TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 274, Number 1, November 1982

REGULARIZING EFFECTS FOR $u_t = \Delta\varphi(u)$

BY

MICHAEL G. CRANDALL AND MICHEL PIERRE[1]

ABSTRACT. One expression of the fact that a nonnegative solution of the initial-value problem

$$\text{(IVP)} \qquad \begin{cases} u_t - \Delta u^m = 0, & t > 0, x \in R^N, \\ u(0, x) = u_0(x), \end{cases}$$

where $m > 0$, is more regular for $t > 0$ than a rough initial datum u_0 is the remarkable pointwise inequality $u_t = \Delta u^m \geqslant -(N/(N(m-1)+2)t)u$ obtained by Aronson and Bénilan for $t > 0$ and $m > \max((N-2)/N, 0)$. This inequality was used by Friedman and Caffarelli in proving that solutions of (IVP) are continuous for $t > 0$. The main results of this paper generalize the Aronson-Bénilan inequality and show the extended inequality is valid for a much broader class of equations of the form $u_t = \Delta\varphi(u)$. In particular, the results apply to the Stefan problem which is modeled by $\varphi(r) = (r-1)^+$ and imply $((u-1)^+)_t \geqslant -((u-1)^+ + N/2)/t$ in this case.

This paper concerns nonnegative solutions of initial-value problems of the form

$$\text{(IVP)} \qquad \begin{cases} u_t - \Delta\varphi(u) = 0, & \text{for } t > 0, x \in R^N, \\ u(0, x) = u_0(x), & x \in R^N. \end{cases}$$

Hereafter, it is assumed that

$$(1) \qquad \varphi\colon R \to R \text{ is continuous, nondecreasing and } \varphi(0) = 0.$$

It is known that (see, e.g., [5] and its references) if $u_0 \in L^1(R^N) \cap L^\infty(R^N)$, then there is exactly one $u \in C([0, \infty); L^1(R^N))$ which satisfies

$$(2) \qquad \operatorname{ess\,inf} u_0 \leqslant u \leqslant \operatorname{ess\,sup} u_0,$$

$u(0) = u_0$ and $u_t = \Delta\varphi(u)$ in $\mathcal{D}'((0, \infty) \times R^N)$ (i.e., in the sense of distributions). We refer to this u simply as the solution of (IVP). The mapping $u_0 \to u$ is nonexpansive from $L^1(R^N)$ into $C([0, \infty); L^1(R^N))$. It follows from (2) that u is nonnegative and bounded whenever u_0 is nonnegative and bounded and it is hereafter assumed that $u_0 \geqslant 0$.

Received by the editors April 7, 1981 and, in revised form, September 4, 1981.

1980 *Mathematics Subject Classification*. Primary 35K15, 35K55.

Key words and phrases. Regularizing effect, porous media equations, strong solution, degenerate parabolic equations, Stefan problem.

[1]Sponsored by the United States Army under Contract No. DAAG29-80-C-0041 and supported in part by the National Science Foundation under Grant MCS-8002946.

0002-9947/82/0000-0015/$03.25

We will prove that if φ satisfies (1), is twice continuously differentiable and

$$(3) \qquad r(\varphi'(r))^2 \leqslant K[\varphi(r) + a][r\varphi''(r) + 2\varphi'(r)/N] \quad \text{for } 0 < r \leqslant \sup u_0$$

("ess sup" is hereafter abbreviated to "sup") for some $K > 0$ and $a \geqslant 0$, then the solution u of (IVP) satisfies

$$(4) \qquad \varphi(u)_t \geqslant (K/t)(\varphi(u) + a) \quad \text{in } \mathcal{D}'((0, \infty) \times R^N).$$

In fact, we will establish (4) for nonlinearities φ which satisfy integrated forms of (3) and thus φ need not be twice differentiable.

The pointwise inequality (4) is a generalization of the result of Aronson and Bénilan [**1**] for the case $\varphi(r) = r^m$, $m > 0$. Indeed, for this φ, (3) with $a = 0$ becomes

$$m^2 \leqslant K(m(m-1) + 2m/N)$$

which is equivalent to $N(m-1) + 2 > 0$ and

$$K \geqslant m/((m-1) + 2/N).$$

Hence if $\varphi(r) = r^m$ and $N(m-1) + 2 > 0$, Theorem 1 implies

$$(5) \qquad (u^m)_t \geqslant -(mN/N(m-1) + 2)(u^m/t).$$

Formally $(u^m)_t = mu^{m-1}u_t$ and (5) thus becomes

$$(6) \qquad u_t \geqslant -(N/N(m-1) + 2)(u/t)$$

which is a main result of [**1**].

