DIMENSION OF STRATIFIABLE SPACES

BY

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Abstract. We define a subclass, denoted by $EM_3$, of the class of stratifiable spaces, and obtain several dimension theoretical results for $EM_3$ including the coincidence theorem for dim and Ind. The class $EM_3$ is countably productive, hereditary, preserved under closed maps and, furthermore, the largest subclass of stratifiable spaces in which a harmonious dimension theory can be established.

1. Introduction. Beyond metric spaces, the following line of generalized metric spaces has been established by many authors [S, C, B, H, Ok]:

metric $\rightarrow$ Lašnev$^1$ $\rightarrow$ $M_1$ $\rightarrow$ stratifiable $\rightarrow$ paracompact $\sigma$.

After Katětov and Morita’s work for metric spaces, the first attack to this line in dimension theory was done by Leibo [L1] who proved the equality $\dim X = \mathrm{Ind} X$ for any Lašnev space $X$. Nagami extended this result by defining $L$-spaces [N3] and free $L$-spaces [N4]. Free $L$-spaces form a countably productive and hereditary class containing every Lašnev space and included in the class of $M_1$-spaces. It is now desired to develop a satisfactory dimension theory of a still larger class of generalized metric spaces, say, $M_1$-spaces or stratifiable spaces.

In this direction we define a subclass of stratifiable spaces in terms of a special kind of $\sigma$-closure-preserving collection.

Definition 1.1. Let $X$ be a space. A collection $\mathcal{S}$ of subsets of $X$ is called an encircling net (or, for short, $E$-net) if for any point $x$ and any open neighborhood $U$ of $x$, there exists a subcollection $\mathcal{S}'$ of $\mathcal{S}$ such that $x \in X - \mathcal{S}' \subseteq U$ and $\mathcal{S}'$ is a closed set of $X$ (where $\mathcal{S}'$ denotes the union of the members of $\mathcal{S}$).

By $EM_3$ we denote the class of stratifiable spaces with $\sigma$-closure-preserving $E$-nets, and by $M_3$ the class of stratifiable spaces.

The class $EM_3$ is countably productive, hereditary and preserved under closed maps as well as perfect maps (Corollary 3.9).

Our first main result is a characterization of members of $EM_3$ as those spaces which are the perfect (closed) images of zero-dimensional stratifiable spaces (Theorem 3.8). This means that $EM_3$ is just the maximal perfect subclass of $M_3$ in the sense of Nagami [N1].

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1 A space is called a Lašnev space if it is the closed image of a metric space.
The second main results appear in Theorems 4.2 and 4.3 and consist of the following theorems for $EM_{3}$:
(a) the equidimensional $G_{6}$-envelope theorem,
(b) the dimension raising theorem,\(^2\)
(c) the decomposition theorem,
(d) the coincidence theorem for $\text{dim}$ and $\text{Ind}$.
These theorems for $EM_{3}$ extend the corresponding theorems for free $L$-spaces \([N_{4}]\) as well as those for Lašnev spaces \([L_{1}, L_{2}, O_{1}]\).
It is an open problem whether the inclusion $EM_{3} \subset M_{3}$ is proper. But the characterization above implies that $EM_{3}$ is the largest\(^3\) subclass of $M_{3}$ in which the dimension raising theorem holds. We also see in Corollary 4.5 that $EM_{3}$ is the largest\(^3\) subclass of $M_{3}$ in which the decomposition theorem and the equidimensional $G_{6}$-envelope theorem simultaneously hold.
Our arguments are based on Gruenhage and Junnila's result that a stratifiable space is an $M_{2}$-space \([G, J]\). Indeed, though we use the word "stratifiable" in view of its significance, what we need is only the existence of a $\sigma$-closure-preserving quasi-base.
**Conventions.** Throughout this paper a space is a Hausdorff topological space, and a map means an onto continuous one. Let $X, Y$ be spaces and let $f : X \to Y$ be a map. For a collection $\mathcal{F}$ of subsets of $X$, the symbol $\mathcal{F}^{*}$ denotes the union of all members of $\mathcal{F}$, and $f(\mathcal{F})$ means the collection of subsets of $Y$ of the form $\{f(F) : F \in \mathcal{F}\}$. For a subset $Z$ of $X$ we denote by $\overline{Z}$ (or $\text{Cl} \ Z$) the closure of $Z$, by $\text{Int} \ Z$ the interior of $Z$, and by $\text{Bd} \ Z$ the boundary of $Z$.

2. Encircling nets and large encircling nets. Encircling nets are naturally strengthened as follows:

**Definition 2.1.** Let $X$ be a space. A collection $\mathcal{S}$ of subsets of $X$ is called a large encircling net (or, simply, an LE-net) if for any disjoint closed sets $C$ and $K$ of $X$, there exists a subcollection $\mathcal{S}'$ of $\mathcal{S}$ such that $C \subseteq \mathcal{S}' \subseteq X - K$ and $\mathcal{S}'$ is a closed set of $X$.

**Remarks.** Since an LE-net is a net in the usual sense, it follows from Siwiec-Nagata \([SN]\) that a space with a $\sigma$-closure-preserving LE-net is a $\sigma$-space. But a space with a $\sigma$-closure-preserving $E$-net is not necessarily a $\sigma$-space as will be seen in Example 2.8. On the other hand it is trivial that a regular $\sigma$-space $X$ with $\text{ind} \ X = 0$ admits a $\sigma$-closure-preserving $E$-net, and that a normal $\sigma$-space $X$ with $\text{dim} \ X \leqslant 0$ admits a $\sigma$-closure-preserving LE-net.

