AMPLENESS AND CONNECTEDNESS IN COMPLEX $G/P$

BY

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ABSTRACT. This paper determines the “ampleness” of the tangent bundle of the complex homogeneous space, $G/P$, by calculating the maximal fibre dimension of the desingularization of a nilpotent subvariety of the Lie algebra of $G$.

1. Introduction. Let $G$ be a connected semisimple complex Lie Group. Let $P$ be a parabolic subgroup of $G$, and define the homogeneous space $Z = G/P$. Let $\phi: \mathbb{P}(T^*Z) \to \mathbb{P}^N$ be the map determined by the global sections of $TZ$, the tangent bundle of $Z$; see §3. Define the ampleness of $TZ$, $\text{amp}(TZ)$, to be the maximum fibre dimension of $\phi$, and the coampleness, $\text{ca}(Z)$, to be $\dim Z - \text{amp}(TZ)$.

In this paper, I calculate $\text{ca}(Z)$. The results are in Table 1 at the end of this section. I have also determined the number of irreducible components in a largest fibre of the ampleness map $\phi$. This result is described in §5.

The following theorem of Sommese [21A, §3] is a generalization of the Barth-Larsen Lefschetz theorems.

**Lefschetz theorem.** Let $f: X \to Z$ be a regular immersion of the connected compact complex manifold $X$ into $Z$. Let $Y$ be a connected compact complex submanifold of $Z$. Let $k$ be the ampleness of $NY$, the normal bundle of $Y$, i.e. $k$ is the largest fibre dimension of the restriction of $\phi$ to $\mathbb{P}(N^*Y)$. Sommese [21C, Corollary 1.4] shows that $Z \setminus Y$ is $k + \text{cod} Y$ convex in the sense of Andreotti-Grauert. Assume that $2 \cdot \dim Y \geq k + \dim Z$ and $\dim X \leq \dim Y + 1$. Then $\pi_i(X, f^{-1}(Y), x) = 0$ for $i < \dim X - \text{cod} Y - k$.

It is difficult to compute $k$ exactly, but the inequality $k \leq \text{amp}(TZ)$ may be used.


**Connectedness theorem.** Let $f: X \to Z \times Z$ be a regular map of the compact irreducible variety $X$ to $Z \times Z$ and let $\Delta \subset Z \times Z$ be the diagonal embedding of $Z$. Let $l = \min\{rk(g_i)\}$ where $g = \bigoplus g_i$ is the decomposition of the Lie algebra of $G$ into...
simple ideals. Then

(i) \( \dim f(X) \geq 2 \dim(Z) - l = f^{-1}(\Delta) \neq \phi \).

(ii) \( \dim f(X) > 2 \dim(Z) - l = f^{-1}(\Delta) \) is connected.

From Table 1, we see that \( l \leq \text{ca}(Z) \). In [10A], Faltings explains that, in the above theorem, "\( l \)" may be replaced by the better bound "\( \text{ca}(Z) \)".

For \( Z \) other than projective space, it is not clear what are the higher homotopy results implied by the connectedness theorem; see Fulton and Lazarsfeld [12, §10.1].

I should mention, also, that part (i) of the connectedness theorem follows from work of Sommese [21, Proposition 1.1].

Example 1. \( G = \text{SL}(n + 1, \mathbb{C}) \), \( l = n = \text{rk}(G) \), and \( \text{ca}(G/P) = n \) for each parabolic \( P \). See Hansen [15, p. 3], for examples that the results (i), (ii) are sharp.

Example 2. \( G = \text{O}(2n, \mathbb{C}) \), \( l = n = \text{rk}(G) \), and \( \text{ca}(G/P) = 2n - 3 \) for each parabolic \( P \). Let \( Z \subset \mathbb{P}^{2n-1} \) be the \( 2n - 2 \) dimensional quadric given by the equation \( \sum_{i=1}^{2n} z_i^2 = 0 \). The above results imply that \( X \cap Y \neq \emptyset \) whenever \( X \) and \( Y \) are closed subvarieties of \( Z \) satisfying

\[
\dim(X) + \dim(Y) \geq 2(2n - 2) - (2n - 3) = 2n - 1.
\]

This result is sharp:

Let \( X \) and \( Y \) be the images, respectively, of the maps \( \mathbb{P}^{n-1} \rightarrow Z \) given by \( x \mapsto (x, ix) \) and \( y \mapsto (y, -iy) \). Clearly, \( X \cap Y = \emptyset \), while \( \dim(X) + \dim(Y) = 2n - 2 \).

Example 3. \( G = \text{O}(2n + 1, \mathbb{C}) \), \( l = n = \text{rk}(G) \), and

\[
\text{ca}(G/P) = \begin{cases} 
2n - 1 & \text{for one special } P, \\
2n - 2 & \text{for every other parabolic } P.
\end{cases}
\]

Let \( Z \) be the \( 2n - 1 \) dimensional quadric. The above results imply that

\[
X \cap Y \neq \emptyset \quad \text{whenever } X \text{ and } Y \text{ are closed subvarieties of } Z \text{ satisfying}
\]

\[
(*) \quad \dim(X) + \dim(Y) \geq 2n.
\]

In fact, as pointed out to me by Sommese, one has a stronger (and sharper) result by replacing \( 2n \) by \( 2n - 1 \) in (\(*\)). Sharpness follows from Example 2 since the \( 2n - 2 \) dimensional quadric is contained in \( Z \). The validity follows from the fact that \( H^\bullet(Z, \mathbb{C}) \) is ring isomorphic to \( H^\bullet(\mathbb{P}^{2n-1}, \mathbb{C}) \).

