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BROWNIAN MOTION AND A GENERALISED LITTLE PICARD'S THEOREM

BY

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ABSTRACT. Goldberg, Ishihara, and Petridis have proved a generalised little Picard's theorem for harmonic maps; if a harmonic map of bounded dilatation maps euclidean space, for example, into a space of negative sectional curvatures bounded away from zero then that map is constant. In this paper a probabilistic proof is given of a variation on this result, requiring in addition that the image space has curvatures bounded below. The method involves comparing asymptotic properties of Brownian motion with the asymptotic behaviour of its image under such a map.

1. Introduction. The application of the theory of Brownian motion to prove results in harmonic and complex analysis is now widespread. The purpose of this paper is to contribute an application to geometric function theory; a probabilistic proof of a version of the generalised little Picard theorem proved in **[5]** using geometric methods. The little Picard theorem of complex analysis already has a probabilistic proof due to Burgess Davis **[2]**; the general strategy remains the same in higher dimensions, but the actual tactics require considerable modification.

The reason for this is that the maps we will consider do not preserve Brownian motion up to time changes. Maps which do so are very special; essentially when composed with harmonic functions they must yield harmonic functions. Such maps are characterized in **[4]**.

The tactics of the proof inhabit §2 in the form of a sequence of lemmas concerning the behaviour of images of Brownian motion by harmonic maps of bounded dilatation. The theorem, given in §3, follows immediately from these lemmas for they show that if a nontrivial harmonic map of bounded dilatation existed between two particular kinds of manifold then the image of Brownian motion would have properties incompatible with properties known to hold for the Brownian motion on the domain manifold.

The main theorem is slightly different from the result proved geometrically. More conditions are required on the image manifold, while the domain can be slightly more general.

Pinsky, in **[8]**, gives an introduction to Brownian motion on manifolds. He also discusses the results of Prat **[9]** which inspire the lemmas of §2. The survey **[3]**

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discusses the research field of harmonic maps. An introduction to the work of this paper, without proofs, can be found in [7].

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2. A collection of lemmas. Let $F\colon M \to N$ be a harmonic map, defined between complete Riemannian manifolds M and N. Suppose further that N is simply connected and of nonpositive sectional curvatures. So $\mathrm{Sect}(N) \leqslant 0$. By the Cartan-Hadamard theorem [**1**, 1.33], given a particular point $\mathbf{o}$ in N, the exponential map

$$\mathrm{Exp}\colon T_{\mathbf{o}}N \to N \quad \text{is a diffeomorphism.}$$

Given such a point $\mathbf{o}$ the maps $\rho\colon n \mapsto {}^N d(n, \mathbf{o})$ defined on N, and $\Theta\colon n \mapsto \mathrm{Exp}^{-1}(n)/\rho(n)$ defined on $N \setminus \{\mathbf{o}\}$, provide a smooth polar coordinate system for points in $N \setminus \{\mathbf{o}\}$.

Suppose that X is Brownian motion on M. We shall suppose that M is stochastically complete, so that X_t is defined all times $t \geqslant 0$. This section will discuss the behaviour of $Y = F(X)$ and $Z = \rho(Y)$, and thus obtain the main result of the paper. Note that X implicitly depends on X_0, its initial position. The estimates to be obtained will hold uniformly in the choice of X_0 in M, and of the point $\mathbf{o}$ in N.

An application of Itô's lemma to the random process Z shows the existence of a Brownian motion B on the real line such that

$$dZ = dB_\sigma + \Lambda\, dt \tag{2.1}$$

on the time intervals where Z is nonzero. Here σ is given by

$$d\sigma/dt = \xi^{\alpha\beta}(X)\left[\frac{\partial\rho}{\partial y^\alpha}\frac{\partial\rho}{\partial y^\beta}\right](Y), \tag{2.2}$$

and Λ by

$$\Lambda = \frac{1}{2}\xi^{\alpha\beta}(X)\left[\frac{\partial^2\rho}{\partial y^\alpha \partial y^\beta} - {}^N\Gamma^k_{\alpha\beta}\frac{\partial\rho}{\partial y^k}\right](Y), \tag{2.3}$$

and ξ by

$$\xi^{\alpha\beta}(x) = {}^M g^{ij}\left[\frac{\partial F^\alpha}{\partial x^i}\frac{\partial F^\beta}{\partial x^j}\right](x) \quad \text{for } x \text{ in } M. \tag{2.4}$$

These definitions do not depend on the particular coordinate systems used. To obtain (2.1) one uses the fact that F is harmonic.

