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CONTRACTIVE PROJECTIONS ON $C_0(K)$

BY

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ABSTRACT. We show that the range of a norm one projection on a commutative C^*-algebra has a ternary product structure (Theorem 2). We describe and characterize all such projections in terms of extreme points in the unit ball of the image of the dual (Theorem 1). We give necessary and sufficient conditions for the range to be isometric to a C^*-algebra (Theorem 4) and we show that the range is a C_σ-space (Theorem 5).

Introduction. In his Ph.D. dissertation **[2]**, the first named author, jointly with Arazy, gave a complete description of all contractive projections (and their ranges) on the algebra C_∞ of all compact operators on a separable complex Hilbert space, and also on its dual, the space C_1 of trace class operators. As suggested by Effros and Størmer **[9]**, many of the spaces enumerated by Friedman and Arazy as the range of a contractive projection are either Jordan algebras of operators or Lie algebras of operators. Effros and Størmer also prove in **[9]** that the range of a positive unital (therefore contractive) projection on any unital C^*-algebra has the structure of a Jordan algebra, thus explaining the appearance of Jordan algebras in the work of Friedman and Arazy.

The present paper arose out of an attempt to explain, in the context of a general C^*-algebra, the appearance of the Lie algebras in the work of Friedman and Arazy. We are able to explain this phenomenon completely in the case of a commutative C^*-algebra. Indeed, Theorem 2 below states that the range of an arbitrary contractive projection on $C_0(K)$, K locally compact, has the structure of an (associative) ternary algebra. This result is based on the complete description of arbitrary contractive projections on $C_0(K)$ given in Theorem 1. Theorems 1 and 2 were announced in **[11]**.

Throughout this paper, $C_0(K)$ will denote the collection of all continuous functions on a locally compact Hausdorff space K with values in a field F which is either the real or complex numbers. All results in this paper, as well as the proofs given here, are valid for both real and complex scalars.

While the role of C^*-algebras and Jordan algebras in mathematical physics (quantum statistical mechanics) is well known, it is only recently that ternary algebras, in particular the type that occur here as the range of a contractive

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projection on a C^*-algebra, have been used to construct Lie algebras and Lie superalgebras (Bars-Günaydin [4]). The latter play important roles in mathematical physics, e.g., in gauge theories of elementary particles. The fact that ternary algebras appear naturally as the range of contractive projections suggests a connection between two different models of mathematical physics.

The significance of the present paper is that it suggests that the range of a contractive projection on an *arbitrary* C^*-algebra has an algebraic structure which includes, to some extent, Jordan algebras and Lie algebras, i.e., a Jordan triple system.[1] Thus contractive projections should be a key tool in characterizing the "state space" of a Jordan triple system. This is physically significant because of the fact that the quadratic formulation of quantum mechanics, which includes octonionic quantum mechanics, can be expressed in terms of Jordan triple products [12].

A basic problem in Banach space theory is to determine the structure of a (bounded linear) projection and of its range. In this paper we consider contractive (i.e., norm one) projections on the Banach space $C_0(K)$ of all continuous functions vanishing at infinity on a locally compact Hausdorff space K, i.e., a commutative C^*-algebra. The contractive projections and their ranges on complex $L^q(\mu)$, $1 \leqslant q < \infty$, were described by Douglas [8] for $q = 1$ and by Ando [1] for other values of q. The range of a contractive projection on $C^{\mathbf{R}}(K)$, K compact, was considered primarily by Wulbert and Lindenstrauss [20].

This paper is organized as follows. In §1 we prove the basic structure theorem for contractive projections. To facilitate its statement we need some notation. If $\{\mu_i\}_{i\in I}$ is a family of Borel measures on a locally compact space K with polar decompositions $\mu_i = \varphi_i \cdot |\mu_i|$ and mutually disjoint supports, then for $f \in C_0(K)$ we shall let Qf be a function on $S \equiv \bigcup \operatorname{supp}|\mu_i|$ which is equal to $\langle f, \mu_i\rangle\bar{\varphi}_i$, $|\mu_i|$-almost everywhere on $\operatorname{supp}|\mu_i|$. In Theorem 1 and Proposition 1.1 we show that a linear map $P\colon C_0(K) \to C_0(K)$ is a contractive projection if and only if there exist norm one Borel measures $\{\mu_i\}_{i\in I}$ with mutually disjoint supports such that Qf can be chosen to be *continuous* on S and $P = EQ$ where E is a linear isometric operator from $Q(C_0(K))$ to $C_0(K)$ such that Eg extends g from S to K for all $g \in Q(C_0(K))$.

We call $\{\mu_i\}_{i\in I}$ the *atoms* of P and $Q\colon C_0(K) \to C_0(S)$ the *essential part* of P. Theorem 1 has many applications due to the simple form of Q and that fact that $\bar{S}$ is a (Shilov) boundary for the range of P. For example, it implies that a bicontractive projection (i.e., $\|I - P\| = 1$ also) on $C_0(K)$ has the form $Pf = \frac{1}{2}(f + \lambda \cdot f \circ \sigma)$ where σ is an involutive homeomorphism of K and $\lambda\colon K \to \mathbf{C}$ satisfies $\lambda \circ \sigma = \bar{\lambda}$, $|\lambda| = 1$.

We also show in §1 that every contractive projection on $C_0(K)$ with finite-dimensional range is a sum of disjoint one-dimensional contractive projections and we give some simple consequences of Theorem 1.

In §2 we establish the algebraic properties of the range of a contractive projection on $C_0(K)$. We begin by defining a C^*-ternary algebra and proving a Gelfand-Naimark type representation theorem. We then show that the essential part Q of P

[1] See the authors' forthcoming paper *Contractive projections on operator triple systems* (Math. Scand.).

has the property that its range is a ternary algebra of functions. It follows that the range of P is a commutative C^*-ternary algebra (Theorem 2). These results are then applied to a study of averaging projections (Theorem 3) and to necessary and sufficient conditions for the range to be isometric to some $C(H)$ (Theorem 4).

In our final §3 we prove that the range of a contractive projection on $C_0(K)$ is a C_σ-space (Theorem 5).

Some of the results in this paper are actually stronger than the statements. For example, it is often true that a statement about a projection onto a space can be replaced by a statement about operators that leave that space invariant. This is applicable to the ergodic theory of nonpositive contractions on $C(K)$ (see [**22**]).

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1. The structure theorem. Let $A = C_0(K)$ be the algebra of all continuous functions vanishing at infinity on a locally compact Hausdorff space K. We identify the dual A' with the space of Borel measures on K and denote the duality between A'' and A' by $\langle \xi, \mu \rangle$, for $(\xi, \mu) \in A'' \times A'$. We shall be using the following facts:

(i) Any bounded Borel function φ on K defines an element $\hat{\varphi}$ of A'' by the rule $\langle \hat{\varphi}, \mu \rangle = \int \varphi \, d\mu$, for $\mu \in A'$. The map $\varphi \to \hat{\varphi}$ extends the canonical map of A into A''. We shall identify φ with $\hat{\varphi}$.

(ii) A'' is an involutive Banach algebra (Arens multiplication and involution, see Bonsall-Duncan [**7**]) containing the canonical image of A as a weak-star dense subalgebra. We shall identify A with its canonical image in A''.

(iii) A' is a left A''-module under the map $A'' \times A' \ni (\xi, \mu) \to \xi \cdot \mu$ defined by $\langle a, \xi \cdot \mu \rangle = \langle a\xi, \mu \rangle$ for $a \in A$.

(iv) For each $\mu \in A'$, there is a polar decomposition $\mu = \varphi \cdot |\mu|$, where $|\mu|$ is a positive element of A' and φ is a bounded Borel function vanishing on the complement of the support of $|\mu|$ (denoted by $\operatorname{supp}|\mu|$ or $\operatorname{supp}\mu$) and satisfying $\varphi^2\bar{\varphi} = \varphi$ (cf. Rudin [**16**], $\bar{\varphi}$ is the complex conjugate of φ).

In order to describe contractive projections, we shall use the following concept, which is a slight generalization of the definition in Bade [**3**]. Let X be a locally compact Hausdorff space, let Y be any subset of X and let W be a linear subspace of $C_b(Y)$, the space of bounded continuous complex valued functions on Y. A linear operator E: $W \to C_0(X)$ is called a *simultaneous extension operator* from W to $C_0(X)$ if Ef is an extension of f for all f in W. Obviously $\|Ef\| \geq \|f\|$. We call E *isometric* if $\|Ef\| = \|f\|$ for all f in W.