**Proposition 2.2.** A metric space admits a $\sigma$-locally finite LE-net.

**Proof.** Let $M$ be a metric spaces and \(\{\mathcal{S}_{i} : i = 1, 2, \ldots\}\) a sequence of locally finite closed covers of $M$ such that, for each $i$, the diameter of each member of $\mathcal{S}_{i}$ is smaller than $1/i$. Let $C, K$ be disjoint closed sets of $M$ and put
\[
\mathcal{F}_{i} = \{ E \in \mathcal{S}_{i} : E \cap C \neq \emptyset \text{ and } E \cap K = \emptyset \}.
\]
\(^2\) The dimension raising theorem for a topological class $\mathcal{C}$ is: If $X \in \mathcal{C}$ and $\text{dim} \ X \leqslant n$, then $X$ is the image of a space $X_0 \in \mathcal{C}$ with $\text{dim} \ X_0 \leqslant 0$ under a perfect map of order not greater than $n + 1$.
\(^3\) When using this word we take no account of infinite-dimensional spaces in the sense of $\text{dim}$. License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
It is then clear that $\bigcup_{i=1}^{\infty} \mathcal{S}_i^*$ is a closed set of $X$ including $C$ but not meeting $K$. Hence $\bigcup_{i=1}^{\infty} \mathcal{S}_i$ is a $\sigma$-locally finite $LE$-net on $M$, which completes the proof.

**Proposition 2.3.** The property of having a $\sigma$-closure-preserving $LE$-net is preserved under closed maps.

We thus have

**Proposition 2.4.** A Lašnev space admits a $\sigma$-closure-preserving $LE$-net, and hence it is a member of $EM_3$.

**Lemma 2.5.** If $\mathcal{S}$ is an $E$-net (resp. $LE$-net) on a space, then $\{ \mathcal{E}: E \in \mathcal{S} \}$ is an $E$-net (resp. $LE$-net) on the space.

**Proposition 2.6.** The property of having a $\sigma$-closure-preserving $E$-net is countably productive, hereditary and preserved under perfect maps.

**Proof.** Let $X_i, i = 1, 2, \ldots, $ be spaces with $\sigma$-closure-preserving $E$-nets $\mathcal{S}_i$. It is then clear that

$$\left\{ E_j \times \prod_{i=1, i \neq j}^{\infty} X_i: E_j \in \mathcal{S}_j, j = 1, 2, \ldots \right\}$$

is a $\sigma$-closure-preserving $E$-net on $\prod_{i=1}^{\infty} X_i$.

By the preceding lemma it is obvious that the property is hereditary.

Let $X$ be a space with a $\sigma$-closure-preserving $E$-net $\mathcal{S}$ and let $f: X \to Y$ be a perfect map onto a space $Y$. By Lemma 2.5 we may assume that every finite intersection of members of $\mathcal{S}$ is again a member of $\mathcal{S}$. To show that $f(\mathcal{S})$ is an $E$-net on $Y$ let $y \in Y$ and let $U$ be an open neighborhood of $y$. There exist subcollections $\mathcal{S}_i, 1 \leq i \leq k,$ of $\mathcal{S}$ such that $f^{-1}(y) \cap X - \bigcup_{i=1}^{k} \mathcal{S}_i^* \subset f^{-1}(U)$ and $\mathcal{S}_i^*$ is a closed set of $X$. It then follows from assumption that $f(\bigcap_{i=1}^{k} \mathcal{S}_i^*)$ is a closed set of $Y$ written as a union of members of $f(\mathcal{S})$ such that $y \in Y - f(\bigcap_{i=1}^{k} \mathcal{S}_i^*) \subset U$. This completes the proof.

**Proposition 2.7.** Let $X$ be a space (resp. a semistratifiable space). Then the following statements are equivalent:

1. $X$ admits a $\sigma$-closure-preserving $LE$-net (resp. $E$-net).
2. $X$ admits a $\sigma$-locally finite $LE$-net (resp. $E$-net).
3. $X$ admits a $\sigma$-discrete $LE$-net (resp. $E$-net).

**Proof.** It follows from Lemma 2.5 and a remark above that a space with a $\sigma$-closure-preserving $LE$-net admits a $\sigma$-closure-preserving net of closed sets, and therefore it is semistratifiable. Hence the proposition is immediate from Lemma 2.5 and the following fact, which is essentially due to Siwiec and Nagata [SN]: Let $X$ be a semistratifiable space and $\mathcal{S}$ a $\sigma$-closure-preserving collection of closed sets of $X$. Then there exists a $\sigma$-discrete collection $\mathcal{F}$ of closed sets of $X$ such that each member of $\mathcal{S}$ is a union of members of $\mathcal{F}$.

As for famous pathological spaces, we have the following results which imply particularly that the existence of $\sigma$-closure-preserving $E$-nets does not mean, in general, that of $\sigma$-closure-preserving $LE$-nets (but, for stratifiable spaces, the former means the latter as will be seen in Theorem 3.8).
Examples 2.8. (1) The Michael line \( I(M) \) has a \( \sigma \)-discrete \( E \)-net, but does not have a \( \sigma \)-closure-preserving \( LE \)-net.

(2) The same is true for the Sorgenfrey line \( R(S) \).

(3) \([0, \omega_1]\) does not admit a \( \sigma \)-closure-preserving \( E \)-net.

(4) The quotient space \( I(M)/Q \) obtained by identifying the rational points in \( I(M) \) does not admit a \( \sigma \)-closure-preserving \( E \)-net. In particular the property of having a \( \sigma \)-closure-preserving \( E \)-net is not preserved under closed maps.

Proof. (1) and (2) (simultaneously). Let \( \mathcal{F} \) be a \( \sigma \)-discrete net of closed sets in the unit interval \( I \) (resp. the real line \( R \)) with the usual topology. It is easy to see that \( \mathcal{F} \) is a \( \sigma \)-discrete \( E \)-net on \( I(M) \) (resp. \( R(S) \)). But \( I(M) \) (resp. \( R(S) \)) does not admit a \( \sigma \)-closure-preserving \( LE \)-net because it is not a \( \sigma \)-space.

(3) For any \( \sigma \)-closure-preserving collection \( \mathcal{F} \) of \([0, \omega_1]\), \( \mathcal{F} \) fails to be an \( E \)-net at \( \omega_1 \); indeed, \( \text{Cl}(\{F: F \in \mathcal{F}, \omega_1 \notin F\}) \cap \{\omega_1\} = \emptyset \).