The results to be established for Y and Z depend on analysis of the three entities σ, Λ, and ξ.

For x in M the quadratic form $\xi(x)$ on $T_{F(x)}M$ has eigenvalues

$$\lambda_1(x) \geqslant \lambda_2(x) \geqslant \cdots \geqslant \lambda_n(x) \geqslant 0.$$

We define $\mu\colon t \mapsto \int_0^t \lambda_1(X_s)\, ds$ and note that

$$d\sigma/dt \leqslant d\mu/dt. \tag{2.5}$$

In the remainder of the section the map F will be taken to have K-bounded dilatation, and so

(2.6) $$\lambda_1(x) \leqslant K^2\lambda_2(x) \quad \text{for all } x \text{ in } M.$$

Using this notation we now state and prove a number of preliminary results. The first lemma gives an explicit lower bound for the drift of Z, using geometrical arguments.

LEMMA 1. *Suppose that* $\operatorname{Sect}(N) \leqslant -H^2 < 0$. *Then the lower bound*

(2.7) $$\Lambda \geqslant \tfrac{1}{2}\lambda_1(X)K^{-2}H\coth(HZ)$$

holds whenever Z is nonzero.

PROOF. Given x in M we chose normal geodesic coordinates $z^1, z^2,\dots,z^n$ about x to correspond to the eigenvectors associated with the eigenvalues $\lambda_1(x), \lambda_2(x),\dots,\lambda_n(x)$ of the quadratic form $\xi(x)$. Then by change of coordinates

$$\xi^{\alpha\beta}\left[\frac{\partial^2\rho}{\partial y^\alpha\partial y^\beta} - {}^N\Gamma^k_{\alpha\beta}\frac{\partial\rho}{\partial y^k}\right](F(x)) = \sum_{i=1}^n \lambda_i(x)\frac{d^2}{ds^2}\rho(z^i(s))\,|_{s=0}.$$

The second derivatives here can be bounded using the Hessian comparison theorem [**6**, Theorem A] to compare with hyperbolic space of constant curvature $-H^2$. Hyperbolic geometry as described in [**6**, Exercise 3.15] then allows the bound

$$\frac{d^2}{ds^2}\rho(z^i(s))\,|_{s=0} \geqslant \left(1 - \langle \operatorname{grad}\rho, \operatorname{grad} z^i\rangle^2_{F(x)}\right)H\coth(H\rho\circ F(x)).$$

The terms $\langle \operatorname{grad}\rho, \operatorname{grad} z^i\rangle$ are the terms in an orthogonal expansion of the unit vector $\operatorname{grad}\rho$. This fact, together with the bounded dilatation condition (2.6), enables us to show

$$\sum_{i=1}^n \lambda_i(x)\frac{d^2}{ds^2}\rho(z^i(s))\,|_{s=0} \geqslant K^{-2}\lambda_1(x)H\coth(H\rho\circ F(x)),$$

which establishes the result. □

If X satisfies a suitable 0-1 law then much more can be said about the behaviour of Z. To state this 0-1 law we must first define the invariant σ-field for the Brownian motion X. This is a family $\mathcal{I}$ of subsets of $C([0,\infty); M)$, the space of continuous paths on M. The set A belongs to $\mathcal{I}$ if it is true that the path ω belongs to A if and only if

$$t \mapsto \omega(s+t) \quad \text{belongs to } A \text{ for all } s \geqslant 0.$$

Examples of such sets are the set of $F(\omega)$ tending to infinity, the set of $F(\omega)$ with at least two limit points, the set of $F(\omega)$ converging to a given point.

The Brownian motion X satisfies a 0-1 law on $\mathcal{I}$ if $P\{X \in A\}$ is 0 or 1 whenever A belongs to $\mathcal{I}$. A martingale argument can be used to show that this 0-1 law holds if M supports no nonconstant bounded harmonic functions. Thus by [**11**] we see that if the Ricci curvatures of M are all nonnegative then X satisfies this 0-1 law.