PROPOSITION 1.1. *Let K be a locally compact Hausdorff space and let $\{\mu_i\}_{i\in I}$ be a famly of complex Borel measures on K with polar decompositions $\mu_i = \varphi_i \cdot |\mu_i|$ satisfying these properties*:

(1.1) $\|\mu_i\| = 1, i \in I$;

(1.2) $\operatorname{supp}|\mu_i| \cap \operatorname{supp}|\mu_j| = \varnothing$ *for* $i \neq j,\ i,j \in I$;

(1.3) *For each f in $C_0(K)$, there is a unique element Qf in $C_b(S)$ where $S = \bigcup_{i\in I} S_i$, $S_i = \operatorname{supp}|\mu_i|$, such that, for each i in I,*

$$Qf\,|\,S_i = \langle f, \mu_i \rangle \bar{\varphi}_i\,|\,S_i, \qquad |\mu_i|\text{-}a.e.$$

Then for each isometric simultaneous extension operator E from $Q(C_0(K))$ to $C_0(K)$, the operator $P = EQ$ is a contractive projection on $C_0(K)$.

PROOF. P is clearly linear and contractive. Let $S_i = \operatorname{supp}|\mu_i|$. For f in A, $\langle Pf, \mu_i\rangle = \langle EQf, \mu_i\rangle = \int_{S_i} Qf\,d\mu_i = \langle f, \mu_i\rangle\langle\bar{\varphi}_i, \mu_i\rangle = \langle f, \mu_i\rangle$. Therefore

$$\begin{aligned} P^2f\,|\,S_i &= EQ(Pf)\,|\,S_i = Q(Pf)\,|\,S_i = \langle Pf, \mu_i\rangle\bar{\varphi}_i\,|\,S_i \\ &= \langle f, \mu_i\rangle\bar{\varphi}_i\,|\,S_i = Pf\,|\,S_i. \end{aligned}$$

Thus $P^2f\,|\,S = Pf\,|\,S$, $P(Pf - f)\,|\,S = 0$, $Q(Pf - f) = 0$, $EQ(Pf - f) = 0$, so P is a projection.

For a discussion of the existence of simultaneous extension operators see Bade **[3]**.

The main result of this section is the converse to Proposition 1.1. In the following lemmas, P denotes a projection of norm one on $A = C_0(K)$ with adjoint $P'\colon A' \to A'$ which is of course also a projection of norm one. The unit ball of a Banach space X will be denoted by $X_1 = \{x \in X : \|x\| \leqslant 1\}$.

Lemma 1.2 is due to Robert E. Atalla. The authors are indebted to him for correcting their original proof of Lemma 1.3 and for simplifying their proof of Lemma 1.4.

LEMMA 1.2. *Let μ be an extreme point of the convex set $P'(A_1')$, with polar decomposition $\mu = \varphi\cdot|\mu|$. Then for each f in $C_0(K)$,*

$$Pf = \langle f, \mu\rangle\bar{\varphi}, \qquad |\mu|\text{-a.e.} \tag{1.4}$$

PROOF. Suppose that $(Pf)\varphi$ is not constant $|\mu|$-a.e. Then we may assume that there exists real α such that $E = \{x : \operatorname{Re}(Pf(x)\varphi(x)) \geqslant \alpha\}$ and $F = \{x : \operatorname{Re}(Pf(x)\varphi(x)) < \alpha\}$ both have positive $|\mu|$-measure. Let $t = |\mu|(E)$, $1 - t = |\mu|(F)$ so that $\mu = t[(\mu\,|\,E)/t] + (1-t)[(\mu\,|\,F)/(1-t)]$. Then

$$\begin{aligned} \mu = P'\mu &= tP'\big[(\mu\,|\,E)/t\big] + (1-t)P'\big[(\mu\,|\,F)/(1-t)\big] \\ &\equiv t\mu_1 + (1-t)\mu_2. \end{aligned}$$

We shall prove that $\mu_1 \neq \mu_2$, contradicting the extremality of μ. Now $\langle f, \mu_1\rangle = \langle f, P'((\mu\,|\,E)/t)\rangle = t^{-1}\langle Pf, \mu\,|\,E\rangle = t^{-1}\int_E Pf\,d\mu = t^{-1}\int_E (Pf)\varphi\,d\,|\mu|$. Therefore $\operatorname{Re}\langle f, \mu_1\rangle \geqslant \alpha$ and similarly $\operatorname{Re}\langle f, \mu_2\rangle < \alpha$, so $\mu_1 \neq \mu_2$. Thus $Pf = k\bar{\varphi}$, $|\mu|$-a.e. for some constant k. But $\langle f, \mu\rangle = \langle Pf, \mu\rangle = k\langle\bar{\varphi}, \mu\rangle = k\langle\bar{\varphi}, \varphi|\mu|\rangle = k$.

REMARK. There is at least one f in $C_0(K)$ such that $\langle f, \mu\rangle \neq 0$. Therefore, for any extreme point μ of $P'(A_1')$, with polar decomposition $\mu = \varphi\cdot|\mu|$, we may and *shall* choose φ to be continuous on the support of $|\mu|$, and to vanish off this set. Thus (1.4) holds everywhere on $\operatorname{supp}|\mu|$, $|\varphi|\,Pf = \langle f, \mu\rangle\bar{\varphi}$, and by the weak-star density of A in A'' we have $|\varphi|\,P''\xi = \langle\xi, \mu\rangle\bar{\varphi}$, for $\xi \in A''$.

LEMMA 1.3. *Let μ be an extreme point of the convex set $P'(A_1')$ with polar decomposition $\mu = \varphi\cdot|\mu|$. Then for each $\nu \in A'$,*

$$P'(|\varphi|\cdot\nu) = \langle\bar{\varphi}, \nu\rangle\mu. \tag{1.5}$$

PROOF. For f in A, $\langle P'(|\varphi|\cdot\nu), f\rangle = \int Pf\,|\varphi|\,d\nu = \langle f, \mu\rangle\langle\bar{\varphi}, \nu\rangle = \langle\langle\bar{\varphi}, \nu\rangle\mu, f\rangle$.

LEMMA 1.4. *Let μ_1, μ_2 be extreme points of the convex set $P'(A_1')$. Then either μ_1 is a scalar multiple of μ_2 or* $\operatorname{supp}|\mu_1| \cap \operatorname{supp}|\mu_2| = \varnothing$.

PROOF. If $y \in \operatorname{supp}|\mu_1| \cap \operatorname{supp}|\mu_2|$, then by Lemma 1.3, $P'(\delta_y) = \overline{\varphi_1(y)}\mu_1 = \overline{\varphi_2(y)}\mu_2$ *so that* $\mu_1 = \varphi_1(y)\overline{\varphi_2(y)}\mu_2$.

LEMMA 1.5. *Let* $\{\mu_i\}_{i\in I}$ *be a maximal family of extreme points of* $P'(A_1')$ *such that* $|\mu_i| \neq |\mu_j|$ *for* $i \neq j$, i,j *in* I. *Let* $\mu_i = \varphi_i \cdot |\mu_i|$ *be the polar decomposition of* μ_i *and let* χ *be the characteristic function of* $\bar{S}$, *where* $S = \bigcup_{i\in I}\operatorname{supp}|\mu_i|$. *Then for* ξ *in* A'',

$$P''\xi = P''(\chi\xi). \tag{1.6}$$

PROOF. By maximality, if μ is an extreme point of $P'(A_1')$ then $|\mu| = |\mu_j|$ for some j, so by Lemma 1.4, μ is a scalar multiple of μ_j. Thus every convex combination of extreme points of $P'(A_1')$ is supported on S. By Kreĭn-Mil'man every element of $P'(A_1')$ is supported in $\bar{S}$. Thus, if $(\xi, \mu) \in A'' \times A_1'$, $\langle P''\xi, \mu\rangle = \langle \xi, P'\mu\rangle = \langle \xi, \chi \cdot P'\mu\rangle = \langle \chi\xi, P'\mu\rangle = \langle P''(\chi\xi), \mu\rangle$.

THEOREM 1. *Let* $A = C_0(K)$ *with* K *a locally compact Hausdorff space, and let* P *be a contractive projection on* A: $P^2 = P$, $\|P\| = 1$. *Then there exist measures* $\{\mu_i\}_{i\in I} \subset A'$ *satisfying* (1.1)–(1.3) *and there exists an isometric simultaneous extension operator* E *from* $Q(A)$ *to* A *such that* $P = EQ$.