(4) If \( I(M)/Q \) had a \( \sigma \)-closure-preserving \( E \)-net, then every point in \( I(M)/Q \), in particular the quotient image of \( Q \), would be a \( G_\delta \)-set of \( I(M)/Q \); but this is impossible because \( Q \) is not a \( G_\delta \)-set of \( I(M) \).

3. Characterizations of \( EM_3 \).

Lemma 3.1 [O2, Lemma 3.1]. Let \( X \) be a submetrizable space (that is, \( X \) admits a weaker metric topology), and let \( \mathcal{U} \) be a \( \sigma \)-discrete collection of cozero sets of \( X \). Then there exist a metric space \( M \) and a one-to-one map \( f: X \to M \) such that \( f(U) \) is an open set of \( M \) for every \( U \in \mathcal{U} \).

The following lemma plays a fundamental role in this paper.

Lemma 3.2. Let \( X \) be a paracompact \( \sigma \)-space and let \( \mathcal{F} = \bigcup_{i=1}^{\infty} \mathcal{F}_i \) be a collection of closed sets of \( X \) such that \( \mathcal{F}_i \) is closure-preserving for each \( i \). Then there exist a metric space \( M \) and a one-to-one map \( f: X \to M \) such that \( f(F) \) is a closed set of \( M \) for every \( F \in \mathcal{F} \) and such that \( f(\mathcal{F}_i) \) is closure-preserving in \( M \) for every \( i \).

Proof. Let \( \mathcal{B} = \bigcup_{j=1}^{\infty} \mathcal{B}_j \) be a net of \( X \) consisting of closed sets such that \( \mathcal{B}_j \) is discrete for each \( j \). For each \( i \) let \( \mathcal{V}_i = \{V_i(B): B \in \mathcal{B}_i\} \) be a discrete collection of open sets of \( X \) such that \( B \subset V_i(B) \) for each \( B \in \mathcal{B}_i \). For \( i, j = 1, 2, \ldots \), \( B \in \mathcal{B}_j \), put

\[ W_i(B) = V_i(B) \cap \left( X - \{F: F \in \mathcal{F}_j, F \cap B = \emptyset\}\right) \ast. \]

Then \( W_i(B) \) is an open set of \( X \), and \( \{W_i(B): B \in \mathcal{B}_i\} \) is discrete in \( X \). Hence Lemma 3.1 applies to give a metric space \( M \) and a one-to-one map \( f: X \to M \) such that \( f(W_i(B)) \) is an open set of \( M \) for every \( B \in \mathcal{B}_j \), \( i, j = 1, 2, \ldots \). It is then obvious that for each \( i \), \( f(\mathcal{F}_i) \) is a closure-preserving collection of closed sets of \( M \). This completes the proof.

Definition 3.3. Let \( X \in EM_3 \) and let \( \{\mathcal{F}, \mathcal{V}, \mathcal{G}, \mathcal{S}\} \) be a quartet of collections of subsets of \( X \). The quartet is called an \( E \)-quartet if we can write \( \mathcal{F} = \bigcup_{j=1}^{\infty} \mathcal{F}_j \), \( \mathcal{V} = \bigcup_{j=1}^{\infty} \mathcal{V}_j \), \( \mathcal{G} = \bigcup_{j=1}^{\infty} \mathcal{G}_j \), \( \mathcal{S} = \bigcup_{j=1}^{\infty} \mathcal{S}_j \), and if the following four conditions are satisfied:

\[ (1_q) \] \( \mathcal{F} \) is a net on \( X \) consisting of closed sets.
(2q) For each i, \( V_i \) is a discrete collection of open sets of \( X \) written as \( V_i = \{ V_i(F) : F \in \mathcal{T}_i \} \) in such a manner that \( F \subset V_i(F) \) for each \( F \in \mathcal{T}_i \).

(3q) \( \mathcal{S} \) is an E-net on \( X \) consisting of closed sets and \( \mathcal{S}_i \) is closure-preserving for each \( i \).

(4q) \( \mathcal{S} \) is a quasi-base\(^4\) for \( X \) consisting of closed sets and \( \mathcal{S}_i \) is closure-preserving for each \( i \).

By Heath [H], Gruenhage [G] and Junnila [J], each member of \( EM_3 \) admits an E-quartet.

**Definition 3.4.** Let \( X \) be a member of \( EM_3 \) with an E-quartet \( \{ \mathcal{G}, \mathcal{V}, \mathcal{S}, \mathcal{S} \} \). A map \( f: X \to Y \) onto a normal space \( Y \) is called an E-map with respect to the E-quartet if the following five conditions are satisfied:

(0f) \( f \) is one-to-one.

(1f) \( f(F) \) is a closed set for every \( F \in \mathcal{G} \).

(2f) \( f(V) \) is an open set for every \( V \in \mathcal{V} \).

(3f) \( f(E) \) is a closed set for every \( E \in \mathcal{S} \), and \( f(\mathcal{S}_i) \) is closure-preserving in \( Y \) for every \( i \).

(4f) \( f(S) \) is a closed set for every \( S \in \mathcal{S} \), and \( f(\mathcal{S}_i) \) is closure-preserving in \( Y \) for every \( i \).

Noting that \( \{ X - V : V \in \mathcal{V} \} \) is a \( \sigma \)-closure-preserving collection of closed sets of \( X \), we have the following result by virtue of Lemma 3.2.

**Proposition 3.5.** Let \( X \) be a member of \( EM_3 \). Then for any E-quartet of \( X \) there exist a metric space \( M \) and an E-map \( f: X \to M \) with respect to the E-quartet.

The following lemma is well known (see, for example, [E, 2.3.16]).

**Lemma 3.6.** Let \( X \) be a space and let \( C, K \) be disjoint closed sets of \( X \). Let \( \mathcal{U} \) be a countable open cover of \( X \) such that for each \( U \in \mathcal{U} \), either \( U \cap C = \emptyset \) or \( U \cap K = \emptyset \). Then \( C \) and \( K \) are separated by a closed set \( S \) such that \( S \subset \{ \text{Bd } U : U \in \mathcal{U} \}^* \).

Now we have the following result frequently used later.

