C*-ALGEBRA FIBRE BUNDLES

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Abstract. It will be shown in this paper that for any C*-algebra fibre bundle with
base space X and fibre A, a C*-algebra, the Jacobson spectrum of the C*-algebra of
sections of the fibre bundle can be identified as a topological fibre bundle with the
same base space X and fibre the Jacobson spectrum of A.

A C*-algebra fibre bundle $\Sigma = (E, X, A, p, U, \phi_u, G)$ is specified up to equiva-
lence [3] by a topological bundle space $E$, a locally compact Hausdorff base space $X$,
a C*-algebra fibre $A$, a continuous projection $p: E \to X$, an open covering $U$ of $X$,
and homeomorphisms $\phi_u: u \times A \to p^{-1}(u)$ for $u \in U$. The $\phi_u$ are fibre-preserving:
$\phi_u(x, A) = p^{-1}(x)$. Furthermore, there is an effective topological group $G$ of *-auto-
morphisms of the fibre $A$. The mappings $\phi_u$ and the fibre $A$ are related as follows:

For $x \in u \cap v$, $u,v \in U$, and $a \in A$, let

$$\phi_u^{-1}(x, a) = (x, g_{uv}(x)(a)).$$

Then $g_{uv}(x) \in G$, and the map $g_{uv}: u \cap v \to G$ is continuous. If $y \in p^{-1}(x)$,
x $\in u \in U$, the relation is described by writing $\phi_u^{-1}(y) = (x, t_u(y))$. Let $S$ be the set
of continuous sections $\gamma: X \to E$ ($p\gamma(x) = x$) and $D = \{\gamma \in S: |\gamma(x)| =
\|t_u(\gamma(x))\|_A$ vanishes at infinity}. Note that $|\gamma(x)| = \|t_u(\gamma(x))\|_A$ is independent of
the choice of $u$ in $U$ containing $x$, since $g_{uv}(x) \in G$ is an isometry.

For $\gamma_1, \gamma_2 \in D$, if $x \in u \in U$, define

$$(\gamma_1 + \gamma_2)(x) = \phi_u(x, t_u(\gamma_1(x)) + t_u(\gamma_2(x))),$$

and

$$(\gamma_1 \gamma_2)(x) = \phi_u(x, t_u(\gamma_1(x))t_u(\gamma_2(x))).$$

If $x \in v \in U$ also, then

$$(\gamma_1 + \gamma_2)(x) = \phi_v(x, t_v(\gamma_1(x)) + t_v(\gamma_2(x)))$$

$$= \phi_u(x, g_{uv}(x)(t_u(\gamma_1(x)) + t_u(\gamma_2(x))))$$

$$= \phi_u(x, t_u(\gamma_1(x)) + t_u(\gamma_2(x))).$$

Thus $(\gamma_1 + \gamma_2)(x)$ is well defined as is $(\gamma_1 \gamma_2)(x)$; clearly, $\gamma_1 + \gamma_2$ and $\gamma_1 \gamma_2$ belong to
$D$.

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For $\gamma \in D$, since $| \gamma(x) |$ is continuous on $X$,

$$\| \gamma \| \equiv \sup_{x \in X} | \gamma(x) | < \infty.$$ 

Direct verification shows that $\| \cdot \|$ is a norm on $D$ and that with respect to this norm and the operations as defined above, and the involution $\gamma^*(x) \equiv \phi_u(x, (t_u(\gamma(x)))^*)$, $D$ is a $C^*$-algebra, called the $C^*$-algebra of sections of the fibre bundle $\Sigma$.

**Lemma 1.** Let $\Sigma = (E, X, A, p, U, \phi, G)$ be a $C^*$-algebra fibre bundle and let $D$ be its $C^*$-algebra of sections. If $u \in U$, and $f \in C_c(X, A)$ with support$(f) \subset u$, define

$$\gamma(x) = \phi_u(x, f(x)), \quad \text{if } x \in u,$$

$$= O(x), \quad \text{if } x \not\in u,$$

where $O(x) = \phi_v(x, 0)$, if $x \in v \in U$. Then $\gamma \in D$ and $t_u(\gamma(x)) = f(x)$, whence $A = \{ t_u(\gamma(x)): \gamma \in D \}$ for each $x \in X, u \in U$ with $x \in u$.