If X satisfies the 0-1 law then it is easy to show that $Y = F(X)$ satisfies a corresponding 0-1 law.

LEMMA 2. *In addition to the assumptions of Lemma* 1 *suppose that the process* X *satisfies a* 0-1 *law on its invariant* σ*-field. Then either* F *is trivial or else* $Z_t \to \infty$ *as* $t \to \infty$.

PROOF. We define three events concerning the asymptotic behaviour of Z;

$\Omega_1 = \{Z \text{ converges to infinity}\}$;

$\Omega_2 = \{\text{the path of } Y \text{ has at least two limit points as time } t \text{ tends to infinity}\}$;

$\Omega_3 = \{Y \text{ converges to some limit in } N\}$.

The events $\Omega_1, \Omega_2, \Omega_3$ make up a decomposition of the underlying probability space Ω and are measurable with respect to the invariant σ-field of X. By the 0-1 law it follows that exactly one of them has probability 1 and the others have probability 0. Thus there are three cases to consider. In case $P\Omega_1 = 1$ then there is nothing left to prove. We shall dispose of the other two cases after a preliminary discussion of the behaviour of Z.

Suppose that for some $c > 0$ and some v $P(Z_v > c) > 0$. Given c' in $(0, c)$ let $\tilde{Z}$ solve

$$d\tilde{Z} = dB_\sigma + \frac{1}{2}\frac{d\sigma}{dt}K^{-2}H\,dt$$

after time v, with $\tilde{Z}_v = c$. If $Z_v > c$ then we know $\tilde{Z} < Z$ until Z next hits 0. This is because $\cosh(w) \geqslant 1$ and because until Z is zero the relations (2.1), (2.5), (2.7) hold. On the other hand $\tilde{Z}$ is a time-change of a Brownian motion of constant positive drift. So $\tilde{Z}$ has a positive chance of never sinking from c to the level c'. This is true even when we condition $\tilde{Z}$ on $Z_v > c$. So a similar fact is true for Z and $P(Z_s \geqslant c'$ for all $s \geqslant v) > 0$. Since c' was arbitrary in $(0, c)$, and the 0-1 law holds, if $P(Z_v > c) > 0$ then $P(\liminf Z_s \geqslant c) = 1$.

Suppose that $P\Omega_2 = 1$. There must be two disjoint open sets in N both visited by Y at arbitrarily large times. (By the 0-1 law if this is possible then it is certain.) A covering argument shows that we may take one of these sets to be a geodesic open ball. Since the choice of $\mathbf{o}$ was arbitrary we may take $\mathbf{o}$ as the centre of this ball. But if Y is to visit the other open set then Z must exceed the radius of the ball by a certain amount. From the preliminary remark we may deduce that $\liminf Z_s >$ radius of ball, with probability 1. This contradicts the presumption that the geodesic ball is visited by Y at arbitrarily large times. So it is not possible for Ω_2 to have probability 1.

Suppose that $P\Omega_3 = 1$. We shall show that F must be trivial. For by the 0-1 law the limit of Y is nonrandom, and we may take $\mathbf{o}$ to be that limit. But then $\liminf Z_s = 0$ with probability 1 and so by the preliminary discussion $P(Z_t > 0) = 0$ for all t. Thus by the continuity of Y we know

$$P(Y_t = \mathbf{o} \text{ for all } t) = 1.$$

It can be shown that X has a positive chance of visiting a given nonvoid open subset of M, by an application of [**10**, Theorem 3.1], so if $Y = F(X)$ is to be fixed at $\mathbf{o}$ then $F(M) = \{\mathbf{o}\}$, and F is trivial.

This concludes the proof of the lemma. □

In particular, under the assumptions of Lemmas 1 and 2 if F is nontrivial then it must have unbounded range.

LEMMA 3. *In addition to the assumptions of Lemmas* 1 *and* 2 *suppose that F is nontrivial. Then*

$$\liminf P\big(Z_t \geqslant R/2 + c\mu(t) \quad \textit{for all } t \geqslant 0\big) = 1 \tag{2.8}$$

where the infimum is taken over possible choices of $\mathbf{o}$ *in N and starting points m in M for X such that*

$${}^N d(\mathbf{o}, F(m)) = \rho(F(m)) = R,$$

and the limit is as $R \to \infty$, and c is any number less than $\frac{1}{2}K^{-2}H$.