When φ satisfies (3) with $a = 0$, the estimate (4) formally yields

$$(7) \qquad u_t \geqslant -(K/t)(\varphi(u)/\varphi'(u)).$$

Although (7) is not an immediate consequence of (4) due to regularity questions, the possibility that $\varphi'(u)$ may have zeros, etc...., we will show this inequality holds under assumptions that guarantee it is meaningful. If, moreover, $\varphi(u) \leqslant cu\varphi'(u)$ for some constant c (as is implied by (3) when $a = 0$, $N \geqslant 3$), an inequality of type (6) can also be obtained.

These last considerations are clearly not relevant when $a \neq 0$ and $\varphi'(u)$ has zeros. In this case (4) is a different sort of estimate which can be obtained for a much larger class of functions φ than the $u_t \geqslant -cu/t$ estimates. In particular, if $\varphi(u) = \max(u-1, 0)$ we have a model of the Stefan problem (see, e.g., [**6, 9**]) and the integrated form of (3) holds with $a = 1$, $K = N/2$.

The methods used in this work are similar to those of [**1**]. First (IVP) is approximated appropriately. Next it is shown that if u is a solution of the approximate problem, then $p = \Delta\psi(u)$, where $\psi'(r) = \varphi'(r)/r$, satisfies a parabolic inequality which allows one to conclude that $\Delta\psi(u) \geqslant -h(\psi(u))/t$ for certain functions h constructed in the argument. The consequences for u are as stated above. Refinements and implications of (4) are also discussed. These primary arguments are given in §1. The question of "strong solutions" is addressed in §2.

Other works in which related estimates appear are [**3** and **8**], to which we refer for further references. In particular, [**8**] contains results which apply to (IVP) when the sign condition $u_0 \geqslant 0$ is dropped.

1. The main results. Preparatory to formulating the main results, we restate the basic assumption

$$r(\varphi'(r))^2 \leqslant K(\varphi(r) + a)(r\varphi''(r) + 2\varphi'(r)/N), \tag{1.1}$$

in which $K > 0$ and $a \geqslant 0$, in integrated form. Assume that $\varphi(r) + a > 0$, and $\varphi'(r) > 0$ on some interval $(0, M)$ for the moment, so that (1.1) may be rewritten (on $(0, M)$) as

$$k\varphi'/(\varphi + a) \leqslant \varphi''/\varphi' + 2N^{-1}r^{-1}, \qquad k = 1/K,$$

or

$$0 \leqslant (\ln\varphi')' - k(\ln(\varphi + a))' + 2(\ln r)'/N,$$

or

$$\ln\left(\left[\varphi'/(\varphi + a)^k\right]r^{2/N}\right) \text{ is nondecreasing}$$

or, finally, $\varphi'(r)r^{2/N}/(\varphi(r) + a)^k$ is nondecreasing. This final monotonicity property is conveniently regarded as the convexity of

$$\begin{cases} \tau \to (1-k)^{-1}\left[\varphi(\tau^{N/(N-2)}) + a\right]^{1-k}, & N \geqslant 3, \\ \tau \to (1-k)^{-1}\left[\varphi(e^{\tau}) + a\right]^{1-k}, & N = 2, \\ \tau \to (1-k)^{-1}\left[\varphi(-1/\tau) + a\right]^{1-k}, & N = 1. \end{cases} \tag{1.2}$$

(If $k = 1$ above, $(\varphi + a)^{1-k}/(1-k)$ means $\ln(\varphi + a)$.)

A main result is

THEOREM 1. *Let* $0 \leqslant u_0 \in L^1(R^N) \cap L^\infty(R^N)$ *and assume that* (1.2) *is convex on*

$$\begin{cases} 0 \leqslant \tau \leqslant (\sup u_0)^{(N-2)/N} & \text{if } N \geqslant 3, \\ -\infty < \tau < \ln(\sup u_0) & \text{if } N = 2, \\ -\infty < \tau < -1/\sup u_0 & \text{if } N = 1. \end{cases}$$

Let u be the solution of (IVP).

Then for $K = 1/k$,

$$\varphi(u)_t \geqslant -(K/t)(\varphi(u) + a) \quad \text{in } \mathcal{D}'((0,\infty) \times R^N). \tag{1.3}$$

The proof of Theorem 1 will be augmented to obtain the following refinement in the case in which $a = 0$ and $\varphi(r)/\varphi'(r)$ is a "good" function.

