SMOOTHNESS OF THE BOUNDARY VALUES OF FUNCTIONS BOUNDED AND HOLOMORPHIC IN THE DISK

BY

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ABSTRACT. The non-Euclidean counterparts of Hardy-Littlewood’s theorems on Lipschitz and mean Lipschitz functions are considered. Let \( 0 < p < \infty \) and \( 0 < \alpha < 1 \). For \( f \) holomorphic and bounded, \(| f | < 1\), in \( | z | < 1 \), the condition that

\[
|f'(z)| = O\left(\left(1 - |z|\right)^{\alpha - 1}\right)
\]

is necessary and sufficient for \( f \) to be continuous on \( |z| < 1 \) with the boundary function \( f(e^{it}) \in \sigma_{\alpha} \), the hyperbolic Lipschitz class. Furthermore, the condition that the \( p \)-th mean of \( f^* \) on the circle \( |z| = \frac{1}{2} \) is \( O\left(1 - \frac{r}{2}\right)^{-\alpha - 1} \) is necessary and sufficient for \( f \) to be of the hyperbolic Hardy class \( H_{\alpha} \) and for the radial limits to be of the hyperbolic mean Lipschitz class \( \sigma_{\alpha} \).

1. Introduction. We shall prove the non-Euclidean counterparts of the following Theorems A and B due to G. H. Hardy and J. E. Littlewood [2, Theorem 4, p. 627 and Theorem 3, p. 625] (see [1, Theorem 5.1, p. 74 and Theorem 5.4, p. 78]).

Let \( \Phi \) be the family of complex-valued functions \( \varphi \) defined on the real axis such that \( \varphi \) is periodic with period \( 2\pi \). We say that \( \varphi \in \Phi \) is of Lipschitz class \( \Lambda_{\alpha} \) \((0 < \alpha \leq 1)\) if

\[
\sup_{|t - s| < \tau} |\varphi(t) - \varphi(s)| = O(\tau^{\alpha}) \quad \text{as} \quad \tau \to +0.
\]

Let \( D = \{|z| < 1\} \) and let \( D^* = \{|z| \leq 1\} \) in the plane.

**Theorem A.** Let \( f \) be a function holomorphic in \( D \) and let \( 0 < \alpha \leq 1 \). Then \( f \) is continuous on \( D^* \) and the function \( f(e^{it}) \) is of class \( \Lambda_{\alpha} \) if and only if

\[
|f'(z)| = O\left(\left(1 - |z|\right)^{\alpha - 1}\right) \quad \text{as} \quad |z| \to 1 - 0.
\]

We say that \( \varphi \in \Phi \) is of mean Lipschitz class \( \Lambda_{p,\alpha} \) \((1 < p < \infty, 0 < \alpha \leq 1)\) if the restriction of \( \varphi \) to \([0, 2\pi]\) is of \( L^p[0, 2\pi] \) and if

\[
\sup_{0 < h < \tau} \left[ \int_{0}^{2\pi} |\varphi(t + h) - \varphi(t)|^p \, dt \right]^{1/p} = O(\tau^{\alpha})
\]

as \( \tau \to 0 \). For \( 0 < r < 1 \), \( 0 < p < \infty \), and for \( v \) nonnegative and subharmonic in \( D \), we set

\[
\mu_p(r, v) = \left[ \frac{1}{2\pi} \int_{0}^{2\pi} v(re^{it})^p \, dt \right]^{1/p};
\]
this is an increasing function of $r$. The Hardy class $H^p \ (0 < p < \infty)$ consists of $f$ holomorphic in $D$ such that $\mu_p(r, |f|) = O(1)$ as $r \to 0$, or equivalently, the subharmonic function $|f|^p$ has a harmonic majorant in $D$. By the boundary value of a complex-valued function $g$ in $D$ at the point $e^{it}$ of the unit circle we mean the radial limit $g(e^{it}) = \lim_{r \to 1} g(re^{it})$. Each function $f \in H^p \ (0 < p < \infty)$ admits the boundary value $f(e^{it})$ at a.e. point $e^{it}$, and $f(e^{it}) \in L^p[0, 2\pi]$. 

