0 < \alpha \leq 1$) if $|\varphi(t)| < 1$ a.e., if the restriction of $\sigma(\varphi(t)) \equiv \sigma(\varphi(t))$ to $[0, 2\pi]$ is of $L^p[0, 2\pi]$, and if

\[\sup_{0 < h < \tau} \left[\int_0^{2\pi} \sigma(\varphi(t + h), \varphi(t))^p dt\right]^{1/p} = O(\tau^\alpha)\]

as $\tau \to 0$. For $f \in B$ set $\sigma(f)(z) \equiv \sigma(f(z))$, the hyperbolic counterpart of $|f(z)|$ ($z \in D$). Then $\log \sigma(f)$ is subharmonic in $D$ because $X(x) \equiv \log \sigma(e^x)$ is a convex and increasing function of $x \in (-\infty, 0)$, with $-\infty = X(-\infty) \equiv \lim_{x \to -\infty} X(x)$, and $\log \sigma(f) = X(\log |f|)$. For each $a \in D$, the identity $\sigma(g) = \sigma(f, a)$ holds, where $g = (f - a)/(1 - \overline{a}f) \in B$ for $f \in B$. Therefore log $\sigma(f, a)$ and $\sigma(f, a)^p = \exp(p \log \sigma(f, a))[0 < p < \infty]$ are subharmonic in $D$. Let $H^p_a$ be the set of all $f \in B$ such that $\mu_p (r, \sigma(f)) = O(1)$ as $r \to 1$, or equivalently, the subharmonic function $\sigma(f)^p$ admits a harmonic majorant in $D$. The hyperbolic Hardy class $H^p_a (0 < p < \infty)$ is the counterpart of $H^p$. We are now ready to propose a hyperbolic analogue of Theorem B.
THEOREM 2. Let \( f \in B \), and let \( 1 \leq p < \infty \), \( 0 < \alpha \leq 1 \). Then \( f \in H^p_\alpha \) and the function \( f(e^{it}) \) is of class \( \sigma \Lambda^p_\alpha \) if and only if
\[
\mu_p(r, f^*) = O\left( (1 - r)^{\alpha - 1} \right) \quad \text{as } r \to 1.
\]

In the case \( \alpha = 1 \) in (1.4), the subharmonic function \( f^p \) admits a harmonic majorant.

The proof of Theorem 1 is not difficult and depends on Theorem A; we need comparisons of the non-Euclidean distance and the Euclidean distance. The proof of the “if” part of Theorem 2 is, in a sense, routine. Not easy is the proof of the “only if” part of Theorem 2. There is no relation between \( \sigma(f) \) and \( f^* \) like that between \( |f| \) and \( |f'| \), namely, one cannot assert that \( \sigma(f') = f^* \) even if \( |f'| < 1 \).

2. Proof of Theorem 1. Consider the two inequalities
\[
\begin{align*}
|z - w| &\leq \sigma(z, w), &z, w \in D, \\
\sigma(z, w) &\leq 2 |z - w|/|1 - \bar{z}w|
\end{align*}
\]
for \( z, w \in D \) with \( |z - w|/|1 - \bar{z}w| \leq 1/\sqrt{2} \). The inclusion formula \( \sigma \Lambda^\alpha \subseteq \Lambda^\alpha \) follows from (2.1). If \( \varphi \in \Lambda^\alpha \) and if \( |\varphi(t)| < 1 \) for all \( t \in (-\infty, \infty) \), then \( \varphi \in \sigma \Lambda^\alpha \).

To observe this we set \( \max |\varphi(t)| = M < 1 \) because \( \varphi \) is continuous. Then there exist two positive constants \( K \) and \( \delta \) such that
\[
K\delta^\alpha \leq (1 - M^2)/\sqrt{2} \quad \text{and} \quad |\varphi(t) - \varphi(s)| \leq K\tau^\alpha
\]
for all \( \tau, 0 < \tau < \delta \), and for all \( t, s \) with \( |t - s| \leq \tau \). Since
\[
|\varphi(t) - \varphi(s)| \leq (1 - M^2)/\sqrt{2},
\]

it follows that
\[
|\varphi(t) - \varphi(s)|/|1 - \overline{\varphi(t)} \varphi(s)| \leq 1/\sqrt{2},
\]
whence, by (2.2),
\[
\sigma(\varphi(t), \varphi(s)) \leq \left[ 2/(1 - M^2) \right] |\varphi(t) - \varphi(s)| \leq K_1 \tau^\alpha
\]
for all \( t, s \) with \( |t - s| \leq \tau < \delta \) \((K_1 = 2K/(1 - M^2))\). Therefore \( \varphi \in \sigma \Lambda^\alpha \).

To prove the “only if” part of Theorem 1, we notice first that if \( f(e^{it}) \in \Lambda^\alpha \). Since \( |f(e^{it})| < 1 \) for all \( t \), it follows from the maximum modulus principle that \( A = \max\{|f(z)|; z \in D^\#\} < 1 \). Since \( f^* \leq |f'|/(1 - A^2) \), the conclusion (1.3) follows from (1.1).