THEOREM 2. *In addition to the assumptions of Theorem* 1, *assume* $a = 0$, $r_0 \geqslant 0$ *and* $\varphi(r) = 0$ *for* $0 \leqslant r \leqslant r_0$. *Assume* $\varphi \in C^1(0, \sup u_0)$, $\varphi'(r) > 0$ *for* $r > r_0$ *and that* $\lim_{r \downarrow r_0} \varphi(r)/\varphi'(r) = 0$. *Then the solution u of* (IVP) *satisfies*

$$u_t \geqslant -K\alpha(u)/t \quad \text{in } \mathcal{D}'((0,\infty) \times R^N) \tag{1.4}$$

where

$$\alpha(r) = \begin{cases} \varphi(r)/\varphi'(r), & r > r_0, \\ 0, & r \leqslant r_0. \end{cases}$$

We point out explicitly the special cases $\varphi(r) = (r-1)^{+m} = (\max(r-1,0))^m$ on $[0, \infty)$ in which we may take $r_0 = 1$. Now (1.1) will be satisfied with $a = 0$ if

$$rm \leqslant K\big((m-1)r + 2N^{-1}(r-1)\big) \quad \text{for } 1 \leqslant r \leqslant \sup u_0.$$

Thus if $m > 1$ we may use Theorem 2 and $K = m/(m-1)$ to conclude that

$$u_t \geqslant \frac{m}{m-1}\frac{1}{t}\frac{1}{m}(u-1)^+ = -\frac{1}{(m-1)t}(u-1)^+$$

no matter what u_0 is. If $m = 1$, corresponding to the Stefan problem, we cannot satisfy (1.1) with $a = 0$ and require instead

$$r \leqslant K\big((r-1)^+ + a\big)(2/N), \qquad r > 1,$$

or $K \geqslant 1$ and $Ka \cdot 2/N \geqslant 1$. This is achieved by $K = 1$ and $a = N/2$. (The reader may check that the integrated condition holds in this case, erasing doubt as to the validity of the cavalier treatment of φ'' above.) Hence we conclude, by Theorem 1, only that

$$\big((u-1)^+\big)_t \geqslant -t^{-1}\big((u-1)^+ + N/2\big).$$

It is interesting to notice that if φ is convex and Lipschitz continuous on $(0, \sup u_0)$ and satisfies (1), then (1.1) (or its integrated form) holds with

$$Ka = \frac{N}{2} \sup_{r \in (0, \sup u_0)} (r\varphi'(r)).$$

Since K can be chosen as small as we desire, Theorem 1 yields $\varphi(u)_t \geqslant -Ka/t$.

We begin the proofs of Theorems 1 and 2. To this end we consider the problem (IVP) under the strong assumptions that

$$u_0 \in C_0^\infty(R^N), \quad u_0 \geqslant 0, \tag{1.5}$$

and that φ satisfies (1), and for some $K, a, \delta \geqslant 0$,

$$\begin{cases} \text{(i) } \varphi \in C^\infty(0,\infty), \\ \text{(ii) } r(\varphi'(r))^2 \leqslant K(\varphi(r) + a)(r\varphi''(r) + 2\varphi'(r)/N) \\ \qquad\qquad\qquad\qquad \text{for } 0 < r \leqslant \sup u_0 + \delta, \\ \text{(iii) } \varphi'(r) > 0 \quad \text{for } 0 < r \leqslant \sup u_0 + \delta. \end{cases} \tag{1.6}$$

Standard methods [**9**, **10**] then guarantee that for $0 < \varepsilon < \delta$ the problem

$$w_{\varepsilon t} - \Delta\varphi(w_\varepsilon) = 0, \qquad w_\varepsilon(0,x) = u_0(x) + \varepsilon \tag{1.7}$$

has a unique solution $w_\varepsilon \in C^\infty([0, \infty) \times R^N)$ satisfying

$$\sup u_0 + \varepsilon \geqslant w_\varepsilon \geqslant \varepsilon, \tag{1.8}$$

and the derivatives of w_ε are bounded for bounded $t \geqslant 0$. Moreover,

$$w_\varepsilon \leqslant w_\mu \quad \text{for } 0 < \varepsilon \leqslant \mu < \delta \tag{1.9}$$

and

$$\int_{R^N} (w_\varepsilon - \varepsilon)\, dx = \int_{R^N} u_0\, dx. \tag{1.10}$$

It follows that $\lim_{\varepsilon\downarrow 0}(w_\varepsilon - \varepsilon) = \lim_{\varepsilon\downarrow 0} w_\varepsilon$ is the solution of (1.1) and that in order to prove $\varphi(u)_t \geqslant -(K/t)(\varphi(u) + a)$ it suffices to show $\varphi(w_\varepsilon)_t \geqslant -(K/t)(\varphi(w_\varepsilon) + a)$.