**Proof.** Cf. [2, Lemma 1.1].

**Lemma 2.** Let $\Sigma = (E, X, A, p, U, \phi, G)$ be a $C^*$-algebra fibre bundle and let $D$ be its $C^*$-algebra of sections. If $I$ is a closed two-sided ideal of $D$, for each $x \in X$ and $u \in U$ such that $x \in u$, set $I_u(x) = \{ t_u(\gamma(x)): \gamma \in I \}$. Then $I = \{ \gamma \in D: \text{for every } x \in X, \text{there exists in } U a u \text{ with } x \in u \text{ and such that } t_u(\gamma(x)) \in I_u(x) \}.$

**Proof.** If $\gamma \in I$, for every $x \in X$ and $u \in U$ with $x \in u$, then $t_u(\gamma(x)) \in I_u(x)$ by definition of $I_u(x)$.

Conversely, if $\gamma \in D$ and if for every $x \in X$, there exists a $u$ in $U$ with $x \in u$ such that $t_u(\gamma(x)) \in I_u(x)$. If $e > 0$, $K = \{ x \in X: | \gamma(x) | \geq e \}$ is compact. For every $y$ in $K$, there exists a $v$ in $U$ with $y \in v$ and there exists a $z_y$ in $I$ such that $t_v(\gamma(y)) = t_v(z_y(y))$. Since $A$ has an approximate identity, there exists an $a_y$ in $A$ such that

$$| t_v(z_y(y)) - a_y(t_v(z_y(y))) | \leq e.$$

Let $f \in C_c(X)$ be such that $f(y) = 1$ and support$(f) \subset v$. Define

$$w_y(x) = \phi_v(x, f(x)a_y), \quad \text{if } x \in v,$$

$$= O(x), \quad \text{if } x \not\in v.$$

Then $w_y \in D$ and $t_v(w_y(y)) = f(y)a_y = a_y$ (Lemma 1), and

$$| \gamma(y) - w_y z_y(y) | \leq \| t_v(\gamma(y)) - t_v(w_y(y))t_v(z_y(y)) \|$$

$$= \| t_v(z_y(y)) - a_y t_v(z_y(y)) \| < e.$$ 

By the continuity of $x \to \gamma(x)$ on $X$, $| \gamma(x) - (w_y z_y(x)) | < e$ on a neighborhood of $y$. Therefore, there is a finite open covering $O_1, O_2, \ldots, O_n$ of $K$ such that $| \gamma(x) - (w_y z_y(x)) | < e$ on $O_i$ for $i = 1, 2, \ldots, n$, and there are $g_1, g_2, \ldots, g_n$ constituting a partition of unity subordinate to $O_1, O_2, \ldots, O_n$, i.e., $g_i \in C_c(X), g_i$ vanishes outside $O_i$ and $g_1 + g_2 + \cdots + g_n = 1$ on $K$. Then $\sum_{i=1}^n g_i w_i z_i \in I$, since $g_i w_i \in D$ and $z_i \in I$. 

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and
\[ |\gamma(x) - \sum_{i=1}^{n} (g_j w_i z_i)(w) | < \epsilon \quad \text{on} \ K, \]
thus \( \| \gamma - \sum_{i=1}^{n} g_j w_i z_i \| < \epsilon \). Whence \( \gamma \in I \), and the proof is complete.

Let \( A \) be a \( C^* \)-algebra. We denote by \( \hat{A} \) the set of equivalence classes of irreducible representations of \( A \) endowed with the Jacobson topology [1, 3.1.1].