PROOF. By Zorn's lemma there is a family $\{\mu_i\}_{i\in I}$ of extreme points of $P'(A_1')$ which is maximal with respect to the condition $|\mu_i| \neq |\mu_j|$ for $i \neq j$, $i,j \in I$. Then (1.1) is satisfied and (1.2) follows from Lemma 1.4. Let $S = \bigcup_{i\in I}\operatorname{supp}|\mu_i|$ and let χ be the characteristic function of $\bar{S}$. Define operators M, $\tilde{Q}$, $\tilde{T}$ on A'' to A'' by

$$M\xi = \chi\xi, \qquad \xi \in A''; \qquad \tilde{Q} = MP''; \qquad \tilde{T} = (I - M)P''.$$

By Lemma 1.5, $\tilde{T} = (I - M)P'' = (I - M)P''MP'' = \tilde{T}\tilde{Q}$ and by Lemma 1.2, if we define $Qf \in C_b(S)$, for $f \in A$, by $Qf = Pf\,|\,S$ then (1.3) holds. We now simply define E: $Q(A) \to C_0(K)$ by $EQf = Pf$ and verify that E is well defined and a linear isometric simultaneous extension operator from $Q(A)$ to $C_0(K)$. Since Pf is an extension of Qf it will suffice to prove that $\|Pf\| = \|Qf\|$. We have $Pf = \tilde{Q}f + \tilde{T}f$, $\tilde{Q}f$ and $\tilde{T}f$ are bounded Borel functions with disjoint supports, $\|\tilde{T}\| \leq 1$, and $\tilde{T} = \tilde{T}\tilde{Q}$. Therefore

$$\|Pf\| = \max(\|\tilde{Q}f\|, \|\tilde{T}f\|) = \max(\|\tilde{Q}f\|, \|\tilde{T}\tilde{Q}f\|) = \|\tilde{Q}f\|.$$

Since $\tilde{Q}f$ agrees with Qf on S and vanishes off $\bar{S}$, $\|Qf\| = \|\tilde{Q}f\| = \|Pf\|$. The theorem is proved.

REMARKS. If $\{\mu_j'\}_{j\in J}$ is another maximal family of extreme points of $P'(A_1')$, then Lemma 1.4 implies that each μ_j' is a scalar multiple of some μ_i. Therefore the operator Q: $A \to C_b(S)$ is uniquely determined by P. We shall refer to any extreme point of $P'(A_1')$ as an *atom* of P and we shall call Q the *essential part* of P. Note that $Qf = \tilde{Q}f\,|\,S$ and that $\tilde{Q}$ is a projection on A''. Also, $\bar{S}$ is a boundary for the range of P in the sense of Šilov: each f in $P(A)$ assumes its norm at a point of $\bar{S}$.

Our first application of Theorem 1 will be to a decomposition of contractive projections into "irreducible" ones.

Let P be a contractive projection on $A = C_0(K)$ with atoms $\{\mu_i\}_{i\in I}$ and let $S_i = \operatorname{supp}|\mu_i|$. Let $Z = \bigcap_{f\in B} f^{-1}(0)$ where $B = P(A)$ is the range of P. For $J \subset I$, set $S_J = \bigcup_{i\in J} S_i$. Note that $S_J \cup S_{J^c} = S_I\,(= S)$, and $\bar{S} = \bar{S}_J \cup \bar{S}_{J^c}$. We call $J \subset I$ a *divisor* of I if $\bar{S}_J \cap \bar{S}_{J^c} \subset Z$. For each divisor J of I we can define a contractive projection P_J on A (called a *divisor of* P) as follows: if $f \in A$, choose (by Tietze) $\tilde{f} \in A$ to satisfy $\|\tilde{f}\| \leq \|f\|$ and

$$\tilde{f} = \begin{cases} Pf & \text{on } \bar{S}_J, \\ 0 & \text{on } \bar{S}_{J^c}. \end{cases}$$

Set $P_J f = P(\tilde{f})$. It is easy to see that P_J is well defined and is a contractive projection on A.

If J is a divisor, then so is J^c and $P = P_J + P_{J^c}$. For this it suffices to observe that $P_J f$ is zero on $\bar{S}_{J^c}$. Note that $P_J P_{J^c} = 0 = P_{J^c} P_J$.

PROPOSITION 1.6. *Let P be a contractive projection on $A = C_0(K)$ and let $\{\mu_i\}_{i \in I}$ be the atoms of P. If* $\operatorname{int}\operatorname{supp}|\mu_{i_0}| \neq \varnothing$ *then $\{i_0\}$ is a divisor of I.*

PROOF. Since $\operatorname{supp}|\mu_{i_0}|$ has an interior point we can choose $f \in A$ such that $\langle f, \mu_{i_0}\rangle \neq 0$ and $f(K \setminus \operatorname{supp}\mu_{i_0}) = 0$. Now let $x \in \operatorname{supp}|\mu_{i_0}|$. Then $Pf(x) = \langle f, \mu_{i_0}\rangle \overline{\varphi_{i_0}(x)} \neq 0$ and $Pf(S_{\{i_0\}^c}) = 0$. Thus $x \notin \bar{S}_{\{i_0\}^c}$ and we have proved that $S_{i_0} \cup \bar{S}_{\{i_0\}^c} = \varnothing \subset Z$.

If we call J a *prime divisor* of I when no nontrivial divisors of I are contained in J we obtain the following consequence of Proposition 1.6.

COROLLARY 1.7. *If J is a prime divisor of cardinality* 2 *or more, then* $\operatorname{supp}|\mu_i|$ *is nowhere dense for all $i \in J$.*

LEMMA 1.8. *Let T: $A \to A$ be a linear map with range in $B = P(A)$ and suppose there is a subset $J \subset I$ such that*

$$Tf(x) = \begin{cases} \langle f, \mu_i\rangle \overline{\varphi_i(x)}, & x \in S_J, \\ 0, & x \in S_{J^c}. \end{cases}$$

Then J is a divisor and $T = P_J$.

PROOF. Each member of $T(A)$ vanishes on S_{J^c}, therefore on $\overline{S_{J^c}}$, and each member of $(P - T)(A)$ vanishes on S_J, hence on $\bar{S}_J$. Since $P(A)$ is spanned by $T(A)$ and $(P - T)(A)$ we have $\bar{S}_J \cap \bar{S}_{J^c} \subset Z$.

PROPOSITION 1.9. *Let J_1, J_2 be divisors of I. Then $J_1 \cap J_2$ and $J_1 \cup J_2$ are divisors and*

$$P_{J_1 \cap J_2} = P_{J_1} P_{J_2}, \tag{1.7}$$

$$P_{J_1 \cup J_2} = P_{J_1} + P_{J_2} - P_{J_1 \cap J_2}. \tag{1.8}$$

PROOF. $P_{J_1} P_{J_2}$ is a contraction on A with range in B and

$$P_{J_1} P_{J_2} = \begin{cases} \langle f, \mu_i\rangle \bar{\varphi}_i & \text{on } S_{J_1 \cap J_2}, \\ 0 & \text{on } S_{(J_1 \cap J_2)^c}. \end{cases}$$

This proves (1.7). Similarly, applying Lemma 1.8 to $P_{J_1} + P_{J_2} - P_{J_1 \cap J_2}$ will prove (1.8).

The set of divisors of P forms a lattice with P as largest element and the prime divisors as minimal elements.

COROLLARY 1.10. *If $P(A)$ is finite-dimensional, then P is a finite sum of one-dimensional contractive projections P_i ($i \in I$) on A which are disjoint*: $P_iP_j = P_jP_i = 0$ *for* $i \neq j$; $i,j \in I$.

It is natural to try to decompose the index set I into a disjoint union of prime divisors. Example 3 below shows that this is not possible.

We shall now give some applications of Theorem 1 to the case when the atoms are point masses.

PROPOSITION 1.11. *Let P be a contractive projection on $A = C_0(K)$ and let $Y = \{x \in K$: δ_x is an atom of $P\}$. Then $Pf \mid \overline{Y} = f \mid \overline{Y}$ for each f in A. In particular, if Y is dense, then $P = I =$ the identity.*

PROOF. Clearly $Y \subset S$ and, for $y \in Y$, we can choose μ_i such that $\delta_y = \mu_i$. Thus $Pf(y) = Qf(y) = \langle f, \mu_i\rangle\overline{\varphi_i(y)} = f(y)$. Since f and Pf are continuous, $Pf \mid \overline{Y} = f \mid \overline{Y}$.

Let B be a linear subspace of $A = C_0(K)$. We say B is *weakly separating* (cf. **[17]**), if for any two distinct points x, $y \in K$ such that $\|\delta_x \mid B\| = \|\delta_y \mid B\| = 1$, there exists $f \in B$ such that $|f(x)| \neq |f(y)|$.

