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ON GROUP C^*-ALGEBRAS OF BOUNDED REPRESENTATION DIMENSION

BY

IAIN RAEBURN

ABSTRACT. We consider the structure of group C^*-algebras whose irreducible representations have bounded dimension. We give some general results, including a description of the topology on the spectrum, and then calculate explicitly the C^*-algebras of two specific groups, one of them a nonsymmorphic space group.

The structure theory of C^*-algebras whose irreducible representations are all of the same finite dimension n is complete and useful; they can all be described as the full section algebras of bundles with fibre $M_n(\mathbf{C})$ (this result is due to Fell [**10**] and Tomiyama and Takesaki [**21**]). If there are irreducible representations of more than one dimension, the situation changes dramatically and any structure theory or classification becomes very complicated (see, for example, [**8**] and [**22**]). Our goal here is to see how much can be said if the C^*-algebra in question is the group C^*-algebra of a locally compact group. We have accumulated some general structural results, and for two specific groups realised their C^*-algebras as algebras of sections of appropriate matrix algebra bundles.

We shall consider, then, locally compact groups whose irreducible representations have bounded dimension. A well-known theorem of Moore [**16**] implies that these are precisely the groups G with an open normal abelian subgroup N of finite index. There are many interesting examples of such groups; in particular, by a theorem of Thoma [**20**], the class includes all discrete type I groups, and among these are the space groups, which have been extensively studied because of their relevance to solid-state physics (see, for example, [**4**]). The representation theory of such a group G is relatively straightforward; the Mackey machine gives all the irreducibles in terms of the dual $\hat{N}$ of N, the action of G on $\hat{N}$, and the multiplier representations of the little groups. Our first result shows how to obtain the topology on the dual $\hat{G}$ in terms of Mackey's construction. The answer is not particularly surprising: a programme to do precisely this has been developed by Fell and Baggett, and a specific conjecture has been made by Baggett [**1**]. Although the strongest results obtained so far (due to Schochetman [**18**]) do not apply directly in our setting, routine arguments using Mackey's results show that Baggett's conjecture holds for these groups. Our other general results concern groups with Hausdorff spectrum; we have collected various relevant bits and pieces which are implicit in the literature. The most

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interesting of our observations is that, when $\hat{G}$ is Hausdorff, $C^*(G)$ is the direct sum of homogeneous algebras (this is a consequence of work of Liukkonen [**14**] and Sund [**19**]). This material is contained in §1.

The two explicit calculations are the content of our second section. The first group is the Heisenberg-type group G of 3×3 upper triangular matrices of integers with the corner entry 0 or 1. This has Hausdorff spectrum, and its C^*-algebra is the sum of an abelian algebra and a 2-homogeneous algebra. An interesting feature of this example is that the 2-homogeneous part contains a nontrivial twist—that is, is not isomorphic to $C(X, M_2(\mathbf{C}))$ for any space X. The second example is a space group: these are the symmetry groups of infinite lattices in Euclidean space, and are extensions of an integer group $\mathbf{Z}^n$ by a finite subgroup of $O(n)$. The one we consider is $p4gm$, which is an extension of $\mathbf{Z}^2$ by the dihedral group D_8. It is nonsymmorphic (cannot be written as a semidirect product). The spectrum of this group is highly non-Hausdorff, and its C^*-algebra a rather complicated subalgebra of the continuous functions from a triangle into $M_8(\mathbf{C})$.

Our notation will be as follows. If G is a locally compact group (always assumed separable) and N a closed normal subgroup, then we write q: $G \to G/N$ for the quotient map and G_γ for the stabiliser of $\gamma \in \hat{N}$ under the action of G. If ω is a multiplier for G and T is an ω-representation of a closed subgroup H, we denote the Hilbert space where T acts by $\mathcal{H}(T)$, and we write $\omega - \operatorname{Ind}_H^G T$ for the induced ω-representation of G; we merely suppress the ω when we deal with ordinary representations. We hope everything else is standard.

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1. General results. Let G be a separable locally compact group such that $\{\dim T: T \in \hat{G}\}$ is bounded. It follows from Theorem 1 of Moore [**16**] that there is an open abelian subgroup A of finite index in G, and, by intersecting A with its (finitely many) conjugates, that G has an open normal abelian subgroup N of finite index. It is easily established that N is regularly embedded in G, so that the results of Mackey [**15**, §8] show how to construct the irreducible representations of G.

Thus if T is an irreducible representation of G, then there is a character $\gamma \in \hat{N}$ and a multiplier ω of G_γ/N such that γ extends to an $(\omega \circ (q \times q))^{-1}$-representation L of G_γ, and such that there is an irreducible ω-representation M of G_γ/N with T equivalent to $\operatorname{Ind}_{G_\gamma}^G L \otimes (M \circ q)$. An examination of the proof of Theorem 8.2 of [**15**] shows that if c: $G_\gamma/N \to G_\gamma$ is a cross section for q, then we can define L and ω by

$$(1) \quad L(g) = \gamma\big(gc(Ng)^{-1}\big), \qquad \omega(Ng, Nh) = \frac{\gamma\big(c(Ngh)h^{-1}g^{-1}\big)}{\gamma\big(c(Ng)g^{-1}\big)\gamma\big(c(Nh)h^{-1}\big)};$$

in fact, it is very easy to verify directly that L and ω are as required.

PROPOSITION 1. *Let G be a separable locally compact group, and let N be an open normal abelian subgroup of finite index in G. Let $T \in \hat{G}$ be represented as $\mathrm{Ind}_{G_\gamma}^G[L \otimes (M \circ q)]$ with L and ω given by* (1), *and let V be a subset of $\hat{G}$. Then $T \in \overline{V}$ if and only if there are a closed subgroup H of G with $N \leqslant H \leqslant G_\gamma$, an irreducible ω-representation M_0 of H/N and a sequence $\{\gamma_i\} \subset \hat{N}$ satisfying*:

(a) $\gamma_i \to \gamma$ *in* $\hat{N}$ *and* $G_{\gamma_i} = H$ *for all* i;

(b) *each* γ_i *extends to an* $(\omega \circ (q \times q))^{-1}$*-representation* L_i *of* H *in such a way that* $L_i \to L|_H$ *uniformly on compacta in* H;

(c) $\mathrm{Ind}_H^G[L_i \otimes (M_0 \circ q)] \in V$ *for all* i;

(d) $M \subset \omega - \mathrm{Ind}_{H/N}^{G_\gamma/N} M_0$.

We shall several times have to use the following lemma. It is a consequence of Theorem 4.6 of [**15**] applied to the pair (H, G) of regularly related subgroups of G; alternatively, it is not hard to come up with a direct proof by realising both representations on the space $\mathcal{K} = \{\phi\colon H\backslash G \to \mathcal{H}(T)\}$.

LEMMA 2. *Let G and N be as in Proposition* 1, *let $\gamma \in \hat{N}$ and define L and ω by* (1). *If H is a closed subgroup of G_γ containing N, and if M is a finite-dimensional ω-representation of H/N, then*

$$L \otimes \left[\left(\omega - \mathrm{Ind}_{H/N}^{G_\gamma/N} M\right) \circ q\right] \sim \mathrm{Ind}_H^{G_\gamma}[L|_H \otimes (M \circ g)].$$

PROOF OF PROPOSITION 1. We first suppose that there are H, γ_i, L_i and M_0 satisfying (a), (b), (c) and (d). Since $|G\colon H| < \infty$ and $\mathcal{H}(M_0) < \infty$, (b) implies that

$$\mathrm{Ind}_H^G[L_i \otimes (M_0 \circ q)] \to \mathrm{Ind}_H^G[L|_H \otimes (M_0 \circ q)]$$

uniformly on compact subsets of G. By Lemma 2 and (d),

$$L \otimes (M \circ q) \subset L \otimes \left[\left(\omega - \mathrm{Ind}_{H/N}^{G_\gamma/N} M_0\right) \circ q\right] \sim \mathrm{Ind}_H^{G_\gamma}[L|_H \otimes (M_0 \circ q)],$$

and so

$$T \sim \mathrm{Ind}_{G_\gamma}^G[L \otimes (M \circ q)] \subset \mathrm{Ind}_H^G[L|_H \otimes (M_0 \circ q)].$$

In particular, it follows that $\mathrm{Ind}_H^G[L_i \otimes (M_0 \circ q)] \to T$ in $\hat{G}$, and hence that $T \in \overline{V}$ by (c).