**Proposition 3.7.** Let \( X \) be a member of \( EM_3 \) with an E-quartet \( \{ \mathcal{G}, \mathcal{V}, \mathcal{S}, \mathcal{S} \} \). Let \( f: X \to Y \) be an E-map with respect to the E-quartet onto a normal space \( Y \). Then \( \text{Ind } X \leq \text{Ind } Y \).

**Proof.** The proof is by induction on \( \text{Ind } Y \). If \( Y = \emptyset \) then the proposition is trivial. Suppose that the proposition is valid when \( \text{Ind } Y < n - 1 \) and consider the case of \( \text{Ind } Y = n \). To show \( \text{Ind } X \leq n \), let \( C, K \) be disjoint closed sets of \( X \). For the time being, fix a point \( x \) in \( X - C \) arbitrarily. We show that there exists an open neighborhood \( W \) of \( x \) such that \( \overline{W} \cap C = \emptyset \) and \( \text{Ind } \text{Bd } W \leq n - 1 \). Let \( \mathcal{S}(x) \) be a subcollection of \( \mathcal{S} \) such that \( x \in X - \mathcal{S}(x)^* \subset X - C \) and \( \mathcal{S}(x)^* \) is a closed set. Write \( \mathcal{S}(x) \) is the closure of \( \mathcal{S}_i(x) \) where \( \mathcal{S}_i(x) \subset \mathcal{S}_i \). Put \( \mathcal{S}_i(x) = \{ S \in \mathcal{S}_i : S \cap \mathcal{S}_i(x)^* = \emptyset \} \).

\(^4\) A collection \( \mathcal{S} \) of subsets of a space \( X \) is called a quasi-base for \( X \) if for any point \( x \) and any open neighborhood \( U \) of \( x \) there exists a member \( S \) of \( \mathcal{S} \) such that \( x \in \text{Int } S \subset S \subset U \).
and $S(x) = \bigcup_{i=1}^{\infty} S_i(x)$. Fix $i_0$ so that $x \in \text{Int } S_{i_0}(x)^*$. By $(3_1)$ and $(4_1)$ there exist open sets $O_j, j = 1, 2, \ldots$, of $Y$ such that

$$f\left(\bigcup_{j=1}^{j} S_i(x)^*\right) \cup f(S_{i_0}(x)^*) \subset O_j \subset \overline{O}_j \subset Y - f(S_j(x)^*)$$

and

$$\text{Ind Bd } O_j \leq n - 1.$$ Define $W = \bigcap_{j=1}^{\infty} f^{-1}(O_j)$. Then

$$x \in W \subset \overline{W} \subset \bigcap_{j=1}^{\infty} f^{-1}(\overline{O}_j) \subset X - \delta(x)^* \subset X - C.$$ To show that $W$ is open, let $x' \in W$. Since $x' \in X - \delta(x)^*$ and $\delta(x)^*$ is a closed set, it follows from $(4_2)$ that $x' \in \text{Int } S_m(x)^*$ for some $m$. Then

$$x' \in \bigcap_{j=1}^{m-1} f^{-1}(O_j) \cap \text{Int } S_m(x)^* \subset W,$$

which implies that $W$ is open. To show $\text{Ind Bd } W \leq n - 1$, note that, for any subset $Z$ of $X$, $f| Z : Z \to f(Z)$ is again an $E$-map with respect to the $E$-quartet $\{\overline{F} | Z, \forall | Z, S | Z, S | Z\}$ on $Z$. Hence we may apply induction hypothesis to obtain $\text{Ind } f^{-1}(\text{Bd } O_j) \leq n - 1, j = 1, 2, \ldots$, which yields

$$\text{Ind Bd } W \leq \text{Ind} \left(\bigcup_{j=1}^{\infty} \text{Bd } f^{-1}(O_j)\right)$$

$$= \max\{\text{Ind Bd } f^{-1}(O_j) : j = 1, 2, \ldots\}$$

$$\leq \max\{\text{Ind } f^{-1}(\text{Bd } O_j) : j = 1, 2, \ldots\} \leq n - 1.$$ Hence $W$ is a required open neighborhood of $x$; we have thus finished “local” separation.

Now put

$$\mathcal{F}_i(C) = \{F \in \mathcal{F}_i : F \subset W \text{ for some open set } W \text{ with } \overline{W} \cap C = \emptyset \text{ and } \text{Ind Bd } W \leq n - 1\}.$$ Then by $(1_4)$ and by the “local” separation above, we have $\bigcup_{i=1}^{\infty} \mathcal{F}_i(C)^* = X - C$. For each $F \in \mathcal{F}_i(C)$, fix such a $W$ and denote it by $W_i(C, F)$. On the other hand, by $(1_4)$ and $(2_4)$, there exist open sets $H_i(F), F \in \mathcal{F}_i$, of $Y$ such that $f(F) \subset H_i(F) \subset \overline{H_i(F)} \subset f(V_i(F))$ and $\text{Ind Bd } H_i(F) \leq n - 1$ (where the set $V_i(F)$ is as in Definition 3.3(2_4)). By induction hypothesis again,

$$\text{Ind Bd } f^{-1}(H_i(F)) \leq \text{Ind } f^{-1}(\text{Bd } H_i(F)) \leq n - 1.$$ Put for each $F \in \mathcal{F}_i(C)$,

$$D_i(C, F) = W_i(C, F) \cap f^{-1}(H_i(F)).$$ Then

$$\text{Ind Bd } D_i(C, F) \leq \max\{\text{Ind Bd } W_i(C, F), \text{Ind Bd } f^{-1}(H_i(F))\} \leq n - 1.$$
Put $D_i(C) = \{D_i(C, F): F \in \mathcal{F}(C)\}$. Since $D_i(C, F) \subseteq V_i(F)$, $(2_q)$ implies that $\{D_i(C, F): F \in \mathcal{F}(C)\}$ is discrete. Thus $\text{Ind} \text{ Bd } D_i(C) < n - 1$, $i = 1, 2, \ldots$. By the same discreteness and by the fact $D_i(C, F) \subseteq W_i(C, F) \subseteq \text{Cl} W_i(C, F) \subseteq X - C$, we have $C \cap \text{Cl} D_i(C) = \emptyset$ for every $i = 1, 2, \ldots$. We also obtain $\bigcup_{i=1}^{\infty} D_i(C) = X - C$ because $\bigcup_{i=1}^{\infty} \mathcal{F}(C)^* = X - C$.

