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DIFFERENTIABLE GROUP ACTIONS ON HOMOTOPY SPHERES: III. INVARIANT SUBSPHERES AND SMOOTH SUSPENSIONS

BY

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ABSTRACT. A linear action of an abelian group on a sphere generally contains a large family of invariant linear subspheres. In this paper the problem of finding invariant subspheres for more general smooth actions on homotopy spheres is considered. Classification schemes for actions with invariant subspheres are obtained; these are formally parallel to the classifications discussed in the preceding paper of this series. The realizability of a given smooth action as an invariant codimension two subsphere is shown to depend only on the ambient differential structure and an isotopy invariant. Applications of these results to specific cases are given; for example, it is shown that every exotic 10-sphere admits a smooth circle action.

In our previous papers in this series [**45, 44**], we have considered the theory of semifree actions on homotopy spheres as formulated by W. Browder and T. Petrie [**10**] and M. Rothenberg and J. Sondow [**34**]. Specifically, in the first paper a method was presented for describing (at least formally) those exotic spheres admitting such semifree actions—a problem first posed explicitly by Browder in [**3**, Problem 1, p. 7] —and the second paper extended the whole theory to handle certain actions that are not semifree. This paper will treat another problem posed in Browder's paper [**3**, Problem 3] regarding invariant subspheres of homotopy spheres with group actions.

One motivation for considering this question is that linear actions on spheres generally admit a great assortment of invariant linear subspheres (e.g., if the group is abelian and the dimension is much larger than the group's order), and from this viewpoint the existence of invariant subspheres reflects the extent to which an arbitrary smooth action resembles some natural linear model. In particular, this idea is central to the work of Browder and Livesay on free involutions [**8**] (compare also [**3**]).

The existence of such subspheres is directly related to the realizability of actions as equivariant smooth suspensions, providing basic necessary conditions for such a realization. We shall also consider a dual problem in this paper; namely, the description of those group actions that can be smoothly equivariantly suspended. Questions of this sort first arose in the study of free involutions [**16**], and their close

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relationship with the invariant subsphere problem was clarified in later work of Browder and Livesay [**8**], Lopez de Medrano [**29**], and others.

Of course, there is no topological obstruction to suspending an action, and given that exotic spheres roughly represent the simplest differences between the DIFF and TOP categories, it is fairly predictable that the obstructions should involve groups of homotopy spheres. Actually, these obstructions already arose in [**45** and **49**]; for circle actions, the differential structure on the homotopy sphere is the obstruction to smooth suspendability, while for cyclic group actions one must supplement a differential structure condition with an assumption that the action is isotopically trivial.

We shall find it convenient to divide invariant homotopy subspheres into two types. The Type I (homotopy) subspheres are those invariant $K \subseteq \Sigma$ such that the action is free on $\Sigma - K$, while the Type II subspheres are more or less the others. An important advantage of Type I subspheres is that the methods of [**50**] adapt easily to discuss them.

Aside from its intrinsic interest, the invariant subsphere question—especially for Type I subspheres—and the equivariant suspension construction figure importantly in answering some basic questions about the smooth S^1 and $\mathbf{Z}_p$ actions that can exist on a given exotic sphere. To illustrate the applicability of these notions, we shall construct a smooth S^1 action on the generator of $\Theta_{10} = \mathbf{Z}_6$, which previously had not been shown to admit such an action (compare [**41**]). The results on invariant subspheres in this paper also suffice to complete the proofs of some assertions in [**44**] about realizing certain classes in $\pi_1(\mathrm{Diff}(S^n))$ by continuous representations of the circle group (see §4). More comprehensive applications will appear in papers IV–VI of this series, including (i) a purely homotopy-theoretic characterization of those exotic spheres admitting $\mathbf{Z}_p$ (p an odd prime) actions with a fixed point set of given codimension and (ii) an inductive method for constructing smooth circle actions on exotic spheres with Pontrjagin-Thom invariants $\beta_2, \ldots, \beta_{p-1} \in \pi_{*(p)}$ (see [**48**] for a partial summary).

There have been several previous studies of invariant subspheres, particularly in the free case (e.g., see [**3, 8, 9, 12, 13, 17, 18, 19, 25, 29, 55**]). For the most part, such studies have dealt with a class of invariant subspheres called *characteristic* (see §2 for a description in our context; also compare [**3**], for example). Characteristic subspheres of semifree and more general actions have also been studied in [**3, 9,** and **56**].

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1. Type I subspheres and their knot invariants. We are mainly interested in the theory and applications of invariant subspheres for semifree actions. However, it is just as easy to formulate everything for ultrasemifree actions as defined in [**50**, p. 267]. If G is abelian, an effective action is ultrasemifree if the set of isotropy subgroups $G_x \neq \{1\}$ has a unique minimal element (called the *subprincipal isotropy*

subgroup). In particular, if p is prime then every action of $\mathbf{Z}_{p^r}$ is ultrasemifree. If G acts semifreely, then G itself is the subprincipal isotropy subgroup; in general, if H is the subprincipal isotropy subgroup, then G acts freely on the complement of the fixed point set of H.

As stated in the introduction, we shall be interested in triples $\Sigma \supseteq K \supseteq F$ of smooth homotopy spheres, where

(a) $G = \mathbf{Z}_{p^r}$ (p prime), Q_{2^r} (generalized quaternion group), S^1, or S^3 acts smoothly and ultrasemifreely on Σ with subprincipal isotropy bound (or group) H.

(b) K is a G-invariant homotopy subsphere.

(c) $F = K^H = \Sigma^H$.

This will be the precise definition of an ultrasemifree *action* on Σ *with an invariant Type* I (*homotopy*) *subsphere*. In order to organize our thoughts more efficiently, we assume $x \in F$ is a fixed point and denote the equivariant tangent spaces at x in F, K, and Σ by α, $\alpha + W$, and $\alpha + V$ respectively; the G-modules W and V are free G-modules, and we make the allowable assumption $W \subseteq V$. Denote the quotient module V/W by Ω. Since the triples in question all admit invariant Riemannian metrics with K a totally geodesic submanifold and the decomposition $\alpha + W + \Omega$ may be assumed orthogonal, we shall assume such metrics have been chosen when necessary.

Of course, a triple $\Sigma \supseteq K \supseteq F$ defines two separate ultrasemifree G-actions, one on Σ and one on K. Therefore by the results of [**45**] one has *two* knot invariants associated to $F = \Sigma^H = K^H$. It is natural to look for a relationship between these two knot invariants. An obvious model for such a relationship is given by the normal bundles associated to $\Sigma \supseteq K \supseteq F$; namely, the normal bundle $\nu(F, \Sigma)$ is the sum of $\nu(F, K)$ plus the restriction of $\nu(K, \Sigma)$ to F (compare [**21**]).

In analogy with the situation for normal bundles, we need a *third* knot invariant, that of K in Σ. Let ζ denote the normal bundle of K; then the arguments of [**45, 50**] show that the composite

$$(1.1) \qquad S(\Omega) \to S(\zeta) \subseteq \Sigma - K \quad (S = \text{unit sphere or unit sphere bundle})$$

is a G-homotopy equivalence if $\dim_R \Omega \geqslant 3$ and a G-homotopy retract if $\dim_R \Omega = 2$. Thus, as in [**45, 50**], we may again define an equivariant fiber retraction $S(\zeta) \to S(\Omega)$, and we shall denote this by $\omega_0(K \subseteq \Sigma)$. This invariant takes values in the set

$$(1.2) \qquad F/0_{G,\Omega,\text{free}}(K) \quad (\text{see } [\mathbf{41}, \mathbf{44}] \text{ for notation}).$$

REMARK. If F, K, and Σ are merely $\mathbf{Z}_p$-homology spheres, in analogy with [**45**] one may define a knot invariant in $F_{(p)}/0_{G,\Omega,\text{free}}$ rather than $F/0_{G,\Omega,\text{free}}$.

We can now state and prove a Sum Formula for knot invariants which parallels the well-known identity for normal bundles mentioned above.

THEOREM 1.3. *Suppose that G as above acts ultrasemifreely on the triple $\Sigma \supseteq K \supseteq F$ with $\Sigma^H = F = K^H$. Let i: $F \to K$ denote inclusion. Then the knot invariants of F in Σ and K are related by the formula $\omega(F \subseteq \Sigma) = \omega(F \subseteq K) \oplus i^*\omega_0(K \subseteq \Sigma)$, where direct sum is defined as in* [**50**, §2].