PROOF. From the conclusion to Lemma 2 we know that for sufficiently large R there must be m in M with $\rho(F(m)) = R$. Given such an R and m, start X at m. By Lemma 1 until Z hits 0 the equation (2.1) is satisfied, together with the lower bound (2.7).

As a consequence of this, and the fact that $\cosh(w) \geqslant 1$, we can argue as in the proof of Lemma 2 that until Z first hits 0 it remains above Z^*, where Z^* is given by

$$dZ^* = dB_\sigma + \tfrac{1}{2}\lambda_1(X)K^{-2}H\,dt$$

and $Z^*(0) = R$. So it is enough to show that

$$\lim_{R\to\infty} P\big(Z_t^* \geqslant R/2 + c\mu(t) \text{ for all } t \geqslant 0\big) = 1$$

whenever c satisfies the required condition. But by well-known facts concerning real Brownian motions begun at zero, such as B,

$$P\big(B_t \geqslant -R/2 + \big[c - \tfrac{1}{2}K^{-2}H\big]t \text{ for all } t \geqslant 0\big) \to 1 \quad \text{as } R \to \infty$$

whenever the term in square brackets is negative. The result follows by making a time-change in the equation for Z^* and using $d\sigma/dt \leqslant \lambda_1(X)$. $\square$

The next lemma gives an explicit upper bound for the drift of Z, given an additional lower bound on the sectional curvatures. This bound is necessary in order to control the behaviour of $Y = F(B)$ over short time intervals.

LEMMA 4. *If* $-L^2 \leqslant \mathrm{Sect}(N) \leqslant -H^2 < 0$ *then the upper bound*

$$\Lambda \leqslant \tfrac{1}{2}(n-1)\lambda_1(X)L\coth(LZ) \tag{2.9}$$

holds whenever Z is nonzero.

PROOF. The argument is very similar to that involved in the proof of Lemma 1. By choosing normal geodesic coordinates $z^1, z^2, \ldots, z^n$ as above, and applying the Hessian comparison theorem using hyperbolic space of constant curvature $-L^2$ we find

$$\Lambda \leqslant \frac{1}{2}\sum_{i=1}^{n} \lambda_i(X)\big(1 - \langle \operatorname{grad}\rho, \operatorname{grad} z^i\rangle_Y^2\big)L\coth(LZ).$$

The result follows from observing that $\lambda_1(X)$ is the largest eigenvalue, and that

$$\sum_{i=1}^{n} \langle \operatorname{grad} \rho, \operatorname{grad} z^i \rangle^2 = 1. \quad \square$$

We note that neither the above lemma, nor the lemma following require the existence of a 0-1 law for the invariant σ-field of $BM(M)$.

Lemma 4 enables control of the short-time behaviour of $Y = F(X)$.

LEMMA 5. *Suppose the conditions of Lemma 4 hold.*

Fix $\gamma \in (0,1)$. Then, whatever choice is made of $\mathbf{o}$ in N and X_0 in M, there is a positive constant q such that with probability at least q

$$\sup_{r \leqslant \mu(t) \leqslant r+1} {}^N d\big(Y(\mu^{-1}(r)), Y(t)\big) \leqslant r^\gamma \quad \textit{for all } r = 0, 1, 2, \ldots. \tag{2.10}$$

Moreover with probability 1 *all but finitely many of these inequalities hold.*

[Note; here $\mu^{-1}(r) = \inf\{t: \mu(t) \geqslant r\}$.]

PROOF. Suppose that r is a nonnegative integer. Then, by an extension of the argument of Prat [**9**, 1541 following],

$$P\Big(\sup_{\mu(t) \leqslant 1} d(Y_t, Y_0) > r^\gamma\Big) \leqslant P\Big(\sup_{s \leqslant 1} |B_s| > r^\gamma - \delta - \tfrac{1}{2}(n-1)L\coth(L\phi)\Big),$$

where δ, ϕ are arbitrary constants satisfying $r^\gamma > \delta > \phi > 0$ and can be chosen regardless of r, of $\mathbf{o} \in N$ and of X_0.