**Theorem B.** Let $f$ be a function holomorphic in $D$, and let $1 \leq p < \infty$, $0 < \alpha \leq 1$.

Then $f \in H^p$ and the function $f(e^{it})$ is of class $\Lambda^p_\alpha$ if and only if

\begin{align*}
\mu_p(r, |f'|) = O((1 - r)^{\alpha - 1}) \quad &\text{as } r \to 1. \tag{1.2}
\end{align*}

In the case $\alpha = 1$, (1.2) says that $f' \in H^p$.

The non-Euclidean hyperbolic distance between $z$ and $w$ in $D$ is defined by

$$
\sigma(z, w) = \frac{1}{2} \log \frac{|1 - \bar{z}w| + |z - w|}{|1 - \bar{z}w| - |z - w|}.
$$

We set $\sigma(z) \equiv \sigma(z, 0)$, the hyperbolic counterpart of $|z|$, $z \in D$. We say that $\varphi \in \Phi$ is of class $\sigma \Lambda_\alpha$ ($0 < \alpha \leq 1$) if $\varphi$ is bounded, $|\varphi| < 1$, and if

$$
\sup_{|r - z| \leq \tau} \sigma(\varphi(t), \varphi(s)) = O(\tau^\alpha) \quad &\text{as } \tau \to +0.
$$

Let $B$ be the family of functions $f$ holomorphic and bounded, $|f| < 1$, in $D$. Then, apparently, $f(e^{it})$ exists a.e. For $f \in B$, the Schwarz-Pick lemma reads

$$
f^*(z) \equiv |f'(z)| / (1 - |f(z)|^2) \leq (1 - |z|^2)^{-1}, \quad z \in D.
$$

We note that $\log f^*$ is subharmonic in $D$, so that $f^{*p} = \exp(p \log f^*) \ (0 < p < \infty)$ is subharmonic in $D$. The hyperbolic analogue of Theorem A is

**Theorem 1.** Let $f \in B$, and let $0 < \alpha \leq 1$. Then $f$ is continuous on $D^\#$ and the function $f(e^{it})$ is of class $\sigma \Lambda^p_\alpha$ if and only if

\begin{align*}
f^*(z) = O\left((1 - |z|)^{\alpha - 1}\right) \quad &\text{as } |z| \to 1 - 0. \tag{1.3}
\end{align*}

We say that $\varphi \in \Phi$ is of class $\sigma \Lambda^p_\alpha$ ($1 \leq p < \infty$, $0 < \alpha \leq 1$) if $|\varphi(t)| < 1$ a.e., if the restriction of $\sigma(\varphi) \equiv \sigma(\varphi(t))$ to $[0, 2\pi]$ is of $L^p[0, 2\pi]$, and if

$$
\sup_{0 < h < r < \tau} \left[ \int_0^{2\pi} \sigma(\varphi(t + h), \varphi(t))^{p} dt \right]^{1/p} = O(\tau^\alpha)
$$

as $\tau \to 0$. For $f \in B$ set $\sigma(f)(z) \equiv \sigma(f(z))$, the hyperbolic counterpart of $|f(z)|$ ($z \in D$). Then $\log \sigma(f)$ is subharmonic in $D$ because $X(x) \equiv \log \sigma(e^x)$ is a convex and increasing function of $x \in (-\infty, 0)$, with $-\infty = X(-\infty) \equiv \lim_{x \to -\infty} X(x)$, and $\log \sigma(f) = X(\log |f|)$. For each $a \in D$, the identity $\sigma(g) = \sigma(f, a)$ holds, where $g = (f - a)/(1 - \bar{a}f) \in B$ for $f \in B$. Therefore $\log \sigma(f, a)$ and $\sigma(f, a)^p = \exp(p \log \sigma(f, a)) \ (0 < p < \infty)$ are subharmonic in $D$. Let $H^p_\alpha$ be the set of all $f \in B$ such that $\mu_p(r, \sigma(f)) = O(1)$ as $r \to 1$, or equivalently, the subharmonic function $\sigma(f)^p$ admits a harmonic majorant in $D$. The hyperbolic Hardy class $H^p_\alpha \ (0 < p < \infty)$ is the counterpart of $H^p$. We are now ready to propose a hyperbolic analogue of Theorem B.
THEOREM 2. Let \( f \in B \), and let \( 1 < p < \infty, 0 < \alpha \leq 1 \). Then \( f \in H^p_\alpha \) and the function \( f(e^{it}) \) is of class \( \sigma \Lambda_\alpha \) if and only if

\[
\mu_p(r, f^*) = O\left( (1 - r)^{\alpha - 1} \right) \text{ as } r \to 1.
\]

In the case \( \alpha = 1 \) in (1.4), the subharmonic function \( f^p \) admits a harmonic majorant.