LEMMA 1.12. *Let P be a contractive projection on $A = C_0(K)$ and suppose $B = P(A)$ is weakly separating. Then each atom μ_i of P is supported by a single point.*

PROOF. Let $x \in \operatorname{supp}|\mu_i|$. Choose $g \in C_0(K)$ such that $\|g\| = 1$ and $g = \bar{\varphi}_i$ on $\operatorname{supp}\mu_i$ (Tietze extension and (1.3)). If $f = Pg$ then $\|f\| \leq 1$ and $f(x) = Pg(x) = \langle g, \mu_i\rangle\overline{\varphi_i(x)} = \langle\bar{\varphi}_i, \mu_i\rangle\overline{\varphi_i(x)} = \overline{\varphi_i(x)}$. Thus for each $x \in \operatorname{supp}|\mu_i|$, $\|\delta_x \mid B\| = 1$. Suppose now that x, $y \in \operatorname{supp}|\mu_i|$ and that $x \neq y$. Then for each $h \in B$, $|h(x)| = |\langle h, \mu_i\rangle\overline{\varphi_i(x)}| = |\langle h, \mu_i\rangle| = |\langle h, \mu_i\rangle\overline{\varphi_i(y)}| = |h(y)|$, a contradiction.

By combining Proposition 1.11 and Lemma 1.12 we obtain

PROPOSITION 1.13. *Let P be a contractive projection on $A = C_0(K)$ with essential part Q and suppose $B = P(A)$ is weakly separating. Then*

(i) $Qf = \chi f, f \in A$ *where* $\chi = \chi_{\overline{S}}$;

(ii) *the operator of restriction to $\overline{S}$ is an isometric isomorphism of B onto $C_0(\overline{S})$.*

Part (ii) of Proposition 1.13 was proved by Wulbert **[17]** in case K is compact and $A = C^{\mathbf{R}}(K)$.

LEMMA 1.14. *Let P be a contractive projection on $A = C_0(K)$ and let $x_0 \in S$ be a peaking point for the subspace $B = P(A)$, i.e., there exists $f_0 \in B$ such that $f_0(x_0) = \|f_0\|$ and $|f_0(x)| < \|f_0\|$ for all x in K, $x \neq x_0$. Then δ_{x_0} is an atom of P, i.e., $\chi_0 \in Y$* (*cf. Proposition* 1.11).

PROOF. Since $x_0 \in S$ there exists $i_0 \in I$ such that $x_0 \in \operatorname{supp}|\mu_{i_0}|$ for some atom μ_{i_0}. Then for any $f \in A$, and $x \in \operatorname{supp}|\mu_{i_0}|$,

$$Pf(x) = \langle f, \mu_{i_0}\rangle\,\overline{\varphi_{i_0}(x)}\,, \qquad |Pf(x)| = |\langle f, \mu_{i_0}\rangle|\,.$$

In particular for $f = f_0$, $|f_0(x)| = |Pf_0(x)| = |\langle f_0, \mu_{i_0}\rangle| = |f_0(x_0)| = \|f_0\|$. This shows that $x = x_0$ so that $\operatorname{supp}|\mu_{i_0}| = \{x_0\}$ and therefore δ_{x_0} is an atom of P.

COROLLARY 1.15. *Let B be any closed subspace of $C(\mathbf{T})$ which contains the disk algebra $\mathfrak{D}$ ($=$ the algebra of functions on the unit circle $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$ having a continuous extension to $|z| \leqslant 1$, analytic on $|z| < 1$). Then B cannot be the range of a contractive projection $P \neq I$ on $C(\mathbf{T})$.*

PROOF. If $z_0 \in \mathbf{T}$, $f \in \mathfrak{D}$ defined by $f(z) = z_0/(2z_0 - z)$ satisfies $f(z_0) = 1$, $|f(z)| < 1$ if $z \neq z_0$, $z \in \mathbf{T}$. Thus the set of peaking points for $\mathfrak{D}$, and hence for B, is $\mathbf{T}$. Since $\bar{S}$ is a boundary, $\bar{S} = \mathbf{T}$, so by Lemma 1.14 and Proposition 1.11, $P = I$.

It is known that the disk algebra $\mathfrak{D}$ is even uncomplemented in $C(\mathbf{T})$ (Theorem of Rudin, Curtis, and Arens [**10**]).

COROLLARY 1.16. *Let $A = C[0, 1]$ and suppose B is a closed subspace of A, and P is a contractive projection on A with range B.*

(i) *If $1, x, x^2 \in B$, then $B = A$ and $P =$ the identity.*

(ii) *If $x, x^2 \in B$, then $B|_{[\alpha,1]} = C[\alpha, 1]$ and $Pf(t) = f(t)$ for all $f \in A$, $t \in [\alpha, 1]$, where $\alpha = 2^{1/2} - 1 = .41421\ldots$.*

PROOF. (i) For each $t \in [0, 1]$ there is a member f of B which peaks at t. Since $\bar{S}$ is a boundary for B, S is dense in $[0, 1]$. Therefore we can apply 1.11 and 1.14.

(ii) As in (i), S is dense in $[\alpha, 1]$ where $\alpha^2 - 2\alpha - 1 = 0$.

REMARKS. Part (i) of 1.16 has been known for a long time. The analogous result for $A = L^1[0, 1]$ and B four-dimensional is proved in Wulbert [**19**]. Part (ii) seems to be new and does not involve the assumption that P be Markovian, i.e., positive. For each $\beta \in [0, \alpha]$ and $f \in A$ let $b_{f,\beta}$ be a parabola passing through the points $(0, 0)$, $(\beta, f(\beta))$, $(1, f(1))$. Then

$$b_{f,\beta}(t) = \frac{f(\beta) - \beta f(1)}{\beta^2 - \beta} t^2 + \frac{f(\beta) - \beta^2 f(1)}{\beta - \beta^2} t.$$

Now define P_β: $A \to A$ by $P_\beta f = f$ on $[\beta, 1]$ and $P_\beta f = b_{f,\beta}$ on $[0, \beta]$. Then P_β fixes x and x^2 and is a contractive projection on A with $P_\beta \neq$ identity for $\beta \neq 0$. We cannot drop the assumption $x^2 \in B$; see Example 1.

We shall now use Theorem 1 to describe the structure of any *bicontractive* projection on $C_0(K)$, i.e., a linear operator P satisfying $P^2 = P$, $\|P\| = 1$, $\|I - P\| = 1$.

LEMMA 1.17. *Let $A = C_0(K)$ and let $\mu = \varphi \cdot |\mu|$ be the polar decomposition of μ in A' with $\|\mu\| = 1$ and φ continuous on* $\operatorname{supp}\mu$. *Suppose that the one-dimensional projection $Q\xi = \langle \xi, \mu\rangle\bar{\varphi}$ on A'' is bicontractive. Then (up to a scale factor) either*

(a) *$\mu = \delta_x$ for some $x \in K$; or*

(b) *$\mu = \frac{1}{2}(\delta_x + \lambda\delta_y)$ for some $x, y \in K$ and $\lambda \in \mathbf{T}$.*

PROOF. Suppose that $\operatorname{supp}|\mu|$ had 3 or more points or that $\operatorname{supp}|\mu| = \{x, y\}$ with $|\mu|(\{x\}) < \frac{1}{2}$. In either case there would be a point $x \in \operatorname{supp}|\mu|$ and an open neighborhood U of x such that $|\mu|(U) < \frac{1}{2}$. We may assume that $\varphi(x) = 1$. Now pick $f \in A$ such that $\|f\| = 1$, $f(x) = 1$ and $f|[(K \setminus U) \cap \operatorname{supp}|\mu|] = -\overline{\varphi}$. Then $Qf(x) = \overline{\langle f, \mu\rangle\varphi(x)} = \int_U f\,d\mu + \int_{K\setminus U} f\,d\mu = \int_U f\,d\mu - |\mu|(K \setminus U)$ so that $\operatorname{Re} Qf(x) < 0$. But $((I - Q)f)(x) = f(x) - Qf(x)$ so $\operatorname{Re}((I - Q)f)(x) > 1$, a contradiction. The alternatives to our presumption at the beginning of the proof are (a) and (b).