**Theorem 3.** Let \( \Sigma = (E, X, A, p, U, \phi_u, G) \) be a \( C^* \)-algebra fibre bundle and let \( D \) be its \( C^* \)-algebra of sections. Then to each \( (x, u, \pi) \in X \times U \times \hat{A} \) with \( x \in u \), there corresponds a \( \sigma \) in \( \hat{D} \) such that \( \sigma(\gamma) = \pi(t_u(\gamma(x))) \), for \( \gamma \in D \). Conversely, every \( \sigma \) in \( \hat{D} \) is of this form.

**Proof.** If \( (x, u, \pi) \in X \times U \times \hat{A} \) with \( x \in u \), then since \( A = \{ t_u(\gamma(x)): \gamma \in D \} \) (Lemma 1), \( \sigma(\gamma) = \pi(t_u(\gamma(x))) \), \( \gamma \in D \), is an irreducible representation of \( D \). Conversely, if \( \sigma \in \hat{D} \), put \( I = \ker \sigma \) and set \( S = \{ x \in X: I_u(x) \neq A, \text{for every} \ u \in U \text{with} \ x \in u \} \). If \( S = \emptyset \), then for every \( x \) in \( X \), there exists in \( U \) a \( u \) with \( x \in u \) such that \( I_u(x) = A \). It follows from Lemma 2 that \( I = A \), a contradiction, hence \( S \neq \emptyset \).

Suppose \( x \) and \( y \) are distinct elements in \( S \). Let \( u \) and \( v \) be in \( U \) such that \( x \in u \) and \( y \in v \). Choose two open subsets \( u_1 \) and \( v_1 \) in \( X \) such that \( u_1 \cap v_1 = \emptyset \) and \( x \in u_1 \subset u, \ y \in v_1 \subset v \). Define \( M = \{ \gamma \in D: \gamma(z) = \phi_u(z, f(z)), \text{if} \ z \in u, = \phi_v(z), \text{if} \ z \in v_1 \} \). Similarly, let \( N = \{ \gamma \in D: \gamma(z) = \phi_v(z, g(z)), \text{if} \ z \in v, = \phi_v(z), \text{if} \ z \in v, \text{where} \ g \in C_{00}(X, A) \text{with support} \ (g) \subset (v_1) \} \). Then \( M \) and \( N \) are closed two-sided ideals of \( D \), and if \( \gamma_1, \gamma_2 \in M \cdot N \),
\[
\| \gamma_1 \gamma_2 \| = \sup_{x \in X} |(\gamma_1 \gamma_2)(x)| = \sup_{x \in X} \| t_u(\gamma_1(x))t_u(\gamma_2(x)) \| \\
\leq \sup_{x \in X} \| t_u(\gamma_1(x)) \| \cdot \| t_u(\gamma_2(x)) \| = 0,
\]
hence \( 0 = M \cdot N \subset I \). Since \( I = \ker \sigma \) is a primitive ideal of \( D \), by [1, 2.11.4], \( M \subset I \) or \( N \subset I \). Suppose \( M \subset I \), then \( M_u(x) \subset I_u(x) \). Note that \( A = M_u(x) \), since if \( a \in A \), by the construction of the ideal \( M \) there exists a \( \gamma \) in \( M \) such that \( a = t_u(\gamma(x)) \), and hence \( A = I_u(x) \) which contradicts \( x \) in \( S \). Therefore \( S = \{ x_0 \} \) is a one-point set. Owing to Lemma 1, it follows that \( \sigma \) induces a \( \pi \) in \( \hat{A} \) according to
\[
\pi(t_u(\gamma(x_0))) = \sigma(\gamma), \quad \text{for} \ \gamma \in D.
\]
This shows that for every \( \sigma \) in \( \hat{D} \) there corresponds \( (x, u, \pi) \) in \( X \times U \times \hat{A} \) such that \( \sigma(\gamma) = \pi(t_u(\gamma(x_0))) \) for \( \gamma \in D \).