We now suppose that $T_i \to T$ in $\hat{G}$ and $\{T_i\} \subset V$. It is not hard to see that for each $S \in \hat{G}$ there is a character $\chi \in \hat{N}$ and an integer m such that $S|_N$ is equivalent to $\bigoplus\{\delta \cdot 1_m\colon \delta \in G \cdot \chi\}$, where 1_m denotes the $m \times m$ identity matrix. We choose such characters χ_i for the representations T_i; for T, of course, we have

$$T|_N = \bigoplus \{\delta \cdot 1_m\colon \delta \in G \cdot \gamma\}.$$

By the corollary to Theorem 1.3 of [**9**] we have $T|_N$ weakly contained in $\{T_i|_N\}$, and this implies that γ is weakly contained in $\{g \cdot \chi_i\colon g \in G,\ i \in \mathbf{N}\}$. Since the usual topology on $\hat{N}$ agrees with that given by weak containment [**9**, Theorem 1.3], it follows that there is a sequence of characters $\gamma_{i_j} = g_j \cdot \chi_{i_j}$ such that $T_{i_j}|_N$ lives on $G \cdot \gamma_{i_j}$ and $\gamma_{i_j} \to \gamma$ in $\hat{N}$. We relabel $\{T_{i_j}\}$ by $\{T_i\}$ for convenience.

Because there are only finitely many subgroups of G which contain N, we may by passing to a subsequence assume that all the stabilisers G_{γ_i} are equal to a fixed subgroup H; the continuity of the action of G on $\hat{N}$ implies that H is a subgroup (possibly proper) of G_γ. According to Mackey's construction [**15**, §8] and (1) above, each γ_i extends to a $(\omega_i \circ (q \times q))^{-1}$-representation J_i of H as follows:

$$(2)\quad J_i(h) = \gamma_i\big(hc(Nh)^{-1}\big), \qquad \omega_i(Nk, Nh) = \frac{\gamma_i\big(c(Nkh)h^{-1}k^{-1}\big)}{\gamma_i\big(c(Nk)k^{-1}\big)\gamma_i\big(c(Nh)h^{-1}\big)},$$

and there are irreducible ω_i-representations M_i of H/N such that T_i is equivalent to $\operatorname{Ind}_H^G[J_i \otimes (M_i \circ q)]$. Since the multiplier group $H^2(H/N, S^1)$ is finite (see, for example, [**6**, Theorem 53.3]), we can by passing to another subsequence assume that each ω_i is equivalent to a fixed multiplier τ of H/N; that is, that there are functions β_i: $H/N \to S^1$ satisfying $\beta_i(e) = 1$ and

$$(3)\qquad \omega_i(Nk, Nh) = \frac{\beta_i(Nkh)}{\beta_i(Nk)\beta_i(Nh)}\tau(Nk, Nh).$$

The ω_i only depend on the values of the γ_i at finitely many points, and so as a consequence of (1) and (2) we have $\omega_i \to \omega$ on $H/N \times H/N$. By passing to a subsequence we may suppose there is a function β: $H/N \to S^1$ such that $\beta_i \to \beta$, and then

$$\omega(Nk, Nh) = \frac{\beta(Nk, Nh)}{\beta(Nk)\beta(Nh)}\tau(Nk, Nh).$$

We may therefore take our fixed multiplier to be ω; that is, we suppose (3) with τ replaced by the restriction of ω to $H/N \times H/N$ and with $\beta_i \to 1$. Then for each i, $Nh \to \beta_i(Nh)^{-1}M_i(Nh)$ is an irreducible τ-representation of H/N, and so we may suppose by going to yet another subsequence that each $\beta_i^{-1}M_i$ is equivalent to a fixed ω-representation M_0 of H/N. Thus $L_i = J_i(\beta_i \circ q)$ is a one-dimensional $(\omega \circ (q \times q))^{-1}$-representation of H, satisfying

$$L_i|_N = \gamma_i \quad \text{and} \quad T_i \sim \operatorname{Ind}_H^G\big[L_i \otimes (M_0 \circ q)\big] \quad \text{for all } i.$$

The representations J_i and L are defined using the same cross-section c, and $\gamma_i \to \gamma$ in $\hat{N}$, so it follows that $J_i \to L|_H$ uniformly on compact subsets of H. Since $\beta_i \to 1$, we therefore have $L_i \to L|_H$ uniformly on compacta, and we have established everything except (d).

It remains, then, to verify that $M \subset \omega - \operatorname{Ind}_{H/N}^{G_\gamma/N} M_0$. Let

$$\omega - \operatorname{Ind}_{H/N}^{G_\gamma/N} M_0 = M_0' \oplus \cdots \oplus M_p'$$

be the decomposition of $\omega - \operatorname{Ind} M_0$ into irreducible ω-representations. Then for each j, $\operatorname{Ind}_{G_\gamma}^G L \otimes (M_j' \circ q)$ is an irreducible representation of G (see [**15**, Theorem 8.4]), and

$$(4)\qquad \operatorname{Ind}_{G_\gamma}^G\Big[L \otimes \Big(\big(\omega - \operatorname{Ind}_{H/N}^{G_\gamma/N} M_0\big) \circ q\Big)\Big] = \bigoplus_j \operatorname{Ind}_{G_\gamma}^G\big[L \otimes (M_j' \circ q)\big].$$

It follows from Lemma 2 and the theorem on induction in stages that

$$\operatorname{Ind}_H^G\left[L|_H \otimes (M_0 \circ q)\right] \sim \operatorname{Ind}_{G_\gamma}^G\left[L \otimes \left(\left(\omega - \operatorname{Ind}_{H/N}^{G_\gamma/N} M_0\right) \circ q\right)\right],$$

and so (4) must be the decomposition of $\operatorname{Ind}_H^G L|_H \otimes (M_0 \circ q)$ into irreducibles. Since $L_i \to L|_H$ uniformly on compacta, it follows that

$$T_i \sim \operatorname{Ind}_H^G[L_i \otimes (M_0 \circ g)] \to \operatorname{Ind}_H^G[L|_H \otimes (M_0 \circ q)]$$

uniformly on compacta, and hence pointwise on $C^*(G)$. By Theorem 2.1 of [**9**] the sequence $\{T_i\}$ in $\hat{G}$ can only converge to members of the set

$$\mathbb{S} = \left\{\operatorname{Ind}_{G_\gamma}^G\left[L \otimes (M_j' \circ q)\right] : 1 \leqslant j \leqslant p\right\},$$

and we conclude that $T \in \mathbb{S}$. Thus M must be equivalent to some M_j', and so contained in $\omega - \operatorname{Ind} M_0$. This completes the proof of Proposition 1.