For notational convenience we now drop the subscript ε in w_ε. Thus w is to solve $w_t - \Delta\varphi(w) = 0$, $w(0, x) = u_0(x) + \varepsilon$. The main steps in the proof involve the function

$$\psi(\zeta) = \int_{\varepsilon/2}^{\zeta} \frac{\varphi'(s)}{s}\, ds \quad \text{for } \zeta > 0, \tag{1.11}$$

and the standard change of dependent variable

$$v = \psi(w). \tag{1.12}$$

LEMMA 1. *Let* (1.5), (1.6) *hold*, $0 < \varepsilon < \delta$, w *be the classical solution of* (1.7) *and* v *be given by* (1.12), (1.11). *Then*

$$v_t = g(v)\Delta v + |\nabla v|^2 \quad \textit{for } g(v) = \varphi'(\psi^{-1}(v)), \tag{1.13}$$

$$w_t = w\Delta v + (\varphi'(w)/w)\,|\nabla w|^2, \tag{1.14}$$

and $p = \Delta v$ *satisfies*

$$\begin{aligned} p_t \geqslant\ & g(v)\Delta p + 2(\nabla g(v) + \nabla v)\cdot\nabla p \\ & + g''(v)\,|\nabla v|^2 p + (g'(v) + 2/N)p^2. \end{aligned} \tag{1.15}$$

In the statement of Lemma 1, $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_N)$ and $a \cdot b = a_1 b_1 + \cdots + a_N b_N$. The proof consists of direct calculations. E.g., substituting the identities

$$v_t = \psi'(w)w_t, \quad \nabla v = \psi'(w)\nabla w,$$

$$\begin{aligned} \Delta v &= \frac{\varphi'(w)}{w}\Delta w + \left(\frac{\varphi''(w)}{w} - \frac{\varphi'(w)}{w^2}\right)|\nabla w|^2 \\ &= \frac{\varphi'(w)}{w}\Delta w + \left(\frac{\varphi''(w)w - \varphi'(w)}{(\varphi'(w))^2}\right)|\nabla v|^2 \end{aligned}$$

into $w_t = \Delta\varphi(w) = \varphi'(w)\Delta w + \varphi''(w)\,|\nabla w|^2$ yields (1.13) and (1.14). To obtain (1.15) one applies Δ to the equation $v_t = g(v)\Delta v + |\nabla v|^2$ and uses

$$\Delta|\nabla v|^2 = 2\sum_{i,j=1}^{N}(v_{x_i x_j})^2 + 2\nabla v\cdot\nabla(\Delta v) \geqslant (2/N)(\Delta v)^2 + 2\nabla v\cdot\nabla(\Delta v).$$

To continue, let $\mathfrak{L}$ be the nonlinear operator

$$\begin{aligned} \mathfrak{L}(Z) = \ & g(v)\Delta Z + 2(\nabla g(v) + \nabla v)\cdot\nabla Z \\ & + g''(v)\,|\nabla v|^2 Z + (g'(v) + 2/N)Z^2, \end{aligned} \tag{1.16}$$

so the parabolic inequality (1.15) satisfied by $p = \Delta v$ can be rewritten $p_t \geqslant \mathfrak{L}(p)$. We now seek a comparison function Z in the form $Z = -h(v)/t$ with the property $Z_t \leqslant \mathfrak{L}Z$. If $h(r) > 0$ for $r > 0$, we then have $Z \leqslant p$ for small $t > 0$ (since "$Z = -\infty$" at $t = 0$) and so $Z \leqslant p$ for all $t > 0$ by standard comparison results [**9**].

LEMMA 2. *Let w be the solution of* (1.7) *and $v = \psi(w)$ as above. Let $a \geqslant 0$, $K \geqslant 0$,*

$$h(r) = K\frac{\varphi(\psi^{-1}(r)) + a}{\varphi'(\psi^{-1}(r))\psi^{-1}(r)} \tag{1.17}$$

and $Z = -h(v)/t$. Then

$$Z_t = \mathfrak{L}(Z) + (1/h(v) - (g'(v) + 2/N))Z^2. \tag{1.18}$$

In particular, if

$$1/h(v) \leqslant g'(v) + 2/N \tag{1.19}$$

then $Z \leqslant \Delta v$ for $t > 0$ and

$$\varphi(w)_t \geqslant -(K/t)(\varphi(w) + a) \quad \textit{for } t > 0. \tag{1.20}$$

PROOF. We set aside the verification of (1.18) for the moment. Once this is established and (1.19) holds we have $Z_t \leqslant \mathfrak{L}(Z)$ and then $Z \leqslant \Delta v$ by the maximum principle as remarked above. Returning to (1.14) we have

$$w_t = w\Delta v + \frac{\varphi'(w)}{w}|\nabla w|^2 \geqslant w\Delta v \geqslant -\frac{wh(v)}{t} = -K\frac{(\varphi(w) + a)}{\varphi'(w)t}, \tag{1.21}$$

or $\varphi(w)_t = \varphi'(w)w_t \geqslant -K(\varphi(w) + a)/t$ as claimed in (1.20).