To prove the “if” part of Theorem 1 we first note that (1.1) holds by \( |f'| \leq f^* \). By Theorem A, \( f \) is continuous on \( D^\# \) and \( f(e^{it}) \in \Lambda^\alpha \). Now, if \( |f(e^{it})| = 1 \) for a certain \( t \), then
\[
\infty = \lim_{r \to 1} \sigma(f(re^{it}), f(0)) \leq \lim_{r \to 1} \int_0^r f^*(re^{it}) \, dp < \infty
\]
by (1.3); this is a contradiction. Therefore \( \max |f(e^{it})| < 1 \), which, together with \( f(e^{it}) \in \Lambda^\alpha \), shows that \( f(e^{it}) \in \sigma \Lambda^\alpha \).
3. Proof of Theorem 2. For the proof of the “if” part we assume that
\[ \mu_p(r, f^*) \leq K(1 - r)^{a-1} \quad \text{for } 0 < r < 1, \]
where \( K > 0 \) is a constant. To prove that \( f \in H_p^0 \) we apply the continuous form of the Minkowski inequality (see [3, (7), p. 20]) to
\[ \sigma(f(re^{i\theta}), f(0)) \leq \int_0^r f^*(\rho e^{i\theta}) \, d\rho \]
for \( 0 \leq t \leq 2\pi \, (0 < r < 1) \). Then
\[ \mu_p(r, \sigma(f, f(0))) \leq \int_0^r \mu_p(\rho, f^*) \, d\rho \leq K/\alpha < \infty \]
by (3.1). Since \( \sigma(f) \leq \sigma(f, f(0)) + \sigma(f(0), 0) \), the Minkowski inequality in the usual form yields that \( \mu_p(r, \sigma(f)) = O(1) \), or \( f \in H_p^0 \). Since \( \mu_p(r, \sigma(f)) \) is bounded for \( 0 < r < 1 \), the Fatou lemma shows that \( |f(e^{i\theta})| < 1 \) a.e. and \( \sigma(f(e^{i\theta})) \in L^p[0, 2\pi] \).

Now, let \( 0 < h < \alpha < 1/2 \), and set \( s = t + h \) for \( t \in (-\infty, \infty) \). Let \( (h < 1) - h < r < 1 \), and set \( \rho = r - h \). Then
\[ \sigma(f(re^{i\theta}), f(re^{i\theta})) \leq \int_0^r f^*(\lambda e^{i\theta}) \, d\lambda + \int_0^r f^*(\lambda e^{is}) \, d\lambda \]
\[ + \int_0^r f^*(\rho e^{is}) \, dx. \]
The third term in the right-hand side is not greater than \( Kh(1 - \rho)^{a-1} \) by (3.1). Applying the Minkowski inequality first in the usual and then in the continuous form we obtain
\[ \left[ \frac{1}{2\pi} \int_0^{2\pi} \sigma(f(re^{i\theta+h}), f(re^{i\theta}))^p \, dt \right]^{1/p} \leq 2 \int_0^r \mu_p(\lambda, \lambda^*) \, d\lambda + Kh(1 - \rho)^{a-1}. \]
The first term in the right-hand side is not greater than \( (2K/\alpha)h^a \) by (3.1), together with \( (1 - \rho)^a \leq (1 - r)^a + h^a \), while the second term is not greater than \( K(1 - \rho)^a \leq 2^aKh^a \). Therefore the left-hand side of (3.2) is not greater than \( K_1\tau^a \), where \( K_1 > 0 \) is a constant. Letting \( r \to 1 \) and considering the Fatou lemma one finds that
\[ \left[ \frac{1}{2\pi} \int_0^{2\pi} \sigma(f(e^{i\theta+h}), f(e^{i\theta}))^p \, dt \right]^{1/p} \leq K_1\tau^a, \]
which completes the proof of \( f(e^{i\theta}) \in \Lambda_\alpha^p \).