COROLLARY 3. *Baggett's conjecture* [**1**, *Conjecture* 2, *p.* 176] *is valid for groups whose irreducible representations have bounded dimension.*

PROOF. (To save space, we shall use Baggett's notation without explanation.) We observe first that the sufficiency of the conjecture follows from Fell's results on the continuity of induction [**11**]. So we have to prove that if $W \in \bar{B}$ is catalogued by (K, S), then there are subgroup representations (J, T) and (J^i, T^i) satisfying Baggett's conditions (i)–(iii). We apply our proposition with $T = W$, $(G_\gamma, L \otimes (M \circ q)) = (K, S)$ and $V = B$, and then set

$$(J, T) = (H, L|_H \otimes (M_0 \circ q)), \qquad (J^i, T^i) = (H, L_i \otimes (M_0 \circ q)).$$

Then (i) is our (c), (iii) follows from the fact that $L_i \otimes (M_0 \circ q) \to L|_H \otimes (M_0 \circ q)$ uniformly on compacta, thus a fortiori in the inner-hull-kernel topology on the space of representations of H, and hence also in the space of subgroup representations by Lemma 2.6 of [**11**]. We have already observed that our $H \subset G_\gamma$, so that in Baggett's notation $J \subset K$, and (d) together with Lemma 2 implies that

$$\begin{aligned} S = L \otimes (M \circ q) &\subset L \otimes \left[\left(\omega - \operatorname{Ind}_{H/N}^{G_\gamma/N} M_0\right) \circ q\right] \\ &\sim \operatorname{Ind}_H^{G_\gamma}\left[L|_H \otimes (M_0 \circ q)\right] = \operatorname{Ind}_J^K T, \end{aligned}$$

which is stronger than (iii) of [**1**] (but which is only to be expected since we are dealing with finite-dimensional representations—see Baggett's own comments in 6.2.B of [**1**]). This completes the proof.

Our remaining general results concern groups whose spectrum is Hausdorff. Such groups have been extensively studied, and the comments we make here are mostly specialisations of known work to the case where there is an abelian subgroup of finite index, and where it is sometimes possible to get a little extra information cheaply. The first of these observations is probably the most interesting; several of the remarks in §5 of [**3**] are immediate consequences.

PROPOSITION 4. *Let G be a separable locally compact group such that the dimensions of the irreducible representations of G are bounded, and suppose that $\hat{G}$ is Hausdorff. Then $C^*(G)$ is the direct sum of homogeneous C^*-algebras.*

PROOF. Let N be an open normal abelian subgroup of finite index, and let U be a compact neighbourhood of the identity in N (and hence also in G). If $\{g_i\}$ is a set of coset representatives for $N\backslash G$, then $V = \bigcap_i g_i^{-1}Ug_i$ is also a compact neighbourhood of the identity in G, and it is not hard to see that V is invariant under inner automorphisms of G. Thus G is an $[IN]$-group, and hence by Theorem 5.2 of [**14**] also an $[FC]^-$-group. Since G is clearly type I, Proposition 3.1 of [**14**] implies that G has a compact normal subgroup K such that G/K is abelian. Sund has shown [**19**, Lemma 1.2] that $\hat{G}$ is the union of the open and closed subsets

$$\hat{G}_{T,K} = \{S \in \hat{G}\colon S|_K \text{ is quasi-equivalent to } T|_K\}.$$

Mackey's analysis, together with Baggett's and Kleppner's work [**2**] on multiplier representations of abelian groups, implies that the representations in $\hat{G}_{T,K}$ all have the form $(\gamma \circ q)T$ for some $\gamma \in (G/K)\hat{}$ [**19**, Lemma 1.1] and, in particular, all have the same dimension. Thus the set $\hat{G}_n = \{T \in \hat{G}\colon \dim T = n\}$ is a discrete union of $\hat{G}_{T,K}$'s, so that G is the union of the open and closed subsets $\hat{G}_n$. It follows that $C^*(G)$ is the direct sum of the ideals

$$I_n = \{a \in C^*(G)\colon \pi(a) = 0 \text{ for all } \pi \in \hat{G} \setminus \hat{G}_n\},$$

and I_n is n-homogeneous since its irreducible representations are precisely those in $\hat{G}_n$ by [**7**, 3.2.1]. The result is now proved.

COROLLARY 5. *Let G be a separable locally compact group whose irreducible representations are of bounded dimension. Then $\hat{G}$ is Hausdorff if and only if G has a compact normal subgroup K such that G/K is abelian.*

PROOF. We showed in the course of proving the previous proposition that if $\hat{G}$ is Hausdorff, then it has such a subgroup. The converse follows from Sund's analysis of $\hat{G}$ [**19**, Theorem 1.3].

REMARK. Baggett and Sund [**3**] have conjectured that Corollary 5 is true for arbitrary type I groups.

PROPOSITION 6. *Let G be a countable discrete group. The following are equivalent*:

(a) *$\hat{G}$ is Hausdorff.*

(b) *G is type* I, *and the commutator subgroup G' is finite.*

(c) *$|G'| < \infty$ and G has an abelian subgroup of finite index.*

(d) *G has finite conjugacy classes and an abelian subgroup of finite index.*

(e) *The centre of G has finite index.*

PROOF. (a) $\Rightarrow$ (b) is contained in Theorem 5.2 of [**14**]. (b) $\Rightarrow$ (c) is Satz 4 of [**20**]. (c) $\Rightarrow$ (d) is elementary since the conjugacy class of $g \in G$ is contained in $G'g$. (d) $\Rightarrow$ (e). Suppose that A is abelian and $G = \bigcup_{i=1}^n Ag_i$. Then the centralisers $C_G(g_i)$ have finite index (equal to the size of the conjugacy class of g_i), and so therefore does $A \cap (\bigcap_{i=1}^n C_G(g_i))$. But this latter group is contained in the centre of G. (e) $\Rightarrow$ (a) is a special case of Corollary 5.2 of [**12**]. Alternatively, it can be deduced from Proposition 1 and the fact that if $T_i \to T$ and $\dim T = \dim T_i$, then $\{T_i\}$ cannot converge to anything else (Corollary 1 to Theorem 2.2 of [**9**]).

REMARK. That $|G: Z(G)|$ finite implies $|G'|$ finite is an old result of Neumann [**17**, Theorem 5.3]. The converse is known to be false in general and true for finitely generated groups, but to our knowledge the partial converse contained in (b) $\Rightarrow$ (e) has not been pointed out before. The following example (lifted from §5.6 of [**23**]) shows what can happen if G' is finite but G is not type I. Let $A = \oplus_{n=1}^{\infty} \mathbf{Z}/3\mathbf{Z}$, and define a multiplier τ on A by

$$\tau((v_1, v_2, \ldots), (w_1, w_2, \ldots)) = \exp i\frac{2\pi}{3}\left(\sum_i (v_{2i-1}w_{2i} - v_{2i}w_{2i-1})\right).$$

It is easy to check that τ is a multiplier and that $\tau^3 \equiv 1$, and simple calculations show that τ is totally skew in the sense of [**2**]. Thus by [**2**, Lemma 3.1] the regular τ-representation R_τ of A is a factor representation of type II_1. If we let G be the central extension of $\mathbf{Z}/3\mathbf{Z}$ by A corresponding to the cocycle $\tau \in H^2(A, \mathbf{Z}/3\mathbf{Z})$, then R_τ lifts to an ordinary factor representation R_τ^0 of G which is also of type II_1. In particular, G is not type I. The commutator $G' = \mathbf{Z}/3\mathbf{Z}$ is finite, but so is $Z(G) = \mathbf{Z}/3\mathbf{Z}$.

