THE CATALAN EQUATION OVER FUNCTION FIELDS

BY

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Abstract. Let $K$ be the function field of a projective variety. Fix $a, b, c \in K^*$. We show that if $\max\{m, n\}$ is sufficiently large, then the Catalan equation $ax^m + by^n = c$ has no nonconstant solutions $x, y \in K$.

The Cassels-Catalan conjecture states that for fixed nonzero integers $a, b, c$, the equation $ax^m + by^n = c$ has only finitely many solutions in integers $x, y, m, n$ satisfying $m \geq 3$ and $n, \lvert x \rvert, \lvert y \rvert \geq 2$. At present, the only known case of this conjecture is $a = -b = c = 1$, due to Tijdeman [6]. In this paper we prove a strengthened version of the Cassels-Catalan conjecture in the case that the number field $\mathbb{Q}$ is replaced by an arbitrary function field. The proof uses only elementary algebraic geometry, the principal tool being the Riemann-Hurwitz formula.

Theorem. Let $k$ be a field of characteristic $p$ (possibly with $p = 0$), and let $K/k$ be the function field of a nonsingular projective variety. Fix $a, b, c \in K^*$.

Then there are only finitely many pairs of integers $m, n \geq 2$ (prime to $p$ if $p \neq 0$) for which the Cassels-Catalan equation $ax^m + by^n = c$ has even a single nonconstant solution $x, y \in K$, $x, y \not\in k$.

Further, for any particular pair $m, n$ as above, there will be only finitely many solutions $x, y \in K$ unless either:

(i) $a/c$ is an $m$th power and $b/c$ is an $n$th power in $K$, in which case there may be infinitely many solutions $(x, y) = (\alpha(a/c)^{1/m}, \beta(b/c)^{1/n})$ with $\alpha, \beta \in k$ satisfying $\alpha^m + \beta^n = 1$; or

(ii) $(m, n) \in \{(2, 2), (2, 3), (3, 2), (2, 4), (4, 2), (3, 3)\}$, in which case the Cassels-Catalan equation defines a curve of genus 0 or 1 over $K$.

Proof. We first note that taking “generic” hyperplane sections of the variety whose function field is $K$, we are reduced by Bertini’s theorem to the case that $K = k(C)$ is the function field of a nonsingular projective curve $C$. (See, e.g., [5] for the details of this standard reduction.) Second, we may replace $k$ by its algebraic closure, since at worst this will create extra solutions. Third, dividing the equation by $c$ and replacing $a$ and $b$ by $a/c$ and $b/c$, we may assume that our equation is

\begin{equation}
ax^m + by^n = 1.
\end{equation}

Let $D = \max\{\deg(a), \deg(b)\}$. (Note $a$ and $b$ are now functions on the curve $C$, so have degrees.) By symmetry, we may assume that $m \geq n$.

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Consider the desingularization of the covering of $C$ given by the equations $u^m = a, v^n = b$. Since this is a composition of cyclic coverings (note that $k$ contains both an $m$th and an $n$th root of unity), it will consist of a union of isomorphic curves. Let $C'$ be any one of these curves, and $f: C' \to C$ the natural map. We note for future reference that the degree of $f$ divides $mn$; in particular, $\deg(f)$ is prime to $p$ if $p > 0$.

Let $V/C$ be the projective surface given by the equation $ax^m + by^n z^{m-n} = z^m$. Let $V' = V \times_{C'} C'$ be base extension of $V$ by $C'$. If $V_0$ is the projective curve $x^m + y^n z^{m-n} = z^m$ then we have a natural map

$$V' \to V_0,$$

$$[x, y, z] \to [ux, vy, z].$$

Further let $V_0^*$ be the desingularization of $V_0$ and $h: V_0^* \to V_0$ the natural map.

Now suppose that we are given functions $x, y \in k(C)$ which satisfy equation (\ast). Then $P = [s, y, 1]$ gives a section $P: C \to V$. This extends to a section $C' \to V'$, and composed with the map $V' \to V_0$ from above, gives a map of curves $C' \to V_0$. But $C'$ is nonsingular, so it factors through the normalization map $h$ to yield a map $\phi: C' \to V_0^*$. All of this is summarized in the following diagram.

Assume for now that $\phi$ is surjective. (The other case is dealt with later.) We apply the Riemann-Hurwitz formula twice, once to $\phi$ and once to $f$. Note that since $\deg(f)$ is prime to $p = \text{char}(k)$, there is no wild ramification in $f$ even if $p > 0$. (See, e.g., [3, Chapter IV. 2] for basic facts about the Riemann-Hurwitz formula.)

\[ [2g(V_0^*) - 2] \deg(\phi) \leq 2g(C') - 2 \]

\[ = [2g(C) - 2] \deg(f) + \sum_{t \in C} [e_t(f) - 1], \]

where $e_t(f) = \deg(f) - \# f^{-1}(t) + 1$. Now $e_t(f) \leq \deg(f)$, and the only points $t \in C$ at which $f$ can possibly ramify are zeros and poles of $a$ and $b$. Thus $e_t(f) = 1$ except possibly at $4D$ points of $C$, hence

\[ \sum_{t \in C} [e_t(f) - 1] \leq 4D \deg(f). \]

It is an elementary exercise to resolve the singularity of $V_0$ at $[0, 1, 0]$ (which is singular if and only if $m - n > 1$) and compute the Euler characteristic of $V_0^*$. One finds

\[ 2g(V_0^*) - 2 = mn - m - n - (m, n), \]

where $(m, n)$ is the greatest common divisor of $m$ and $n$. 