REMARK. The identical argument also works for quadruples $F \subseteq N \subseteq K \subseteq \Sigma$ with $F = N^H = K^H = \Sigma^H$ to show that $\omega_0(N \subseteq \Sigma) = \omega_0(N \subseteq K) \oplus i^*\omega_0(K \subseteq \Sigma)$.

PROOF. It will be helpful to introduce some notation. Let ξ, η, and ζ denote the equivariant normal bundles of F in Σ, F in K, and K in Σ respectively. As noted before, we have $\xi = \eta \oplus i^*\zeta$. Therefore the following diagram is an equivariant fiberwise pushout:

$$\begin{array}{ccc} S(\eta) \times_F S(i^*\zeta) & \to & S(\eta) \\ \downarrow & & \downarrow \\ S(i^*\zeta) & \to & S(\xi) \end{array} \tag{1.4}$$

Consider next the following commutative diagram:

$$\begin{tikzcd}
 & & K - F \arrow[dr, "\subseteq"] & & \\
 & S(W) \subseteq S(\eta) \arrow[ur, "\subseteq"] \arrow[rr, "\subset"] & & (\Sigma - K) - S(\Omega) & \\
S(W) \times S(\Omega) \subseteq S(\eta) \times_F S(i^*\zeta) \arrow[uu] \arrow[ur] \arrow[rr, "\subset"] \arrow[dd] & & & (\Sigma - K) - [S(W) \cup S(\Omega)] \arrow[uu] \arrow[ul] \arrow[dl] & \\
 & S(\Omega) \subseteq S(i^*\zeta) \arrow[dr, "\subseteq"'] \arrow[rr, "\subset"] & & (\Sigma - K) - S(W) & \\
 & & \Sigma - K \arrow[ur, "\subseteq"'] & &
\end{tikzcd} \tag{1.5}$$

By duality the three horizontal composites are equivariant homotopy equivalences. However, $S(V) = S(W) * S(\Omega)$ is the pushout of $S(W) \leftarrow S(W) \times S(\Omega) \to S(\Omega)$, and it follows that the equivariant retraction $S(\eta \oplus i^*\zeta) \to S(V)$ is given by a pushout construction; specifically, from the fiberwise join over F defined by

$$S(\eta \oplus i^*\zeta) = S(\eta) *_F S(i^*\zeta)$$

into the representation of $S(V)$ as an ordinary pushout above. □

In our subsequent work we shall be especially interested in the special case $\dim_R \Omega = 2$. Therefore we shall derive an important simplification that occurs in this case.

PROPOSITION 1.6. *If Ω is a free 2-dimensional G-module and X is an invariantly pointed finite G-equivariant cell complex, then $F/0_{G,\Omega,\text{free}}(X)$ is trivial.*

PROOF. By induction and an exact sequence argument, it suffices to show $F/0_{H,\Omega,\text{free}}(X)$ is a point where $X = S^n$ or S^{n+} and $H \subseteq G$ is an arbitrary closed subgroup (compare [**45**, (2.9)] and the arguments of [**50**, §2–3]). But in these cases

$$F/0_{H,\Omega,\text{free}}(X) = \pi_n(SF_H(\Omega), SC_H(\Omega)),$$

and, as noted in [**45**], the latter groups are trivial. □

COROLLARY 1.7. *Suppose that G acts smoothly and ultrasemifreely on the homotopy sphere Σ with subprincipal isotropy bound H, and let $F = \Sigma^H$. Assume in addition there is an invariant homotopy subsphere K with* $\dim \Sigma - \dim K = 2$. *Then the knot invariants satisfy $\omega(F \subseteq \Sigma) = \omega(F \subseteq K) \oplus \Omega$.* □

We shall conclude this section by outlining some consequences of 1.6 and 1.7 involving the exotic 8-sphere. The starting point is the following result from [**50**, §8].

Suppose that S^1 acts smoothly and effectively on the exotic sphere Σ^8. Then the fixed point set of $\mathbf{Z}_2$ is 4-dimensional and the knot invariant of this fixed point set is nontrivial for the associated $\mathbf{Z}_4$ action.

Using this, one may prove the following.

PROPOSITION 1.8. *Suppose that S^1 acts smoothly and effectively on the exotic sphere Σ^8. Then there is no invariant subsphere K^6 such that S^1 acts freely on $\Sigma - K$.*

As explained in [**39**, Example 2.6], this result implies that certain homotopy classes in $\pi_1(\mathrm{Diff}\, S^6)$ cannot be represented by representations of the circle group.

PROOF OF 1.8 (OUTLINE). Suppose such an action exists on a homotopy 8-sphere. By [**50**, Proposition 8.3], the fixed point set F of $\mathbf{Z}_2$ is 4-dimensional, and it follows that K^6 is an invariant Type I codimension 2 subsphere for the associated $\mathbf{Z}_4$ action. By 1.3, this implies that the knot invariant of the $\mathbf{Z}_4$ action is a direct sum of two pieces. Specifically, if V represents the 4-dimensional normal representation to the fixed point of $\mathbf{Z}_2$, then this splitting corresponds to a splitting of $V = W \oplus \Omega$ with W and Ω both 2-dimensional. By 1.6, each of these summands must be trivial. Therefore the knot invariant of the given action is trivial. Therefore [**50**, Theorem 8.12] implies that the ambient homotopy 8-sphere must be the standard sphere. □

2. Groups of actions with Type I subspheres. We now consider the problem posed in the introduction: Given an ultrasemifree G-homotopy $(\alpha + V)$-sphere, when does it admit a smooth invariant $(\alpha + W)$-sphere of Type I? We shall approach this problem using the groups $\Theta^G_{\alpha+V}$ as defined in [**50**, §6]. This will involve a systematic generalization of the groups $\Theta^G_{\alpha+V}$ and the exact sequences into which they fit. Not surprisingly, many of the necessary arguments are virtually word for word adaptations of the arguments described in [**50**, §6]. In the interests of avoiding lengthy duplications, we shall omit many of the particulars that involve word for word generalization, concentrating instead on the necessary changes of notation. Therefore, some familiarity with [**50**, §6] is probably very useful for understanding this section, and it might be helpful to have that article available for quick reference while reading the rest of this section.

The first step is to define groups of pairs of G-homotopy spheres

$$\Theta^G_{(\alpha+W\leqslant\alpha+V)} \tag{2.1}$$

consisting of equivariant h-cobordism classes of triples (Σ, K, F) with Σ an oriented G-homotopy $(\alpha + V)$-sphere and K an oriented $(\alpha + W)$-subsphere of Type I.[2] As usual, connected sum along the fixed point set and Type I subsphere defines an addition (commutative if $\dim \Sigma^G \geqslant 1$) and inverses are defined by taking the

[2]Such actions are frequently called "semilinear" in the literature.

opposite orientation. We may also define relative groups

$$\Theta^G_{(\alpha+W\leqslant\alpha+V,\alpha+V)} \tag{2.2}$$

consisting of a homotopy $(1+\alpha+V)$-disk Δ with a Type I invariant $(\alpha+W)$-sphere $K\subseteq\Sigma=\partial\Delta$. The following result is then elementary.

PROPOSITION 2.3. *There is an exact sequence*

$$\to \Theta^G_{(\alpha+W\leqslant\alpha+V,\alpha+V)} \xrightarrow{B} \Theta^G_{(\alpha+W\leqslant\alpha+V)} \xrightarrow{F} \Theta^G_{\alpha+V} \xrightarrow{R} \Theta^G_{(\alpha+W-1,\alpha+V-1;\alpha+V-1)},$$

where B denotes taking the boundary, F denotes forgetting the subsphere, and R denotes removing a linear disk about a fixed point. □

Thus we can solve the question at the beginning of this section *up to h-cobordism* if we can calculate the image of F. If one is willing to introduce equivariant torsion as in [**33**] and deal with groups of s-cobordism classes as in [**50**], then a result of Rothenberg [**33**] allows an exact answer in many cases. In the specific cases we consider later, the appropriate torsion groups vanish and the two notions coincide.

By itself, and even with the exact sequence of [**50**] for $\Theta^G_{\alpha+V}$, the exact sequence of 2.3 gives no evidence of being effectively computable. However, the method of proof for [**50**, (6.2)] immediately yields the following additional sequence.