The correspondence between \( X \times U \times \hat{A} \) and \( \hat{D} \) in Theorem 3 is not necessarily bijective, although to each \( \sigma \) in \( \hat{D} \) there corresponds a unique \( x \) in \( X \). It will be shown that in the Jacobson topology on \( \hat{D}, \hat{D} \) is a topological fibre bundle over \( X \) with fibre \( \hat{A} \) and covering \( I \).

**Lemma 4.** Let \( \Sigma = (E, X, A, p, U, \phi_u, G) \) be a \( C^* \)-algebra fibre bundle and let \( D \) be its \( C^* \)-algebra of sections. For \( \sigma \) in \( \hat{D} \) there exists according to Theorem 3 a unique \( x \) in \( X \) such that \( \sigma \) corresponds to \( (x, u, \pi) \). Define \( p \) on \( \hat{D} \) into \( X \) by \( \bar{p}(\sigma) = x \), then \( \bar{p} \) is a continuous projection.
Proof. It is immediate that \( \tilde{\rho} \) so defined is surjective.

Assume the net \( \{\sigma_\lambda\} \) converges to \( \sigma \) in \( \tilde{D} \). Choose \( (x_\lambda, u_\lambda, \pi_\lambda) \) and \( (x, u, \pi) \) in \( X \times U \times \tilde{A} \) such that

\[
\sigma_\lambda(\gamma) = \pi_\lambda(\tau_u(\gamma(x_\lambda))) \quad \text{and} \quad \sigma(\gamma) = \pi(\tau_u(\gamma(x))), \quad \gamma \in D.
\]

Note that \( x_\lambda \) and \( x \) are uniquely determined.

If \( x_\lambda \xrightarrow{\lambda} x \) in \( X \), there exist a subnet \( \{x_\lambda^*\} \) and a function \( f \in C_0(X) \) with support \( \{ f \} \subset u \) such that \( f(x_\lambda^*) = 0 \) and \( f(x) = 1 \). Furthermore, pick an \( a \) in \( A \) such that \( \pi(a) \neq 0 \). Define

\[
\gamma(y) = \phi_u(y, f(y)a), \quad \text{if} \ y \in u,
\]

\[
= O(y), \quad \text{if} \ y \notin u,
\]

then \( \gamma \in D \) and \( \tau_u(\gamma(x)) = f(x)a = a \),

\[
\tau_u(\gamma(x_\lambda)) = f(x_\lambda)a = 0, \quad \text{if} \ x_\lambda \in u,
\]

\[
= 0, \quad \text{if} \ x_\lambda \notin u,
\]

hence \( \tau_u(\gamma(x_\lambda)) = 0 \). Moreover,

\[
\tau_u(\gamma(x_\lambda)) = g_{u \cap A}(x_\lambda)(\tau_u(\gamma(x_\lambda))) = g_{u \cap A}(x_\lambda)(0) = 0, \quad \text{if} \ x_\lambda \in u,
\]

\[
= 0, \quad \text{if} \ x_\lambda \notin u,
\]

Thus

\[
\tau_u(\gamma(x_\lambda)) = 0 \quad \text{and} \quad \pi_\lambda(\tau_u(\gamma(x_\lambda))) = 0,
\]

therefore, \( \gamma \in \cap_\lambda \ker \pi_\lambda \). Since \( \sigma_\lambda \xrightarrow{\lambda} \sigma \) in \( \tilde{D} \), it follows that \( \cap_\lambda \ker \pi_\lambda \subset \ker \sigma \), by \([1, 3.4.4 \text{ and } 3.4.10]\). Whence \( \gamma \in \ker \sigma \), but \( \sigma(\gamma) = \pi(\tau_u(\gamma(x))) = \pi(a) \neq 0 \), a contradiction. Therefore \( \tilde{\rho} \) is continuous.

For each \( u \) in \( U \), define \( \tilde{\phi}_u : u \times \tilde{A} \to \tilde{p}^{-1}(u) \) by \( \tilde{\phi}_u(x, \pi)(\gamma) = \pi(\tau_u(\gamma(x))) \), \( \gamma \in D \).