For the proof of the “only if” part in the case \( 0 < \alpha < 1 \) we remember [1, p. 74] that
\[ \int_{-\pi}^{\pi} \frac{|t|^a \, dt}{1 - 2r \cos t + r^2} = O((1 - r)^{a-1}). \]
Fix \( z = re^\theta \neq 0 \) in \( D \) for a moment, and set
\[ g(w) = \frac{(f(w) - f(z))}{(1 - \overline{f(z)}f(w))}, \quad w \in D. \]
Since \( g \in B \), the Cauchy integral formula of \( g - g(e^{i\theta}) \) yields
\[
g'(z) = \frac{1}{2\pi i} \int_{|z|=1} \frac{g(\zeta) - g(e^{i\theta})}{(\zeta - z)^2} \, d\zeta,
\]
whence
\[
f^*(z) = |g'(z)| \leq \frac{1}{2\pi} \int_{-\pi}^\pi \frac{|g(e^{i(t+\theta)}) - g(e^{i\theta})|}{1 - 2r \cos t + r^2} \, dt.
\]
Since
\[
|g(e^{i(t+\theta)}) - g(e^{i\theta})| \leq \sigma(g(e^{i(t+\theta)}), g(e^{i\theta}))
\]
\[
= \sigma(f(e^{i(t+\theta)}), f(e^{i\theta})),
\]
it follows from (3.5) that
\[
f^*(re^{i\theta}) \leq \frac{1}{2\pi} \int_{-\pi}^\pi \frac{\sigma(f(e^{i(t+\theta)}), f(e^{i\theta}))}{1 - 2r \cos t + r^2} \, dt.
\]
Now, it is an easy exercise to observe that
\[
\int_0^{2\pi} \sigma(f(e^{i(t+\theta)}), f(e^{i\theta}))^p \, d\theta \leq K_2 |t|^p\alpha
\]
for all \( t, |t| < \pi \), where \( K_2 > 0 \) is a constant. The Minkowski inequality, together with (3.3), asserts from (3.6) that, for \( 0 < r < 1 \),
\[
\mu_p(r, f^*) = O((1 - r)^{\alpha-1}).
\]
To prove that \( \mu_p(r, f^*) = O(1) \) if \( f \in H^p \) and if \( f(e^{it}) \in \sigma\Lambda_p \) we need some properties of \( F \in H^p \) with \( F(e^{it}) \in \sigma\Lambda_p \). Since \( \sigma\Lambda_{p_1} \subset \sigma\Lambda_1 \subset \Lambda_1 \), \( F(e^{it}) \) is equal a.e. to a function of bounded variation on \([0, 2\pi]\) (see [1, Lemma 1, p. 72]). Since \( F \in B \subset H^1 \), \( F(e^{it}) \) can be considered as an absolutely continuous function on \([0, 2\pi]\) by [1, Theorem 3.10, p. 42]. Furthermore, by [1, Theorem 3.11, p. 42],
\[
F'(e^{it}) = \frac{d}{dt} F(e^{it}) = ie^{it} \lim_{r \to 1} F'(re^{it}) = e^{it} F'(e^{it})
\]
exists a.e. on \([0, 2\pi]\); this derivative \( F'(e^{it}) \) is of class \( L^1[0, 2\pi] \). The principal point we need is the fact that
\[
F'(e^{it}) \equiv F'(e^{it}) \bigg/ (1 - |F(e^{it})|^2)
\]
for \( t \in [0, 2\pi] \) is of class \( L^p[0, 2\pi] \). In effect, since \( F(e^{it}) \in \sigma\Lambda_p \), there exist constants \( K_3 > 0 \) and \( \delta > 0 \) such that
\[
\int_0^{2\pi} \left[ \frac{\sigma(F(e^{i(t+h)}), F(e^{it}))}{|h|} \right]^p \, dt \leq K_3
\]
for all \( h \) with \( 0 < |h| < \delta \). Letting \( h \to 0 \) and considering the Fatou lemma, one obtains that
\[
\int_0^{2\pi} F'(e^{it})^p \, dt \leq K_3.
\]
Now, consider $g$ of (3.4). Since $f \in H_0^p$ and $f(e^{it}) \in \sigma \Lambda^1$, it follows that $g \in H_0^p$ and $g(e^{it}) \in \sigma \Lambda^p$. Therefore $g$ is absolutely continuous and $g'(e^{it})$ is of $L^1[0, 2\pi]$. Differentiating the Poisson integral

$$g(w) = \frac{1}{2\pi} \int_0^{2\pi} P(R, s - t)g(e^{it}) \, dt$$

with respect to $s$, where $w = Re^{is} \neq 0$, and $P(R, s - t) = (1 - R^2)/|e^{it} - Re^{is}|^2$, one observes that

$$iwg'(w) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial s} P(R, s - t)g(e^{it}) \, dt$$

(3.7)

$$= -\frac{1}{2\pi} \int_0^{2\pi} \left[ \frac{\partial}{\partial t} P(R, s - t) \right] g(e^{it}) \, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} P(R, s - t)g'(e^{it}) \, dt.$$ 

On the other hand,

$$|g'(e^{it})| = \frac{|f'(e^{it})|}{|1 - f(z)f(e^{it})|^2} \leq f'(e^{it}).$$

It then follows from (3.7), together with $f'(e^{it}) \in L^p[0, 2\pi]$ that

$$|w|^p |g'(w)|^p \leq \frac{1}{2\pi} \int_0^{2\pi} P(R, s - t)f'(e^{it})^p \, dt.$$ 

On setting $w = z = re^{i\theta}$, one obtains that

$$|z|^pf^*(z)^p \leq \frac{1}{2\pi} \int_0^{2\pi} P(r, \theta - t)f^*(e^{it})^p \, dt,$$

so that $\mu_p(r, f^*) = O(1)$.

**REFERENCES**


**DEPARTMENT OF MATHEMATICS, TOKYO METROPOLITAN UNIVERSITY, FUKAZAWA, SETAGAYA, TOKYO 158, JAPAN**