The proof of Theorem 1 is not difficult and depends on Theorem A; we need comparisons of the non-Euclidean distance and the Euclidean distance. The proof of the “if” part of Theorem 2 is, in a sense, routine. Not easy is the proof of the “only if” part of Theorem 2. There is no relation between \( \sigma(f) \) and \( f^* \) like that between \( |f| \) and \( |f'| \), namely, one cannot assert that \( \sigma(f') = f^* \) even if \( |f'| < 1 \).

2. Proof of Theorem 1. Consider the two inequalities

\[
|z - w| \leq \sigma(z, w), \quad z, w \in D, \\
\sigma(z, w) \leq 2 |z - w| / |1 - \bar{z}w|
\]

for \( z, w \in D \) with \( |z - w| / |1 - \bar{z}w| \leq 1 / \sqrt{2} \). The inclusion formula \( \sigma \Lambda_\alpha \subset \Lambda_\alpha \) follows from (2.1). If \( \varphi \in \Lambda_\alpha \) and if \( |\varphi(t)| < 1 \) for all \( t \in (-\infty, \infty) \), then \( \varphi \in \sigma \Lambda_\alpha \).

To observe this we set \( \max |\varphi(t)| = M < 1 \) because \( \varphi \) is continuous. Then there exist two positive constants \( K \) and \( \delta \) such that

\[
K\delta^\alpha \leq (1 - M^2) / \sqrt{2} \quad \text{and} \quad |\varphi(t) - \varphi(s)| \leq K\tau^\alpha
\]

for all \( \tau, 0 < \tau < \delta \), and for all \( t, s \) with \( |t - s| \leq \tau \). Since

\[
|\varphi(t) - \varphi(s)| \leq (1 - M^2) / \sqrt{2},
\]

it follows that

\[
|\varphi(t) - \varphi(s)| / |1 - \overline{\varphi(t)} \varphi(s)| \leq 1 / \sqrt{2},
\]

whence, by (2.2),

\[
\sigma(\varphi(t), \varphi(s)) \leq \left[ 2 / (1 - M^2) \right] |\varphi(t) - \varphi(s)| \leq K_1 \tau^\alpha
\]

for all \( t, s \) with \( |t - s| \leq \tau < \delta \) \( (K_1 = 2K/(1 - M^2)) \). Therefore \( \varphi \in \sigma \Lambda_\alpha \).

To prove the “only if” part of Theorem 1, we notice first that if \( f(e^{it}) \in \Lambda_\alpha \). Since \( |f(e^{it})| < 1 \) for all \( t \), it follows from the maximum modulus principle that \( A = \max \{|f(z)|; z \in D^\#\} < 1 \). Since \( f^* \leq |f'|/(1 - A^2) \), the conclusion (1.3) follows from (1.1).

To prove the “if” part of Theorem 1 we first note that (1.1) holds by \( |f'| \leq f^* \). By Theorem A, \( f \) is continuous on \( D^\# \) and \( f(e^{it}) \in \Lambda_\alpha \). Now, if \( |f(e^{it})| = 1 \) for a certain \( t \), then

\[
\infty = \lim_{r \to 1} \sigma(f(re^{it}), f(0)) \leq \lim_{r \to 1} \int_0^r f^*(pe^{it}) \, dp < \infty
\]

by (1.3); this is a contradiction. Therefore \( \max |f(e^{it})| < 1 \), which, together with \( f(e^{it}) \in \Lambda_\alpha \), shows that \( f(e^{it}) \in \sigma \Lambda_\alpha \).
3. Proof of Theorem 2. For the proof of the “if” part we assume that

\[ \mu_p(r, f^*) \leq K(1-r)^{a-1} \quad \text{for} \quad 0 < r < 1, \]

where \( K > 0 \) is a constant. To prove that \( f \in H^p_\alpha \) we apply the continuous form of the Minkowski inequality (see \([3, (7), p. 20]) to