2. Two specific group C^*-algebras.

2.1. Let G denote the group $\mathbf{Z} \times \mathbf{Z} \times (\mathbf{Z}/2\mathbf{Z})$ with multiplication defined by

$$(n, m, \alpha)(n_1, m_1, \alpha_1) = (n + n_1, m + m_1, \alpha + \alpha_1 + nm_1 \pmod 2);$$

we think of G as 3×3 upper triangular matrices of the form

$$\begin{pmatrix} 1 & n & \alpha \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix} \qquad (n, m \in \mathbf{Z}, \alpha = 0 \text{ or } 1).$$

The centre of h is $2\mathbf{Z} \times 2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, which is of index 4 in G, and so we conclude from Proposition 6 that $\hat{G}$ is Hausdorff. The group C^*-algebra is given by

PROPOSITION 7. *Let G be as above, and let $\mathbf{T}^2$ denote the torus $S^1 \times S^1$. Then $C^*(G)$ is isomorphic to $C(\mathbf{T}^2) \oplus B$, where B is the algebra of continuous functions b: $[0, \pi] \times [0, \pi] \to M_2(\mathbf{C})$ satisfying*

$$b(\pi, t) = b(0, t), \qquad \mathrm{Ad}\begin{pmatrix} \cos s & i\sin s \\ i\sin s & \cos s \end{pmatrix}(b(s, \pi)) = b(s, 0) \quad \textit{for all } s, t.$$

PROOF. The subgroup $K = \{(0, 0, \alpha)\}$ is normal and $G/K \cong \mathbf{Z} \times \mathbf{Z}$ is abelian; we shall analyse the representations of G in terms of K. The space $\hat{K}$ consists of two points, namely the trivial character ε: $K \to \{1\}$ and the character δ defined by $\delta(0, 0, \alpha) = (-1)^\alpha$, and the action of G on $\hat{K}$ is trivial. We deduce (see the proof of Proposition 4) that $\hat{G}$ is the disjoint union of two sets $\hat{G}_1$ and $\hat{G}_2$, consisting of those irreducible representations of G whose restrictions to K are multiples of, respectively, ε and δ. Those representations in $\hat{G}_1$ are just the characters of $G/K = \mathbf{Z} \times \mathbf{Z}$; $\hat{G}_1$ is therefore homeomorphic to the two-dimensional torus $\mathbf{T}^2 = S^1 \times S^1$.

We define a multiplier ω on $\mathbf{Z} \times \mathbf{Z}$ by

$$\omega((n, m), (n_1, m_1)) = (-1)^{nm_1} \qquad (n, m, n_1, m_1 \in \mathbf{Z});$$

then $L(n, m, \alpha) = (-1)^\alpha$ is an $(\omega \circ (q \times q))$-representation of G which extends δ. One irreducible $\bar{\omega}$-representation of $\mathbf{Z} \times \mathbf{Z}$ is given by (note that $\bar{\omega} = \omega$)

$$R(2n, 2m) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad R(2n+1, 2m+1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$R(2n, 2m+1) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad R(2n+1, 2m) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $S\colon G \to U_2$ defined by

$$S(n, m, \alpha) = (L \otimes (R \circ q))(n, m, \alpha) = (-1)^\alpha R(n, m)$$

is an irreducible representation of G whose restriction to K is a multiple of δ. It follows from [**19**, Proposition 1.1] that every representation in $\hat{G}_2$ has the form $(\gamma \circ q) \otimes S$ for some $\gamma \in (\mathbf{Z} \times \mathbf{Z})\hat{}$, and from [**19**, p. 316] that $\gamma \to (\gamma \circ q) \otimes S$ induces a homeomorphism of

$$(\mathbf{Z} \times \mathbf{Z})\hat{}\,/\,\{\gamma \in (\mathbf{Z} \times \mathbf{Z})\hat{}\colon (\gamma \circ q) \otimes S \text{ is equivalent to } S\}$$

onto the open and closed subset $\hat{G}_2$ of $\hat{G}$. Now if $(\gamma \circ q) \otimes S \sim S$, then since $S(2n, 2m, 0)$ is the identity we must have $\gamma(2n, 2m) = 1$ for all n, m. Conversely, it is not hard to see that if we denote by $\gamma_{s,t}$ the character $(n, m) \to \exp(isn)\exp(itm)$, then the unitaries

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

respectively, intertwine $(\gamma_{\pi,0} \circ q) \otimes S$, $(\gamma_{0,\pi} \circ q) \otimes S$ and $(\gamma_{\pi,\pi} \circ q) \otimes S$ with S. Thus $(\gamma \circ q) \otimes S$ is equivalent to S exactly when γ annihilates $2\mathbf{Z} \times 2\mathbf{Z}$; we deduce that $\hat{G}_2$ is homeomorphic to $(2\mathbf{Z} \times 2\mathbf{Z})\hat{} = (\mathbf{Z} \times \mathbf{Z})\hat{}\,/(2\mathbf{Z} \times 2\mathbf{Z})^\perp$, which is another torus.

We shall denote by A the C^*-algebra of continuous functions a from $[0, \pi] \times [0, \pi]$ into $M_2(\mathbf{C})$ satisfying

$$\mathrm{Ad}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}(a(\pi, t)) = a(0, t), \qquad \mathrm{Ad}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}(a(s, \pi)) = a(s, 0)$$

for all s and t. Then it follows from the preceding paragraph that

$$\Phi(f)(s, t) = ((\gamma_{s,t} \circ q) \otimes S)(f) \qquad (f \in C^*(G))$$

defines a homomorphism of $C^*(G)$ onto A, and that the kernel of Φ is the ideal corresponding to the open subset $\hat{G}_1$ of $\hat{G}$. Thus Φ is an isomorphism of the 2-homogeneous part of $C^*(G)$ (the part corresponding to the open set $\hat{G}_2$ in $\hat{G}$) onto A, and we have $C^*(G) \cong C(\mathbf{T}^2) \oplus A$. However, it is easy to check that

$$\Psi(a)(s, t) = \mathrm{Ad}\begin{pmatrix} e^{it}\cos\frac{s}{2} & ie^{it}\sin\frac{s}{2} \\ i\sin\frac{s}{2} & \cos\frac{s}{2} \end{pmatrix}(a(s, t))$$

defines an isomorphism Ψ of A onto B, and we have now proved the proposition.

REMARK. The algebra B appearing here is 2-homogeneous with spectrum $\mathbf{T}^2$, and it is natural to ask if it is isomorphic to $C(\mathbf{T}^2, M_2(\mathbf{C}))$. The structure theory of homogeneous C^*-algebras says that B is the algebra of sections of a bundle E over $\mathbf{T}^2$ with fibre $M_2(\mathbf{C})$ and structure group $PU(2) = \mathrm{Aut}\, M_2(\mathbf{C})$; the bundle E in this

case is obtained from the trivial bundle $([0,\pi]\times S^1)\times M_2$ on the cylinder by pasting $\{0\}\times S^1\times M_2$ and $\{\pi\}\times S^1\times M_2$ via the map $\phi\colon S^1\to PU(2)$ given by

$$\phi\colon s\in[0,\pi]\to \mathrm{Ad}\begin{pmatrix}\cos s & i\sin s\\ i\sin s & \cos s\end{pmatrix}$$

(here we are identifying S^1 with $[0,\pi]/\{0,\pi\}$; note that $\phi(0)=\phi(\pi)$ so ϕ is well defined on S^1). If there were an isomorphism Φ of B onto $C(\mathbf{T}^2, M_2)$, then by composing with a suitable automorphism of $C(\mathbf{T}^2, M_2)$ we could assume that $\hat{\Phi}\colon \mathbf{T}^2=\hat{B}\to\mathbf{T}^2=C(\mathbf{T}^2,M_2)\hat{}$ was the identity. Then Φ would have to be a $C(\mathbf{T}^2)$-module isomorphism, and hence come from a bundle isomorphism $\psi\colon E\to\mathbf{T}^2\times M_2(\mathbf{C})$. Pulling this isomorphism back to $[0,\pi]\times S^1$, where both bundles are trivial, shows that such a ψ would give a continuous path of loops ψ_t ($t\in[0,\pi]$) in $PU(2)$ joining the trivial loop ψ_0 to the loop $\psi_\pi=\phi$. Since $\mathrm{Ad}\colon SU(2)\to PU(2)$ is a (2-sheeted) covering, this would imply that ϕ lifts to a loop in $SU(2)$; however the unique lift of ϕ to $SU(2)$ is

$$s\in[0,\pi]\to\begin{pmatrix}\cos s & i\sin s\\ i\sin s & \cos s\end{pmatrix}$$

which is not a loop. Hence there cannot be an isomorphism of B onto $C(\mathbf{T}^2, M_2(\mathbf{C}))$.