LEMMA 1.18. *Let P be a bicontractive projection on A. Then Pf is supported on $\bar{S}$, where $S = \bigcup_{i\in I}\operatorname{supp}|\mu_i|$ and $\{\mu_i\}_{i\in I}$ are the atoms of P.*

PROOF. Let $Pf = \tilde{Q}f + \tilde{T}f$ as in the proof of Theorem 1. It suffices to prove that $\tilde{T}f(x) = 0$ for all $f \in A$ and $x \in K \setminus \bar{S}$. Suppose $\tilde{T}f(x) \neq 0$ for some $(f, x) \in A \times (K \setminus \bar{S})$. We may assume that $\tilde{T}f(x) < 0$. Choose $g \in A$ such that $g|\bar{S} = f|\bar{S}$, $g(x) = 1$ and $\|g\| = 1$. Then $\tilde{T}g = \tilde{T}\tilde{Q}g = \tilde{T}\tilde{Q}Mg = \tilde{T}\tilde{Q}Mf = \tilde{T}\tilde{Q}f = \tilde{T}f$ and $((1 - P)g)(x) = g(x) - \tilde{Q}g(x) - \tilde{T}g(x) = 1 - \tilde{T}f(x) > 1$, a contradiction.

PROPOSITION 1.19. *Let P be a contractive projection on $A = C_0(K)$. Then P is bicontractive if and only if there exists a continuous function λ: $K \to \mathbf{T}$ and a homeomorphism σ of K such that*

$$Pf = \tfrac{1}{2}(f + \lambda f \circ \sigma), \qquad f \in A, \tag{1.9}$$

and

$$\lambda \circ \sigma = \bar{\lambda}, \qquad \sigma^2 = \textit{identity}. \tag{1.10}$$

PROOF. (1.9) and (1.10) easily imply that P is a bicontractive projection. Conversely, if P is bicontractive with atoms $\{\mu_i\}_{i\in I}$, then $Q_i = \langle\cdot\,, \mu_i\rangle\overline{\varphi}_i$ is bicontractive on A'' since

$$(I - Q_i)f = \begin{cases} (I - P)f & \text{on } \operatorname{supp}|\mu_i|, \\ f & \text{on } K \setminus \operatorname{supp}|\mu_i|. \end{cases}$$

Therefore, by Lemma 1.17 there are subsets Y and X of S and functions λ: $X \to \mathbf{T}$, σ: $X \to X$, $\sigma^2 =$ identity, such that $S = Y \cup X$, $Y \cap X = \varnothing$, and the collection of atoms of P is

$$\left\{\tfrac{1}{2}\left(\delta_x + \lambda(x)\delta_{\sigma(x)}\right) : x \in X\right\} \cup \left\{\delta_y : y \in Y\right\}.$$

We can extend the functions λ and σ to $K \setminus (\bar{S} \setminus S)$ as follows:

$$\lambda \equiv 1 \text{ on } Y, \qquad \lambda \equiv -1 \text{ on } K \setminus \bar{S},$$
$$\sigma = \text{identity on } Y \cup (K \setminus \bar{S}).$$

With these extended functions we have, for $f \in A$, and x in the dense set $K \setminus (\bar{S} - S)$,

$$Qf(x) = \left(f(x) + \lambda(x)f(\sigma(x))\right)/2. \tag{1.11}$$

It is easy to check that λ and σ are continuous on $K \setminus (\bar{S} \setminus S)$ and have continuous extensions to K such that (1.10) and (1.11) hold. By Lemma 1.18, (1.9) holds too. This completes the proof.

Bernau and Lacey prove in **[6]** that a bicontractive projection P on $C_0(K)$ has the property that $2P - I$ is an involutive isometry of $C_0(K)$ onto itself. This fact, together with the Banach-Stone Theorem **[5**, p. 138**]** gives another proof of Proposition 1.19.

In particular, if P is an M-projection on $C_0(K)$, i.e., if

$$\|f\| = \max(\|Pf\|, \|(I-P)f\|)$$

for all $f \in C_0(K)$, then P is bicontractive and it is not difficult to see that σ is the identity and therefore $\lambda \equiv 1$ in Proposition 1.19. Therefore, by Lemma 1.18 $Pf = \chi_{\bar{S}} f$ for $f \in C_0(K)$. This is proved in Behrends [**5**, p. 13], using the Banach-Stone Theorem.

The following result shows that the extension operator E, though it appears to be somewhat arbitrary, is actually uniquely determined by the range and essential part of P.

PROPOSITION 1.20. *Let P_1 and P_2 be contractive projections on $A = C_0(K)$ with essential parts Q_1 and Q_2. Suppose $P_1(A) = P_2(A)$ and $Q_1 = Q_2$. Then $P_1 = P_2$.*

PROOF. Let $B = P_1(A) = P_2(A)$. Then for any $f \in A$, $g \equiv P_1 f - P_2 f \in B$. Since $\bar{S}$ is a boundary for B and $g \mid \bar{S} = 0$ we have $g = 0$.

EXAMPLE 1. Let $A = C[0,1]$ and for $f \in A$ let $Pf(t) = f(0) + (f(1) - f(0))t$, $t \in [0,1]$. Then P is a contractive projection whose range consists of all linear functions. One has $Pf = f(0)G + f(1)F$ where $F(t) = t$, $G(t) = 1 - t$. Hence $P'\mu = \langle \mu, G\rangle \delta_0 + \langle \mu, F\rangle \delta_1$, so the atoms of P are $\{\delta_0, \delta_1\}$. We have $Qf = f(0)\chi_{\{0\}} + f(1)\chi_{\{1\}}$ for $f \in A$. $P(A)$ is not a subalgebra of A but is isometric to $C(\{0,1\})$.

EXAMPLE 2. Let $A = C_0(R)$ and for $f \in A$ let P be the bicontractive projection defined by $Pf(x) = (f(x) - f(-x))/2$. Then $P'\mu = (\mu - \check{\mu})/2$ where $\int_{\mathbf{R}} f(x)\, d\check{\mu}(x) = \int_{\mathbf{R}} f(-x)\, d\mu(x)$. The atoms are $\mu_t = (\delta_t - \delta_{-t})/2$, $t \in (0,\infty)$. $P(A)$ consists of the odd functions in A so is not a subalgebra but is closed under triple products. Also, since $(0,\infty)$ is connected, P is a prime projection, i.e., a prime divisor of itself.

EXAMPLE 3. Let $K = \bigcup_{n=0}^{\infty} L_n$ where L_0 is the x-axis and L_n $(n \geqslant 1)$ is the line $y = \frac{1}{n}x$. For $f \in C_0(K)$ let $Pf(x, y) = (f(x, y) - f(-x, -y))/2$. Then $I = \{0, 1, 2, \dots\} \times (0, \infty)$ and if we set $J_n = \{n\} \times (0,\infty)$, for $n \geqslant 0$, then J_n is a prime divisor of I for $n \geqslant 1$ but J_0 is not a divisor of I.

2. Algebraic properties of the range. Let P be a contractive projection on $A = C_0(K)$. As seen in §1 the range $B = P(A)$ is not always a subalgebra of A. In the present section we show that B is always a ternary algebra in the sense of M. R. Hestenes [**13**], and in some cases, a ternary subalgebra of A. This implies that B has the structure of an (associative) algebra. We shall also give necessary and sufficient conditions for B to have the structure of a C^*-algebra with unit, i.e., a Banach $*$-algebra such that $\|a^*a\| = \|a\|^2$ holds.

A *ternary algebra* is a linear space X over a field such that to each ordered triple of vectors (a, b, c) in X there corresponds a vector, to be denoted by $[a, b, c]$, in X. The mapping $[\cdot,\cdot,\cdot]\colon X \times X \times X \to X$ is subject to the following conditions:

$$(2.1)\qquad \begin{cases} \text{(i)} & [a,a,a] = 0 \text{ if and only if } a = 0; \\ \text{(ii)} & [[a,b,c],d,e] = [a,[d,c,b],e] = [a,b,[c,d,e]]; \\ \text{(iii)} & [\alpha a, b, c] = \alpha[a,b,c] = [a, b, \alpha c], \quad \alpha \text{ scalar}; \\ \text{(iv)} & [a+b, c, d] = [a,c,d] + [b,c,d], \\ & [a, b, c+d] = [a,b,c] + [a,b,d]. \end{cases}$$

A ternary algebra is said to be $*$-linear if

(2.1) (v) $[a, \alpha b + c, d] = \bar{\alpha}[a, b, d] + [a, c, d]$, α scalar.

See Hestenes [**13**] for examples and a detailed study of finite-dimensional ternary algebras.