The computational part of the paper concerns a desingularization of the unipotent variety of \( G \). This map has been studied by Springer [22] and Steinberg [25]. Their results on fibre dimensions deal largely with the case \( P = \text{a Borel subgroup of } G \). R. Elkik [8] has also noticed the connection between this desingularization and the cotangent bundle of \( G/P \) (see Remark 3.3 for more details). In [26], Steinberg explains that, following my letter to him, he did the computations for determining \( \text{ca}(G/P) \). For the case \( P = B \) a Borel subgroup, he gives the succinct formula \( \text{ca}(G/B) = |\rho^*| \), where \( \rho \) is a highest root for the simple group \( G \) and \( \rho^* \) is its dual, or coroot. This formula is equivalent to Theorem 4.2.

The notion of \( k \)-ampleness is due to Andrew Sommese [20, §1], and I would like to thank him for suggesting that I consider doing these computations for \( G/P \). The preprint of Faltings provided further motivation to compare ampleness and rank in the simple groups.
I would like to thank James Carrell, Hanspeter Kraft, George Maxwell, Peter Slodowy and Robert Steinberg for helpful letters and conversations. A special thanks goes to Bomshik Chang for discussions on Weyl groups.

Table 1

| $G$ (all roots long) | $\text{ca}(G/P) = |\rho|$ for every parabolic subgroup $P$. |
|----------------------|---------------------------------------------------------------|
| $A_n$                | $n$                                                           |
| $D_n$                | $2n - 3$                                                      |
| $E_6$                | 11                                                            |
| $E_7$                | 17                                                            |
| $E_8$                | 29                                                            |

$G$ Coxeter-Dynkin diagram (The short roots are darkened)

| $G$ | $\text{ca}(G/B)$ | $|\rho|$ |
|-----|------------------|--------|
| $B_n$ | $2n - 2$ | $2n - 1$ |
| $C_n$ | $n$ | $2n - 1$ |
| $G_2$ | 3 | 5 |
| $F_4$ | 8 | 11 |

$\text{ca}(G/P_\Theta) = \text{ca}(G/B) + \min\{d(\alpha) : \alpha \in \Theta\}$

$\text{ca}(Z_1 \times Z_2) = \min(\text{ca}(Z_1), \text{ca}(Z_2))$ (cf. (3.4))

$|\rho|$ is the height of the highest root

$d(\alpha) =$ no. of nodes from $\alpha$ to the long roots

2. Background material. In this section, I describe the notation that is used in the paper. The references Borel \[3, 4, 5\], Carter \[6\], Humphreys \[16\], Samelson \[18\], Serre \[19\] and Steinberg \[24\] contain proofs and elaborations.

Let $G$ be a connected complex semisimple Lie group. The centralizer of a subset, $A$, of $G$ is $Z(A) = \{g \in G : ga = ag \ \forall a \in A\}$. The normalizer of $A$ is $N(A) = \{g \in G : gAg^{-1} \subset A\}$.

Let $\mathfrak{g}$ denote the Lie algebra of $G$, with Lie bracket $[ , ]$. For each $b \in \mathfrak{g}$, $\text{ad}(b) \in \text{End}(\mathfrak{g})$ is defined by $\text{ad}(b)(c) = [b, c]$. The Killing form on $\mathfrak{g}$ is given by $(b, c) = \text{Trace}(\text{ad}(b) \circ \text{ad}(c))$. This pairing is nondegenerate and induces an identification, $\mathfrak{g}^* \cong \mathfrak{g}$, of the vector space dual of $\mathfrak{g}$, with $\mathfrak{g}$.

For $g \in G$, let $C(g)(x) = gxg^{-1}$, and denote by $\text{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ the differential of $C(g)$ at the identity element of $G$. Let $\exp : \mathfrak{g} \to G$ be the exponential map. For each $a \in \mathfrak{g}$, $t \mapsto \exp(ta)$ is the 1-parameter subgroup of $G$ whose tangent vector at $t = 0$ is $a$. One has

$$\text{Ad}(\exp(a)) = e^{\text{ad}(a)} = \sum_{m=0}^{\infty} (\text{ad}(a))^m / m!.$$
Fix a maximal torus $H$ in a Borel subgroup $B$ of $G$. Let $\frak{h}$ be the Lie algebra of $H$. Let $R \subseteq \frak{h}^*$ be the root system with respect to $H$, and $R^+$ (resp. $R^-$) the positive (resp. negative) roots with respect to $B$. We write $\alpha > 0$ (resp. $\alpha < 0$) for $\alpha \in R^+$ (resp. $\alpha \in R^-$). Let $\Sigma = \{\alpha_1, \ldots, \alpha_n\}$ denote the simple roots, i.e. a basis for $R$; $n = \dim(H) = \text{rank}(G)$.