In fact, a slightly more careful analysis along these lines would show that there are precisely two isomorphism classes of 2-homogeneous C^*-algebras with spectrum $\mathbf{T}^2$ —the trivial one and the one we have here.

2.2. The second example which we shall consider in depth is the space group $p4gm$. Before we give a more precise definition, we explain our notation, which is one of those common in the physics literature. If $T\in O(2)$ and $v\in\mathbf{R}^2$, then we shall denote by $\{T\mid v\}$ the transformation $w\to Tw+v$ of $\mathbf{R}^2$. The symbols C_{4z}^+, C_{2z} and C_{4z}^- denote anticlockwise rotation through, respectively, $\pi/2$, π and $3\pi/2$, and σ_x, σ_y, σ_1, σ_2 denote reflection in the lines $x=0$, $y=0$, $y+x=0$, $y-x=0$. The C's and σ's form the symmetry group of the square with centre 0 and sides parallel to the axes, which is known as the dihedral group D_8 of order 8. The space group $G=p4gm$ is the group of motions in $\mathbf{R}^2$ generated by the elements $\{E\mid 1,0\}$, $\{E\mid 0,1\}$, $\{C_{4z}^+\mid 0\}$, $\{\sigma_x\mid \frac{1}{2},\frac{1}{2}\}$. The pure translations form a normal subgroup N isomorphic to $\mathbf{Z}\times\mathbf{Z}$, and the quotient G/N is isomorphic to D_8. The extension $0\to\mathbf{Z}\times\mathbf{Z}\to G\to D_8\to 0$ is not split, so that, in physicists' language, G is nonsymmorphic.

The irreducible representations of G are obtained by Mackey's method applied to the normal abelian subgroup N. It is traditional to identify the dual of N with a square centred at $(0,0)$ of side 1 parallel to the axes (called the Brillouin zone of G); the point (α,β) corresponds to the representation

$$\{E\mid n,m\}\to\exp\{-2\pi i(\alpha n+\beta m)\}.$$

The action of G on $\hat{N}$ is then just the usual action of D on $\mathbf{R}^2$; thus the orbit space is parametrised by the set

$$\Omega=\{(\alpha,\beta)\in[0,\tfrac{1}{2}]\times[0,\tfrac{1}{2}]\colon \beta\leqslant\alpha\}. \tag{5}$$

We label the points of the boundary of Ω as follows: $\Gamma = (0,0)$, $X = (\frac{1}{2},0)$, $M = (\frac{1}{2},\frac{1}{2})$, $\Sigma(\alpha) = (\alpha,\alpha)$, $Y(\beta) = (\frac{1}{2},\beta)$, $\Delta(\alpha) = (\alpha,0)$. We denote the respective stabilisers by G_Γ, G_Σ, etc. The irreducible representations of G are obtained by inducing appropriate irreducible representations of the stabilisers; thus, for example, those $T \in \hat{G}$ whose restrictions to N live on the orbit $G \cdot \Sigma(\alpha)$ are obtained by inducing representations in $\hat{G}_\Sigma$ which themselves restrict to multiples of $\Sigma(\alpha)$. These representations of the stabilisers have been tabulated by Cracknell [**5**], and so all we have to do to find $\hat{G}$ is to induce Cracknell's representations up to G. We shall be concerned with the way these representations behave as they approach the boundary, so we shall write (say) $\operatorname{Ind}_N^G(\alpha,\beta)$ for any $(\alpha,\beta) \in \Omega$ and $\operatorname{Ind}_{G_\Sigma}^G \Sigma_1(\alpha)$ for any $\alpha \in [0,\frac{1}{2}]$, even though $\operatorname{Ind}_N^G(\alpha,\beta)$ is only irreducible for (α,β) in the interior of Ω and $\operatorname{Ind} \Sigma_1(\alpha)$ is only irreducible for $\alpha \in]0,\frac{1}{2}[$. The representations we shall use are listed in Table 1. A complete set of irreducible representations for G is

$$\begin{aligned}&\{\operatorname{Ind}_N^G(\alpha,\beta) : \alpha \in]0,\tfrac{1}{2}[, 0 < \beta < \alpha\} \cup \{\operatorname{Ind}_{G_\Sigma}^G \Sigma_i(\alpha) : i = 1,2, \alpha \in]0,\tfrac{1}{2}[\}\\ &\qquad \cup \{\operatorname{Ind}_{G_Y}^G Y_i(\beta) : i = 1,2, \beta \in]0,\tfrac{1}{2}[\} \cup \{\operatorname{Ind}_{G_\Delta}^G \Delta_i(\alpha) : i = 1,2, \alpha \in]0,\tfrac{1}{2}[\}\\ &\qquad \cup \{M_i, \Gamma_i : 1 \leqslant i \leqslant 5\} \cup \{X\}.\end{aligned}$$

We now give our description of $C^*(p4gm)$.

PROPOSITION 8. *There are smooth maps P_Σ, P_Y, P_Δ of $[0,\frac{1}{2}]$ into the manifold $P_4(\mathbf{C}^8)$ of rank 4 projections, and orthonormal bases $\{e_i^\Gamma : 1 \leqslant i \leqslant 8\}$, $\{e_i^M : 1 \leqslant i \leqslant 8\}$, $\{e_i^X : 1 \leqslant i \leqslant 8\}$ for $\mathbf{C}^8$ such that the ranges of*

$$\begin{aligned}&P_Y(\tfrac{1}{2}) = \operatorname{sp}\{e_1^M, e_2^M, e_3^M, e_4^M\}; \quad P_\Sigma(\tfrac{1}{2}) = \operatorname{sp}\{e_5^M, e_2^M, e_3^M + e_7^M, e_4^M + e_8^M\};\\ &P_\Delta(\tfrac{1}{2}) = \operatorname{sp}\{e_1^X, e_2^X, e_3^X, e_4^X\}; \quad P_Y(0) = \operatorname{sp}\{e_1^X + e_5^X, e_2^X + e_6^X, e_3^X + e_7^X, e_4^X + e_8^X\};\\ &P_\Delta(0) = \operatorname{sp}\{e_1^\Gamma, e_2^\Gamma, e_3^\Gamma, e_4^\Gamma\}; \quad P_\Sigma(0) = \operatorname{sp}\{e_1^\Gamma, e_6^\Gamma, e_3^\Gamma + e_7^\Gamma, e_4^\Gamma + e_8^\Gamma\};\end{aligned}$$

and such that $C^*(p4gm)$ is isomorphic to the C^*-algebra of continuous functions f: $\Omega \to M_8(\mathbf{C})$ (with Ω given by (5)) satisfying