Quite similarly we can obtain open subsets $D_i(K)$, $i = 1, 2, \ldots$, such that $\text{Ind} \text{ Bd } D_i(K) < n - 1$, $K \cap \text{Cl} D_i(K) = \emptyset$ and $\bigcup_{i=1}^{\infty} D_i(K) = X - K$. Hence, applying Lemma 3.6, we have a closed set $B$ separating $C$ and $K$ such that

$$B \subseteq \left( \bigcup_{i=1}^{\infty} \text{ Bd } D_i(C) \right) \cup \left( \bigcup_{i=1}^{\infty} \text{ Bd } D_i(K) \right).$$

By the countable sum theorem for Ind, we have $\text{Ind } B \leq n - 1$. Thus $\text{Ind } X \leq n$, which completes the proof of Proposition 3.7.

We can now prove a characterization theorem for $EM_3$.

**Theorem 3.8.** The following statements about a space $X$ are equivalent:

1. $X$ is a stratifiable space with a $\sigma$-closure-preserving $E$-net.
2. $X$ is the perfect image of a stratifiable space $X_0$ with $\dim X_0 \leq 0$.
3. $X$ is the closed image of a stratifiable space $X_0$ with $\text{ind } X_0 \leq 0$.
4. $X$ is a stratifiable space with a $\sigma$-closure-preserving $LE$-net.

**Proof.** The implications (2) $\rightarrow$ (3) and (4) $\rightarrow$ (1) are obvious. To show (1) $\rightarrow$ (2) let $X$ be a member of $EM_3$ with an $E$-quartet $\{\mathcal{F}, \mathcal{V}, \mathcal{S}, \mathcal{S}\}$. By Proposition 3.5 there exists an $E$-map $f: X \to M$ onto a metric space $M$ with respect to $\{\mathcal{F}, \mathcal{V}, \mathcal{S}, \mathcal{S}\}$. By Morita [M], $M$ is the image of a metric space $P$ with $\dim P \leq 0$ under a perfect map $g$. Now let $T$ be the fiber product of $P$ and $X$ with respect to $g$ and $f$, that is,

$$T = \{(p, x) \in P \times X: g(p) = f(x)\}$$

with the topology induced from $P \times X$. Let $t_p, t_x$ be the restrictions to $T$ of the projections from $P \times X$ onto $P$ and $X$, respectively. We thus have the following commutative diagram:

$$\begin{array}{ccc}
X & \xrightarrow{t_x} & T \\
\downarrow f & & \downarrow t_p \\
M & \xleftarrow{g} & P
\end{array}$$

It is a well-known property of fiber products that the perfectness of $g$ implies the perfectness of $t_X$ (see [Pe, Lemma 7.5.13]). $T$ is stratifiable by [C, Theorems 2.3, 2.4]. Hence what should be proved is the zero-dimensionality of $T$. By Proposition 2.2, $P$ admits an $E$-quartet $\{\mathcal{F}_p, \mathcal{V}_p, \mathcal{S}_p, \mathcal{S}_p\}$. Now define

$$\begin{align*}
\mathcal{F}_T &= \{t_p^{-1}(F_p) \cap t_X^{-1}(F): F_p \in \mathcal{F}_p, F \in \mathcal{F}\}, \\
\mathcal{V}_T &= \{t_p^{-1}(V_p) \cap t_X^{-1}(V): V_p \in \mathcal{V}_p, V \in \mathcal{V}\}, \\
\mathcal{S}_T &= \{t_p^{-1}(S_p) \cap t_X^{-1}(S): S_p \in \mathcal{S}_p, S \in \mathcal{S}\}, & \text{and} \\
\mathcal{E}_T &= \{t_p^{-1}(E_p): E_p \in \mathcal{E}_p\} \cup \{t_X^{-1}(E): E \in \mathcal{E}\}.
\end{align*}$$

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Then it is easy to see that the quartet \((\mathcal{S}_T, \mathcal{V}_T, \mathcal{S}_T, \mathcal{S}_T)\) is an \(E\)-quartet of \(T\). Furthermore, the map \(t_p\) is an \(E\)-map with respect to \(\{\mathcal{S}_T, \mathcal{V}_T, \mathcal{S}_T, \mathcal{S}_T\}\) because, in general, \(t_p (t^{-1}_p (P') \cap \mathcal{V}_X (X')) = P' \cap g^{-1} \circ f(X')\) for any \(P' \subset P\) and \(X' \subset X\), and because \(f\) is an \(E\)-map with respect to \(\{\mathcal{S}, \mathcal{V}, \mathcal{S}, \mathcal{S}\}\). Hence, applying Proposition 3.7, we have \(\text{Ind} \ T \leq 0\). Thus the implication \((1) \rightarrow (2)\) has been proved.

To show \((3) \rightarrow (4)\) let \(X_0\) be a stratifiable space with \(\text{ind} \ X_0 \leq 0\) and let \(f: X_0 \rightarrow X\) be a closed map. Note that every net on \(X_0\) is an \(E\)-net; hence \(X_0\) is a member of \(EM_3\) by Heath [H]. It now follows from the implication \((1) \rightarrow (2)\) that \(X_0\) is the image of a stratifiable space \(X_1\) with \(\dim X_1 \leq 0\) under a perfect map \(h\). Since every net on \(X_1\) is an \(LE\)-net, it follows from Heath [H] again that \(X_1\) admits a \(\sigma\)-closure-preserving \(LE\)-net. Hence, applying Proposition 2.3 to the closed map \(f \circ h\), we see that \(X\) admits a \(\sigma\)-closure-preserving \(LE\)-net. On the other hand \(X\) is stratifiable by Borges [B, Theorem 3.1]. This completes the proof of Theorem 3.8.