\[ \sigma(f(re^{it}), f(0)) \leq \int_0^r f^*(\rho e^{it}) \, d\rho \]

for \( 0 \leq t \leq 2\pi (0 < r < 1) \). Then

\[ \mu_p(r, \sigma(f, f(0))) \leq \int_0^r \mu_p(\rho, f^*) \, d\rho \leq K/\alpha < \infty \]

by (3.1). Since \( \sigma(f) \leq \sigma(f, f(0)) + \sigma(f(0), 0) \), the Minkowski inequality in the usual form yields that \( \mu_p(r, \sigma(f)) = O(1) \), or \( f \in H^p_\alpha \). Since \( \mu_p(r, \sigma(f)) \) is bounded for \( 0 < r < 1 \), the Fatou lemma shows that \( |f(e^{it})| < 1 \) a.e. and \( \sigma(f(e^{it})) \in L^p[0, 2\pi] \).

Now, let \( 0 < h < \tau < 1/2 \), and set \( s = t + h \) for \( t \in (-\infty, \infty) \). Let \( (h < 1) \ 1 - h < r < 1 \), and set \( \rho = r - h \). Then

\[ \sigma(f(re^{is}), f(re^{it})) \leq \int_\rho^r f^*(\lambda e^{is}) \, d\lambda + \int_\rho^s f^*(\lambda e^{ix}) \, d\lambda \]

\[ + \int_s^r f^*(\rho e^{ix}) \, dx. \]

The third term in the right-hand side is not greater than \( Kh(1-\rho)^{a-1} \) by (3.1). Applying the Minkowski inequality first in the usual and then in the continuous form we obtain

\[ \left[ \frac{1}{2\pi} \int_0^{2\pi} \sigma(f(re^{i(t+h)}), f(re^{it}))^p \, dt \right]^{1/p} \leq 2 \int_0^r \mu_p(\lambda, f^*) \, d\lambda + Kh(1-\rho)^{a-1}. \]

(3.2)

The first term in the right-hand side is not greater than \((2K/\alpha)h^a\) by (3.1), together with \((1-\rho)^a \leq (1-r)^a + h^a\), while the second term is not greater than \(K(1-\rho)^a \leq 2K^a\). Therefore the left-hand side of (3.2) is not greater than \( K_1\tau^a \), where \( K_1 > 0 \) is a constant. Letting \( r \to 1 \) and considering the Fatou lemma one finds that

\[ \left[ \frac{1}{2\pi} \int_0^{2\pi} \sigma(f(e^{i(t+h)}), f(e^{it}))^p \, dt \right]^{1/p} \leq K_1\tau^a, \]

which completes the proof of \( f(e^{it}) \in \Lambda^p_\alpha \).

For the proof of the “only if” part in the case \( 0 < \alpha < 1 \) we remember [1, p. 74] that

\[ \int_{-\pi}^\pi \frac{|t|^\alpha \, dt}{1 - 2r \cos t + r^2} = O((1-r)^{a-1}). \]

(3.3)

Fix \( z = re^\theta \neq 0 \) in \( D \) for a moment, and set

\[ g(w) = \left( f(w) - f(z) \right) / \left( 1 - \overline{f(z)} f(w) \right), \quad w \in D. \]

(3.4)

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Since $g \in B$, the Cauchy integral formula of $g - g(e^{i\theta})$ yields

$$g'(z) = \frac{1}{2\pi i} \int_{|\zeta| = 1} \frac{g(\zeta) - g(e^{i\theta})}{(\zeta - z)^2} \, d\zeta,$$

whence

$$f^*(z) = |g'(z)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|g(e^{i(t+\theta)}) - g(e^{i\theta})|}{1 - 2r \cos t + r^2} \, dt. \tag{3.5}$$

Since

$$|g(e^{i(t+\theta)}) - g(e^{i\theta})| \leq \sigma(g(e^{i(t+\theta)}), g(e^{i\theta})) = \sigma(f(e^{i(t+\theta)}), f(e^{i\theta})),$$

it follows from (3.5) that

$$f^*(re^{i\theta}) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sigma(f(e^{i(t+\theta)}), f(e^{i\theta}))}{1 - 2r \cos t + r^2} \, dt. \tag{3.6}$$