Rather than verify (1.18) by direct computation, we consider the result of computing $Z_t - \mathfrak{L}(Z)$ when $Z = -h(v)/t$. The identities

$$Z_t = -\frac{h'(v)}{t}v_t + \frac{h(v)}{t^2} = \frac{-h'(v)}{t}v_t + \frac{1}{h(v)}Z^2,$$

$$\nabla Z = -\frac{h'(v)}{t}\nabla v, \qquad \Delta Z = -\frac{h'(v)}{t}\Delta v - \frac{h''(v)}{t}|\nabla v|^2$$

imply that $v_t = g(v)\Delta v + |\nabla v|^2$ becomes, upon multiplication by $-h'(v)/t$,

$$\begin{aligned}
Z_t - \frac{1}{h(v)}Z^2 = Z_t - \frac{h(v)}{t^2} &= g(v)\left(\Delta Z + \frac{h''(v)}{t}|\nabla v|^2\right) + \nabla v \cdot \nabla Z \\
&= g(v)\Delta Z + 2(\nabla g(v) + \nabla v)\cdot\nabla Z + g''(v)|\nabla v|^2 Z + \left(g'(v) + \frac{2}{N}\right)Z^2 \\
&\quad + \frac{h''(v)}{t}g(v)|\nabla v|^2 - (2\nabla g(v) + \nabla v)\cdot\nabla Z - g''(v)|\nabla v|^2 Z \\
&\quad - (g'(v) + 2/N)Z^2 \\
&= \mathfrak{L}(Z) + (h''(v)g(v) + 2g'(v)h'(v) + h'(v) + g''(v)h(v))\frac{|\nabla v|^2}{t} \\
&\quad - (g'(v) + 2/N)Z^2.
\end{aligned}$$

Simplifying the above a bit, it follows that

$$Z_t = \mathfrak{L}(Z) + \left(\frac{1}{h(v)} - \left(g'(v) + \frac{2}{N}\right)\right)Z^2 + ((gh)''(v) + h'(v))\frac{|\nabla v|^2}{t}. \tag{1.22}$$

The general solution h of $(gh)'' + h' = 0$ is

$$h(r) = \frac{C_1}{g(r)}\exp\left(-\int\frac{1}{g}\right) + \frac{C_2}{g}\left(\exp\left(-\int\frac{1}{g}\right)\right)\left(\int\exp\int\frac{1}{g}\right) \tag{1.23}$$

where $\int f$ denotes a function whose derivative is f and C_1, C_2 are constants. Now recall that $g(r) = \varphi'(\psi^{-1}(r))$, $\psi'(r) = \varphi'(r)/r$ and so $\int 1/g = \ln\psi^{-1}$, $\int\psi^{-1} = \varphi(\psi^{-1})$. Thus (1.23) can be written

$$h(r) = \frac{C_1}{\varphi'(\psi^{-1}(r))}\frac{1}{\psi^{-1}(r)} + \frac{C_2}{\varphi'(\psi^{-1}(r))}\frac{1}{\psi^{-1}(r)}\varphi(\psi^{-1}(r)). \tag{1.24}$$

Set $C_1 = Ka$ and $C_2 = K$ in (1.24) to find (1.17). Now (1.18) follows from (1.22) and $(gh)'' + h' = 0$. This completes the proof of Lemma 2.

We now can prove

LEMMA 3. *In addition to the assumptions of Theorem* 1, *let* φ *satisfy* (1.6) *with* $\delta = 0$. *Then* (1.3) *holds.*

PROOF. If $u_0 \in C_0^\infty(R^N)$, $u_0 \geqslant 0$, (1.6) is satisfied with $\delta > 0$ and w_ε solves (1.7), then by Lemma 2, (1.20) holds provided (1.19) is satisfied with h given by (1.17) and $v = \psi(w_\varepsilon)$. Recalling again that $g(r) = \varphi'(\psi^{-1}(r))$, etc., (1.19) reduces to

$$w_\varepsilon(\varphi'(w_\varepsilon))^2 \leqslant K(\varphi(w_\varepsilon) + a)(w_\varepsilon\varphi''(w_\varepsilon) + (2/N)\varphi'(w_\varepsilon)).$$

This holds by (1.6), (1.8) and $0 < \varepsilon < \delta$. Since w_ε decreases to the solution u as ε decreases to zero, (1.20) will hold (in $\mathcal{D}'((0,\infty) \times R^N)$) for u. The mapping $u_0 \to u$ is nonexpansive from $L^1(R)$ into $C([0,\infty); L^1(R^N))$ and hence (1.3) holds for general $0 \leqslant u_0 \in L^1(R^N) \cap L^\infty(R^N)$ satisfying the hypotheses of the lemma when it holds for the special $u_0 \in C_0^\infty(R^N)$ just treated. (To allow $\delta = 0$, approximate u_0 suitably from below.)