**Corollary 3.9.** The class \(EM_3\) is countably productive, hereditary and preserved under closed maps.

**Proof.** This is immediate from Theorem 3.8, Proposition 2.6 and the analogous result for \(M_3\) due to Ceder [C] and Borges [B].

A topological class \(\mathcal{C}\) is called \textit{perfect} (Nagami [N], also see [N2]) if it is countably productive, hereditary, preserved under perfect maps, included in the class of normal spaces, and every member of \(\mathcal{C}\) is the perfect image of a zero-dimensional (in the sense of \(\dim\)) member of \(\mathcal{C}\). Theorem 3.8 and Corollary 3.9 say

**Corollary 3.10.** The class \(EM_3\) is the maximal perfect subclass of \(M_3\).

Recently Itô [I] has presented a free \(L\)-space, a certain closed image of which is not a free \(L\)-space. But we have

**Corollary 3.11.** Every closed image of a free \(L\)-space is a member of \(EM_3\).

**Proof.** By Nagami [N4, Theorem 2.10] and Theorem 3.8, every free \(L\)-space is a member of \(EM_3\) (it is also easy to directly prove that every free \(L\)-space admits a \(\sigma\)-closure-preserving \(E\)-net). Hence this corollary is immediate from Corollary 3.9.

4. Dimension for \(EM_3\). We begin with the equidimensional \(G_\delta\)-envelope theorem. To show this, the following lemma is useful.

**Lemma 4.1 (Oka [O4, Lemma 3.3]).** Let \(X\) be a hereditarily normal space and let \(f: X \rightarrow L\) be a map onto a metric space \(L\). Then for any subset \(Y \subset X\), there exist a \(G_\delta\)-set \(Z\) of \(X\), a metric space \(M\) and maps \(g: Z \rightarrow M\), \(h: M \rightarrow f(Z)\) such that

(i) \(Y \subset Z\),

(ii) \(\dim g(Y) \leq \dim Y\) and

(iii) \(f \mid Z = h \circ g\).

**Theorem 4.2.** Let \(X \in EM_3\) and let \(Y\) be a subset of \(X\) with \(\dim Y \leq n\). Then there exists a \(G_\delta\)-set \(G\) of \(X\) such that \(Y \subset G\) and \(\dim G \leq n\).

**Proof.** Let \(f: X \rightarrow L\) be an \(E\)-map onto a metric space \(L\) with respect to an \(E\)-quartet, say \(\{\mathcal{S}, \mathcal{V}, \mathcal{S}, \mathcal{S}\}\), on \(X\). By the above lemma there exist a \(G_\delta\)-set \(Z\) of \(X\), a
metric space $M$ and maps $g: Z \to M$, $h: M \to f(Z)$ satisfying (i), (ii), (iii) above. Since $\dim g(Y) \leq n$ and $M$ is metrizable, we can find a $G_\delta$-set $H$ of $M$ such that $g(Y) \subset H$ and $\dim H \leq n$ (see, for example, [E, 4.1.19]). Define $G = g^{-1}(H)$. Then $G$ is a $G_\delta$-set of $Z$, and hence of $X$. To show $\dim G \leq n$, note that $g|G$ is an $E$-map with respect to $\{\mathcal{F}|G, \mathcal{V}|G, \mathcal{S}|G, \mathcal{S}|G\}$ because $f|G$ is so and because $f|G = h \circ g|G$ by (iii). Hence by Proposition 3.7 we have $\text{Ind} G \leq \text{Ind} H$. Consequently

$$\dim G \leq \text{Ind} G \leq \text{Ind} H = \dim H \leq n,$$

as required. This completes the proof.

The following theorem occupies the central position in dimension theory of $EM_3$. The key argument of the proof has already appeared in the proof of Theorem 3.8.

**Theorem 4.3.** The following statements about a space $X$ are equivalent:

1. $X \in EM_3$ and $\dim X \leq n$.
2. $X$ is the image of a stratifiable space $X_0$ with $\dim X_0 \leq 0$ under a perfect map of order not greater than $n + 1$.
3. $X$ is a stratifiable space which is the union of $G_\delta$-sets $X_i$, $1 \leq i \leq n + 1$, with $\dim X_i \leq 0$.
4. $X \in EM_3$ and $\text{Ind} X \leq n$.

**Proof.** $(1) \rightarrow (2)$. Let $X$ be a member of $EM_3$ such that $\dim X \leq n$. Let $\{\mathcal{F}, \mathcal{V}, \mathcal{S}, \mathcal{S}\}$ be an $E$-quartet of $X$. By Proposition 3.5 there exist a metric space $L$ and an $E$-map $f: X \to L$ with respect to the $E$-quartet. By Pasynkov's factorization theorem [P, Theorem 29], there exist a metric space $M$ and maps $g: X \to M$, $h: M \to L$ such that $\dim M \leq n$ and $f = h \circ g$. It then follows from Morita [M] that $M$ is the image of a metric space $P$ with $\dim P \leq 0$ under a perfect map $r$ such that $\text{ord} r \leq n + 1$. Let $T$ be the fiber product of $P$ and $X$ with respect to $r$ and $g$, and let $t_P, t_X$ be the restrictions to $T$ of the projections from $P \times X$ onto $P$ and $X$, respectively. We thus obtain the following commutative diagram:

$$
\begin{array}{ccc}
X & = & X \\
\downarrow f & & \downarrow g \\
L & \leftarrow & M \\
\downarrow h & & \downarrow t_P \\
T & \leftarrow & P
\end{array}
$$

It is obvious that $t_X$ is a perfect map of order not greater than $n + 1$ and that $T$ is a stratifiable space. Note that $g$ is an $E$-map with respect to $\{\mathcal{F}, \mathcal{V}, \mathcal{S}, \mathcal{S}\}$ because $f$ is so and $f = h \circ g$. Now, as in the proof of Theorem 3.8, $t_P$ is also an $E$-map with respect to a certain $E$-quartet of $T$, and hence $\dim T \leq 0$ by Proposition 3.7.