Each $\alpha \in R$ may be expressed as $\alpha = \Sigma n_i \alpha_i$ with either all $n_i \geq 0$ (i.e. $\alpha \in R^+$) or all $n_i \leq 0$ (i.e. $\alpha \in R^-$). The height of $\alpha$ is $|\alpha| = \Sigma n_i$. An element $\rho$ of $R$ is a highest root when $\rho + \alpha \not\in R$ whenever $\alpha > 0$. For simple $G$ there is a unique such root. Also, $R$ is a reduced root system, i.e. for each $\alpha \in R$, the only multiples of $\alpha$ that again belong to $R$ are $\pm \alpha$.

One has the decomposition
$$
\frak{g} = \frak{h} \oplus \{\frak{g}^\alpha: \alpha \in R\}
$$
where $\frak{g}^\alpha = \{a \in \frak{g}: \text{ad}(b)(a) = \alpha(b)a \forall b \in \frak{h}\}$ is the 1-dimensional root space associated to $\alpha$. Moreover, $[\frak{g}^\alpha, \frak{g}^\beta] \subseteq \frak{g}^{\alpha + \beta}$, so that $[\frak{g}^\alpha, \frak{g}^\alpha] = 0$ when $\rho$ is a highest root and $\alpha > 0$. Also $[\frak{g}^\alpha, \frak{g}^\beta] = 0$ when $\alpha + \beta \neq 0$ while $\frak{g}^\alpha$ and $\frak{g}^{-\alpha}$ are perfectly paired by the Killing form.

The homomorphism
$$
x_\alpha = \exp(\alpha) : \frak{g}^\alpha \to X_\alpha \subset G
$$
is an isomorphism onto $X_\alpha$, the root subgroup associated to $\alpha$. One has $X_\rho \subset Z(X_\alpha)$ when $\rho$ is a highest root and $\alpha > 0$.

For each subset $\Theta \subseteq \Sigma$, one has the standard parabolic subgroup
$$
(2.1) \quad P = P_\Theta = H \cdot \prod \{X_\alpha: \alpha \in \langle \Theta \rangle\} \cdot \prod \{X_\alpha: \alpha > 0, \alpha \not\in \langle \Theta \rangle\}.
$$
The products are taken in any fixed order, and $\langle \Theta \rangle$ denotes the span of $\Theta$ in $R$. The third factor is the unipotent radical, $U_\rho$ of $P$, and $P \subseteq N(U_\rho)$. Every parabolic subgroup of $G$ is conjugate to some standard parabolic.

Similarly, the Lie algebra of $P_\Theta$ is
$$
p = \frak{h} \oplus \{\frak{g}^\alpha: \alpha \in \langle \Theta \rangle\} \oplus \{\frak{g}^\alpha: \alpha > 0, \alpha \not\in \langle \Theta \rangle\}.
$$
The third summand is the nilpotent radical $N_\rho$ of $P$ and is also the annihilator of $\frak{p}$ in $\frak{g}$ with respect to the Killing form.

(2.2) We denote $U = U_B = \prod \{X_\alpha: \alpha > 0\}$. Then $X_\rho \subset Z(U)$ when $\rho$ is a highest root. The map
$$
H \times C^N \to B \quad (N = \#(R^+)),
$$
$$(t, c) \mapsto t \cdot \prod \{x_\alpha(c_\alpha): \alpha > 0\}$$
is an isomorphism of varieties.

(2.3) The finite group $W = N(H)/H$ is the Weyl group of $G$ with respect to $H$. For each $\omega \in W$, let $n_\omega$ be a fixed representative in $N(H)$. The group $W$ embeds as a group of linear transformations on $\frak{h}^*$, leaving $R$ invariant. One has $n_\omega X_\alpha n_\omega^{-1} = X_{\sigma_\alpha}(\omega)$ for each root subgroup $X_\alpha$ and $\omega \in W$.

For each $\alpha \in R$, let $\sigma_\alpha \in W$ be the reflection through $\alpha$. Let $\sigma_i = \sigma_{\alpha_i}$ for each of the simple roots $\alpha_i$. Then, $W$ is generated by the simple reflections $\{\sigma_1, \ldots, \sigma_n\}$. The length, $l = l(\omega)$, of an element $\omega \in W$ is defined as the least number of simple
reflections by which one may write \( \omega = \sigma_i \cdots \sigma_j \sigma_i \). One has also that \( l(\omega) = \#\{\alpha > 0 : \omega(\alpha) < 0\} \). The simple reflection \( \sigma_i \) permutes the elements of \( R^+ \backslash \{\alpha_i\} \). It follows that

\[
l(\omega \sigma_i) = \begin{cases} l(\omega) + 1, & \omega(\alpha_i) > 0, \\ l(\omega) - 1, & \omega(\alpha_i) < 0. \end{cases}
\]

There is a unique element, \( \omega_0 \), of \( \mathfrak{W} \) taking \( R^+ \) to \( R^- \). One has \( l(\omega \omega_0) = \#(R^+) - l(\omega) \) for each \( \omega \in \mathfrak{W} \).