Now, it is an easy exercise to observe that

$$\int_{0}^{2\pi} \sigma(f(e^{i(t+\theta)}), f(e^{i\theta}))^p \, d\theta \leq K_2 |t|^\alpha$$

for all $t$, $|t| < \pi$, where $K_2 > 0$ is a constant. The Minkowski inequality, together with (3.3), asserts from (3.6) that, for $0 < r < 1$,

$$\mu_p(r, f^*) = O((1 - r)^{\alpha - 1}).$$

To prove that $\mu_p(r, f^*) = O(1)$ if $f \in H^p_\sigma$ and if $f(e^{it}) \in \sigma \Lambda^p_1$ we need some properties of $F \in H^p_\sigma$ with $F(e^{it}) \in \sigma \Lambda^p_1$. Since $\sigma \Lambda^p_1 \subset \sigma \Lambda_1 \subset \Lambda_1$, $F(e^{it})$ is equal a.e. to a function of bounded variation on $[0, 2\pi]$ (see [1, Lemma 1, p. 72]). Since $F \in B \subset H^1$, $F(e^{it})$ can be considered as an absolutely continuous function on $[0, 2\pi]$ by [1, Theorem 3.10, p. 42]. Furthermore, by [1, Theorem 3.11, p. 42],

$$F'_*(e^{it}) = \frac{d}{dt} F(e^{it}) = ie^{it} \lim_{r \to 1} F''(re^{it}) = e^{it} F'(e^{it})$$

exists a.e. on $[0, 2\pi]$; this derivative $F'_*(e^{it})$ is of class $L^1[0, 2\pi]$. The principal point we need is the fact that

$$F^*(e^{it}) \equiv \lim_{t \to 0} F'_*(e^{it}) \left| \frac{1}{t} - F(e^{it}) \right|^2$$

for $t \in [0, 2\pi]$ is of class $L^p[0, 2\pi]$. In effect, since $F(e^{it}) \in \sigma \Lambda^p_1$, there exist constants $K_3 > 0$ and $\delta > 0$ such that

$$\int_{0}^{2\pi} \left[ \frac{\sigma(F(e^{i(t+h)}), F(e^{it}))}{|h|} \right]^p \, dt \leq K_3$$

for all $h$ with $0 < |h| < \delta$. Letting $h \to 0$ and considering the Fatou lemma, one obtains that

$$\int_{0}^{2\pi} F^*(e^{it})^p \, dt \leq K_3.$$
Now, consider $g$ of (3.4). Since $f \in H^p_0$ and $f(e^{it}) \in \sigma \Lambda^p_{	heta}$, it follows that $g \in H^p_0$ and $g(e^{it}) \in \sigma \Lambda^p_{	heta}$. Therefore $g$ is absolutely continuous and $g'_*(e^{it})$ is of $L^1[0, 2\pi]$. Differentiating the Poisson integral

$$g(w) = \frac{1}{2\pi} \int_0^{2\pi} P(R, s - t) g(e^{it}) \, dt$$

with respect to $s$, where $w = Re^{is} \neq 0$, and $P(R, s - t) = (1 - R^2)/|e^{it} - Re^{is}|^2$, one observes that

$$iwg'(w) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial s} P(R, s - t) g(e^{it}) \, dt$$

(3.7)

$$= -\frac{1}{2\pi} \int_0^{2\pi} \left[ \frac{\partial}{\partial t} P(R, s - t) \right] g(e^{it}) \, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} P(R, s - t) g'_*(e^{it}) \, dt.$$

On the other hand,

$$|g'_*(e^{it})| = \frac{|f'_*(e^{it})| |(1 - |f(z)|^2)|}{|1 - f(z)f(e^{it})|^2} \leq f^*(e^{it}).$$

It then follows from (3.7), together with $f^*(e^{it}) \in L^p[0, 2\pi]$ that

$$|w|^p |g'(w)|^p \leq \frac{1}{2\pi} \int_0^{2\pi} P(R, s - t) f^*(e^{it})^p \, dt.$$

On setting $w = z = re^{i\theta}$, one obtains that

$$|z|^p f^*(z)^p \leq \frac{1}{2\pi} \int_0^{2\pi} P(r, \theta - t) f^*(e^{it})^p \, dt,$$

so that $\mu_p(r, f^*) = O(1)$.

REFERENCES