PROOF OF THEOREM 1. According to Lemma 3, the assertion of the theorem is correct if also φ is smooth and $\varphi'(r) > 0$ on $r > 0$. Indeed, the convexity assumed in the theorem is equivalent to (1.6)(ii) (with $\delta = 0$) in this case. It is also known that if $\{\varphi_n\}$ is a sequence of nondecreasing continuous functions with $\varphi_n \to \varphi$ everywhere on $[0, \sup u_0]$, then the solutions u_n of

$$u_{n_t} - \Delta\varphi_n(u_n) = 0, \qquad u_{n|_{t=0}} = u_0,$$

satisfy $\lim_{n\to\infty} u_n = u$ in $C([0,\infty); L^1(B))$ for every compact $B \subset R^N$ [4]. Thus the result will follow at once if we can approximate functions φ satisfying the hypotheses of Theorem 1 by sequences satisfying (1.6) with $K = 1/k$ in (1.6)(ii). This may be done as follows if $N \geqslant 3$, $k \neq 1$. Let $(\varphi(\tau^{N/(N-2)}) + a)^{1-k}/(1-k) = \Phi(\tau)$ be convex on $0 \leqslant \tau \leqslant (\sup u_0)^{(N-2)/N}$. Choose C^∞ convex functions Φ_n on this interval with $\Phi_n(0) = a^{1-k}/(1-k)$ and $\Phi_n(\tau) > 0$ for $0 < \tau$ so that $\Phi_n \to \Phi$ uniformly on $[0, \sup u_0]$. Set $\varphi_n(r) = ((1-k)\Phi_n(r^{(N-2)/N}))^{1/(1-k)} - a$. Then $\varphi_n \to \varphi$ uniformly on $[0, \sup u_0]$ and φ_n satisfies (1.6) with $K = 1/k$. Obvious modifications need to be made for $k = 1$ or $N = 1,2$, but the arguments are basically the same and we omit them.

PROOF OF THEOREM 2. Again appropriate approximations need to be made and again we explicitly discuss only $N \geqslant 3$. Recall that now $a = 0$ and set $\Phi(\tau) = \varphi(\tau^{N/(N-2)})^{1-k}/(1-k)$. Define φ_λ by

$$\frac{1}{(1-k)}\varphi_\lambda(\tau^{N/(N-2)})^{1-k} = \Phi_\lambda(\tau) = \Phi(\tau) + e^{-\pi^2}(e^{\lambda\tau} - 1).$$

Now $\varphi_\lambda \in C^1$ and $\varphi'_\lambda(r) > 0$ on $(0, \sup u_0]$. We may solve (1.7) with $\varepsilon > 0$ and φ replaced by φ_λ for $w = w_\lambda$ (with the dependence on ε suppressed). It suffices to assume that u_0, φ and ψ are smooth in the process and remove this subsequently by approximation. Then, by Lemma 2, $\varphi'_\lambda(w_\lambda)w_{\lambda t} \geqslant -(K/t)\varphi_\lambda(w_\lambda)$ and, since w_λ is smooth and $\varphi'_\lambda(w_\lambda) > 0$,

$$w_{\lambda t} \geqslant -(K/t)(\varphi_\lambda(w_\lambda)/\varphi'_\lambda(w_\lambda)). \tag{1.25}$$

But

$$\frac{\varphi_\lambda(\tau^{N/(N-2)})}{\varphi'_\lambda(\tau^{N/(N-2)})} = \frac{(1-k)N}{N-2}\tau^{2/(N-2)}\frac{\Phi_\lambda(\tau)}{\Phi'_\lambda(\tau)}$$
$$= \frac{(1-k)N}{N-2}\tau^{2/(N-2)}\frac{\left(\Phi(\tau) + e^{-\lambda^2}(e^{\lambda\tau} - 1)\right)}{\left(\Phi'(\tau) + e^{-\lambda^2}\lambda e^{\lambda\tau}\right)}.$$