$(2) \rightarrow (3)$. Let $t: X_0 \to X$ be a perfect map from a stratifiable space $X_0$ with $\dim X_0 \leq 0$ onto a space $X$ such that $\text{ord} t \leq n + 1$. Put $Y_i = \{x \in X: |t^{-1}(x)| = i\}$, $1 \leq i \leq n + 1$. It then follows from Nagami [N2, Lemma 4] that $\dim Y_i \leq 0$ for each $i = 1, 2, \ldots, n + 1$. Since $X$ is a member of $EM_3$ by Theorem 3.8, we may apply Theorem 4.2 to obtain $G_\delta$-sets $X_i$, $1 \leq i \leq n + 1$, such that $\dim X_i \leq 0$ and $Y_i \subset X_i$.

The implication $(4) \rightarrow (1)$ is trivial.
Finally the implication (3) \(\Rightarrow\) (4) is assured by the following theorem (but the fact \(\text{Ind } X \leq n\) only is direct from (3) as a consequence general for hereditarily normal spaces).

**Theorem 4.4.** Let \(X\) be a normal \(\sigma\)-space expressed as the finite union of \(G_\delta\)-sets \(X_i\), \(1 \leq i \leq k\), such that \(\dim X_i \leq 0\). Then \(X\) admits a \(\sigma\)-closure-preserving \(LE\)-net.

**Proof.** The proof is by induction on \(k\). When \(k = 1\), the theorem is trivial. Now suppose that the theorem is valid when \(k = m - 1\), and consider the case \(k = m\). Put \(Y_m = X - X_m\). Then by induction hypothesis and Lemma 2.5, the normal \(\sigma\)-space \(Y_m\) admits a \(\sigma\)-closure-preserving \(LE\)-net, say \(\mathcal{E}\), consisting of closed sets of \(Y_m\). Write \(Y_m = \bigcup_{i=1}^{\infty} C_i\) with closed sets \(C_i\) such that \(C_i \subset C_{i+1}\), and put \(\mathcal{E}_i = \mathcal{E} \mid C_i\). Let \(\mathcal{F}\) be a \(\sigma\)-locally finite net of \(X\). Now consider the \(\sigma\)-closure-preserving collection \(\bigcup_{i=1}^{\infty} \mathcal{E}_i \cup \mathcal{F}\) of \(X\). To show that the collection is an \(LE\)-net on \(X\), let \(C, K\) be disjoint closed sets of \(X\). Since \(X\) is hereditarily normal and \(\text{Ind } X_m \leq 0\), there exists a closed set \(S\) separating \(C\) and \(K\) such that \(S \cap X_m = \emptyset\). Represent \(X\) as the disjoint union \(V \cup S \cup W\), where \(V\) and \(W\) are open sets of \(X\) including \(C\) and \(K\) respectively. Write \(V = \bigcup_{i=1}^{\infty} V_i\) with open sets \(V_i\) such that \(V_i \subset V_{i+1}\) for every \(i\). For each \(i\) take a subcollection \(\mathcal{E}_i\) of \(\mathcal{E}_j\) such that

\[
(W \cup S) \cap C_i \subset \big|_{i}^{*} \subset C_i - (V_i \cup C)
\]

and \(\big|_{i}^{*}\) is a closed set of \(C_i\). Now put

\[
B = W \cup \left( \bigcup_{i=1}^{\infty} \big|_{i}^{*} \right).
\]

It is easy to see that \(B\) is a closed set of \(X\) including \(K\) and not meeting \(C\). Since \(W\) is the union of some members of \(\mathcal{F}\), \(B\) is the union of some members of \(\bigcup_{i=1}^{\infty} \mathcal{E}_i \cup \mathcal{F}\). Thus \(\bigcup_{i=1}^{\infty} \mathcal{E}_i \cup \mathcal{F}\) is a \(\sigma\)-closure-preserving \(LE\)-net on \(X\). This completes the proof of Theorem 4.4 and, therefore, of Theorem 4.3.

**Remark.** Slightly modifying the above proof, we can weaken the condition "\(X_i\) is \(G_\delta\)" in Theorem 4.4 to "\(X_i\) is either \(G_\delta\) or \(F_\sigma\)."

As a trivial version of Theorem 4.4, we have the following result which tells us that the dimension theory does not work well in the remainder \(M_3 - EM_3\).

**Corollary 4.5.** Let \(X\) be a normal \(\sigma\)-space not admitting a \(\sigma\)-closure-preserving \(LE\)-net. Then either

1. \(X\) cannot be decomposed into finitely many zero-dimensional (in the sense of \(\dim\)) subsets, or
2. there exists a zero-dimensional (in the sense of \(\dim\)) subset of \(X\) not admitting an equidimensional \(G_\delta\)-envelope.

As an immediate consequence of Theorem 4.3, we have

**Corollary 4.6.** Let \(X\) be a stratifiable space with \(\text{ind } X \leq 0\). Then \(\dim X = \text{Ind } X\).

**Remark.** This result, however, is generalized to paracompact \(\sigma\)-spaces in my recent paper [Oka].
We conclude this section with the following result, an immediate consequence of Corollary 3.11 and Theorem 4.3.

**Corollary 4.7.** Let $X$ be the closed image of a free $L$-space. Then $\dim X = \text{Ind } X$.

5. Other spaces admitting $σ$-closure-preserving $E$-nets. Let $C$ be a topological property. A space is called *peripherally* $C$ if every point in the space admits an open neighborhood base, the boundary of each member of which is $C$.

**Theorem 5.1.** (1) A peripherally $σ$-discrete, paracompact $σ$-space admits a $σ$-closure-preserving $E$-net.

(2) A peripherally $σ$-compact, stratifiable space admits a $σ$-closure-preserving $E$-net.

**Proof.** We shall prove (1) and (2) simultaneously. Let $\mathcal{F}$ be a $σ$-locally finite net (resp. a $σ$-closure-preserving quasi-base) of $X$ consisting of closed sets. To show that $\mathcal{F}$ itself is an $E$-net on $X$ let $x$ be a point of $X$ and $V$ an open neighborhood of $x$.