PROPOSITION 2.4. *There is an exact sequence*

$$hS^G_{\alpha+W+1}(S(\Omega)) \to \Theta^G_{(\alpha+W\leqslant\alpha+V)} \xrightarrow{(K,\omega)} \begin{matrix}\Theta^G_{\alpha+W}\\ \oplus \\ \pi^G_{\alpha+W}(F(\Omega)/0(\Omega))\end{matrix} \to hS^G_{\alpha+W}(S(\Omega))$$

if $\dim\Omega\geqslant 3$. *If* $\dim\Omega=2$, *then a similar statement is true provided* $G=S^1$.

The symbol K denotes taking the Type I subsphere, while ω denotes the knot invariant of K in Σ.

PROOF (SKETCH). The case $W=0$ is just [**50**, (6.2)], and the proof of 2.4 is essentially a word forward generalization of the proof for this special case. The crucial fact needed to make this generalization is the homotopy equivalence of $S(\Omega)$ and $\Sigma-K$. If $\dim\Omega\geqslant 3$, this is just the usual general position argument, while if $\dim\Omega=2$ and $G=S^1$ this follows by the argument used in [**9**]. □

REMARK. If G is finite and $\dim\Omega=2$ one can formulate a similar result using the methods of [**12**].

To study the image of F in Proposition 2.3 more clearly, we must interlock these sequences with those from [**50**, §6]. The following exact ladder contains the crucial information.

PROPOSITION 2.5. *There exists a commutative diagram with exact rows as follows*:

$$\begin{array}{ccccccc}
 & & & & \pi^G_{\alpha+W}(F(\Omega)/0(\Omega)) & & \\
 & & & & \oplus & & \\
\to & hS^{G,\mathrm{WS}}_{\alpha+1}(S(W)\subseteq S(V)) & \to \Theta^G_{(\alpha+W\subseteq\alpha+V)} & \xrightarrow{\chi} & \pi^G_\alpha(F(W)/0(W)) & \to & \\
 & & & & \oplus & & \\
 & & & & \Theta^{G/H}_\alpha & & \\
 & A'\downarrow & A\downarrow & & \sigma\downarrow & & \\
 & & & & \pi^G_\alpha(F(V)/0(V)) & & \\
\to & hS^G_{\alpha+1}(S(V)) & \to \Theta^G_V & \xrightarrow{\chi'} & \oplus & \to & \\
 & & & & \Theta^{G/H}_\alpha & &
\end{array}$$

Here is an explanation of the notation: The superscript WS means "weakly split"; in other words, a homotopy equivalence that is Poincaré transverse to the submanifold such that the induced map of triads is a homotopy equivalence of triads. There are three components of the map χ, the first being the knot invariant of K in Σ, the second being the knot invariant of F in K, and the third being the equivariant h-cobordism class of F. Since the proof offers no surprises, we shall not present it (the second row is already in [**50**]). □

To conclude this section, we shall describe the relationship between the results presented above and the more specialized notion of a *characteristic* subsphere, which has been studied in several previous papers (for example, see [**3**, **20**, or **56**]). An invariant subsphere $K \subseteq \Sigma$ of Type I is characteristic if the equivariant degree 1 collapse map $\Sigma' \to S^{\alpha+V}$ can be made equivariantly *smoothly* transverse to $S^{\alpha+W}$ such that K is the latter's transverse inverse image. The smoothness must be emphasized because the knot invariant construction implies that one can always make this collapsing map equivariantly *Poincaré* transverse. In fact, we have the following elementary criterion for characteristic subspheres.

PROPOSITION 2.6. *An invariant subsphere $K \subseteq \Sigma$ is characteristic if and only if the knot invariant of K in Σ vanishes.* □

An extensive study of semifree actions on spheres with characteristic subspheres appears in the thesis of V. Vijums [**56**]; since his methods generalize immediately to ultrasemifree actions, we shall present everything in that context. Vijums defines groups of G-homotopy spheres with characteristic Type I subspheres that are analogous to (2.1) and will be denoted by

$$\Theta^G_{(\alpha+W\leqslant\alpha+V),c}$$

in this paper. Elaborating on Proposition 2.6, we can relate Vijums' groups to ours with the following exact sequence, in which ω denotes the knot invariant:

(2.7)

$$\to \pi^G_{\alpha+W+1}(F(\Omega)/0(\Omega)) \to \Theta^G_{(\alpha+W\leqslant\alpha+V),c} \to \Theta^G_{(\alpha+W\leqslant\alpha+V)} \xrightarrow{\omega} \pi^G_{\alpha+W}(F(\Omega)/0(\Omega)) \to$$

If we combine (2.7) and Proposition 1.6, we obtain the following easy observation.

COROLLARY 2.8. *Suppose that* $\dim_R \Omega = 2$. *Then* $\Theta^G_{(\alpha+W\leq\alpha+V),c}$ *and* $\Theta^G_{(\alpha+W\leq\alpha+V)}$ *are isomorphic.* □

In general the map ω in (2.7) is nonzero (numerous examples will appear in Part IV), and thus there is a substantial difference between characteristic subspheres and subspheres that are merely invariant. Each is worth studying for separate reasons.

3. Existence theorem for Type I subspheres. In this section we shall apply the machinery of §2 to establish a partial converse to Corollary 1.7. We shall use this result in Part IV to construct circle actions on certain exotic spheres. The discussion here is limited to circle actions; for a treatment of finite group actions in a similar spirit see [**56**].

THEOREM 3.1. *Let* S^1 *act ultrasemifreely on* Σ, *where* $\dim \Sigma$ *is even* (*we adopt the usual notation from* §§1 *and* 2 *henceforth*). *Suppose that the tangent space at a fixed point decomposes as* $\alpha + V = \alpha + W + \Omega$ *with* Ω *an irreducible free* S^1*-module. Then there is an invariant Type* I *subsphere* $K^{\alpha+W}$ *in* Σ *up to h-cobordism if and only if the knot invariant of* Σ *lies in the image of*

$$(\oplus\Omega)_*\colon \pi^G_\alpha(F(W)/0(W)) \to \pi^G_\alpha(F(V)/0(V)) \qquad (G = S^1 \textit{ here}).$$

Invariant codimension 2 subspheres have been much studied, particularly in the free case; papers of Browder [**3**], Cappell and Shaneson [**12, 13**], Homer [**17, 18**], H. T. and M. C. Ku [**20**], Montgomery and Yang [**30**] and Stoltzfus [**55**] give a representative sampling of such results.

PROOF. We shall use the exact ladder of Proposition 2.5: The "only if" part is a restatement of Corollary 1.7, so we shall consider only the "if" part henceforth. By construction $\chi'(\Sigma, S^1)$ lies in the image of σ, say as $\sigma(\omega, F)$; recall that $F(\Omega)/0(\Omega)$ is equivariantly contractible by 1.6. We claim that (ω, F) lies in the image of χ.

The main step is to construct an S^1-homotopy $(\alpha + W)$-sphere K with $K^H = F$ and knot invariant W with K nonequivariantly diffeomorphic to the standard sphere. Once this is known we may proceed as in [**44**] to construct a "smooth S^1-suspension" as follows: As in [**44**], there is an equivariant diffeomorphism $K \times S^1 \simeq S^{|\alpha+W|} \times S^1$, where S^1 acts trivially on the standard sphere $S^{|\alpha+W|}$. Then we have the manifold

$$\Sigma' = K \times D^2 \cup_2 D^{|\alpha+W+1|} \times S^1, \tag{3.2}$$

which defines an element of $\Theta^G_{(\alpha+W\leq\alpha+V)}$ with $\chi(\Sigma', S^1) = (\omega, F)$. By the exact sequence of [**42**, (1.1)], the existence of the K depends upon whether the image of (ω, F) in $hS^G_\alpha(S(W))$ is zero. By the surgery exact sequence and the triviality of $L_{\text{odd}}(1)$, it is enough to show that the associated normal invariant is zero. This will follow from the commutativity of the diagram below (for semifree actions this was discussed in [**45**, Proposition 2.7]):

$$\begin{array}{ccccc} \pi^G_\alpha(F(W)) & \to & hS^G_\alpha(S(W)) & \to & [S^\alpha \wedge_G (S(W)^+), F/0] \\ \downarrow & & & & \text{restriction} \uparrow \\ \pi^G_\alpha(F(V)) & \to & hS^G_\alpha(S(V)) & \to & [S^\alpha \wedge_G (S(V)^+), F/0] \end{array} \tag{3.3}$$