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Next we express $\deg(\phi)$ in terms of $x$ and $y$. Consider the following commutative diagram.

\[
\begin{array}{ccc}
V_0^* & \xrightarrow{\text{degree 1}} & V_0 \\
\phi & \downarrow \text{degree } mn & \Rightarrow \text{degree } mn \\
C' & \xrightarrow{f} & C \\
& \downarrow h & \Rightarrow [x, y, z] \\
& & \Rightarrow [x^m, z^m] \\
& & \Rightarrow [ax^m, 1]
\end{array}
\]

From this we read off

\[
\deg(\phi) = \frac{\deg(ax^m)\deg(f)}{mn} \geq \frac{(\deg(x^m) - D)\deg(f)}{mn}.
\]

Using a similar diagram we obtain

\[
\deg(\phi) = \frac{\deg(by^n)\deg(f)}{mn} \geq \frac{(\deg(y^n) - D)\deg(f)}{mn}.
\]

Now putting all of these computations into the inequality (**) and dividing by $\deg(f)$, we obtain

\[
(5 - m^{-1} - n^{-1} - [m, n]^{-1})D + 2g(C) - 2 \geq (1 - m^{-1} - n^{-1} - [m, n]^{-1})\max\{\deg(x^m), \deg(y^n)\},
\]

where $[m, n]$ is the least common multiple of $m$ and $n$. In particular we obtain the fundamental inequality

\[
(+) \quad 5D + 2g(C) - 2 \geq (1 - m^{-1} - n^{-1} - [m, n]^{-1})\max\{\deg(x^m), \deg(y^n)\}.
\]

We now check the case that $\phi$ is a constant map. This will mean that $ax$ and $by$ are constant functions on $C'$, so raising to the $m$th (respectively $n$th) power we find that $ax^m$ and $by^n$ are in the constant field $k$. Hence, since $ab \neq 0$, $\deg(x^m) \leq \deg(a) \leq D$ and $\deg(y^n) \leq \deg(b) \leq D$. These are stronger than the inequality (+) except in the one trivial case $D = g(C) = 0$, in which case they imply that $\deg(x) = \deg(y) = 0$, while (+) would yield the impossibility $-2 \geq 0$. In any case, (+) holds whenever $x$ and $y$ are not constant functions.

It is easy to check that if $m, n \geq 2$ are integers other than the six pairs listed in (ii) of the theorem, then

\[
1 - 1/m - 1/n - 1/[m, n] \geq 1/6
\]

(attaining this minimum for $(m, n) = (6, 2)$). Now since our solution $x, y \in k(C)$ to (+) was assumed to be nonconstant, we have $\deg(x), \deg(y) \geq 1$, so the inequality (+) yields

\[
5D + 2g(C) - 2 \geq \frac{1}{6}\max\{m, n\}.
\]
This bounds the possibilities for $m$ and $n$. Then, for any particular $m$ and $n,$

$$5D + 2g(C) - 2 \geq \frac{1}{6} \max \{ m \deg(x), n \deg(y) \}$$

bounds $\deg(x)$ and $\deg(y)$. But a curve of genus at least 2 over a function field $K$ can have only finitely many points of bounded degree unless the curve is birational, over $K,$ to a curve defined over the field of constants $k.$ Further, all but finitely many of those points will come from $k$-valued points on the new curve. (See [5, Corollary on p. 42]. Note that the result is true also for $\text{char}(k) = p > 0,$ since we have taken $k$ to be algebraically closed. In general, one must take a purely inseparable extension of $k.$) One easily checks that the only way for the Cassels-Catalan equation (*) to reduce to an equation over $k$ is for $a$ to be a $m$th power and $b$ to be an $n$th power, which is the case covered by (i) of the theorem.

The inequality (+) derived in the course of proving the above theorem is of independent interest, since it gives effective bounds for $m,$ $n,$ $\deg(x),$ $\deg(y)$ in the case that $K$ is the function field of a curve. We therefore state it separately.

**Theorem.** Let $k$ be a field and $C/k$ a nonsingular projective curve with function field $k(C).$ Fix two functions $a, b \in k(C)^*.$ Suppose $m, n \geq 2$ are integers [prime to $\text{char}(k)$ if $\text{char}(k) \neq 0$] and $x, y \in k(C)$ are functions satisfying $ax^m + by^n = 1.$ Then

$$5 \max \{ \deg(a), \deg(b) \} + 2g(C) - 2 \geq (1 - m^{-1} - n^{-1} - [m, n]^{-1}) \max \{ \deg(x^m), \deg(y^n) \}.$$  

(Except in the trivial case $g = \deg(x) = \deg(y) = \deg(a) = \deg(b) = 0.$)

It is likely that the coefficient 5 in the left-hand side of this inequality can be improved, and one might ask what the best possible result is. In the special case that $g = 0$ (so $k(C) = k(t)$ is a rational function field), if one restricts $x$ and $y$ to be polynomials, then Davenport has used very different arguments to obtain a similar inequality [1]. If $g = 0,$ $a = b = 1,$ and $m = n$ (i.e. the Fermat equation over $k(t)$), one can show the nonexistence of nonconstant solutions in $k(t)$ by an easy descent argument [2]. Both of these proofs, however, use the unique factorization of $k[t],$ so do not readily generalize to other functions fields.

Returning to the case of number fields, one is tempted to conjecture that an inequality of this sort should hold (with $2g(C) - 2$ replaced by some function involving the degree of the number field over $\mathbb{Q}$). From this the strengthened version of the Cassels-Catalan conjecture would follow formally as in the above proof. (Notice a corollary would be Fermat's last theorem for sufficiently large exponents!) A similar inequality, but only for integral rather than rational solutions, has been proposed by Lang and Waldschmidt [4, p. 213]. They also show how it would follow from a certain Baker-style diophantine inequality, but this unfortunately seems well beyond current techniques.

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REFERENCES


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