(1) $P_\Sigma(\alpha)f(\alpha,\alpha) = f(\alpha,\alpha)P_\Sigma(\alpha)$, $\quad P_Y(\beta)f(\frac{1}{2},\beta) = f(\frac{1}{2},\beta)P_Y(\beta)$ *and* $P_\Delta(\alpha)f(\alpha,0) = f(\alpha,0)P_\Delta(\alpha)$ *for all α and β.*

(2) $\mathbf{C}e_1^\Gamma, \mathbf{C}e_2^\Gamma, \mathbf{C}e_5^\Gamma, \mathbf{C}e_6^\Gamma, \operatorname{sp}\{e_3^\Gamma, e_4^\Gamma\}, \operatorname{sp}\{e_7^\Gamma, e_8^\Gamma\}$ *are invariant for $f(0,0)$;* $\mathbf{C}e_1^M, \mathbf{C}e_2^M, \mathbf{C}e_5^M, \mathbf{C}e_6^M, \operatorname{sp}\{e_3^M, e_4^M\}, \operatorname{sp}\{e_7^M, e_8^M\}$ *are invariant for $f(\frac{1}{2},\frac{1}{2})$;* $\operatorname{sp}\{e_1^X, e_2^X, e_3^X, e_4^X\}, \operatorname{sp}\{e_5^X, e_6^X, e_7^X, e_8^X\}$ *are invariant for $f(\frac{1}{2},0)$.*

(3) *If U_Γ, U_M, U_X are the partial isometries defined on ordered orthonormal bases by*

$$U_\Gamma : \{e_3^\Gamma, e_4^\Gamma\} \to \{e_7^\Gamma, e_8^\Gamma\}; \qquad U_M : \{e_3^M, e_4^M\} \to \{e_7^M, e_8^M\};$$
$$U_X : \{e_1^X, e_2^X, e_3^X, e_4^X\} \to \{e_5^X, e_6^X, e_7^X, e_8^X\},$$

then

$$U_X^* f(\tfrac{1}{2},0)U_X = U_X^* U_X f(\tfrac{1}{2},0), \qquad U_\Gamma^* f(0,0)U_\Gamma = U_\Gamma^* U_\Gamma f(0,0)$$

and

$$U_M^* f(\tfrac{1}{2}, \tfrac{1}{2}) U_M = U_M^* U_M f(\tfrac{1}{2}, \tfrac{1}{2}).$$

PROOF. We define Φ: $L^1(G) \to C(\Omega, M_8(\mathbf{C}))$ by

$$\Phi(a)(\alpha, \beta) = \operatorname{Ind}_N^G(\alpha, \beta)(a) \qquad (a \in L^1(G), (\alpha, \beta) \in \Omega);$$

since each $\operatorname{Ind}_N^G(\alpha, \beta)$ is a nondegenerate *-representation of $L^1(G)$, it is clear that Φ extends to all of $C^*(G)$. We have to show that Φ is an isomorphism and that the range of Φ has the form set out in the proposition.

We shall write $\{e_i\}$ for the usual basis for $\mathbf{C}^8$. We define $P_\Sigma(\alpha)$, $P_\Upsilon(\beta)$ and $P_\Delta(\alpha)$ to be the projections onto the subspaces

$$\begin{gathered}\operatorname{sp}\{e_1 + we_8, e_2 + we_6, e_3 + we_7, e_4 + we_5\},\\ \operatorname{sp}\{e_1 + \zeta e_5, e_2 + \zeta e_8, e_3 + \zeta e_6, e_4 + \zeta e_7\},\\ \operatorname{sp}\{e_1 + \eta e_6, e_2 + \eta e_7, e_3 + \eta e_5, e_4 + \eta e_8\},\end{gathered}$$

respectively, where $w = \exp(-2\pi i\alpha)$, $\zeta = \exp(-\pi i\beta)$ and $\eta = \exp(-\pi i\alpha)$. The bases $\{e_i^M\}$, $\{e_i^X\}$ and $\{e_i^\Gamma\}$ are defined as follows.

$$\begin{gathered}\sqrt{8}\, e_1^M = (1, i, -1, -i, -i, i, -1, 1), \qquad \sqrt{8}\, e_2^M = (1, -i, -1, i, -i, i, 1, -1),\\ 2e_3^M = (0, -i, 0, -i, 0, 0, -1, -1), \qquad 2e_4^M = (1, 0, 1, 0, -i, -i, 0, 0),\\ \sqrt{8}\, e_5^M = (1, i, -1, -i, i, -i, 1, -1), \qquad \sqrt{8}\, e_6^M = (1, -i, -1, i, i, -i, -1, 1),\\ 2e_7^M = (1, 0, 1, 0, i, i, 0, 0), \qquad 2e_8^M = (0, i, 0, i, 0, 0, -1, -1);\\ \sqrt{2}\, e_1^X = ie_3 + e_5, \quad \sqrt{2}\, e_2^X = ie_1 + e_6, \quad \sqrt{2}\, e_3^X = ie_4 + e_8, \quad \sqrt{2}\, e_4^X = ie_2 + e_7,\\ \sqrt{2}\, e_5^X = e_1 + ie_6, \quad \sqrt{2}\, e_6^X = e_3 + ie_5, \quad \sqrt{2}\, e_7^X = e_2 + ie_7, \quad \sqrt{2}\, e_8^X = e_4 + ie_8;\\ \sqrt{8}\, e_1 = (1, 1, 1, 1, 1, 1, 1, 1), \qquad \sqrt{8}\, e_2^\Gamma = (1, -1, 1, -1, 1, 1, -1, -1),\\ 2e_3^\Gamma = (1, 0, -1, 0, -1, 1, 0, 0), \qquad 2e_4^\Gamma = (0, -1, 0, 1, 0, 0, -1, 1),\\ \sqrt{8}\, e_5^\Gamma = (1, 1, 1, 1, -1, -1, -1, -1), \qquad \sqrt{8}\, e_6^\Gamma = (1, -1, 1, -1, -1, -1, 1, 1),\\ 2e_7^\Gamma = (0, 1, 0, -1, 0, 0, -1, 1), \qquad 2e_8^\Gamma = (1, 0, -1, 0, 1, -1, 0, 0).\end{gathered}$$

It is routine to check that the ranges of the P's decompose as asserted in the proposition. Calculations show that $P_\Sigma(\alpha)(\mathbf{C}^8)$ is invariant for $\operatorname{Ind}_N^G(\alpha, \alpha)$, and that the unitary transformation of the range of $P_\Sigma(\alpha)$ onto $\mathbf{C}^4$ defined by

$$e_1 + we_8 \to \sqrt{2}\, e_1, \quad e_2 + we_6 \to \sqrt{2}\, e_2, \quad e_3 + we_7 \to \sqrt{2}\, e_3, \quad e_4 + we_5 \to \sqrt{2}\, e_4$$

intertwines the restriction of $\operatorname{Ind}_N^G(\alpha, \alpha)$ and $\operatorname{Ind} \Sigma_1(\alpha)$. Further, the unitary which sends the basis

$$\{2^{-1/2}(e_1 - we_8), 2^{-1/2}(e_2 - we_6), 2^{-1/2}(e_3 - we_7), 2^{-1/2}(e_4 - we_5)\}$$

TABLE 1. The representations of $G = p4gm$. These were obtained by inducing the representations of Cracknell [5, Table V, p. 117] up to G; our conventions included the use of right cosets and the choice of $\{\{(C_{4z}^{+})^n \mid 0\}: n = 1,2,3,4\}$ as coset representatives for the spaces $G_\Sigma \backslash G$, $G_Y \backslash G$, and $G_\Delta \backslash G$. To save space we have only written down the values of the representations on the generating set $\{E \mid n, m\}$, $\{C_{4z}^{+} \mid 0\}$, $\{\sigma_x \mid \frac{1}{2}, \frac{1}{2}\}$.