If a ternary algebra X is endowed with a complete norm $\|\cdot\|$, satisfying

$$(2.1)\qquad \begin{cases} \text{(vi)} & \|[a, b, c]\| \leqslant \|a\|\,\|b\|\,\|c\| \quad \text{for } a, b, c \text{ in } X; \\ \text{(vii)} & \|[a, a, a]\| = \|a\|^3 \quad \text{for } a \text{ in } X; \end{cases}$$

we call X a *C^*-ternary algebra*. A commutative ternary algebra is one satisfying

(2.1) (viii) $[a, b, c] = [c, b, a]$ for a, b, c in X.

Each C^*-algebra is a C^*-ternary algebra under the triple product $[a, b, c] = ab^*c$. A subspace of a C^*-algebra which is closed under this triple product is a C^*-ternary subalgebra.

The following lemma is a simple version of a "Gelfand-Naimark" theorem for C^*-ternary algebras and was obtained by the second named author in 1965. It will be used in the proof of Theorem 4. A much more general representation theorem is obtained in Zettl [**21**].

LEMMA 2.1. *Let X be a C^*-ternary algebra over the complex field. Suppose there is a fixed element u in X such that $a = [a, u, u] = [u, u, a]$ for all a in X. Then there is a linear isometry Φ of X into the algebra of all continuous linear transformations on a Hilbert space H such that $\Phi(u) = I_H$ and $\Phi([a, b, c]) = \Phi(a)\Phi(b)^*\Phi(c)$.*

PROOF. Define, for a, b in X, the elements $a \circ b = [a, u, b]$ and $a^* = [u, a, u]$. The defining relations for a ternary algebra imply that $(X, +, \circ)$ is a linear associative algebra over the complex field. The condition $\|u\| = \|[u, u, u]\| = \|u\|^3$ implies that $\|u\| = 1$. Hence, $\|a \circ b\| = \|[a, u, b]\| \leqslant \|a\|\,\|b\|$, proving that X is a normed algebra. The $*$-linearity of X and the relation $a = [a, u, u] = [u, u, a]$ imply that $a \to a^*$ is an involution on X. Clearly u is an identity for X. Finally, for $a \in X$,

$$\begin{aligned} \|a\|^3 &= \|[a, a, a]\| = \|[a, a, [u, u, a]]\| = \|[[a, a, u], u, a]\| \\ &= \|[a, a, u] \circ a\| \leqslant \|[a, a, u]\|\,\|a\| \leqslant \|a\|^3. \end{aligned}$$

Hence if $a \neq 0$, $\|a\|^2 = \|[a, a, u]\| = \|[[a, u, u], a, u]\| = \|[a, u, [u, a, u]]\| = \|a \circ a^*\|$. Thus X is a C^*-algebra with identity. By the Gelfand-Naimark Theorem for C^*-algebras there is an algebraic isomorphism Φ of $(X, +, \circ, {}^*)$ into the algebra of bounded operators on some Hilbert space H such that $\Phi(u) = I_H$. Since

$$\begin{aligned} [a, b, c] &= [[a, u, u], b, c] = [a, u, [u, b, c]] = [a, u, [u, b, [u, u, c]]] \\ &= [a, u, [[u, b, u], u, c]] = a \circ [[u, b, u], u, c] = a \circ [u, b, u] \circ c \\ &= a \circ b^* \circ c, \end{aligned}$$

we have

$$\Phi([a, b, c]) = \Phi(a \circ b^* \circ c) = \Phi(a)\Phi(b)^*\Phi(c).$$

This completes the proof.

LEMMA 2.2. *Let P be a contractive projection on $A = C_0(K)$ and let Q be the essential part of P. Then for $f, g, h \in A$,*

$$(2.2) \qquad \begin{aligned} &\text{(i)} \quad Qf\overline{Qg}\,Qh = Q\big(Pf\overline{Pg}\,h\big), \\ &\text{(ii)} \quad Qf\overline{Qg}\,Qh = Q\big(Pf\bar{g}Ph\big). \end{aligned}$$

Therefore $Q(A)$ is a commutative ternary subalgebra of $C_b(S)$.

PROOF. For $x \in S_i$, $Qf\overline{Qg}Qh(x) = \langle f, \mu_i\rangle\overline{\langle g, \mu_i\rangle}\langle h, \mu_i\rangle\bar{\varphi}_i(x)$. Therefore, for $x \in S_i$, *and with* $\chi_i =$ *the characteristic function of* supp μ_i,

$$\begin{aligned} Q\big(Pf\overline{Pg}\,h\big)(x) &= \langle Pf\overline{Pg}\,h, \mu_i\rangle\,\overline{\varphi_i(x)} = \langle(\chi_i Pf)\big(\overline{\chi_i Pg}\big)h, \mu_i\rangle\,\overline{\varphi_i(x)} \\ &= \langle f, \mu_i\rangle\langle\overline{g, \mu_i}\rangle\langle h, \mu_i\rangle\,\overline{\varphi_i(x)} = Qf\overline{Qg}\,Qh(x). \end{aligned}$$

The proof of (ii) is similar—it requires $\mu_i = \varphi_i^2 \cdot \mu_i^*$.

THEOREM 2. *Let P be a contractive projection on $A = C_0(K)$. Then the range $P(A)$ is a C^*-ternary algebra with the ternary product*

$$(2.3) \qquad [f, g, h] = P(f\bar{g}h) \quad \text{for } f, g, h \in P(A).$$

PROOF. Since E is a linear bijection of $Q(A)$ onto $P(A)$ we can transfer the natural ternary product on $Q(A)$ to one on $P(A)$ by the rule $[Pf, Pg, Ph] = E(Qf\overline{Qg}Qh)$ for $f, g, h \in A$. Since $Q(Ph) = Qh$ we have, by Lemma 2.2, $[Pf, Pg, Ph] = E(Qf\overline{Qg}Qh) = E(Qf\overline{Qg}Q(Ph)) = EQ(Pf\overline{Pg}Ph) = P(Pf\overline{Pg}Ph)$, as stated.

REMARK. The following easily verified identities can be used to give a direct proof of Theorem 2:

$$(2.4) \qquad \begin{aligned} &\text{(i)} \quad P\big(Pf\overline{Pg}\,Ph\big) = P\big(Pf\overline{Pg}\,h\big) \quad \text{for } f, g, h \in A; \\ &\text{(ii)} \quad P\big(Pf\overline{Pg}\,Ph\big) = P\big(Pf\bar{g}Ph\big) \quad \text{for } f, g, h \in A. \end{aligned}$$

COROLLARY 2.3. *Let P be a contractive projection on $A = C_0(K)$. Let*

$$S = \bigcup_i \operatorname{supp}|\mu_i|$$

where $\{\mu_i\}$ are the atoms of P. For $f, g, h \in A$ we have

$$(2.5) \qquad P\big(Pf\overline{Pg}\,Ph\big) = Pf\overline{Pg}\,Ph \quad \text{on } \bar{S}.$$

In particular, $P(A)\,|\,\bar{S}$ is a ternary subalgebra of $C_0(\bar{S})$.

PROOF. (2.5) is true on S by (2.2(i)) and both sides of (2.5) are continuous functions.

Corollary 2.3 might be used to prove that a subspace of $C_0(K)$ is not the range of a contractive projection.

As another application of Lemma 2.2 we obtain some results concerned with averaging operators. Recall that an operator T on a Banach algebra is said to be an *averaging operator* if it satisfies the identity

$$(2.6) \qquad T(fTg) = TfTg.$$

THEOREM 3. *Let P be a contractive projection on $A = C_0(K)$, and let Q be the essential part of P. Then Q is positive if and only if*

$$(2.7) \qquad P(PfPg) = P(fPg) \quad \text{for } f, g \in A.$$

PROOF. By the remark following Lemma 1.2, we have, for $(i, \xi, g) \in I \times A'' \times A$,

$$\begin{cases} \chi_i P''(P''\xi Pg) = \langle \xi, \mu_i \rangle \langle g, \mu_i \rangle \langle \bar{\varphi}_i^2, \mu_i \rangle \bar{\varphi}_i, \\ \chi_i P''(\xi Pg) = \langle \xi \bar{\varphi}_i, \mu_i \rangle \langle g, \mu_i \rangle \bar{\varphi}_i. \end{cases} \tag{2.8}$$

If (2.7) holds, then it also holds for P'' on A''. Therefore (2.7) and (2.8) (with $\xi = 1 \in A''$) imply $|\langle 1, \mu_i \rangle| = 1$. Thus there are scalars λ_i, $|\lambda_i| = 1$, such that $\mu_i = \lambda_i |\mu_i|$ and it follows that on $\operatorname{supp} \mu_i$, $Qf = \langle f, \mu_i \rangle \bar{\lambda}_i \chi_i = \langle f, |\mu_i| \rangle \chi_i$, so that $Qf \geqslant 0$ if $f \geqslant 0$.