The right-hand side of the inequality is dominated by

$$2\sqrt{\frac{2}{\pi}}\, a_r^{-1} \exp\Big(-\frac{1}{2} a_r^2\Big)$$

where

$$a_r = r^\gamma - \delta - \tfrac{1}{2}(n-1)L\coth(L).$$

The bound is uniform in X_0 and this and the strong Markov property allow the same bound to hold on the probability of the event

$$A_r = \Big\{\sup_{r \leqslant \mu(t) \leqslant r+1} {}^N d\big(Y(\mu^{-1}(r)), Y(t)\big) > r^\gamma\Big\}$$

even when conditioned on the history of X up to the random time $\mu^{-1}(r)$. For this time is a Markov time. So the process

$$t \mapsto X\big(\mu^{-1}(r) + t\big),$$

conditioned on the history of X up to $\mu^{-1}(r)$ and on $X(\mu^{-1}(r)) = m$, has the same probabilistic properties as has the process X started out at m and not conditioned at all. Because of the uniformity of the bound we can conclude

$$P\big(A_r \mid \text{history of } X \text{ up to time } \mu^{-1}(r)\big) \leqslant 2\sqrt{\frac{2}{\pi}}\, a_r^{-1} \exp\Big(-\frac{1}{2} a_r^2\Big).$$

It is clear that this bound itself is uniform in the choice of $\mathbf{o}$ in N and X_0 in M.

The sum $\Sigma_r a_r^{-1}\exp(-\frac{1}{2}a_r^2)$ converges, and so there are positive r_0 and q_1, not depending on $\mathbf{o} \in N$ nor on X_0, such that

$$P\left\{\bigcap_{r \geqslant r_0} A_r^c \,\middle|\, \text{the history of } X \text{ up to } \mu^{-1}(r_0)\right\} > q_1.$$

The above estimates also show that there is $q_2 > 0$, likewise not depending on $\mathbf{o} \in N$, X_0, such that

$$P\left\{A_r^c \mid \text{the history of } X \text{ up to } \mu^{-1}(r)\right\} > q_2$$

for every r. Therefore there is $q > 0$, not depending on $\mathbf{o} \in N$, X_0, such that

$$P\left\{\bigcap_{r \geqslant 0} A_r^c\right\} > q.$$

This proves the first part of the lemma. The second part is a consequence of the Borel-Cantelli lemma. For

$$P\{\text{all but finitely many of the } A_r^c \text{ are the case}\} = 1 - P\left(\overline{\lim}\, A_r\right) = 1$$

by the Borel-Cantelli lemma, since $\Sigma_r P(A_r) < \infty$. □

The corollary will be applied in the proof of Lemma 7.

COROLLARY TO LEMMA 5. *Suppose the conditions of Lemma* 4 *hold. Suppose also that the process X satisfies a* 0-1 *law on its invariant σ-field, and that F is nontrivial. Then almost surely*

$$\mu(t) \to \infty \quad \text{as } t \to \infty.$$

PROOF. By Lemma 2 we see that Z diverges to infinity. If μ remains bounded then (2.10) implies that Y remains within a compact region, which would contradict the divergence of Z. But the event $\{\mu(t) \to \infty\}$ is invariant, and by Lemma 5 we see that (2.10) holds with positive probability. We conclude that $\mu(t) \to \infty$ almost surely. □

LEMMA 6. *Suppose the conditions of Lemma* 4 *hold. Then there are positive α and β such that, for any x, y in* $N \setminus \{\mathbf{o}\}$, *if*

$${}^N d(x, y) \leqslant \alpha \inf(\rho(x), \rho(y))$$

then

$$\hat{C} \leqslant {}^N d(x, y)\exp(-\beta \inf[\rho(x), \rho(y)]), \tag{2.11}$$

where $\hat{C}$ *is the angle between* $\Theta(x)$ *and* $\Theta(y)$; *that is to say the geodesic length between* $\Theta(x)$ *and* $\Theta(y)$ *on the unit sphere submanifold of* $T_{\mathbf{o}}N$. *Moreover α, β may be chosen without regard to the choice of* **o** *in N.*

PROOF. This geometric lemma is proved in [**9**, Lemma 2], as an application of the Rauch comparison theorem. In fact α and β depend only on the upper bound $-H^2$ of the sectional curvatures. □

The next lemma is the main result of this sequence. From its assertion the theorem follows quickly.