Representation	$\{E \mid n, m\}$	$\{C_{4z}^{+} \mid 0\}$,	$\{\sigma_x \mid \frac{1}{2}, \frac{1}{2}\}$.
$\mathrm{Ind}_N^G(\alpha, \beta)$ $(z = e^{-2\pi i\alpha}, w = e^{-2\pi i\beta})$	$\begin{pmatrix} z^n w^m & & & & & & & & & & & 0 \\ & z^{-m} w^n & & & & & & & & & & \\ & & z^{-n} w^{-m} & & & & & & & & & \\ & & & z^m w^{-n} & & & & & & & & \\ & & & & z^{-n} w^m & & & & & & & \\ & & & & & z^n w^{-m} & & & & & & \\ & & & & & & z^{-m} w^{-n} & & & & & \\ & & & & & & & z^m w^n & & & & \\ & & & & & & & & z^{-n} w^{-m} & & & \\ & & & & & & & & & z^n w^m & & \\ & & & & & & & & & & z^{-m} w^{-n} & \\ 0 & & & & & & & & & & & z^m w^n \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 & & & & & & & & \\ 0 & 0 & 1 & 0 & & & & & & & & \\ 0 & 0 & 0 & 1 & & & & & & & & \\ 1 & 0 & 0 & 0 & & & & & & & & \\ & & & & 0 & 0 & 0 & 1 & & & & \\ & & & & 0 & 0 & 1 & 0 & & & & \\ & & & & 1 & 0 & 0 & 0 & & & & \\ & & & & 0 & 1 & 0 & 0 & & & & \\ & & & & & & & & 0 & 0 & 0 & 1 \\ & & & & & & & & 0 & 0 & 1 & 0 \\ & & & & & & & & 1 & 0 & 0 & 0 \\ & & & & & & & & 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} & & & & 1 & 0 & 0 & 0 & & & & \\ & & & & 0 & 0 & \bar{z} & 0 & & & & \\ & & & & 0 & \bar{z}\bar{w} & 0 & 0 & & & & \\ & & & & 0 & 0 & 0 & \bar{w} & & & & \\ & & & & & & & & w & 0 & 0 & 0 \\ & & & & & & & & 0 & 0 & z & 0 \\ & & & & & & & & 0 & 1 & 0 & 0 \\ & & & & & & & & 0 & 0 & 0 & zw \\ & & & & & & & & & & & \\ & & & & & & & & & & & \\ & & & & & & & & & & & \\ & & & & & & & & & & & \end{pmatrix}$
$\mathrm{Ind}_{G_\Sigma}^G \Sigma_1(\alpha)$	$\begin{pmatrix} w^{n+m} & & & 0 \\ & w^{n-m} & & \\ & & w^{-n-m} & \\ 0 & & & w^{m-n} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & w \\ 0 & 0 & 1 & 0 \\ 0 & \bar{w} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$
$\mathrm{Ind}_{G_\Sigma}^G \Sigma_1(\alpha)$ $(w = e^{-2\pi i\alpha})$	$\begin{pmatrix} w^{n+m} & & & 0 \\ & w^{n-m} & & \\ & & w^{-n-m} & \\ 0 & & & w^{m-n} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & -w \\ 0 & 0 & -1 & 0 \\ 0 & -\bar{w} & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$
M_1	$(-1)^{n+m}$	i	i
M_2	$(-1)^{n+m}$	i	$-i$
M_3	$(-1)^{n+m}$	$-i$	i
M_4	$(-1)^{n+m}$	$-i$	$-i$
M_5	$\begin{pmatrix} (-1)^{n+m} & 0 \\ 0 & (-1)^{n+m} \end{pmatrix}$	$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$	$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

$\mathrm{Ind}_{G_Y}^{G}\, Y_1(\beta)$	$\begin{pmatrix} (-1)^n\zeta^{2m} & & & 0 \\ & (-1)^{-m}\zeta^{2n} & & \\ & & (-1)^{-n}\zeta^{-2m} & \\ 0 & & & (-1)^m\zeta^{-2n} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \zeta & 0 & 0 & 0 \\ 0 & 0 & 0 & -\zeta \\ 0 & 0 & -\bar{\zeta} & 0 \\ 0 & \bar{\zeta} & 0 & 0 \end{pmatrix}$
$\mathrm{Ind}_{G_Y}^{G}\, Y_2(\beta)$ $(\zeta = e^{-\pi i\beta})$	$\begin{pmatrix} (-1)^n\zeta^{2m} & & & 0 \\ & (-1)^{-m}\zeta^{2n} & & \\ & & (-1)^{-n}\zeta^{-2m} & \\ 0 & & & (-1)^m\zeta^{-2n} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -\zeta & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta \\ 0 & 0 & \bar{\zeta} & 0 \\ 0 & -\bar{\zeta} & 0 & 0 \end{pmatrix}$
$\mathrm{Ind}_{G_X}^{G}\, X$	$\begin{pmatrix} (-1)^n & & & 0 \\ & (-1)^n & & \\ & & (-1)^m & \\ 0 & & & (-1)^m \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$
$\mathrm{Ind}_{G_\Delta}^{G}\, \Delta_1(\alpha)$	$\begin{pmatrix} \eta^{2n} & & & 0 \\ & \eta^{-2m} & & \\ & & \eta^{-2n} & \\ 0 & & & \eta^{2m} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \eta & 0 \\ 0 & \bar{\eta} & 0 & 0 \\ \bar{\eta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta \end{pmatrix}$
$\mathrm{Ind}_{G_\Delta}^{G}\, \Delta_2(\alpha)$ $(\eta = e^{-\pi i\alpha})$	$\begin{pmatrix} \eta^{2n} & & & 0 \\ & \eta^{-2m} & & \\ & & \eta^{-2n} & \\ 0 & & & \eta^{2m} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -\eta & 0 \\ 0 & -\bar{\eta} & 0 & 0 \\ -\bar{\eta} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\eta \end{pmatrix}$
Γ_1	1	1	1
Γ_2	1	1	-1
Γ_3	1	-1	1
Γ_4	1	-1	-1
Γ_5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

to the usual basis for $\mathbf{C}^4$ intertwines the restriction of $\operatorname{Ind}(\alpha, \alpha)$ to the complement of $P_\Sigma(\alpha)(\mathbf{C}^8)$ with the representation $\operatorname{Ind}\Sigma_2(\alpha)$, so that $\operatorname{Ind}(\alpha, \alpha) \sim \operatorname{Ind}\Sigma_1(\alpha) \oplus \operatorname{Ind}\Sigma_2(\alpha)$. In the same way, the unitaries which send the ordered bases

$$\begin{aligned}
&\{2^{-1/2}(e_1 + \zeta e_5), 2^{-1/2}(e_2 + \zeta e_8), 2^{-1/2}(e_3 + \zeta e_6), 2^{-1/2}(e_4 + \zeta e_7)\},\\
&\{2^{-1/2}(e_1 - \zeta e_5), 2^{-1/2}(e_2 - \zeta e_8), 2^{-1/2}(e_3 - \zeta e_6), 2^{-1/2}(e_4 - \zeta e_7)\},\\
&\{2^{-1/2}(e_1 + \eta e_6), 2^{-1/2}(e_2 + \eta e_7), 2^{-1/2}(e_3 + \eta e_5), 2^{-1/2}(e_4 + \eta e_8)\},\\
&\{2^{-1/2}(e_1 - \eta e_6), 2^{-1/2}(e_2 - \eta e_7), 2^{-1/2}(e_3 - \eta e_5), 2^{-1/2}(e_4 - \eta e_8)\}
\end{aligned}$$

to the usual basis for $\mathbf{C}^4$, intertwine, respectively,

$$\begin{aligned}
&\operatorname{Ind}_N^G\left(\tfrac{1}{2}, \beta\right)|_{\text{range } P_Y(\beta)} \quad \text{and} \quad \operatorname{Ind}_{G_Y}^G Y_1(\beta);\\
&\operatorname{Ind}_N^G\left(\tfrac{1}{2}, \beta\right)|_{(\text{range } P_Y(\beta))^\perp} \quad \text{and} \quad \operatorname{Ind}_{G_Y}^G Y_2(\beta);\\
&\operatorname{Ind}_N^G(\alpha, 0)|_{\text{range } P_\Delta(\alpha)} \quad \text{and} \quad \operatorname{Ind}_{G_\Delta}^G \Delta_1(\alpha);\\
&\operatorname{Ind}_N^G(\alpha, 0)|_{(\text{range } P_\Delta(\alpha))^\perp} \quad \text{and} \quad \operatorname{Ind}_{G_\Delta}^G \Delta_2(\alpha).
\end{aligned}$$