Conversely, suppose Q is positive. For each $i \in I$, choose $0 \leqslant g_i \in A$ such that $\langle g_i, \mu_i \rangle \neq 0$. Then $\chi_i Q g_i = \langle g_i, \mu_i \rangle \bar{\varphi}_i \geqslant 0$ and therefore each φ_i assumes at most one nonzero value. Multiplying μ_i by a scalar we may assume that $\mu_i \geqslant 0$ and $\varphi_i \geqslant 0$. Then (2.8) implies $Q(PfPg) = Q(fPg)$ and therefore $P(PfPg) = EQ(PfPg) = EQ(fPg) = P(fPg)$.

The following consequence of Theorem 3 was proved in Kelley **[14]** under the assumption that P be positive.

PROPOSITION 2.4. *Let P be a contractive projection on $A = C_0(K)$. Then $P(A)$ is a subalgebra of A if and only if P is averaging.*

PROOF. Assume first that $P(A)$ is a subalgebra of A. If $f \in B \equiv P(A)$ then $\chi_i f = \chi_i Qf = \chi_i \langle f, \mu_i \rangle \bar{\varphi}_i$. Since $f^2 \in B$, $\chi_i f^2 = \chi_i \langle f^2, \mu_i \rangle \bar{\varphi}_i$. Therefore $\chi_i \langle f^2, \mu_i \rangle \bar{\varphi}_i = \chi_i f^2 = (\chi_i f)^2 = \chi_i \langle f, \mu_i \rangle^2 \bar{\varphi}_i^2$, so $\bar{\varphi}_i^2 = \lambda_i \bar{\varphi}_i$. Thus $\bar{\varphi}_i^2 \varphi_i = \lambda_i$ and φ_i assumes only one nonzero value. Thus we may choose $\varphi_i \geqslant 0$ so Q is positive. By Theorem 3, $P(PfPg) = P(fPg)$ and since B is an algebra, $P(fPg) = P(PfPg) = PfPg$. The converse is trivial.

We next consider a property which is more general than weakly separating (cf. **[18]**). The set of extreme points of a convex set S will be denoted by ext S.

Let B be a linear subspace of $C_0(K)$. We say that B has a *weakly separating quotient* if for every two distinct points x, y in K and for each scalar $t \neq 1$ such that $f(x) = tf(y)$ for all $f \in B$, we have that $\delta_x | B \notin \operatorname{ext} B_1'$ (or equivalently $\delta_y | B \notin \operatorname{ext} B_1'$).

LEMMA 2.5. *Let P be a contractive projection on $A = C_0(K)$ and let $B = P(A)$. If $\mu \in \operatorname{ext} P'(A_1')$ then $\mu | B \in \operatorname{ext} B_1'$. In particular if $x \in \bigcup_i \operatorname{supp} |\mu_i|$, then $\delta_x | B \in \operatorname{ext} B_1'$.*

PROOF. Suppose $\mu | B = (\varphi + \Psi)/2$ with $\varphi, \Psi \in B_1'$. Since $\mu = P'\mu$ we have $\mu = \mu \circ P$. Therefore $\mu = \mu \circ P = (\varphi \circ P + \Psi \circ P)/2$ and $\varphi \circ P$, $\Psi \circ P \in P'(A_1')$. By extremality, $\mu = \varphi \circ P = \Psi \circ P$ and $\mu | B = \varphi = \Psi$. The first statement is proved. To prove the second statement, suppose $x \in \overline{\operatorname{supp} |\mu_i|}$. Then for $f \in A$, $Pf(x) = \langle f, \mu_i \rangle \overline{\varphi_i(x)}$ so if $f \in B$, we have $f(x) = \langle f, \mu_i \rangle \overline{\varphi_i(x)}$, i.e., $\delta_x | B = \overline{\varphi_i(x)} \mu_i | B$. By the first part of the lemma, $\delta_x | B \in \operatorname{ext} B_1'$.

PROPOSITION 2.6. *Let P be a contractive projection on $A = C_0(K)$ and let Q be the essential part of P. If $B \equiv P(A)$ has a weakly separating quotient then Q is positive. Therefore* (2.7) *holds.*

PROOF. Suppose Q is not positive. Then there is an i such that φ_i assumes at least two distinct nonzero values, say, $\varphi_i(x) = \alpha$, $\varphi_i(y) = \beta$. For $f \in B$, this implies $f(x) = \langle f, \mu_i \rangle \bar{\alpha}$ and $f(y) = \langle f, \mu_i \rangle \bar{\beta}$. Thus for all $f \in B$, $f(x) = \bar{\alpha}\beta f(y)$, and $\bar{\alpha}\beta \neq 1$. In order to complete this part of the proof it remains to show that $\delta_x \mid B \in \operatorname{ext} B_1'$. This follows from Lemma 2.5, since $\delta_x \mid B = \bar{\alpha}\mu_i \mid B$.

The converse of this proposition is not true. Let $A = C([-2, -1] \cup [1, 2])$ and let $\chi = \chi_{[1,2]}$. Then P defined by $Pf(x) = (\chi(x)f(x) - \chi(-x)f(-x))/2$ has essential part $Q \geqslant 0$ but $P(A)$ does not have a weakly separating quotient.

Proposition 2.6 is contained in a result of Wulbert **[18]**. Wulbert proves that if P is a contractive projection from a subalgebra A of $C(X)$ onto a subspace B with weakly separating quotient then (2.7) holds.

Our next theorem gives necessary and sufficient conditions for the range $P(A)$ of a contractive projeciton to be isometric to $C(H)$. The equivalence of (3) and (6) was proved in the real case by Wulbert and Lindenstrauss **[20]**.

THEOREM 4. *Let P be a contractive projection on $A = C_0(K)$ and let $\{\mu_i\}$ be the atoms of P. The following are equivalent*:

(1) *There exists $u \in P(A)$ such that $|u| = 1$ on $\bigcup_i \operatorname{supp} |\mu_i|$*;

(2) *There exists $u \in P(A)$ such that $P(f|u|^2) = f$ for all $f \in P(A)$*;

(3) *$P(A)$ has the structure of a commutative unital C^*-algebra, i.e., $C(H)$, H compact Hausdorff*;

(4) *$P(A)$ is isometric to a unital C^*-algebra*;

(5) *$P(A)$ is isometric to a unital Banach algebra (i.e., a Banach algebra with unit of norm 1)*;

(6) *There exists an extreme point of the unit ball of $P(A)$.*

PROOF. (1) $\Rightarrow$ (2). $P(f|u|^2) = EQ(f|u|^2) = EQf = Pf = f$.

(2) $\Rightarrow$ (3). By the proof of Lemma 2.1 and Theorem 2, $P(A)$ is a C^*-algebra with identity u. Since it is obviously commutative, there is an isometric algebraic $*$-isomorphism of $P(A)$ onto $C(H)$ for some H compact Hausdorff (Gelfand-Naimark Theorem).

(3) $\Rightarrow$ (4) obvious.

(4) $\Rightarrow$ (5) obvious.

(5) $\Rightarrow$ (6). The identity element of any unital Banach algebra is an extreme point of the unit sphere (theorem of Kakutani).

(6) $\Rightarrow$ (1). Let u be an extreme point of the unit ball of $P(A)$. Suppose $|u(x)| < 1$ for some $x \in \operatorname{supp} |\mu_i|$. Then there is a $\delta > 0$ and a compact neighborhood U of x such that $U \subseteq \{y \in K : |u(y)| \leqslant 1 - \delta\}$. For any $v \in C_0(K)$ such that $\|v\| \leqslant 1$ and $v(K \setminus U) = 0$ we have $\|u \pm \delta v\| \leqslant 1$. If for every such v we had $\langle v, \mu_i \rangle = 0$, then x could not be in $\operatorname{supp} |\mu_i|$. Therefore there is $v \in C_0(K)$ such that $\langle v, \mu_i \rangle \neq 0$ and $\|u \pm \delta v\| \leqslant 1$. Since P is contractive, $\|u \pm P(\delta v)\| \leqslant 1$ and, moreover, $Pv(x) = \langle v, \mu_i \rangle \overline{\varphi_i(x)} \neq 0$. This is a contradiction to the extremality of u. This completes the proof of Theorem 4.

The following is due to Wulbert **[17]** in the real case.