To see the assertion on normal invariants, note that $(\omega \oplus \Omega, F)$ goes to zero in $hS_\alpha^G(S(V))$ (a fortiori in $[S^\alpha \wedge_G (S(V))^+, F/0]$) because it comes from the action on Σ; hence exactness of the sequence in [**50**, §4] implies $(\omega \oplus \Omega_1, F)$ goes to zero on the bottom row. Thus (3.3) implies that (ω, F) goes to zero on the top row. We now need to show that K is the standard sphere. But $(\omega \oplus \Omega, F) \mapsto 0$ in the normal invariant group implies (by [**40**, §3]) that the Pontrjagin-Thom invariant of K goes to zero under the map

$$f^*\colon \left[S^{\alpha+1} \wedge_G \left(S(W)^+\right), F/0\right] \to \pi_{\dim K}(F/0).$$

The former is the normal invariant group associated to $hS_{\alpha+1}^G(S(W))$, and the vanishing of $L_{\text{odd}}(1)$ implies this normal invariant map is onto. Thus if we assume $p(K) = f^*y$, we may take an element of $hS_{\alpha+1}^G$ which projects to $-y$ under the map $hS_{\alpha+1}^G \to \Theta_{\alpha+W}^G$ to obtain a new action on some K' with the same F and the same knot invariant, but $p(K') = 0$. Since the Pontrjagin-Thom invariant is injective for even-dimensional exotic spheres, this means K' is the standard sphere. Thus even if the original K is not standard we can make a simple alteration to obtain a standard sphere.

Therefore, in the notation of 2.5, $\Sigma - A(\Sigma')$ lies in the image of $hS_{\alpha+1}^G(S(V))$. If we can show that A' is *onto*, then it will follow that Σ itself lies in the image of A.

To prove the surjectivity of A', start out with any representative $f\colon Y \to D^{\alpha+1} \times_G S(V)$ of a given class in $hS_{\alpha+1}^G(S(V))$. Without loss of generality, f is already smooth and transverse to $D^{\alpha+1} \times_G S(W)$ with transverse inverse image Y'. Then $f \mid Y'$ is a simply connected odd-dimensional relative normal map, and this is relatively normally cobordant to a homotopy equivalence; let (Y'', g_0) denote such a relatively normally cobordant homotopy equivalence. By a relative version of the normal cobordism extension theorem (see [**6**, p. 221; **4** and **7**, Chapter VI]), (Y, f) is relatively normally cobordant to (Z_1, g_1) with g_1 transverse to $D^{\alpha+1} \times_G S(W)$, $g_1^{-1} = Y''$, and $g_1 \mid Y'' = g_0$ (see Figure 3.1 for the construction of this normal cobordism).

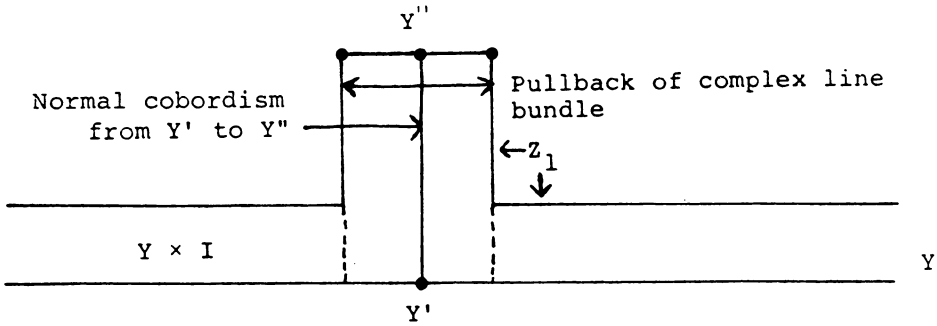

FIGURE 3.1

Let T'' be a suitably small closed tubular neighborhood of Y'' in Z_1 (e.g., half the disk bundle in Figure 3.1), and let E be a tubular neighborhood of $D^{\alpha+1} \times_G S(W)$ in $D^{\alpha+1} \times_G S(V)$ so that $Z_1 - \operatorname{Int} T''$ maps to $D^{\alpha+1} \times_G S(V) - \operatorname{Int} E$ and $\partial T''$ maps to ∂E. Then $g_1 \mid Z_1 - \operatorname{Int} T''$ is a normal map that is a homotopy equivalence

on the boundary, and another application of simply connected surgery implies it is normally cobordant to a homotopy equivalence keeping the boundary pieces fixed. Let W_0 be the extending normal cobordism shown in Figure 3.1, let W_1 be the new normal cobordism with the boundary held fixed, and consider the union $W = W_0 \cup_{Z_1 - \operatorname{Int} T''} W_1$. If Z denotes the manifold obtained by attaching $\partial W_1 - (Z_1 - \operatorname{Int} T'')$ to T'' along $\partial T''$, then W defines a normal cobordism between f and g: $Z \to D^{\alpha+1} \times_G S(V)$, where g is split along $D^{\alpha+1} \times_G S(W)$. It follows from [**11**, Lemma 8.1.a] that (Y, f) lies in the image of A'. □

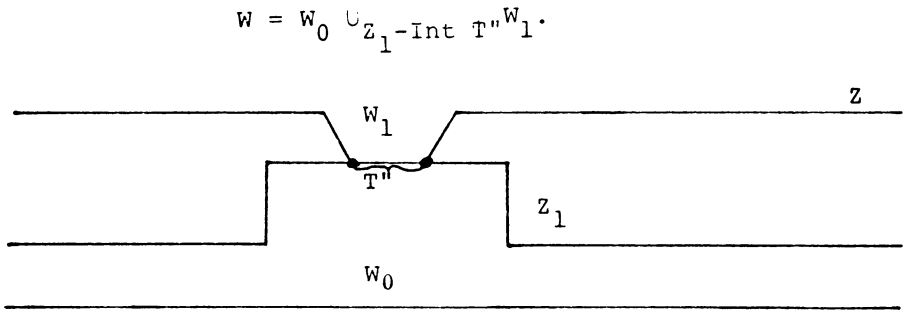

FIGURE 3.2

We now specialize to semifree circle actions and prove a "desuspension theorem in the stable range." This was in fact the starting point for the proof of Theorem 3.1, and I am grateful to Wu-chung Hsiang for bringing this phenomenon to my attention. The techniques of T. Petrie [**31**] may be used to give an alternate proof of this result and similar "desuspension theorems" for finite group actions (also see [**56**, §4]).

PROPOSITION 3.4. *In the notation of Theorem* 3.1, *assume that* S^1 *acts semifreely and* $\dim \alpha \leqslant \dim_R W - 1$. *Then* Σ *admits an invariant Type* I *subsphere.*

PROOF. If the stabilization map $\pi_\alpha^G(F(W)/0(W)) \to \pi_\alpha^G(F(V)/0(V))$ is onto, then the result is true up to h-cobordism by Theorem 3.1. However, the surjectivity of stabilization follows from the spectral sequences of [**40**] under the assumption $\dim \alpha \leqslant \dim_R W - 1$. Finally, in the situation at hand it is elementary to show that all equivariant h-cobordisms are products, so existence up to h-cobordism is equivalent to existence itself. □

4. Smooth suspensions of circle actions. In §3 we discussed the problem of finding an invariant Type I subsphere of codimension two. There is also a dual question: Given a group action on a homotopy sphere, when is it an invariant codimension two subsphere for an action on a larger homotopy sphere? The previously cited papers on free actions give fairly specific information in that case. Also, it has been known for some time that for semifree circle actions it is necessary [**9**] and sufficient [**43**, §2] for an invariant codimension two subsphere of Type I to have the standard differential structure. If we view the larger sphere as a smooth suspension, this condition is a natural one for forming cones in the smooth category. In this section

we shall formulate some smooth S^1-equivariant suspension constructions precisely and apply them to strengthen an observation in [**41**]. We noted there that every exotic sphere of dimension $\leqslant 13$ that bounds a spin manifold admits an effective smooth circle action. For this range, nonspin boundaries occur only in dimensions 9 and 10. The results of [**46**] imply that every exotic 9-sphere admits a smooth semifree circle action, and Theorem 4.4 below completes the picture by constructing circle actions on all exotic 10-spheres. Such actions cannot be semifree because [**38** and **45**] combine to rule out all possible dimensions for the fixed point set for a generator of $\Theta_{10} = \mathbf{Z}_6$, and thus the sort of construction given here is about the simplest possible. The method is also useful for constructing smooth circle actions on elements of Θ_n with order not a prime power; we shall illustrate this with an infinite class of examples generalizing the generators of Θ_{10}.