Let \( (, , ) \) denote a \( \mathfrak{W} \)-invariant positive definite pairing on \( \mathfrak{h}^* \). When \( G \) is simple, at most two root lengths occur in \( R \) (long and short). If only one length occurs, it is called long.

<table>
<thead>
<tr>
<th>( G )</th>
<th>Squared length ratio (long : short)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_n, D_n, E_n )</td>
<td>all roots long</td>
</tr>
<tr>
<td>( B_n, C_n, F_4 )</td>
<td>2 : 1</td>
</tr>
<tr>
<td>( G_2 )</td>
<td>3 : 1</td>
</tr>
</tbody>
</table>

The roots of a given length form an orbit of the Weyl group in \( R \). The reflection \( \sigma_\alpha \) is given by the formula \( \sigma_\alpha(\beta) = \beta - \alpha^*(\beta)\alpha \) where \( \alpha^*(\beta) = 2(\alpha, \beta)/(\alpha, \alpha) \in \mathbb{Z} \).

**Lemma 2.4.** Assume that \( \mathfrak{g} \) is simple. Let \( \alpha, \beta \in R \) such that \( \alpha \neq \pm \beta, (\alpha, \beta) \neq 0 \) and \( (\alpha, \alpha) \leq (\beta, \beta) \). Then

(i) \( \alpha^*(\beta) \cdot \beta^*(\alpha) = 1, 2, \) or 3,

(ii) if \( (\alpha, \beta) > 0 \) then \( \alpha^*(\beta) = (\beta, \beta)/(\alpha, \alpha) \) and \( \beta^*(\alpha) = 1 \), and

(iii) if \( \alpha \) and \( \beta \) are simple then \( (\alpha, \beta) < 0 \) and \( \alpha^*(\beta) = -\beta^*(\alpha) = -1 \).

**Proof.** (i) \( \alpha^*(\beta) \cdot \beta^*(\alpha) = 4(\alpha, \beta)^2/(\alpha, \alpha)(\beta, \beta) \), so \( \alpha^*(\beta) \cdot \beta^*(\alpha) = 0, 1, 2, 3 \) or 4 by Schwarz's inequality, but the hypotheses preclude the values 0 and 4.

(ii) \( \alpha^*(\beta)/\beta^*(\alpha) = (\beta, \beta)/(\alpha, \alpha) \), so the result follows from (i).

(iii) Serre [19, V8, Lemme 3], for example, shows that \( (\alpha, \beta) < 0 \), and the result now follows from (ii).

3. Ampleness of the tangent bundle. With the notation of §2, \( Z = G/P \) is a compact complex homogeneous space with group \( G \). Sommese [20, §1] defines the \( k \)-ampleness of a vector bundle over a compact complex space. This notion is discussed, in particular, for the tangent bundle, \( TZ \), in Goldstein [14, §2] which notation is reviewed below.

The global sections \( S_0, \ldots, S_N \) of \( TZ \) determine the "ampleness map"

\[
\phi: T^*Z \to \mathfrak{g}^*
\]

where \( \mathfrak{g}^* \) is the vector space dual of the Lie algebra, \( \mathfrak{g} \), of \( G \). Explicitly,

\[
\phi(\alpha) = z^* \mathfrak{g}^*(\alpha) \quad \text{where} \quad \alpha \in T^*_zZ, z^* : G \to Z, \\
g \mapsto gz \quad \text{and} \quad z^* : T^*_zZ \to T^*_zG = \mathfrak{g}^*.
\]
The bundle \( TZ \) is \( k \)-ample when
\[
\text{amp}(TZ) := \sup \{ \dim(\phi^{-1}(\alpha)) : \alpha \in g^* \setminus \{0\} \}
\]
has value at most \( k \). The coampleness is \( \text{ca}(Z) := \dim(Z) - \text{amp}(TZ) \).

This section reduces the determination of \( \text{ca}(G/P) \) to a calculation with Weyl groups (Proposition 3.6). First, the ampleness map is translated into Lie group data (Lemma 3.2).

**Notation.** Recall that \( p \subset g \) is the Lie algebra of the parabolic subgroup \( P \). Let \( p^\perp = \{ \alpha \in g^*: p \subset \ker(\alpha) \} \).

Let \( \Phi: G \times p^\perp \to g^* \) be the map
\[
(g, \alpha) \mapsto (\text{Ad}(g))^{-1}(\alpha).
\]
Let \( \Phi: G \times p^\perp \to g^* \) be the restriction of \( \Phi \) to \( G \times p^\perp \).

Define an action of \( P \) on \( G \times p^\perp \) by
\[
h \cdot (g, \alpha) = (gh^{-1}, (\text{Ad}(h))^{-1}(\alpha)).
\]
The map \( \Phi \) is constant on the orbits of \( P \), and so induces a map
\[
\tilde{\Phi}: (G \times p^\perp)/P \to g^*.
\]
The group \( G \) acts on \( G \times p^\perp \) by
\[
h \cdot (g, \alpha) = (hg, \alpha).
\]
This action commutes with the action of \( P \) just described, thus defining a \( G \)-action on the space \( (G \times p^\perp)/P \). The map \( \tilde{\Phi} \) is \( G \)-equivariant, where the action of \( G \) on \( g^* \) is the dual adjoint action.