We have assumed that

$$\frac{\varphi(\tau^{N/(N-2)})}{\varphi'(\tau^{N/(N-2)})} = (1-k)\frac{N}{N-2}\tau^{2/(N-2)}\frac{\Phi(\tau)}{\Phi'(\tau)}, \qquad \tau > r_0^{(N-2)/N},$$

extends continuously to $[0, r_0^{(N-2)/N}]$ as 0, or

$$\lim_{\tau \downarrow r_0^{(N-2)/N}} \frac{\Phi(\tau)}{\Phi'(\tau)} = 0.$$

From this it follows that

$$\lim_{\lambda\to\infty} \frac{\Phi(\tau) + e^{-\lambda^2}(e^{\lambda\tau} - 1)}{\Phi'(\tau) + e^{-\lambda^2}\lambda e^{\lambda\tau}} = \begin{cases} \dfrac{\Phi(\tau)}{\Phi'(\tau)}, & r_0^{(N-2)/N} < \tau \leqslant (\sup u_0)^{(N-2)/N}, \\ 0, & 0 \leqslant \tau \leqslant r_0^{(N-2)/N}, \end{cases}$$

uniformly in $[0, (\sup u_0)^{(N-2)/N}]$. The verification is left to the reader. Hence one may pass to the limit as $\varepsilon \downarrow 0$ in (1.25) and then as $\lambda \to \infty$ to complete the proof of Theorem 2.

COROLLARY. *In addition to the assumptions of Theorem* 2 *let* $N \geqslant 3$. *Then*

$$u_t \geqslant -(N/(N-2))(K-1)(u/t). \tag{1.26}$$

PROOF. The convexity of $\tau \to \varphi(\tau^{N/(N-2)})^{1-k} = \Phi(\tau)$ implies $\Phi(\tau) \leqslant \tau\Phi'(\tau)$ or

$$\varphi(\tau^{N/(N-2)}) \leqslant \tau^{N/(N-2)}(1-k)(N/(N-2))\varphi'(\tau^{N/(N-2)})$$

or

$$\varphi(r)/\varphi'(r) \leqslant N(1-k)r/(N-2), \tag{1.27}$$

provided $\varphi'(r)$ exists and is positive. Since the estimate on φ/φ' depends only on k, it is shared by any smooth approximation $\tilde{\varphi}$ of φ with $\tilde{\varphi}'(r) > 0$ and the same k. We showed above how to make such approximations and then (1.26) holds for the solutions of these approximate problems by Theorem 2 (with $r_0 = 0$) and (1.27). Hence (1.26) follows.

REMARK. (1.27) implies $k < 1$ (unless $\varphi \equiv 0$) since φ/φ' is somewhere positive. If $N = 1$ or 2 we cannot deduce an estimate on $\varphi(r)/r\varphi'(r)$ from our other assumptions. However, we may use Theorem 2 with $r_0 = 0$ if it applies. In particular, if $\varphi(r) = r^m$ with $m > 0$ and $N = 1$ or 2 we recover the estimate of [**1**]. For $N > 2$ and $m > (N-2)/N$, (1.26) with $K = mN/((m-1)N+2)$ also recovers the result of [**1**] for this case.

2. Remarks on strong solutions. We discuss the question of when the results of the previous section imply that the solution u of (IVP) is a "strong solution" in $L^1(R^N)$. By this we mean $u \in C([0,\infty); L^1(R^N))$, $\varphi(u) \in L^1_{\mathrm{loc}}((0,\infty)\times R^N)$, $u_t \in L^1_{\mathrm{loc}}((0,\infty); L^1(R^N))$ and $u_t = \Delta\varphi(u)$.

THEOREM 3. *Let the hypotheses of Theorem* 2 *be satisfied with* $r_0 = 0$. *Assume, moreover, that there is a constant* c *such that* $\varphi(r)/\varphi'(r) \leqslant cr$ *for* $r \in (0, \sup u_0]$. *Then the solution* u *of* (IVP) *is a strong solution. Moreover,* $t\int |u_t(t,x)|\,dx \leqslant 2cK\int u_0(x)\,dx$.

PROOF. This result follows from the arguments given in [**1**]. By Theorem 2, $u_t \geqslant -Kcu/t$ and u_t is therefore a measure on $[0,\infty)\times R^N$. Since $\varphi(u)$ is continuous on $(0,\infty)\times R^N$ by the results of [**11**], $u = \varphi^{-1}(\varphi(u))$ is also continuous and it follows as in [**1**] that $u_t \in L^1_{\mathrm{loc}}((0,\infty)\times R^N)$. Since u_t is bounded below by an integrable function a.e. on $t > 0$, $\int u_t(t,x)\,dx$ is defined for a.e. $t > 0$ (but it might be ∞). However, $t \to \int_{R^N} u(t,x)\,dx$ is nonincreasing. Thus

$$\int_{R^N}\left(\{u_t(t,x)\}^+ - \{u_t(t,x)\}^-\right)dx = \int_{R^N} u_t(t,x)\,dx \leqslant 0$$

and so

$$\begin{aligned}\int |u_t(t,x)|\,dx &= \int\left(|u_t(t,x)^+| + |u_t(t,x)^-|\right)dx \\ &\leqslant 2\int |u_t(t,x)^-|\,dx \leqslant \frac{2cK}{t}\int u(t,x)\,dx \leqslant \frac{2cK}{t}\int u_0(x)\,dx.\end{aligned}$$

It follows that $tu_t \in L^\infty(0,\infty; L^1(R^N)) \subseteq L^1_{\mathrm{loc}}(0,\infty; L^1(R^N))$.