Lemma 5. Use the above notations. For every \( u \) in \( U \), \( \tilde{\phi}_u \) is a homeomorphism from \( u \times \tilde{A} \) onto \( \tilde{p}^{-1}(u) \) and is fibre-preserving.

Proof. If \( \sigma \in \tilde{p}^{-1}(u) \), there exists (Theorem 3) a unique \( x \) in \( X \) and some \( v \in U \) with \( x \in v \cap u \) such that \( \sigma(\gamma) = \pi(\tau_v(\gamma(x))) \), \( \gamma \in D \). Let \( \delta \) be defined on \( A \) by \( \delta(a) = \pi(g_{v \cap A}(x)(a)), a \in A. \) Then \( \delta \in \tilde{A} \) and

\[
\delta(\tau_u(\gamma(x))) = \pi(g_{v \cap A}(x)(\tau_u(\gamma(x)))) = \pi(\tau_v(\gamma(x))) = \sigma(\gamma), \quad \text{for} \ \gamma \in D.
\]

Thus \( \tilde{\phi}_u(x, \delta) = \sigma \), whence \( \tilde{\phi}_u \) is surjective.

To prove \( \tilde{\phi}_u \) is 1-1, assume \( (x, \pi) \neq (y, \sigma) \) in \( u \times \tilde{A} \). If \( x \neq y \), there exists a \( f \) in \( C_0(X) \) with support \( \{ f \} \subset u \) such that \( f(x) = 1 \) and \( f(y) = 0 \). Choose an \( a \) in \( A \) such that \( \pi(a) \neq 0 \). Define

\[
\gamma(z) = \phi_u(z, f(z)a), \quad \text{if} \ z \in u,
\]

\[
= O(z), \quad \text{if} \ z \notin u.
\]
Then
\begin{align*}
\sigma(t_u(y(x))) &= \sigma(f(y)a) = \sigma(a) \neq 0, \\
\pi(t_u(y(x))) &= \pi(f(x)a) = \pi(a) \neq 0.
\end{align*}

Hence \( \tilde{\phi}_u(x, \pi) \neq \tilde{\phi}_u(y, \sigma) \).

If \( x = y \), then \( \pi \neq \sigma \) in \( \mathcal{P}^{-1}(u) \). Hence \( \tilde{\phi}_u(x, \pi) \neq \tilde{\phi}_u(y, \sigma) \). This proves that \( \tilde{\phi}_u \) is bijective.

Suppose the net \( (x_\lambda, \pi_\lambda) \to (x, \pi) \) in \( u \times \mathcal{A} \). If \( \gamma \in \bigcap_\lambda \ker \tilde{\phi}_u(x_\lambda, \pi_\lambda) \), \( \pi_\lambda(t_u(y(x_\lambda))) = 0 \). Since \( \phi_u^{-1}(y) = (x, t_u(y)) \), \( y \to t_u(y) \) is continuous and since \( x_\lambda \to x \) in \( X \) and \( \gamma \) is continuous, \( \gamma(x_\lambda) \to \gamma(x) \) in \( E \). Hence, if \( \epsilon > 0 \), there exists an \( \lambda_0 \) such that if \( \lambda > \lambda_0 \), \( \| t_u(y(x_\lambda)) - t_u(y(x)) \| < \epsilon \),

\begin{align*}
\| \pi_\lambda(t_u(y(x))) \| &= \| \pi_\lambda(t_u(y(x_\lambda))) - \pi_\lambda(t_u(y(x))) \| \\
&\leq \| t_u(y(x_\lambda)) - t_u(y(x)) \| < \epsilon.
\end{align*}