**Lemma 3.1.** There is a commutative diagram (3.1.1), where \( \Gamma \) is a \( G \)-equivariant isomorphism of vector bundles over \( Z \).

\[
\begin{array}{cccccc}
G & \leftarrow & G \times p^\perp & \Phi \\
\pi \downarrow & & b \downarrow & \downarrow \\
G/P & \leftarrow & (G \times p^\perp)/P & \tilde{\Phi} & g^*
\end{array}
\]

(3.1.1)

The maps \( a, b, \pi, \tilde{\rho} \) and \( \rho \) are the natural projections.

**Proof.** It remains to construct \( \Gamma \). Define, first, a map
\[
\hat{\Gamma}: G \times p^\perp \to T^*Z,
\]
\[
(g, \alpha) \mapsto g^{-1}(\pi^{-1}(\alpha)).
\]
Here, let \( z_0 \) denote the point \( \pi(e) \) of \( G/P \) represented by the coset \( P \),
\[
\pi^*: T^*_Z \to T^*_eG = g^*
\]
and

\[ g^* : T^*_{g_0} \to T^*_{\phi g}Z. \]

Note that \( \pi^* \) is 1-1 and has image equal to \( \mathfrak{v}^\perp \). It is elementary to verify that \( \Gamma \) is \( G \)-equivariant, is constant on the orbits of \( P \), that the induced map

\[ \Gamma : (G \times \mathfrak{v}^\perp)/P \to T^*Z \]

is an isomorphism, and that \( \phi \circ \Gamma = \tilde{\phi} \). □

**Lemma 3.2.** There is a commutative diagram (3.2.1).

\[
\begin{array}{ccc}
T^*Z & \xrightarrow{\Lambda_1} & (G \times N_p)/P \\
\phi \downarrow & \simeq & \tilde{\psi} \downarrow \\
g^* & \searrow & \mathfrak{H} \\
\downarrow & \simeq & \downarrow \exp \\
\Upsilon & \searrow & V
\end{array}
\]

The map \( \Lambda_1 \) is a vector bundle isomorphism, and \( \Lambda_2 \) is an isomorphism of varieties.

Here, \( U_p \) and \( N_p \) are the unipotent and nilpotent radicals, respectively, of \( P \). The action of \( P \) on \( U_p \) is given by

\[ h \cdot (g, x) = \left( gh^{-1}, h x h^{-1} \right). \]

The action of \( P \) on \( N_p \) is given by

\[ h \cdot (g, v) = \left( gh^{-1}, \text{Ad}(h)(v) \right). \]

The map \( \mathfrak{H} \) is the Killing identification, and the exponential map, \( \exp \), takes the nilpotent elements \( \mathfrak{H} \) of \( g \) isomorphically onto the unipotent variety \( V \) of \( G \) (cf. Springer [22, §3]). We have \( \psi(g, x) = gxg^{-1} \), and \( \tilde{\psi}(g, v) = \text{Ad}(g)(v) \).

**Proof.** One verifies that \( \mathfrak{H}(\text{ad}(c)*) = -\text{ad}(c) \) for each \( c \in \mathfrak{g} \). Thus, \( \mathfrak{H}(\text{Ad}(g)*) = \text{Ad}(g)^{-1} \). Also, \( \mathfrak{H}(\mathfrak{v}^\perp) = N_p \) and \( \exp(N_p) = U_p \). Using these facts, together with Lemma 3.1, diagram (3.2.1) may be constructed. □

The goal in the remainder of this paper is to calculate the largest fibre dimension of the ampleness map \( \phi \) (or \( \tilde{\phi} \) or \( \psi \)).

**Remark 3.3.1.** The map \( \psi \) has been studied by Springer [22] and Steinberg [25] when \( P = B \). Let

\[ W := \{(gP, u) \in G/P \times V : g^{-1}ug \in U_p\}. \]

The projection \( \pi : W \to V \) is equivalent to the map \( \psi \) of Lemma 3.2. Springer shows, in the case \( P = B \), that \( \pi \) is generically 1-1 and each fibre of \( \pi \) is connected. Steinberg extends these results to general parabolics. In the case \( P = B \), he shows that

\[ \dim(\pi^{-1}(u)) = \frac{1}{2} \times (\dim(Z(u)) - \text{rk}(G)) \]

and obtains some results on the number of irreducible components in each fibre of \( \pi \).

**Remark 3.3.2.** The space \( (G \times N_p)/P \) has also been studied by Elkik [8, §1], the identification with \( T^*(G/P) \) being done in the language of schemes.
We next reduce the computation of $\text{ca}(G/P)$ to the case where $G$ is simple. Without loss of generality, assume that $G$ is simply connected, so $G = \times_{i=1}^{m} G_i$, where the $G_i$ are simple normal subgroups of $G$. From the description of standard parabolics, one has $P = \times_{i=1}^{m} P_i$, so that $G/P \cong \times_{i=1}^{m} (G/P_i)$. Let $Z_i = G/P_i$ and let $Z = G/P \cong \times_{i=1}^{m} Z_i$. Let
\[ \phi_i : T^*Z_i \to g_i^* \]
be the ampleness maps. Then
\[ \phi = \bigoplus_{i=1}^{m} \phi_i : \bigoplus_{i=1}^{m} (T^*Z_i) \to \bigoplus_{i=1}^{m} g_i^* \]
is the ampleness map of $Z$. It follows that
\[ \text{(3.4)} \quad \text{ca}(Z) = \min\{\text{ca}(Z_i) : i = 1, \ldots, m\}. \]

For the remainder of the paper, we assume that $G$ is simple.