LEMMA 7. *Suppose that the conditions of Lemma* 4 *hold, that* Z *tends to infinity with probability* 1 *and that* (2.8) *holds. Then the limiting direction*

$$\Psi = \lim_{t\to\infty} \Theta(Y_t)$$

exists with probability 1 *and has a nontrivial law.*

PROOF. It suffices to establish the following claim: Given $\varepsilon > 0$, and R sufficiently large, if $Z_0 = R$ then $\Theta(Y)$ converges to a limit which has positive probability of being within ε of $\Theta(Y_0)$. Moreover this probability is bounded uniformly away from zero whatever the choices of $\mathbf{o} \in N$, $X_0 \in M$ may be.

This is enough to establish the existence of the limiting direction Ψ for general X_0. For if T_R is the first time that Z hits the level R then $T_R < \infty$ (since Z tends to ∞). Also, when conditioned on $X(T_R) = m$ the process $t \mapsto Y(T_R + t)$ has the probabilistic behaviour of Y when $X_0 = m$ and $Z_0 = R$ (by the Strong Markov property).

The claim can also be used to show the nontriviality of the law of Ψ. Given $\varepsilon > 0$ and a suitable R we can choose x_0, x_1 in M such that ${}^N d(F(x_0), F(x_1)) = 2R$, and choose $\mathbf{o}$ to be the unique point at the midpoint of the geodesic connecting the points $F(x_0)$ and $F(x_1)$ (the geodesic is unique by the Cartan-Hadamard theorem [**1**, 1.33]).

By [**10**, Theorem 3.1] applied to Brownian motion on M there is a positive chance that X will approach close to x_0, and likewise a positive chance that it will approach close to x_1. So the probabilities

P(for some t, $Z_t = R$ and $\Theta(F(X_t))$ is within ε of $\Theta(F(x_0)))$ and

P(for some t, $Z_t = R$ and $\Theta(F(X_t))$ is within ε of $\Theta(F(x_1)))$

are both positive. By an application of the Strong Markov Property, as described above, we can conclude that the chance of Ψ being within 2ε of $\Theta(F(x_0))$ is positive, as is the chance of Ψ being within 2ε of $\Theta(F(x_1))$. But the directions $\Theta(F(x_0))$ and $\Theta(F(x_1))$ are antipodal, so the nondegeneracy of the law of Ψ follows.

The proof of the lemma is therefore completed by the argument below, which establishes the claim.

Pick $\gamma \in (0, 1)$ and let q be the positive probability that (2.10) in Lemma 5 holds. By Lemma 3, given $\varepsilon' > 0$ such that $q - \varepsilon' > 0$, we can pick R so large that with probability $1 - \varepsilon'$

$$Z_t > R/2 + c\mu(t) \quad \text{for all } t \geq 0, \tag{2.12}$$

where $c = \frac{1}{3}K^{-2}H$. Thus with probability at least $q - \varepsilon'$ both (2.10) and (2.12) hold.

Without affecting this last statement we can pick R so large that

$$\alpha(R/2 + c\mu(t)) \geq (\mu(t))^\gamma \quad \text{for all } t \geq 0,$$

and also

$$\sum_r r^\gamma \exp\{-\beta(cr - r^\gamma + R/2)\} < \varepsilon. \tag{2.13}$$

Here α and β are the same constants as appear in Lemma 6.

Suppose that the event of the inequalities of the above paragraphs occurs. We shall show that the limit Ψ exists as a consequence of these inequalities. For, when $r \leqslant \mu(t) \leqslant r + 1$,

$$r^{\gamma} \leqslant \alpha(R/2 + c\mu(t)) \leqslant \alpha Z_t.$$

Thus for sufficiently large R the angle comparison result Lemma 6 can be applied to show that

$$\begin{aligned}\hat{C}(r,t) &= \text{the angle between } \Theta\big(Y(\mu^{-1}(r))\big) \text{ and } \Theta(Y(t))\\ &\leqslant {}^N d\big(Y(\mu^{-1}(r)), Y(t)\big)\exp\big\{-\beta \inf\big(Z(\mu^{-1}(r)), Z(t)\big)\big\}\\ &\leqslant r^{\gamma}\exp\big\{-\beta\big[Z(\mu^{-1}(r)) - r^{\gamma}\big]\big\}\end{aligned}$$

for $r = 0, 1, 2, \ldots$.