Therefore, \( \| t_u(y(x))/\bigcap_{\lambda > \lambda_0} \ker \pi_\lambda \| = \sup_{\lambda > \lambda_0} \| t_u(y(x))/\ker \pi_\lambda \| < \epsilon \). Since \( \pi_\lambda \to \pi \) in \( \mathcal{A} \), hence \( \bigcap_{\lambda > \lambda_0} \ker \pi_\lambda \subset \ker \pi \) [1, 3.4.4 and 3.4.10]. Thus \( \| t_u(y(x))/\ker \pi \| < \epsilon \). Since \( \epsilon \) is arbitrary, \( \gamma(x_\lambda) \in \ker \pi \), \( \tilde{\phi}_u(x_\lambda, \pi_\lambda) \subset \ker \phi_u(x, \pi) \). Hence \( \tilde{\phi}_u(x_\lambda, \pi_\lambda) \to \tilde{\phi}_u(x, \pi) \) in \( \mathcal{P}^{-1}(u) \).

Conversely, assume \( \tilde{\phi}_u(x_\lambda, \pi_\lambda) \to \tilde{\phi}_u(x, \pi) \) in \( \mathcal{P}^{-1}(u) \). If \( x_\lambda \to x \) in \( X \), there exist a subnet \( \{x_{\lambda_0}\} \) and a function \( f \in C_0(X) \) with \( \text{supp}(f) \subset u \) such that \( f(x_{\lambda_0}) = 0 \) and \( f(x) = 1 \). Let \( a \in \mathcal{A} \) be such that \( \pi(a) \neq 0 \). Define

\[ \gamma(y) = \tilde{\phi}_u(y, f(y)a), \quad \text{if } y \in u, \]

\[ = 0(y), \quad \text{if } y \notin u, \]

then \( \gamma \in \mathcal{D} \) and

\begin{align*}
\pi_\lambda(t_u(y(x_\lambda))) &= \pi_\lambda(f(x_\lambda)a) = \pi_\lambda(0) = 0, \\
\pi(t_u(y(x))) &= \pi(f(x)a) = \pi(a) \neq 0.
\end{align*}

Hence \( \gamma \in \bigcap \ker \tilde{\phi}_u(x_\lambda, \pi_\lambda) \), but \( \gamma \notin \ker \tilde{\phi}_u(x, \pi) \), which contradicts \( \tilde{\phi}_u(x_\lambda, \pi_\lambda) \to \tilde{\phi}_u(x, \pi) \). Thus \( x_\lambda \to x \) in \( X \).

Next, it will be shown \( \pi_\lambda \to \pi \) in \( \mathcal{A} \). If \( a \in \bigcap \ker \pi_\lambda \), \( \pi_\lambda(a) = 0 \) for all \( \lambda \). By Lemma 1, there exists a \( \gamma \) in \( \mathcal{D} \) such that \( a = t_u(\gamma(x)) \). It has been shown that \( x_\lambda \to x \) in \( X \), so that for \( \epsilon > 0 \) there exists an \( \lambda_0 \) such that if \( \lambda > \lambda_0 \) then

\[ \| t_u(\gamma(x_\lambda)) - t_u(\gamma(x)) \| < \epsilon, \]

and so

\[ \| \pi_\lambda(t_u(y(x_\lambda))) \| = \| \pi_\lambda(t_u(y(x_\lambda))) - \pi_\lambda(t_u(y(x))) \| \]
\[ \leq \| t_u(y(x_\lambda)) - t_u(y(x)) \| < \epsilon, \]

whence

\[ \| \gamma/\ker \tilde{\phi}_u(x_\lambda, \pi_\lambda) \| = \| \pi_\lambda(t_u(y(x_\lambda))) \| < \epsilon, \]

for \( \lambda > \lambda_0 \).