**Proposition 3.5.** Let $G$ be a (simple) complex Lie group. Let $p$ be the highest root of $G$ with respect to some ordering, and let $x_p$ be any nonidentity element of $X_p$, the root subgroup of $G$ associated to $p$. Let $P$ be a parabolic subgroup of $G$, with unipotent radical $U_p$. Then
\[ \text{ca}(G/P) = \text{cod}_{G}\{g \in G : g^{-1}x_pg \in U_p\}. \]

**Proof.** We use the notation of Lemma 3.2. The projectivization of the ampleness map
\[ P(\psi) : (G \times P(N_p))/P \to P(\mathfrak{U}) \]
is proper and $G$-equivariant. The space $P(\mathfrak{U})$ possesses a unique closed $G$-orbit $\mathfrak{O}$, which is the orbit of a highest root-vector line. Thus, any point of $P(\mathfrak{U})$ may be specialized, within its orbit, to a point of $\mathfrak{O}$ and we have that the maximum $\psi$-fibre dimension (over $\mathfrak{U} \setminus \{0\}$) occurs at $v_p \in \mathfrak{U}$, where $v_p$ is a highest root vector.

Viewing the ampleness map now as
\[ \psi : (G \times U_p)/P \to V, \]
the maximum fibre dimension of $\psi$ is
\[ \dim(\psi^{-1}(x_p)) = \dim(\{ (g, u) \in G \times U_p : gug^{-1} = x_p \}) - \dim(P) \]
\[ = \dim(\{ g \in G : g^{-1}x_pg \in U_p \}) - \dim(P) \]
where $x_p = \exp(v_p)$. Thus,
\[ \text{ca}(G/P) = \dim(G/P) - \dim(\psi^{-1}(x_p)) \]
\[ = \dim(G) - \dim(\{ g \in G : g^{-1}x_pg \in U_p \}). \]

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2 This fact, as related to me by H. Kraft and P. Slodowy, is well known: The adjoint action of $G$ on $\mathfrak{g}$ is irreducible. So, there is a unique line in $\mathfrak{g}$ which is invariant under the Borel group viz. the highest root-vector line (see e.g. [24, §3, 4a]). Using the fact that any two Borel subgroups of $G$ are conjugate, one concludes that the orbit of this line in $P(\mathfrak{U})$ is the unique closed orbit.
The last result of this section expresses $ca(G/P)$ in terms of the length function of the Weyl group of $G$.

**Proposition 3.6.** Let $P$ be the standard parabolic subgroup of the simple group $G$ associated to the subset $\Theta \subset \Sigma$. Then

\[(3.6.1) \quad ca(G/P) = \min \{ l(\omega) : \omega \in \mathfrak{N}, \omega(\rho) < 0 \text{ and } \omega(\rho) \not\in \langle \Theta \rangle \}. \]

Moreover, the irreducible components of a largest fibre of the ampleness map are in 1-1 correspondence with those $\omega$'s which minimize (3.6.1).

**Proof.** By the Bruhat decomposition, each $g \in G$ may be expressed uniquely in the form

$$g = un_\omega b \quad \text{with } u \in U, \omega \in \mathfrak{N} \text{ and } b \in B.$$  

By Proposition 3.5, we have that

$$ca(G/P) = \text{cod}_{G}\{ g \in G : g^{-1}x_{\rho}g \in U_{P} \}.$$  

Now,

$$g^{-1}x_{\rho}g = b^{-1}n_{\omega}^{-1}u^{-1}x_{\rho}un_{\omega}b \in U_{P} \iff n_{\omega}^{-1}x_{\rho}n_{\omega} \in U_{P}$$

$$\iff \omega^{-1}(\rho) \in N_{P} \iff \omega^{-1}(\rho) > 0 \text{ and } \omega^{-1}(\rho) \not\in \langle \Theta \rangle.$$  

The first equivalence follows from $\mathfrak{N} \subset N(U_{P})$ and $x_{\rho} \in Z(U)$. The last two follow from 2.2, 2.3 and 2.1.