To abstractify the smooth suspension construction, we begin with a representation β of S^1 and a finite subgroup $H \subseteq S^1$. As in [**50**] we may define the groups of homotopy β-spheres Θ_β^G for $G = H$ or S^1, and there is a canonical forgetful homomorphism

$$\Theta_\beta^{S^1} \to \Theta_\beta^H. \tag{4.1}$$

As in work of Rothenberg dealing with the semifree case [**32**], these forgetful maps may be fit into a long exact sequence

$$\Theta_{\beta+1}^{H,S^1} \to \Theta_\beta^{S^1} \to \Theta_\beta^H \xrightarrow{j} \Theta_\beta^{H,S^1} \to \cdots, \tag{4.2}$$

where the Θ^{H,S^1} are groups of S^1-homotopy spheres whose H-restrictions bound H-homotopy disks. The map j merely cuts out the interior of a closed linear disk. In order to avoid conflict with [**32** and **42**], we shall denote the relative groups in the exact sequence for the forgetful map $\Theta_\beta^G \to \Theta_{\dim \beta}$ by S_β^G. We then have the following result.

PROPOSITION 4.3. *There is a well-defined homomorphism*

$$\sigma\colon \Theta_\beta^{H,S^1} \to \Theta_{\beta+\psi^H}^{S^1}$$

(if $H = 1$ take $S_\beta^{S^1}$ as the domain) such that $\sigma(\Delta, \partial\Delta = \Sigma)$ has Σ as an invariant codimension two subsphere. The representation ψ^H is the realification of the complex representation $(z, v) \to z^{|H|}v$.

PROOF. We shall simply describe the construction as it applies to representatives of the h-cobordism classes; verification that it passes to a well-defined homomorphism is routine and left to the reader.

Consider the manifold $\Sigma \times D(\psi^H)$, where D denotes the associated disk of the representation. The boundary of this manifold is $\Sigma \times S(\psi^H) = \Sigma \times S^1/H$. It is well known that $\Sigma \times S^1/H$ is canonically S^1 isomorphic to $S^1 \times_H (\Sigma \mid H)$ [**2**, Exercise 9, p. 113], which bounds $S^1 \times_H \Delta$. Therefore we may form the equivariant smooth suspension

$$\sigma_0(\Delta, \Sigma) = \Sigma \times D(\psi^H) \cup_\partial S^1 \times_H \Delta.$$

It is a routine exercise to verify that $\sigma_0(\Delta, \Sigma)$ is an S^1-homotopy $(\beta + \psi^H)$-sphere, and the existence of a codimension two subsphere S^1 isomorphic to Σ is immediate from the definition. $\square$

Using this suspension construction we shall prove that the generator (and in fact every element) of Θ_{10} admits a smooth circle action.

THEOREM 4.4. *Let Σ^{10} generate the image of $\Theta_{10} = \mathbf{Z}_6$. Then (every multiple of) Σ admits a smooth action as an S^1-homotopy $(4 + \mathrm{Re}(t^2 + 2t))$-sphere.*

PROOF. We may decompose Θ_{10} as $\mathbf{Z}_2 \oplus \mathbf{Z}_3$, the first summand being generated by a preimage of $\mu_1\eta$ (in π_{10}) and the second being generated by a preimage of β_1. The first summand admits an action of the type described by results of G. Bredon [**1**], and by taking connected sums along the fixed point set it suffices to prove that the second summand admits such an action. We shall prove this using 4.3 and [**45**, Theorem 3.2] (equivalently, [**50**, Theorem 4.8]); in order to apply these results, we shall require highly detailed information about the exact sequences of type [**42**, (1.1)] (equivalently [**50**, (6.2)]) in which the groups

$$\Theta^{S^1}_{4+3\mathbf{C}},\ \Theta^{S^1}_{4+2\mathbf{C}},\ \Theta^{\mathbf{Z}_2}_{4+4\mathbf{R}}$$

lie. One part of this problem is essentially surgery-theoretic, and another involves explicit calculations for the homotopy groups of equivariant function spaces using the machinery of [**40**].

We shall use some special notation from [**40**] that deviates from our usual notation in this paper, [**45**], and [**50**]; this will make several computational statements in the proof easier to express. The equivariant function spaces $F_{S^1}(\mathbf{C}^n)$ and $F_{\mathbf{Z}_2}(\mathbf{R}^n)$ (free linear actions on the appropriate spheres) will be denoted by $G\mathbf{C}_n$ and $G\mathbf{R}_n$ respectively.

We shall divide the computational proof that "β_1" $\in \Theta_{10}$ has the desired circle action into three steps.

Step 1. Consider a semifree circle action on "β_1" with S^4 as its fixed point set (e.g., take the action constructed in [**41**]). This action has a knot invariant in $\pi_4(G\mathbf{C}_3/U_3)$ that we wish to determine; specifically, we claim it is nontrivial. If an action on Σ^{10} with 4-dimensional fixed point set has trivial knot invariant, by exactness of [**42**, (1.1)] it comes from an element of $hS_5(\mathbf{C}P^2) = hS(\mathbf{C}P^2 \times (S^5, D^4))$ and the differential structure is given by the composite

$$hS_5(\mathbf{C}P^2) \to [S^5(\mathbf{C}P^{2+}), F/0] \xrightarrow{g^*} \pi_{10}(F/0)$$

(compare [**39**]). However, the image of β_1 in $\pi_{10}(F/0)$ is easily seen to be outside the image of g^* (consider the splitting $F/0_{(3)} \simeq BSO_{(3)} \times \mathrm{Cok}\, J_{(3)}$ and the 9-connectivity of $\mathrm{Cok}\, J_{(3)}$).

The group $\pi_4(G\mathbf{C}_3/U_3)$ may be computed by the spectral sequences of [**40**]; it is cyclic of order 24, and in the spectral sequence of [**40**] for $\pi_4(G\mathbf{C}_3)$ it corresponds to $E^\infty_{1,3} = E^2_{1,3} \cong \pi_3$. By taking connected sums along the fixed point set 8 or 16 times we may assume that the action's knot invariant generates the 3-torsion of π_3. But $\pi_6(S^3) \to \pi_3$ is onto at 3, and it follows by 3.1 that Σ admits a Type I invariant subsphere K^8.

REMARKS ON STEP 1. (i) In this particular case one can construct the invariant subsphere directly from the description of the action in [41]. The existence of K was first brought to my attention by Wu-chung Hsiang using an argument as in 3.1.

(ii) The existence of K^8 was mentioned and used in [44, §2].

Step 2. We shall use K^8 to perform an equivariant suspension as in 4.3 with $H = \mathbf{Z}_2$ rather than the choice $H = \{1\}$ which is appropriate to the semifree case. To use 4.3 we must show that the induced involution on K^8 bounds a $\mathbf{Z}_2$-homotopy disk. This is unavoidably computational, the key points being as follows:

(i) The group $hS_5(\mathbf{C}P^1)$ is zero by the surgery exact sequence, and by exactness of [42, (1.1)] $\Theta^{S^1}_{4+2\mathbf{C}}$ is detected by the knot invariant.

(ii) By the spectral sequences of [40] and their morphisms [40, §3, 5], elements of order 3 in $\pi_4(G\mathbf{C}_3/U_3)$ go to zero in $\pi_4(G\mathbf{R}_6/0_6)$.

(iii) By (ii) and exactness of [42, (1.1)], the induced involution on K lies in the image of $hS_5(\mathbf{R}P^3)$.

(iv) By exactness of the surgery sequence, the group $hS_5(\mathbf{R}P^3)$ is finite and 2-primary.

Step 3. Let M^{10} be a smooth suspension of K^8 as in 4.3, with $H = \mathbf{Z}_2$. We claim that M^{10} has order (divisible by) 3, which suffices to complete the proof. To see this, we consider the knot invariant for the *restricted action of* $\mathbf{Z}_3$ on M^{10}. Two applications of 1.7 show that the $\mathbf{Z}_3$ knot invariants of Σ and M are both stabilizations of the knot invariant of K. It follows from [45, 2.7 and 50, 4.3] that the Pontrjagin-Thom invariants of M and Σ have the same images under the map

$$\pi_{10}(F/0)_{(3)} \to [S^4L^5(\mathbf{Z}_3),\ F/0]_{(3)}.$$

To show they are equal, by the Puppe sequence it suffices to show that

$$[S^5L^3(\mathbf{Z}_3),\ F/0]_{(3)} \to \pi_{10}(F/0)_{(3)}$$

is zero; but, as before, this follows from the splitting $F/0_{(3)} \simeq BSO_{(3)} \times \operatorname{Cok} J_{(3)}$ and the 9-connectivity of $\operatorname{Cok} J_{(3)}$. □

The same method of proof yields an infinite family of further examples.