In particular, this proves that each $\Phi(a)$ satisfies condition (1). Further checking shows that the unitaries which send the designated basis to the usual basis for the appropriate $\mathbf{C}^k$ intertwine

$$\begin{aligned}
&\operatorname{Ind}\left(\tfrac{1}{2}, \tfrac{1}{2}\right)|_{\mathbf{C}e_1^M} \text{ and } M_2; \qquad \operatorname{Ind}\left(\tfrac{1}{2}, \tfrac{1}{2}\right)|_{\mathbf{C}e_2^M} \text{ and } M_4;\\
&\operatorname{Ind}\left(\tfrac{1}{2}, \tfrac{1}{2}\right)|_{\mathbf{C}e_5^M} \text{ and } M_1; \qquad \operatorname{Ind}\left(\tfrac{1}{2}, \tfrac{1}{2}\right)|_{\mathbf{C}e_6^M} \text{ and } M_3;\\
&\operatorname{Ind}\left(\tfrac{1}{2}, \tfrac{1}{2}\right)|_{\operatorname{sp}\{e_3^M, e_4^M\}} \text{ and } M_5; \qquad \operatorname{Ind}\left(\tfrac{1}{2}, \tfrac{1}{2}\right)|_{\operatorname{sp}\{e_7^M, e_8^M\}} \text{ and } M_5;\\
&\operatorname{Ind}\left(\tfrac{1}{2}, 0\right)|_{\operatorname{sp}\{e_1^X, e_2^X, e_3^X, e_4^X\}} \text{ and } X; \qquad \operatorname{Ind}\left(\tfrac{1}{2}, 0\right)|_{\operatorname{sp}\{e_5^X, e_6^X, e_7^X, e_8^X\}} \text{ and } X;\\
&\operatorname{Ind}(0, 0)|_{\mathbf{C}e_1^\Gamma} \text{ and } \Gamma_1; \qquad \operatorname{Ind}(0, 0)|_{\mathbf{C}e_2^\Gamma} \text{ and } \Gamma_3;\\
&\operatorname{Ind}(0, 0)|_{\mathbf{C}e_5^\Gamma} \text{ and } \Gamma_2; \qquad \operatorname{Ind}(0, 0)|_{\mathbf{C}e_6^\Gamma} \text{ and } \Gamma_4;\\
&\operatorname{Ind}(0, 0)|_{\operatorname{sp}\{e_3^\Gamma, e_4^\Gamma\}} \text{ and } \Gamma_5; \qquad \operatorname{Ind}(0, 0)|_{\operatorname{sp}\{e_7^\Gamma, e_8^\Gamma\}} \text{ and } \Gamma_5.
\end{aligned}$$

It follows from these statements that every $\Phi(a)$ satisfies conditions (2) and (3) of the proposition, and that each irreducible representation of G is a direct summand of some $\operatorname{Ind}(\alpha, \beta)$. We conclude that Φ is isometric for the C^*-norm, and so defines an isomorphism of $C^*(G)$ into the C^*-algebra

$$B = \{f \in C(\Omega, M_8(\mathbf{C})) : f \text{ satisfies } (1)\text{–}(3)\}.$$

It remains only to check that Φ is surjective.

The irreducible representations of B are all given by evaluation at some point of Ω, followed by restriction to an appropriate common invariant subspace. In view of condition (3), not all invariant subspaces give disjoint representations, and a

complete set of irreducibles is as follows:

$$\begin{aligned}
&\{f \to f(\alpha,\beta) : (\alpha,\beta) \in \operatorname{Int}\Omega\} \\
&\qquad \cup \{f \to P_\Sigma(\alpha) f(\alpha,\alpha), f \to (I - P_\Sigma(\alpha)) f(\alpha,\alpha) : \alpha \in]0, \tfrac{1}{2}[\} \\
&\qquad \cup \{f \to P_Y(\beta) f(\tfrac{1}{2},\beta), f \to (I - P_Y(\beta)) f(\tfrac{1}{2},\beta) : \beta \in]0, \tfrac{1}{2}[\} \\
&\qquad \cup \{f \to P_\Delta(\alpha) f(0,\alpha), f \to (I - P_\Delta(\alpha)) f(0,\alpha) : \alpha \in]0, \tfrac{1}{2}[\} \\
&\qquad \cup \left\{f \to f(0,0)|_{\mathbf{C}e_j^\Gamma} : j = 1,2,5,6\right\} \cup \left\{f \to f(0,0)|_{\operatorname{sp}\{e_3^\Gamma, e_4^\Gamma\}}\right\} \\
&\qquad \cup \left\{f \to f(\tfrac{1}{2},\tfrac{1}{2})|_{\mathbf{C}e_j^M} : j = 1,2,5,6\right\} \cup \left\{f \to f(\tfrac{1}{2},\tfrac{1}{2})|_{\operatorname{sp}\{e_3^M, e_4^M\}}\right\} \\
&\qquad \cup \left\{f \to f(\tfrac{1}{2},0)|_{\operatorname{sp}\{e_1^X, e_2^X, e_3^X, e_4^X\}}\right\}.
\end{aligned}$$

If we compose this set of irreducibles with Φ, then our decompositions of the $\operatorname{Ind}(\alpha,\beta)$ show that we obtain precisely the complete set of irreducibles for $C^*(G)$ which we listed before the statement of the proposition. Hence two irreducible representations of B are equivalent if and only if they give equivalent representations of $C^*(G)$ when composed with Φ; in other words, $\Phi(C^*(G))$ is a rich C^*-subalgebra of B. It follows from [7, Proposition 11.1.6] that $\Phi(C^*(G)) = B$, and we have proved the proposition.

REMARKS. (1) We deliberately chose to study the group $p4gm$ because it appeared to be the most complicated of the 2-dimensional space groups (a list is given in Table 1 of [5]), and would therefore be most representative. We do not foresee that any new problems would arise in doing a similar analysis for a 3-dimensional space group (except, of course, for the scale of the calculations), and we imagine that its C^*-algebra would have properties like this one.

(2) The unpleasant part in proving Proposition 8 was finding the decompositions of $\operatorname{Ind}(\alpha,\beta)$ for $(\alpha,\beta) \in \partial\Omega$. In theory, Proposition 1 gives a way to decide what the decomposition should be, but in practice it did not help much, partly because Cracknell's tables give the representations of the stabilisers as opposed to the multiplier representations of the little groups, and partly because the dimensions were small enough to permit trial and error calculations.

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SCHOOL OF MATHEMATICS, UNIVERSITY OF NEW SOUTH WALES, KENSINGTON, N.S.W. 2033, AUSTRALIA