We conclude that the maximum fibre dimension for the ampleness map is

$$M = \max \{ \dim(Un_{\omega}B) : \omega^{-1}(\rho) > 0, \omega^{-1}(\rho) \not\in \langle \Theta \rangle \} - \dim(P)$$

$$= \dim(B) - \dim(P) + \max \{ l(\omega) : \omega(\rho) > 0, \omega(\rho) \not\in \langle \Theta \rangle \}.$$  

Here we have used the formula

$$\dim(Un_{\omega}B) = \dim(B) + l(\omega)$$  

and then made the substitution $\omega$ for $\omega^{-1}$. Hence

$$ca(G/P) = \dim(G/P) - M$$

$$= \#(R^{+}) - \max \{ l(\omega) : \omega(\rho) > 0, \omega(\rho) \not\in \langle \Theta \rangle \}$$

$$= \min \{ l(\omega_{0}) : \omega(\rho) > 0, \omega(\rho) \not\in \langle \Theta \rangle \}. $$

(Recall that $\dim(G/B) = \#(R^{+})$ and that $l(\omega_{0}) = \#(R^{+}) - l(\omega)$ where $\omega_{0}$ is the Weyl group element taking $R^{+}$ to $R^{-}$.) The proposition now follows by making the substitution $\omega \to \omega_{0}$ and noting that $\omega_{0}(\rho) = -\rho$. □

**4. Computing** $ca(G/P)$. The calculation is divided into three parts (recall that $G$ is simple):

(i) Determine $ca(G/B)$ from the expression of the highest root $\rho = \Sigma n_{i} \alpha_{i}$ in terms of the simple roots $\Sigma = \{ \alpha_{1}, \ldots, \alpha_{n} \}$, as in Theorem 4.2.

(ii) For each standard maximal parabolic $P_{i} := P_{\Sigma \setminus \{ \alpha_{i} \}}$, determine $ca(G/P_{i})$.

Let $d(i)$ be the least number of steps, in the Coxeter-Dynkin diagram of $G$, from $\alpha_{i}$ to the long roots. Then, by Theorem 4.7

$$ca(G/P_{i}) = d(i) + ca(G/B).$$
(iii) Now, let $P = P_\Theta$ be any standard parabolic. By formula (3.6.1), we have
\[
ca(G/P) = \min \{ l(\omega) : \omega\rho < 0 \text{ and } \omega\rho \text{ involves some } \alpha_i, \text{ with } \alpha_i \not\in \Theta \}
\]
\[
= \min \{ ca(G/P_i) : \alpha_i \not\in \Theta \}
\]
\[
= \min \{ d(i) : \alpha_i \not\in \Theta \} + ca(G/B).
\]
The results are summarized in Table 1 in §1.

From formula (3.6.1), we see that, in particular,
\[
(4.1.1) \quad ca(G/B) = \min \{ l(\omega) : \omega(\rho) < 0 \}. \tag{4.1.1}
\]
To each simple root $\alpha_i$ associate the integer
\[
(4.1.2) \quad v_i = \begin{cases} 
1, & \alpha_i \text{ long,} \\
2, & \alpha_i \text{ short, } G \neq G_2, \\
3, & \alpha_i \text{ short, } G = G_2.
\end{cases}
\]
Suppose that $\beta$ is a long root of height at least 2, and that $(\alpha_i, \beta) > 0$ for some $i$. Then
\[
(4.1.3) \quad |\sigma_i(\beta)| = |\beta| - v_i
\]
since $\sigma_i^*(\beta) = v_i$, as in Lemma 2.4.

**Theorem 4.2.** Let $\rho = \sum n_i \alpha_i$ be the expression of the highest root $\rho$ in terms of simple roots. Then
\[
ca(G/B) = \sum n_i/v_i.
\]
(The $v_i$ are defined in (4.1.2).)

**Proof.** It is well known that for each $\beta > 0$ there is some simple root $\alpha_i$ with $(\alpha_i, \beta) > 0$. This, together with (4.1.3) and formula (4.1.1), proves the theorem.

Table 1 contains the results of calculating $ca(G/B)$ for each of the simple Lie groups.

We turn next to calculating $ca(G/P_i)$ where $P_i$ is the maximal parabolic $P_{\Sigma \setminus \{\alpha_i\}}$.

**Lemma 4.3.** Let $\omega \in \Omega$ minimize $l(\omega)$ in formula (4.1.1). Then
(i) $\omega(\rho) = -\alpha_i$ for some (long) simple root $\alpha_i$, and
(ii) $\omega^{-1}(\alpha_j) > 0$ for every other simple root $\alpha_j$.

Let $\alpha_k$ be any long simple root. Then there exists a unique $\omega_k$ minimizing $l(\omega)$ in formula (4.1.1) and satisfying $\omega_k(\rho) = -\alpha_k$.

**Proof.** Suppose that $\omega^{-1}(\alpha) < 0$ and $\omega(\rho) \neq -\alpha_j$. Then $l(\sigma_j \omega) < l(\omega)$. But $\sigma_j$ permutes $R^+ \setminus \{\alpha_i\}$, so that $\sigma_j \omega(\rho) < 0$. This contradicts the minimality of $\omega$, and proves (i) and (ii).