THEOREM 4.5. *Suppose that $p \equiv 3$ (4) is a prime. Then there are smooth S^1 actions on the homotopy $(2p^2 - 2p - 2)$-spheres with Pontrjagin-Thom invariant $\mu_m\eta + \beta_1$ $(m = 4k^2 - 2k - 1)$.*

PROOF. Bredon's results again imply that $\mu_m\eta$ is realized by an S^1-homotopy $((2p-2) + \operatorname{Re}(nt + t^2))$-sphere, so it remains to prove the same is true of β_1. The construction begins with the semifree S^1 actions on such spheres having fixed point set S^{2p-2} [43]. A process of elimination as for $p = 3$ (plus the splitting of $F/0_{(p)}$ and the $(2p^2 - 2p - 3)$-connectivity of $\operatorname{Cok} J_{(p)}$) yields the knot invariant. Specifically, it comes from $E^r_{1,2p-3(p)} \cong \pi_{2p-3(p)}$ in the spectral sequence for $\pi_{2p-2}(G\mathbf{C}_{n+1})$. Furthermore this knot invariant desuspends to $\pi_{2p-2}(G\mathbf{C}_n)$, and consequently the action admits an invariant codimension two Type I subsphere K (compare Remark (ii) to Step 1 in the proof of 4.4 and [44, (2.5)]).

We now claim that a connected sum of q copies of this action along the fixed point set, for some $q \equiv 1 \bmod p$, the induced involution on K bounds a homotopy

disk. These follow as before because $hS_{2p-1}(\mathbf{C}P^{n-1})$ and $hS_{2p-1}(\mathbf{R}P^{2n-1})$ are finite of order prime to p, $\pi_{2p-2}(G\mathbf{C}_n)$ has p-component $\mathbf{Z}_p$, and Θ_{2p-2} has no p-torsion. Using this and the previously mentioned facts about $F/0_{(p)}$ and $\operatorname{Cok} J_{(p)}$, the argument may be concluded as before in the special case $p = 3$. □

5. Smooth suspensions of cyclic actions. As in §4, one can ask whether a given $\mathbf{Z}_k$ action on a homotopy sphere K^n is an invariant Type I subsphere of codimension two in another such manifold Σ^{n+2} with group action. Motivated by the results of [**49**] and those cited in §4 for circle actions, it seems reasonable that one obstruction involves the differential structure on K^n. Since a homotopy n-sphere which embeds in another with codimension two bounds a parallelizable manifold (compare [**24**]), at any rate we know that the differential structure is significantly restricted if n is odd and indeed standard if n is even. Further remarks appear at the end of this section.

On the other hand, as suggested by [**49**, Theorem 2.1], there is also a second obstruction involving the isotopy map $\mathbf{Z}_k \to \pi_0(\operatorname{Diff} S^n) \cong \Theta_{n+1}$ (the isomorphism follows from [**14**] and the generalized Poincaré conjecture, and the map is well defined—and not merely so up to conjugacy—because the codomain is abelian). A brief justification for the assertion regarding isotopy classes and suspension constructions is given in Proposition 5.1. Before doing this, we establish an elementary identity involving the mapping torus of a smooth $\mathbf{Z}_k$ action. Given a smooth $\mathbf{Z}_k$ action on a homotopy sphere A^n, define an associated homotopy sphere B^{n+1} as follows: Let $\Theta_{n+1} = \pi_0(\operatorname{Diff}(D^n, S^{n-1}))$ map to $\pi_0(\operatorname{Diff} A^n)$ by extending a diffeomorphism on $D^n \subseteq A^n$ that is the identity near the boundary to all of A^n, it is well known that this extension homomorphism is onto (compare [**37**, Proposition 2.2]). Let $\gamma \in \pi_0(\operatorname{Diff} A^n)$ be the image of the standard generator $\exp(2\pi i/k) \in \mathbf{Z}_k$ in the homotopy group under the isotopy map of the action; then B^{n+1} is defined to be a homotopy sphere that maps to γ via the extension homomorphism.

LEMMA 5.0. *The manifold $A \times_{\mathbf{Z}_k} S^1$ is diffeomorphic to $A^n \times S^1 \# B^{n+1}$.*

PROOF. The manifold in question is the mapping torus of γ, and γ is the image of B. It follows as in [**5**] that this mapping torus of γ is just $S^1 \times A \# B$ as claimed. □

PROPOSITION 5.1. *Suppose that $\mathbf{Z}_k$ acts smoothly and orientation preservingly on Σ^{n+2} with K^n as an invariant Type* I *subsphere. Assume that K^n is nonequivariantly diffeomorphic to S^n, and the difference of tangent spaces at a fixed point $x \in K$ is a faithful representation of $\mathbf{Z}_k$ in SO_2. Then the isotopy homomorphism $\mathbf{Z}_k \to \Theta_{n+1}$ is trivial.*

PROOF. The normal bundle of K in Σ is trivial by the results of §1. By 5.0, the orbit manifold is $S^n \times S^1 \# H^{n+1}$, where H^{n+1} is the exotic sphere corresponding to the image of a generator of $\mathbf{Z}_k$ in Θ_{n+1}.

Let $\hat{W} = \Sigma - \operatorname{Int}(K \times D^2)$, and let

$$W = \Sigma - \operatorname{Int}(K \times D^2)/\mathbf{Z}_k.$$

It follows from the spectral sequence of the covering pair $(\hat{W}, S^1) \to (W, S^1/\mathbf{Z}_k = S^1)$ that W is an integral homology circle with boundary $S^n \times S^1 \# H^{n+1}$. If we

perform surgery on the embedded circle in the boundary, we may add a 2-handle to W and obtain a manifold X with boundary H^{n+1}. However, calculations using the Seifert-van Kampen theorem and Mayer-Vietoris sequences imply that X is a contractible manifold, and consequently H^{n+1} must be the standard sphere. In other words, the isotopy representation must be trivial. □

There is also an elementary converse to 5.1.

PROPOSITION 5.2. *Suppose that $\mathbf{Z}_k$ acts smoothly on K^n, which is nonequivariantly diffeomorphic to S^n, and assume that the isotopy map Ψ: $\mathbf{Z}_k \to \Theta_{n+1}$ is trivial. Then there exists a smooth $\mathbf{Z}_k$ action on a homotopy sphere Σ^{n+2} with K^n as an invariant Type* I *subsphere.*

PROOF. Consider the manifold $K^n \times_{\mathbf{Z}_k} S^1$; which by 5.0 is just the mapping torus for a generator of the $\mathbf{Z}_k$ action. But this generator is isotopic to the identity, and hence $K^n \times_{\mathbf{Z}_k} S^1$ is diffeomorphic to $S^n \times S^1$ (compare [**49**]). Since the map $S^n \times S^1 \to K(\mathbf{Z}_k, 1)$ giving the connected k-fold covering of $S^n \times S^1$ factors through $D^{n+1} \times S^1$, it follows that $K^n \times S^1$ bounds a free $\mathbf{Z}_k$-manifold W^{n+2} homotopy equivalent to S^1 and such that the diagram below commutes:

$$\begin{array}{ccc} K^n \times S^1 & & \\ & \searrow & \\ \downarrow & & S^1 \\ & \nearrow & \\ W^{n+2} & & \end{array}$$

Therefore we may form the $\mathbf{Z}_k$-manifold $\Sigma = K \times D^2 \cup W$, which is a homotopy sphere with the prescribed sort of $\mathbf{Z}_k$ action. □

As in §4, one can make the smooth suspension construction into a group homomorphism; however, the setup is slightly more delicate than in 4.3 because one also must include explicit isotopies to the identity. For this it is convenient to know that the isotopies may be chosen to have good properties. The following result, which is essentially a routine consequence of the isomorphism $\pi_0(\mathrm{Diff}(D^n, S^{n-1})) \cong \pi_0(\mathrm{Diff}\, S^n)$, provides such information; the proof is left as an exercise.