Thus \( \gamma/\bigcap_{\lambda > \lambda_0} \ker \tilde{\phi}_u(x_\lambda, \pi_\lambda) \| < \epsilon \).
Since \( \tilde{\phi}_u(x, \pi) \to \tilde{\phi}_u(x, \pi) \cap \bigcap_{\lambda > \lambda_0} \ker \tilde{\phi}_u(x, \pi) \subset \ker \tilde{\phi}_u(x, \pi) \). Hence

\[
||\gamma/\tilde{\phi}_u(x, \pi)|| < \epsilon \quad \text{and therefore} \quad \gamma \in \tilde{\phi}_u(x, \pi), \ i.e., \ \pi(t_u(\gamma(x))) = \pi(a) = 0.
\]

Thus \( a \in \ker \pi \), whence \( \bigcap_{\lambda} \ker \pi \subset \ker \pi \). It follows that \( \pi \in (\pi_{\lambda}) \) in \( \hat{A} \) by [1, 3.4.4 and 3.4.10], and the same holds for every subnet of \( \{\pi_{\lambda}\} \). Therefore \( \pi_{\lambda} \to \pi \) in \( \hat{A} \), and this proves that \( \tilde{\phi}_u \) is bicontinuous.

Finally, if \( x \in u \) and \( \sigma \) is in \( \tilde{\rho}^{-1}(x) \), \( \sigma \) corresponds by Theorem 3 to some \( (x, \nu, \pi) \) in \( X \times U \times \hat{A} \) such that

\[
\sigma(\gamma) = \pi(t_u(\gamma(x))), \quad \text{for} \ \gamma \in D.
\]

Since \( \pi \in \hat{A} \), \( \pi g_{uv}(x) \in \hat{A} \), and

\[
\tilde{\phi}_u(x, \pi g_{uv}(x))(\gamma) = \pi(g_{uv}(x)(t_u(\gamma(x)))) = \pi(t_u(\gamma(x))) = \pi(\gamma(x)) = \sigma(\gamma).
\]

Hence \( \tilde{\phi}_u(x, \hat{A}) = \tilde{\rho}^{-1}(x) \) and thus \( \tilde{\phi}_u \) is fibre-preserving.

If \( g \in G \), and if the map \( g^{*}: \hat{A} \to \hat{A} \) is defined by \( g^{*}(\pi)(a) = \pi(g(a)) \), then \( g^{*} \) is bijective. Furthermore, if the net \( \pi_{\lambda} \to \pi \) in \( \hat{A} \), and if \( a \in \bigcap_{\lambda} \ker g^{*}(\pi_{\lambda}) \), \( \pi_{\lambda}(g(a)) = 0 \). Hence \( g(a) \in \bigcap_{\lambda} \ker \pi_{\lambda} \). Since \( \pi_{\lambda} \to \pi \) in \( \hat{A} \), \( \bigcap_{\lambda} \ker \pi_{\lambda} \subset \ker \pi \), by [1, 3.4.4 and 3.4.10]. Whence \( g(a) \in \ker \pi \), \( a \in \ker g^{*}(\pi) \). Thus \( \bigcap_{\lambda} \ker g^{*}(\pi_{\lambda}) \subset \ker g^{*}(\pi) \), \( g^{*}(\pi) \in \{g^{*}(\pi_{\lambda})\} \) in \( \hat{A} \), and the same holds for every subnet of \( \{g^{*}(\pi_{\lambda})\} \). Therefore \( g^{*}(\pi_{\lambda}) \to g^{*}(\pi) \) and thus \( g^{*} \) is continuous. A similar argument shows \( (g^{*})^{-1} \) is continuous. Thus \( g^{*} \) belongs to \( \text{Aut}_0(\hat{A}) \), the group of self-homeomorphisms of \( \hat{A} \).

Consider the group \( \hat{G} \) of self-homeomorphisms of the form \( g^{*} \) where \( g \in G \). The map \( T: G \to \hat{G} \) defined by \( T(g) = g^{*} \) is a group antiepimorphism \( (T(gh) = T(h)T(g)) \). If the quotient group \( G/\ker T \) is given the quotient topology, \( G/\ker T \) is a topological group. Since

\[
g/\ker T \to g^{*}
\]

is an anti-isomorphism from \( G/\ker T \) onto \( \hat{G} \), we topologize \( \hat{G} \) by giving it the topology derived from \( G/\ker T \), i.e., a set \( S^{*} \subset \hat{G} \) is open if and only if the preimage \( S \) is open in \( G/\ker T \). Then \( \hat{G} \) becomes a topological group.