REMARK. This discussion assumed (1.2) convex on the intervals specified in Theorem 3 and $u_0 \in L^\infty(R^N)$. However, if the convexity is global, then $u_t \geqslant -Kcu/t$ holds for all $0 \leqslant u_0 \in L^1(R^N)\cap L^\infty(R^N)$ and, hence, for all $0 \leqslant u_0 \in L^1(R^N)$. Moreover, the results of [**2, 12**] imply $u(t) \in L^\infty(R^N)$ for $t > 0$ under certain conditions (which hold under our assumptions if $N \geqslant 3$). Whenever $u(t) \in L^\infty(R^N)$ for $t > 0$, [**11**] provides a modulus of continuity on each set $[\tau, T]\times B$, $\tau > 0$, B a ball in R^N. Thus the above arguments apply and we have strong solutions for all $u_0 \in L^1(R^N)$, $u_0 \geqslant 0$.

The hypothesis $\varphi'(r) > 0$ for $r > 0$ appears to be nearly necessary for the existence of strong solutions. We remark that if $\varphi(r) = (r-1)^+$ and $N = 1$ then strong solutions need not exist. Indeed, in this model of the Stefan problem (see [**6, 9**]), $\varphi(u)$ is the temperature. If $\varphi(u_0)$ is continuous, smooth on $x > 0$ and $x < 0$ and $\varphi(u_0) > 0$ on $x > 0$, $\varphi(u_0) = 0$ on $x < 0$, then there is a smooth curve $x = s(t)$ such that $\varphi(u) > 0$ for $x > s(t)$ and $\varphi(u) = 0$ for $x \leqslant s(t)$ [**7**]. Moreover, the derivative $\varphi(u)_x$ jumps across $x = s(t)$ and $\Delta\varphi(u) \notin L^1$.

REFERENCES

1. D. G. Aronson and Ph. Bénilan, *Régularité des solutions de l'équation des milieux poreux dans R^N*, C. R. Acad. Sci. Paris Sér. A–B **288** (1979), 103–105.

2. Ph. Bénilan, *Opérateurs accrétifs et semi-groupes dans les espaces L^p* ($1 \leq p \leq \infty$), Functional Analysis and Numerical Analysis Japan-France Seminar, Tokyo and Kyota, H. Fujita (Ed.), Japan Society for the Promotion of Science, 1978, pp. 15–53.

3. Ph. Bénilan and M. G. Crandall, *Regularizing effects of homogeneous evolution equations*, Contributions to Analysis and Geometry (a supplement to Amer. J. Math.), D. N. Clark, G. Pecelli, R. Sacksteder (Eds.), John Hopkins Univ. Press, Baltimore, Md., 1981, pp. 23–30.

4. _____, *The continuous dependence on φ of solutions of $u_t - \Delta\varphi(u) = 0$*, Indiana Univ. Math. J. **30** (1981), 162–177.

5. H. Brézis and M. G. Crandall, *Uniqueness of solutions of the initial-value problem for $u_t - \Delta\varphi(u) = 0$*, J. Math. Pures Appl. **58** (1979), 153–163.

6. L. A. Caffarelli and L. C. Evans, *Continuity of the temperature in the two-phase Stefan problem* (to appear).

7. J. R. Cannon, D. B. Henry and D. B. Kotlow, *Classical solutions of the one dimensional, two-phase Stefan problem*, Ann. Mat. Pura Appl. **107** (1975), 311–341.

8. M. G. Crandall and M. Pierre, *Regularizing effects for $u_t = A\varphi(u)$ in L^1*, J. Funct. Anal. **45** (1982), 194–212.

9. O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural'ceva, *Linear and quasilinear equations of parabolic type*, Transl. Math. Monos., vol. 23, Amer. Math. Soc., Providence, R.I., 1968.

10. E. S. Sabinina, *On the Cauchy problem for the equation of non-stationary gas filtration in several space variables*, Dokl. Akad. Nauk SSSR **136** (1961), 1034–1037.

11. Paul Sacks, Thesis, Univ. of Wisconsin-Madison, 1981.

12. L. Véron, *Coercivité et propriétés régularisantes des semi-groupes nonlinéaires dans les espaces de Banach*, Publ. Math. Fac. Sci. Besançon **3** (1977).

MATHEMATICS RESEARCH CENTER, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706

UNIVERSITÉ SCIENTIFIQUE ET MÉDICALE, SAINT-MARTIN D'HÈRES, FRANCE