LEMMA 5.3. *Suppose that $\mathbf{Z}_k$ acts smoothly on the homotopy sphere K^n, where K^n is nonequivariantly diffeomorphic to S^n and the isotopy map is trivial. Suppose that the local representation of $\mathbf{Z}_k$ at a fixed point $x \in K$ is extended to a representation V^n of S^1 with the same fixed point set. Then there is a smooth map Ψ: $S^1 \times K^n \to K^n$ such that*:

(i) $\Psi \mid \mathbf{Z}_k \times K^n$ *is the group action*,

(ii) Ψ *is $\mathbf{Z}_k$-equivariant*,

(iii) $\Psi \mid \{z\} \times K$ *is a diffeomorphism for all K^n*,

(iv) *there is a closed Ψ-invariant disk D^n in K^n on which Ψ is equivalent to the representation V.* □

REMARK. As in [**44**], it is generally not possible to choose Ψ to be a circle action.

We shall say such mappings Ψ are *good isotopic trivializations of a $\mathbf{Z}_k$ action.*

The construction is now straightforward. Let $\mathcal{X}S_V^{\mathbf{Z}_k}$ consist of all h-cobordism classes of $(V \mid \mathbf{Z}_k)$-homotopy spheres K^n together with

(i) nonequivariant contractible coboundaries for the K^n,

(ii) *good isotopic trivializations* of the group action.

As usual, connected sums at fixed points in "Ψ-linear disks" make these into abelian groups, and by the techniques discussed in this section one can prove the following result.

THEOREM 5.4. *There is a well-defined homomorphism*

$$\sigma\colon \mathcal{X}S_V^{\mathbf{Z}_k} \to \Theta_{V+\mathbf{C}}^{\mathbf{Z}_k}$$

($\mathbf{C}$ = *standard 2-dim rep of* $\mathbf{Z}_k \subseteq S^1$) *such that* $\sigma(K, \Delta, \Psi)$ *has* K *as an invariant codimension two subsphere of Type* I.

PROOF (SKETCH). Consider the manifold $K \times_{\mathbf{Z}_k} S^1$, which we know is homotopy equivalent to $S^n \times S^1$. By the construction of Ψ we can in fact say that there is a homotopy equivalence of triads

$$(K;\, D^n,\, K^n - \operatorname{Int} D^n) \times_{\mathbf{Z}_k} S^1 \to (S^n,\, D_+^n\,,\, D_-^n,) \times S^1$$

which is a diffeomorphism over $D_+^n \times S^1$; in other words this defines a class in $hS_n(S^1) \cong \Theta_n \oplus \Theta_{n+1}$. Since K is a standard sphere and the action is isotopically trivial, the previous arguments show this class must vanish; likewise, the previous construction carries through to give a smooth double suspension of a given action, which is well defined up to the original h-cobordism relation. $\square$

REMARK 5.5. One can attempt a twisted smooth double suspension even if K^n merely bounds a π-manifold as follows: Suppose we have a smooth embedding $K^n \subseteq S^{n+2}$ such that the k-fold cyclic branched cover along K is again a homotopy sphere. Let $C = S^{n+2} - \operatorname{Int} K^n \times D^2$, and let $\hat{C}$ denote its k-fold cyclic covering. Now suppose that $\mathbf{Z}_k$ acts smoothly on K^n, the transformations being isotopic to the identity. It then follows as before that $K^n \times_{\mathbf{Z}_k} S^1$ is diffeomorphic to $K^n \times S^1$. Let $\hat{f}\colon (K^n, \mathbf{Z}_k) \times S^1 \to (K^n, \text{trivial action}) \times S^1$ denote the induced equivariant diffeomorphism of k-fold covering spaces. Then

$$\Sigma^{n+2} = K^n \times D^2 \cup_{\hat{f}} \hat{C}$$

is a homotopy sphere, it has a smooth $\mathbf{Z}_k$ action, and K^n is an invariant Type I subsphere.

Given K^n, one can actually construct such a cyclic branched covering using techniques of Cappell and Shaneson [**11**, **53**] (also compare [**12**, §11]). Consequently, one can always find smooth (twisted) double suspensions of $\mathbf{Z}_k$ actions on boundaries of π-manifolds provided the group transformations are isotopic to the identity. To see that this happens in many cases, consider the realizations of elements in bP_{n+1} as Brieskorn varieties; these yield large special orthogonal group actions, and the induced actions of cyclic subgroups give large families of isotopically trivial actions. In fact, the circle actions themselves are good isotopic trivializations in our sense. For $\mathbf{Z}_k$ actions with a connected positive-dimensional fixed point set, one may also obtain examples as in [**41**, §6]; namely, if $\mathbf{Z}_k$ acts on K^n in this way, then the

connected sum (along the fixed point set) of k copies of this action is isotopically trivial. This is particularly useful if k is relatively prime to the order of bP_{n+1}. $\square$

6. Groups of actions with Type II subspheres. In this section we shall look briefly at the existence and classification problem for group actions on homotopy spheres with Type II homotopy subspheres. To simplify matters, *we limit the discussion to semifree actions*. One outside motivation for interest in such objects comes from $\mathbf{Z}_p \times \mathbf{Z}_p$ actions on homotopy spheres. In this case, if we restrict the action to $\mathbf{Z}_p \times \{1\}$, then usually the fixed point set of $\{1\} \times \mathbf{Z}_p$ is a Type II invariant subsphere. It is possible to proceed further along these lines and refine these ideas to study smooth $\mathbf{Z}_p \times \mathbf{Z}_p$ actions on homotopy spheres in the spirit of [**50**] (an outline of these results appeared in [**51**]).

Of course, the basic examples of invariant Type II subspheres arise from linear representations, and thus we first focus on some linear models. Given a compact Lie group G acting linearly and semifreely, let α and β be nonzero free G-modules with $\alpha \subseteq \beta$, and let $k > l$ be positive integers. Then $S^{l+\alpha}$ (notation of [**52**]) is an invariant Type II subsphere of $S^{k+\beta}$.

The existence question for Type II subspheres is embarrassingly trivial to answer.

PROPOSITION 6.1. *Let k, l, α, β be as above, and let Σ be a G-homotopy $(k+\beta)$-sphere with $x_0 \in \Sigma$ a fixed point. Then Σ admits an invariant Type* II *homotopy $(l+\alpha)$-subsphere K with $x_0 \in K$.*

PROOF. Since an invariant neighborhood of x_0 is G-diffeomorphic to the G-module $\mathbf{R}^k \times E(\beta)$, it suffices to check this for the latter with 0 replacing x_0. But if v is a unit vector in $\mathbf{R}^k = \mathbf{R}^{k-1} \times \mathbf{R}$, we may take K to be the unit sphere in $\mathbf{R}^l \times \mathbf{R} \times E(\alpha)$ about v, composed with the standard inclusion of the latter in $\mathbf{R}^{k-1} \times \mathbf{R} \times E(\alpha) \subseteq \mathbf{R}^k \times E(\beta)$. $\square$

We turn now to classify pairs (Σ, K) of G-homotopy $(k+\beta)$-spheres Σ with invariant Type II $(l+\alpha)$-subspheres K. When using such notation we shall often write $F = \Sigma^G$ and $E = K^G$ without further comment.

As usual, one can define a group structure (almost always abelian) on h-cobordism classes of such objects by connected sum, and the resulting groups will be called $\Theta^G_{(l+\alpha \leq k+\beta)}$. We shall describe how one fits this group into an exact sequence formally parallel to (2.3). For reasons of space the proofs will be either sketched or omitted.

In the classification of semifree actions, the equivariant normal bundle of the fixed point set is a key object. The corresponding item for elements of $\Theta^G_{(l+\alpha \leq k+\beta)}$ is a smooth invariant regular neighborhood of $F \cup K$. It is possible to discuss this concept abstractly using [**15, 19,** and **22**], for example, but in our situation it is probably simpler to view the regular neighborhood R_Σ as the plumbing of the equivariant normal disk bundles of F and K along their intersection E (see the comments below).