Furthermore by the lower bound on Z we see that

$$\hat{C}(r,t) \leqslant r^{\gamma}\exp\{-\beta(cr + R/2 - r^{\gamma})\}.$$

Applying (2.13) and the triangle inequality for the angles,

$$\begin{aligned}&\text{the angle between } \Theta(Y_s) \text{ and } \Theta(Y_t)\\ &\qquad \leqslant \sum_{r \geqslant (\min\{\mu(s),\mu(t)\} - 1)} r^{\gamma}\exp\{-\beta(cr - r^{\gamma} + R/2)\}.\end{aligned}$$

But by (2.13) this is a convergent sum with limit at most ε. Futhermore, the corollary to Lemma 5 may be applied to show that $\mu(t) \to \infty$ as $t \to \infty$. So, if the event mentioned above occurs (and this we have shown to have probability at least $q - \varepsilon'$) then the limit Ψ must exist, and must also be within ε of $\Theta(Y_0)$.

Moreover the bound $q - \varepsilon'$ holds uniformly whatever the choice of $\mathbf{o}$ in N and X_0 in M. It follows that the limit Ψ exists with probability 1. For from Lemma 5 all but finitely many of the inequalities (2.10) hold, and the proof above is easily adapted to use this fact to show that the limit Ψ exists with probability $1 - \varepsilon'$. But ε' was arbitrary. □

3. Conclusion. The main result of this paper appears as a consequence of the sequence of lemmas given in the previous section.

THEOREM. *Suppose N is simply connected, with*

$$-L^2 \leqslant \mathrm{Sect}(N) \leqslant -H^2 < 0.$$

Suppose M is such that $BM(M)$ satisfies a 0-1 law on its invariant σ-field, and M is stochastically complete. Then there is no nonconstant harmonic map F: $M \to N$ of K-bounded dilatation.

PROOF. Suppose there was such a map. The conclusion of Lemma 7 applies; so there would be a limiting asymptotic angle Ψ for the process $F(BM(M))$, and Ψ would have a nontrivial law. But Ψ would also be measurable with respect to the invariant σ-field of $BM(M)$; thus its existence would contradict the 0-1 law. □

If M has nonnegative Ricci curvature then $BM(M)$ certainly satisfies the required 0-1 law, as we remark in the discussion before Lemma 2.

COROLLARY. *The above theorem can be extended to the case when N is not simply connected, and $BM(\tilde{M})$ satisfies a* 0-1 *law on its invariant σ-field. Here $\tilde{M}$ is the universal covering of M.*

PROOF. This follows from lifting arguments. □

By the method of proof it is clear that further generalisation of this result is possible. For example, the curvature conditions on N can be relaxed to hold only off compact sets, as long as N itself is required to be noncompact. The condition of stochastic completeness is not necessary, being imposed only to simplify the exposition. In general a suitable modification of the 0-1 law, to allow for explosions of X, will suffice.

A comparison with the original result **[5]**, proved by geometric means, prompts a question as to whether the lower bound on the curvature of N is necessary for the probabilistic proof. This question remains open. However some progress can be made in a special case.

Suppose that N contains a totally geodesic submanifold $\mathcal{P}$ separating N into two components, and suppose $\mathrm{Sect}(N) \leqslant -H^2 < 0$. Then a generalisation of Lemmas 1 and 2, studying ${}^{N}d(Y, \mathcal{P})$, shows that either F is trivial or eventually Y ends up in one of the components of $N \setminus \mathcal{P}$ and never leaves it. Moreover it has positive chance of ending up in either; thus the 0-1 law for the invariant σ-field is contradicted. This approach uses the method of **[6**, Proposition 7.1].

Recently S. Goldberg and C. Mueller have used probabilistic methods of the type described here to show that, in the theorem stated above, the upper bound on the curvatures can be replaced by a less restrictive condition (personal communication).

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