If \( u, v \in U \) and \( x \in u \cap v \), define

\[
\tilde{g}_{uv}(x): \hat{A} \to \hat{A}
\]

by

\[
\tilde{g}_{uv}(x)(\pi)(a) = \pi(g_{uv}(x)(a)), \quad \text{for} \ \pi \in \hat{A}, a \in A.
\]

Then \( \tilde{g}_{uv}(x) = (g_{uv}(x))^{*} \in \hat{G} \).

Since the map \( g_{uv}: u \cap v \to G \) is continuous, and the canonical map: \( G \to G/\ker T \) is also continuous, it follows that the map \( \tilde{g}_{uv}: u \cap v \to \hat{G} \) is continuous. Moreover,

\[
\tilde{\phi}_u(x, \pi)(\gamma) = \pi(t_u(\gamma(x))) = \pi(g_{uv}(x)(t_u(\gamma(x))))
\]

\[
= \tilde{g}_{uv}(x)(\pi)(t_u(\gamma(x)))
\]

\[
= \tilde{\phi}_u(x, \tilde{g}_{uv}(x)(\pi))(\gamma), \quad \text{for} \ \gamma \in D.
\]

Thus \( \tilde{\phi}_u^{-1}\tilde{\phi}_u(x, \pi) = (x, \tilde{g}_{uv}(x)(\pi)) \).

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Owing to Lemmas 4 and 5, the following result may be demonstrated:

**Theorem 6.** Let $\Sigma = (E, X, A, p, U, \phi_u, G)$ be a C*-algebra fibre bundle over a locally compact Hausdorff base space $X$, with fibre $A$, a C*-algebra. Then $\Sigma$ defines a topological fibre bundle $\tilde{\Sigma} = (\tilde{D}, X, \tilde{A}, \tilde{p}, U, \tilde{\phi}_u, \tilde{G})$ over the same base space $X$, with fibre the Jacobson spectrum $\tilde{A}$ of $A$ and bundle the Jacobson spectrum $\tilde{D}$ of the C*-algebra of sections of $\Sigma$.

**Proof.** By the above argument, the following results are readily derived:

(i) $\tilde{p}$ is a continuous projection from $\tilde{D}$ onto $X$,
(ii) $\tilde{\phi}_u$ is a homeomorphism from $u \times \tilde{A}$ onto $\tilde{p}^{-1}(u)$ and is fibre-preserving in that $\tilde{\phi}_u(x, \tilde{A}) = \tilde{p}^{-1}(x)$,
(iii) $\tilde{\phi}_u \circ \phi_v(x, \pi) = (x, \tilde{g}_{uv}(x)(\pi))$ and $\tilde{g}_{uv}(x) \in \tilde{G}$, and the map $\tilde{g}_{uv} : u \cap v \to \tilde{G}$ is continuous.

Therefore $\tilde{\Sigma} = (\tilde{D}, X, \tilde{A}, \tilde{p}, U, \tilde{\phi}_u, \tilde{G})$ is a topological fibre bundle.

**Remark.** It is shown in [2] that if $\Sigma$ is a Banach algebra fibre bundle over a compact Hausdorff base space $X$ with fibre $A$ a so-called $Q$-uniform Banach algebra then the maximal ideal space (corresponding to $\tilde{D}$) of the Banach algebra of sections of $\Sigma$ with suitable topology can be identified as a topological fibre bundle with base space $X$ and fibre the set of maximal ideals of the Banach algebra $A$. An example is given there to show that if the Jacobson topologies are used for the maximal ideal space of the Banach algebra of sections of $\Sigma$ and the maximal ideal space of $A$, the coordinate functions $\tilde{\phi}_u$ (as in Lemma 5) need not be continuous.

**References**