As for semifree actions, the central themes of our classification are the construction of a G-homotopy equivalence from ∂R_Σ to the corresponding object $\partial R_{\mathrm{LIN}}$ from

the linear action and the observation that G-homotopy equivalence extends to $\Sigma - \operatorname{Int} R_\Sigma \to S^{k+\alpha} - \operatorname{Int} R_{\mathrm{LIN}}$. The G-homotopy equivalence of ∂R_Σ will be defined indirectly through invariants resembling knot invariants. Specifically, to define a homotopy smoothing of $\partial R_{\mathrm{LIN}}$ we first need (i) a homotopy sphere F^k as a candidate for the fixed point set, (ii) a G-homotopy sphere $K^{l+\alpha}$ as a candidate for the Type II subsphere and (iii) a smooth embedding of $E = K^G$ in F. In addition to this we need bundle data to complete the plumbing process. The precise statements and data needed are more easily understood with an explicit description of R, so we shall construct a standard example beginning with $S^l \times D^{k-l} \times D(\beta)$, rounding the corners, and gluing on $D^{l+1} \times S^{k-l-1} \times \frac{1}{3}D(\beta)$ and $D^{l+1} \times \frac{1}{3}D^{k-l} \times S(\alpha) \times \frac{1}{3}D(\beta - \alpha)$ with corners suitably rounded as usual.

Unfortunately, the notation now becomes extremely tedious in complete generality, and therefore *we shall assume $\alpha = \beta$ and $l \leqslant k - 3$ henceforth* (see Addendum 6.8 for the case $\alpha < \beta$). In this case we may take $R = S^k \times D(\alpha) \cup D^k \times S^\alpha$ with rounded corners.

The following result yields the desired key step in setting up the exact sequence.

PROPOSITION 6.2. *Let Σ be a G-homotopy $(k + \beta)$-sphere with invariant Type* II *subsphere $K^{l+\alpha}$. If R_Σ is an invariant closed regular neighborhood of $F \cup K$, then there is a canonical equivariant G-homotopy equivalence $\partial R_\Sigma \to \partial R_{\mathrm{LIN}}$.*

PROOF. We may decompose ∂R_Σ as $S(\nu_F \mid F - D(\nu_{E,F})) \cup S(\nu_K \mid K - D(\nu_{E,K}))$, the intersection being the fiberwise product

$$S(\nu_F \mid E) \times_E S(\nu_K \mid E).$$

A similar decomposition was given before for the linear model, where all normal bundles are trivial.

By the usual homological considerations and the equivariant Whitehead theorem, $\Sigma - K$ has the equivariant homotopy type of S^{k-l-1} (with trivial group action). Consequently, we obtain a canonical equivariant fiber retraction $S(\nu_K) \to S^{k-l-1}$ in addition to the knot invariant retraction $S(\nu_F) \to S(\alpha)$. By construction, the homotopy trivialization of $S(\nu_K)$ extends the homotopy trivialization of $S(\nu_{E,F})$ given in Levine's work [**23**]. We can now plumb together the two fiber homotopy trivializations of $S(\nu_K)$ and $S(\nu_F)$ along $S(\nu_F) \times_E S(\nu_K)$ to form the required homotopy equivalence $\partial R_\Sigma \to \partial R_{\mathrm{LIN}}$. $\square$

The two fiber homotopy trivializations correspond to the knot invariant in previous cases, but a little care is needed to give a precise definition because the homotopy trivializations have a consistency condition built into them as noted above. Specifically, the knot invariant of K is classified by an element of $[S^{k+\alpha}/G, \mathbf{G}_{k-l}/0_{k-l}]$, and its restriction to S^k matches the invariant of F under the map $\Theta_{k,l} \to \pi_k(\mathbf{G}_{k-l}/0_{k-l})$ from [**23**] (here $\mathbf{G}_m$ denotes the selfmaps of S^{m-1}). It follows that the knot invariant of K in Σ and the knot type of E are determined by a

canonical class on the homotopy pullback Y in the following diagram:

$$\begin{array}{ccc} Y & \xrightarrow{j'} & F(S^{\alpha}/G, \mathbf{G}_{k-l}/0_{k-l}) \\ \downarrow & & \downarrow \text{eval. at zero} \\ \widetilde{PD}_{k-l}/0_{k-l} & \xrightarrow{j} & \mathbf{G}_{k-l}/0_{k-l} \end{array} \tag{6.3}$$

This uses the isomorphisms $\Theta_{k,l} \cong \pi_l(\widetilde{PD}_{k-l}/0_{k-l})$ of [35]; as in [23, 35] the fiber of j and hence j' is G/PL because $k - l \geqslant 3$. On the other hand, the inclusion of S^l in $S^{l+\alpha}/G = S^l * [S(\alpha)/G]$ is nullhomotopic, and hence one has that

$$\pi_l(Y) = P_l \oplus \left[S^{l+\alpha}/G, \mathbf{G}_{k-l}/0_{k-l}\right], \quad \text{where } P_l = \pi_l(G/\mathrm{PL}). \tag{6.4}$$

To complete description of the exact sequence, we must describe $\partial R_{\mathrm{LIN}}$ and $S^{k+\alpha} - \operatorname{Int} R_{\mathrm{LIN}}$ explicitly.

PROPOSITION 6.5. *We have the following G-diffeomorphisms*:

(i) $S^{k+\alpha} - S^k \cup S^{l+\alpha} \simeq (\mathbf{R}^l \times E(\alpha) - [R^l \cup E(\alpha)]) \times (\mathbf{R}^{k-l} - \{0\}) \simeq \mathbf{R}^{l+2} \times S^{k-l-1} \times S(\alpha)$.

(ii) $\partial R_{\mathrm{LIN}} \simeq S^{l+1} \times S^{k-l-1} \times S(\alpha)$.

(iii) $S^{k+\alpha} - \operatorname{Int} R_{\mathrm{LIN}} \simeq D^{l+2} \times S^{k-l-1} \times S(\alpha)$. $\square$

These are all routine verifications.

It follows that the homotopy smoothing in 6.2 has codomain $S^{l+1} \times S^{k-l-1} \times S(\alpha)/G$; if one is more careful with the construction, a homotopy smoothing of triads into

$$\left(S^{l+1}, D_+^{l+1}, D_-^{l+1}\right) \times S^{k-l-1} \times S(\alpha)/G$$

is obtained which is a diffeomorphism over D_+^{l+1}. In other words, one gets an element of $hS_{l+1}(S^{k-l-1} \times S(\alpha)/G)$. Furthermore, as in the cases without invariant subspheres, one can extend the homotopy smoothing to a homotopy equivalence from $\Sigma - \operatorname{Int} R_{\Sigma/G}$ to $D^{l+2} \times S^{k-l-1} \times S(\alpha)/G$. Proceeding further along these familiar lines, one finally obtains the desired exact sequence.

THEOREM 6.6. *The following sequence is exact*:

$$\cdots hS_{l+2}\left(S^{k-l-1} \times \frac{S(\alpha)}{G}\right) \xrightarrow{\Gamma} \Theta^G_{(l+\alpha \leqslant k+\alpha)} \xrightarrow{(a,b,c,d)} \begin{array}{c} \Theta_k \\ \oplus \\ \left[S^{l+\alpha}/G, \mathbf{G}_{k-l}/0_{k-l}\right] \\ \oplus \\ P_l \xrightarrow{\pi} hS_{l+1}(S^{k-l-1} \times S(\alpha)/G) \\ \oplus \\ \Theta^G_{l+\alpha} \end{array} \cdots$$

EXPLANATION. The map Γ is a gluing map comparable to the usual map $hS_{k+1}(S(\alpha)/G) \to \Theta^G_{k+\alpha}$, the maps a and d give the invariant subspheres F and K, the maps b and c carry the knot type of $E \subseteq F$ and the knot invariant of K, and π is given by the plumbing construction described in the proof of 6.2. $\square$

For the record we note an immediate consequence of the above discussion.

COROLLARY 6.7. *Given Σ as above with invariant Type* II *subsphere K and $(F, E) = (\Sigma^G, K^G)$, then E bounds a parallelizable submanifold of F.* $\square$

ADDENDUM 6.8. *The case $\alpha < \beta$.* A similar analysis is possible using the splitting $\nu_{(E,\Sigma)} = \nu_{(E,F)} \oplus \nu_{(E,K)} \oplus \xi$ where all three summands are now *positive dimensional.* One can again define an equivariant fiber retraction for $\nu_{(E,\Sigma)}$, and it splits into fiber retractions over (i) an equivariant fiberwise closed regular neighborhood NF of $S(\nu_{(E,F)})$ in $S(\nu_{(E,\Sigma)})$, (ii) the boundary of (i) as given, (iii) and (iv) likewise with K replacing F and (v) the subbundle $S(\nu_{(E,\Sigma)}) - \mathrm{Int}(N_F \cup N_K)$. The fiber restrictions in (i)—(iv) involve restrictions of the knot invariants of F and K to E, and (v) serves as a substitute for an equivariant fiber retraction of the remaining subbundle $S(\xi)$ over E.

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