COROLLARY 2.7. *Let P be a contractive projection on $A = C(K)$, K compact, and suppose $B = P(A)$ has a weakly separating quotient. Then B is isometrically isomorphic to $C(H)$ for some compact Hausdorff space H.*

PROOF. Let Q be the essential part of P. Since B has a weakly separating quotient, Q is positive. Let $u = P1$. For $x \in \operatorname{supp}|\mu_i|$, $u(x) = P1(x) = \langle 1, \mu_i \rangle = 1$. Thus (1) of Theorem 4 is satisfied.

3. An isometric characterization of the range. A compact Hausdorff space X is called a $\mathbf{T}_\sigma$*-space* if there exists a map $\sigma: \mathbf{T} \times X \to X$ such that

$$(3.1) \qquad \begin{cases} \text{(i)} & \sigma \text{ is continuous,} \\ \text{(ii)} & \sigma(\alpha, \sigma(\beta, x)) = \sigma(\alpha\beta, x), \qquad \alpha, \beta \in \mathbf{T}, x \in X, \\ \text{(iii)} & \sigma(1, x) = x. \end{cases}$$

Let X be a $\mathbf{T}_\sigma$-space. Then each $\alpha \in \mathbf{T}$ defines a homeomorphism $\sigma_\alpha: X \to X$ where $\sigma_\alpha(x) = \sigma(\alpha, x)$. A function $f \in C(X)$ is said to be σ-homogeneous if $f(\sigma(\alpha, x)) = \alpha f(x)$ for all $(\alpha, x) \in \mathbf{T} \times X$. The class of σ-homogeneous functions in $C(X)$ is denoted by $C_\sigma(X)$. A complex C_σ*-space* is any complex Banach space which is isometric to a space $C_\sigma(X)$ for some $\mathbf{T}_\sigma$-space X. If $f \in C(X)$, the function

$$(3.2) \qquad (\pi_\sigma f)(x) = \int f(\sigma(\alpha, x))\bar{\alpha}\, d\alpha, \qquad x \in X,$$

where $d\alpha$ is normalized Haar measure on $\mathbf{T}$, is continuous and σ-homogeneous, and the operator π_σ is a contractive projection of $C(X)$ onto $C_\sigma(X)$.

The following construction is from Olsen **[15]**.

Let K be a locally compact Hausdorff space. Let $X = (\mathbf{T} \times K) \cup \{\omega\}$ be the one point compactification of $\mathbf{T} \times K$ and define $\sigma: \mathbf{T} \times X \to X$ by

$$(3.3) \qquad \sigma(\alpha, x) = \begin{cases} (\alpha\alpha_0, y) & \text{if } x = (\alpha_0, y) \in \mathbf{T} \times K, \\ \omega & \text{if } x = \omega. \end{cases}$$

Then X is a $\mathbf{T}_\sigma$-space and the map E: $C_0(K) \to C_\sigma(X)$ defined by $(Ef)(\alpha, y) = \alpha f(y)$, $(\alpha, y) \in \mathbf{T} \times K$, $Ef(\omega) = 0$, is an isometry of $C_0(K)$ onto $C_\sigma(X)$. We state this as

LEMMA 3.1. *For a locally compact Hausdorff space K, $C_0(K)$ is a C_σ-space.*

LEMMA 3.2. *Let P be a contractive projection on $A = C_0(K)$ and let Q be the essential part of P. Let $h \in C_0(K)$. Suppose that for all $i \in I$ and $x, y \in \operatorname{supp}|\mu_i|$,*

$$(3.4) \qquad h(x) = \frac{\varphi_i(y)}{\varphi_i(x)} h(y).$$

Then $Ph \mid \bar{S} = h \mid \bar{S}$ (where $S = \bigcup_i \operatorname{supp}|\mu_i|$).

PROOF. It suffices to prove that h agrees with Ph on $\bigcup_i \operatorname{supp}|\mu_i|$, which is dense in $\bar{S}$. Let $y \in \operatorname{supp}|\mu_i|$. Then $Ph(y) = \langle h, \mu_i \rangle \overline{\varphi_i(y)} = (\int_{S_i} h(x)\, d\mu_i(x))\overline{\varphi_i(y)} = \int_{S_i} h(y)\overline{\varphi_i(x)}\, d\mu_i(x) = h(y)\langle \bar{\varphi}_i, \mu_i \rangle = h(y)$.

THEOREM 5. *Let P be a contractive projection on $A = C_0(K)$. Then $P(A)$ is a C_σ-space.*

PROOF. Let $B = P(A)$ and let $\{\mu_i\}_{i\in I}$ be the atoms of P. Let X be the space given by Lemma 3.1 and consider the diagram

$$\begin{array}{ccccccc} C_0(K) & \xrightarrow{R} & C_0(\bar{S}) & \xrightarrow{E} & C_\sigma(X) & & \\ P\downarrow & & \cup & & \cup & & \\ B & \xrightarrow{R} & B\,|\,\bar{S} & \xrightarrow{E} & \tilde{B} & \xrightarrow{J} & \hat{B} \subset C(Y) \end{array} \tag{3.5}$$

$$f \xrightarrow{R} f\,|\,\bar{S} \xrightarrow{E} \tilde{f} \xrightarrow{J} \hat{f} \tag{3.6}$$

where R denotes restriction, E denotes extension as in Lemma 3.1, Y denotes the space of equivalence classes in X under the equivalence relation $x \sim y$ if $f(x) = f(y)$ for all f in $\tilde{B}$ (with the quotient topology), J is the operator that takes $\tilde{f} \in \tilde{B}$ into the function $\hat{f}(\tilde{x}) = \tilde{f}(x)$ where $\tilde{x}$ is the equivalence class of x ($\hat{f}$ is well defined by definition of Y) and let $\tilde{\sigma}$: $\mathbf{T} \times Y \to Y$ be defined by $\tilde{\sigma}(\alpha, \tilde{x}) = [\sigma(\alpha, x)]^{\tilde{}}$. It is trivial to verify that $\tilde{\sigma}$ is well defined and that Y is a $\mathbf{T}_{\tilde{\sigma}}$-space. It is also easily seen that each map in (3.5) is linear and isometric. It remains to prove that $\hat{B} = C_{\tilde{\sigma}}(Y)$ and it is trivial that $\hat{B} \subset C_{\tilde{\sigma}}(Y)$. To complete the proof, let $g \in C_{\tilde{\sigma}}(Y)$. We must prove there exists $f_1 \in B$ such that $\hat{f}_1 = g$. For this, choose $h \in C_0(K)$ such that $\tilde{h} = g \circ t$ where t: $X \to Y$ is the canonical map. This is possible since $g \circ t \in C_\sigma(X)$ and both R and E are onto. We shall prove that $(Ph)^{\hat{}} = g$ by using Lemma 3.2. Thus we must prove (3.4) of Lemma 3.2. To do this, let $x' = (1, x), y' = (1, y) \in X$ and let $\beta = \beta_i(x, y) = \varphi_i(y)\overline{\varphi_i(x)} \in \mathbf{T}$. We claim that

$$\tilde{\sigma}(\beta, \tilde{y}') = \tilde{x}'. \tag{3.7}$$

Since $\tilde{\sigma}(\beta, \tilde{y}) = \sigma(\beta, y')^{\tilde{}} = \sigma(\beta,(1, y))^{\tilde{}} = (\beta, y)^{\tilde{}}$ (by (3.3)), in order to prove (3.7) we must show that (β, y) and $(1, x)$ are equivalent in X, i.e., for all $f \in B$ we have $\tilde{f}(\beta, y) = \tilde{f}(1, x)$, i.e., $\beta f(y) = f(x)$. But for any $f \in B$, $f(x) = \langle f, \mu_i\rangle\overline{\varphi_i(x)} = \varphi_i(y)\bar{\varphi}_i(x)\langle f, \mu_i\rangle\bar{\varphi}_i(y) = \beta f(y)$. Hence (3.7) is proved. Now to prove (3.4) of Lemma 3.2 we have, for $x, y \in \operatorname{supp}\mu_i$, $h(x) = \tilde{h}(x') = g(\tilde{x}') = g(\tilde{\sigma}(\beta, \tilde{y}')) = \beta g(\tilde{y}') = \beta\tilde{h}(y') = \beta h(y)$. Now by Lemma 3.2, $h\,|\,\bar{S} = Ph\,|\,\bar{S}$, so that $\tilde{h} = (Ph)^{\tilde{}}$ and therefore for $\tilde{x} \in Y$, $(Ph)^{\hat{}}(\tilde{x}) = (Ph)^{\tilde{}}(x) = \tilde{h}(x) = g \circ t(x) = g(\tilde{x})$. The theorem is proved.

For the case of real scalars and compact K, Theorem 5 was proved by Wulbert and Lindenstrauss **[20]**.

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