Take an open set $U$ such that $x \in U \subseteq \overline{U} \subseteq V$ and $\text{Bd } U$ is $σ$-discrete (resp. $σ$-compact). Write $U = \bigcup_{i=1}^{\infty} U_i$ with open sets $U_i$ such that $\overline{U}_i \subseteq U_{i+1}$ for every $i$. Write $\text{Bd } U = \bigcup_{i=1}^{\infty} C_i$ with discrete (resp. compact) closed sets $C_i$, $i = 1, 2, \ldots$. There exists, for each $i$, a discrete (resp. finite) subcollection $\mathcal{F}_i$ of $\mathcal{F}$ such that $C_i \subseteq \mathcal{F}_i \subseteq X - (\overline{U}_i \cup \{x\})$. Then $\bigcup_{i=1}^{\infty} \mathcal{F}_i \cup (X - \overline{U})$ is a closed set of $X$ including $X - V$, not meeting $\{x\}$ and expressed as a union of members of $\mathcal{F}$. Thus $\mathcal{F}$ is an $E$-net of $X$, which completes the proof.

Now we have the following generalization of Corollary 4.6.

**Corollary 5.2.** Let $X$ be a peripherally $σ$-compact (or peripherally $σ$-discrete) stratifiable space. Then $\dim X = \text{Ind } X$.

We next verify a countable sum theorem for $σ$-closure-preserving $LE$-nets.

**Theorem 5.3.** Let $X$ be a normal space expressed as the countable union of closed sets $X_i$, $i = 1, 2, \ldots$, each of which admits a $σ$-closure-preserving $LE$-net. Then $X$ has a $σ$-closure-preserving $LE$-net.

**Proof.** Note that $X$ is perfectly normal because each $X_i$ is. Let $\mathcal{G}_i$ be a $σ$-closure-preserving $LE$-net of $X_i$. It is clear that $\bigcup_{i=1}^{\infty} \mathcal{G}_i$ is $σ$-closure-preserving in $X$. To show that $\bigcup_{i=1}^{\infty} \mathcal{G}_i$ is an $LE$-net, let $C$ and $K$ be disjoint closed sets of $X$. Write $X - C = \bigcup_{i=1}^{\infty} V_i$ with open sets $V_i$ such that $\overline{V}_i \subseteq V_{i+1}$. For each $i$ let $\mathcal{F}_i$ be a subcollection of $\mathcal{G}_i$ such that $\bigcap_{i=1}^{\infty} \mathcal{F}_i$ is a closed set of $X_i$ and $C \cap X_i \subseteq \bigcap_{i=1}^{\infty} \mathcal{F}_i \subseteq X_i - (K \cup \overline{V}_i)$. It is then obvious that $\bigcup_{i=1}^{\infty} \mathcal{F}_i$ is a closed set of $X$ and $C \subseteq \bigcup_{i=1}^{\infty} \mathcal{F}_i \subseteq X - K$. This completes the proof.

The following result is immediate from Theorem 5.3, Proposition 2.2 and Ceder [C, Theorem 8.3].

**Corollary 5.4.** A chunk complex (and hence a $CW$-complex) is a member of $EM_3$.

We list several unsolved problems below.
**Problem 5.5.** (1) Does every stratifiable space admit a $\sigma$-closure-preserving $E$-net? By virtue of Theorem 3.8, this is equivalent to:

(2) (Nagami [N, Problem 4]) Is every stratifiable space a perfect image of a zero-dimensional (in the sense of dim) stratifiable space?

The author also does not know whether the inclusion $EM_3 \subset M_4$ (or $M_4 \subset EM_3$) holds or not.

**Problem 5.6.** Let $X$ be a paracompact $\sigma$-space admitting a $\sigma$-closure-preserving $E$-net. Then:

(1) Does the equality $\dim X = \text{Ind} X$ hold?

(2) Is $X$ a perfect image of a zero-dimensional (in the sense of dim) paracompact $\sigma$-space? More weakly:

(3) Does $X$ admit a $\sigma$-closure-preserving $LE$-net?

In the specific case of $\text{ind} X \leq 0$, (1) admits an affirmative answer by the inequality $\text{Ind} X \leq \dim X + \text{ind} X$ for every nonempty paracompact $\sigma$-space $X$ [O8]; (2) is also affirmative, that is, a paracompact $\sigma$-space of $\text{ind} \leq 0$ is the perfect image of a paracompact $\sigma$-space of dim $\leq 0$.

To outline the proof, let $X$ be a nonempty paracompact $\sigma$-space with $\text{ind} X = 0$. Let $\mathcal{F} = \bigcup_{i=1}^{\infty} \mathcal{F}_i$ and $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$ be as in Definition 3.3. Let $f: X \to M$ be a one-to-one map onto a metric space $M$ such that $f(\mathcal{V}_i^*)$ is open and $f(\mathcal{F}_i^*)$ is closed for every $i$. In [O8, Lemma 5] it is proved that, in general, $\text{Ind} X \leq \text{Ind} M + \text{ind} X$ for any such map $f: X \to M$. The metric space $M$ is the image of a metric space $L$ with $\dim L = 0$ under a perfect map $g$. Let $T$ be the fiber product of $L$ and $X$ with respect to $g$ and $f$. Let $t_L, t_X$ be the restrictions to $T$ of the projections from $L \times X$ onto $L$ and $X$, respectively. Then, since the map $t_L$ is of the “same type” as $f$, we have $\text{Ind} T \leq \text{Ind} L + \text{ind} T = \text{ind} T$. But, in the present case, $\text{ind} T \leq \text{ind}(L \times X) = 0$; hence $\text{Ind} T = 0$. It is clear that $T$ is a paracompact $\sigma$-space and $t_X$ is a perfect map. This completes the proof.

**Problem 5.7.** Let $X$ be a stratifiable space expressed as the union of countably many metrizable $(G_\delta)$ subsets. Does the equality $\dim X = \text{Ind} X$ hold? More strongly, does $X$ admit a $\sigma$-closure-preserving $E$-net? (A space of this type is a natural generalization of a Lašnev space in view of Lašnev's well-known decomposition theorem [La].)

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DIMENSION OF STRATIFIABLE SPACES


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