The subdiagram of the Coxeter-Dynkin diagram for $G$ consisting of the long roots is connected, so we may assume that $(\alpha_i, \alpha_k) \neq 0$. Then, as in Lemma 2.4,
\[
\alpha_i^*(\alpha_k) = \alpha_k^*(\alpha_i) = -1,
\]
so
\[
\sigma_i \sigma_k(\alpha_i) = \sigma_i(\alpha_i + \alpha_k) = -\alpha_i + \alpha_k + \alpha_i = \alpha_k.
\]
To see that $\omega_k = \sigma_i \sigma_k \omega$ is the required Weyl group element, it remains to see that $l(\omega_k) = l(\omega)$. This follows from properties (i) and (ii) of $\omega$:

$$l(\omega_k) = l(\omega) + 1 \quad \text{and} \quad l(\sigma_k \omega) = l(\omega)$$

$$l(\sigma_k \omega) = l(\omega) \quad \text{and} \quad l(\sigma_k \omega) = l(\sigma_k \omega) - 1 = l(\omega).$$

The uniqueness of $\omega_k$ is a standard result on parabolic subgroups of $G$, since $\omega_k$ is the unique element for which $\min\{l(\omega): \omega(\rho) = -\alpha_k\}$ is attained, e.g. see Carter [6, §2.5].

For the maximal parabolics, formula (3.6.1) reads

$$c_a(G/P) = \min\{l(\omega): \omega(\rho) < 0 \text{ and } \omega(\rho) \text{ involves } \alpha_i\}. \tag{4.4.1}$$

For each simple root $\alpha_i$, let

$$k(i) := \min\{l(\omega): \omega(\alpha_j) \text{ involves } \alpha_i, \text{ for some long simple root } \alpha_j\}. \tag{4.4.2}$$

**Remark 4.5.** $k(i) = 0 \iff \alpha_i$ is long, and $k(i) = 1 \iff \alpha_i$ is short and is adjacent to a long root in the Coxeter-Dynkin diagram for $G$.

For each simple root $\alpha_i$, let $d(i)$ be the number of nodes, in the Coxeter-Dynkin diagram for $G$, from $\alpha_i$ to the long roots.

**Lemma 4.6.** For each $i$, we have $k(i) = d(i)$.

**Proof.** The only diagrams having more than one short root are those for $C_n$ and $F_4$. By Remark 4.5, it remains only to verify the lemma for these two cases. Samelson [18, pp. 79–86] contains the descriptions that I will be using of the root systems. The short roots are underlined. In each case, $\Sigma = \{\alpha_1, \ldots, \alpha_n\}$ is the implicit labeling of the simple roots, and $\sigma_i$ is the reflection through $\alpha_i$.

**I.** $G = C_n$.

$$R^+ = \{2a_i, a_i \pm a_j, i < j\}$$

$$\Sigma = \{a_1 - a_2, \ldots, a_{n-1} - a_n, 2a_n\}.$$ 

Figure (4.6.1) shows the only paths (without backtracking) between the long positive roots:

$$2a_n \sigma_{n-1}^{a_{n-2}} \sigma_2 \sigma_1 \rightarrow 2a_{n-1} \rightarrow \cdots \rightarrow 2a_2 \rightarrow 2a_1. \tag{4.6.1}$$

It follows that $k(i) = n - i = d(i)$.

**II.** $G = F_4$.

$$R^+ = \{a_i, a_i \pm a_j, i < j, \frac{1}{2} \times (a_1 \pm a_2 \pm a_3 \pm a_4)\}.$$ 

$$\Sigma = \{\frac{1}{2} \times (a_1 - a_2, a_3 - a_4), a_4, a_3 - a_4, a_2 - a_3\},$$

$$\sigma_1 \sigma_2(a_3) = \sigma_1(a_3 + a_4) = \frac{1}{2} \times (a_1 - a_2 + a_3 + a_4).$$

Thus, $k(1) = 2 = d(1)$. \qed
**Theorem 4.7.** Let $P_i = P_{\Sigma \setminus \{i\}}$ be a maximal parabolic subgroup of the simple group $G$. Then

\[ \text{ca}(G/P_i) = d(i) + \text{ca}(G/B). \]

**Proof.** By Lemma 4.6, it suffices to prove the result with $k(i)$ in place of $d(i)$. Let $\omega = \sigma_{i_1} \cdots \sigma_{i_r} \sigma_i$ be a reduced expression for some $\omega$ minimizing $l(\omega)$ in formula (4.4.1). By considering the sequence,

\[ \beta_r = \sigma_{i_1} \cdots \sigma_{i_r} \sigma_i(\rho), \quad r = 0, \ldots, m, \]

one sees that $\omega$ may be written as $\omega = \omega_2 \omega_1$ where $\omega_1$ minimizes $l(\omega)$ in formula (4.1.1), $\omega_2$ minimizes $l(\omega)$ in formula (4.4.2), and $l(\omega) = l(\omega_1) + l(\omega_2)$. \qed

**5.** Let $L\Sigma$ denote the long roots in the basis $\Sigma$ of the root system $R$ of $G$, a simple complex Lie group. Let $n(P)$ be the number of irreducible components in a largest fibre of the ampleness map $\psi$ (cf. Remark 3.3.1). By Proposition 3.6 and Lemma 4.3, we have that $n(B) = \#(L\Sigma)$. (In fact, when $R$ contains only long roots, then $n(P_\emptyset) = \#(L\Sigma \setminus \emptyset).$) For the maximal parabolics, $n(P_i) = 1$. To see this, one need only verify the uniqueness of minimizing elements for formula (4.4.2). More generally,

\[ n(P_\emptyset) = \max\{1, \#(L\Sigma \setminus \emptyset)\}. \]

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