A GENERAL MAXIMAL OPERATOR AND THE $A_{\rho}^*$-CONDITION

BY

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Abstract. A rearrangement inequality for a general maximal operator $Mf(x) = \sup_{x \in Q} \int_{fQ} dv$ is established. This is then applied to the Hardy-Littlewood maximal operator with weights.

1. Let $\mu, \nu$ be two measures on $\mathbb{R}^n$ and let there be given for each cube $Q \subset \mathbb{R}^n$ a function $\phi_Q$ supported in $Q$. We consider the maximal operator $Mf(x) = \sup_{x \in Q} \int_{fQ} dv$, where the sup is extended over all cubes centered at $x$ and obtain (Theorem 1) the rearrangement inequality $(Mf)_{\mu}^*(x) \leq A \int_{Q} \Phi(t)^{1/\rho} dt$. Here $g^*_{\lambda}$ denotes the non-increasing rearrangement of $g$ with respect to the measure $\lambda$, and $\Phi$ is a non-increasing function given in terms of $\mu, \nu, \phi_Q$. From this one easily sees that $\|Mf\|_{p,u} \leq A \|f\|_{p,v}^{1/\rho}$, and thus the finiteness of this integral, i.e., $\Phi \in L(p', 1)$, gives a weighted norm inequality. This is how the $A_{\rho}^*$-condition comes into play. In fact, if we take $(u, v) \in A_{\rho}$, i.e., $\int_{Q} u \cdot (\int_{Q} v)^{1-\rho} \, dt \leq C \|Q\|^{1/\rho}$ [5], $d\mu = ud\lambda$, $d\nu = v\, d\lambda$, $\phi_Q(x) = \chi_Q(x)/\|Q\| v(x)$, then the above $Mf(x)$ is the usual Hardy-Littlewood maximal operator. Let $\Phi = \Phi_{u,v}$ be the associated $\Phi$. We will show (Theorem 3) that, in the case $u = v$, $\Phi \in L(p', 1)$ if and only if $u \in A_{\rho}$, and in the double weight situation (Theorem 4), $\Phi \in L(p', \infty)$ if and only if there is $(u, v)$ for which $\Phi_{u,v} \sim \Phi$ and $\|Mf\|_{p,u} \leq A \|f\|_{p,v}$.

Finally, we will study the problem when $(u, v) \in A_{\rho}$ implies $\|Mf\|_{p,u} \leq A \|f\|_{p,v}$, and the extrapolation problem, i.e., when does $\|Mf\|_{p,u} \leq A \|f\|_{p,v}$ imply the existence of $\varepsilon > 0$ so that $\|Mf\|_{p-\varepsilon,u} \leq B \|f\|_{p-\varepsilon,v}$? It turns out that the behavior of the iterated maximal operator $M_j$ is crucial here. We will see (Theorem 6) that extrapolation is possible provided the norm of $M_j$ as an operator from $L_p^\rho - L_u^\rho$ grows at most geometrically, a fact which is obvious for $u = v$. All this gives a different, though admittedly long, proof of $u \in A_{\rho}$, and shows that it is the iterated maximal operator that controls this implication.

2. For $v \geq 0$ a Borel measure on $\mathbb{R}^n$ and $f: \mathbb{R}^n \to \mathbb{R}$ a Borel measurable function, let $\lambda_{f,u}(y) = v(x: |f(x)| > y)$, and $f^*_u(t) = \inf\{y: \lambda_{f,u}(y) \leq t\}$, the rearrangement of $f$ with respect to $v$. With each $Q \in \{Q\}$, the collection of cubes in $\mathbb{R}^n$, let there be associated a Borel measurable function $\phi_Q: \mathbb{R}^n \to [0, \infty)$, $\operatorname{supp} \phi_Q \subset Q$. We consider the general maximal operator.

$Mf(x) = \sup_{x \in Q} \int_{fQ} \phi_Q f \, dv$
where the sup is extended over all \( Q \) with center \( x \). If \( \mu \geq 0 \) is another Borel measure on \( \mathbb{R}^n \), finite on compact sets, define

\[
\Phi(t) = \sup_Q \{ \mu(Q) \phi_Q^*(\mu(Q)t) \}.
\]

**Theorem 1.**

\[
(Mf)^*(\xi) \leq A \int_0^\infty \Phi(t) f^*(t\xi) \, dt,
\]

where \( A \) depends only upon the dimension \( n \).

**Proof.** We let \( M_f f(x) = \sup \int \phi_Q f \, dv \), where now the sup is extended over all \( Q \) with center \( x \) and \( \text{diam} \, Q \leq r \). It suffices to prove the theorem for \( M_f f \) and then let \( r \to \infty \).

Let \( E_\tau = \{ x : M_f f(x) > \tau \} \) and \( E_{\tau, r} = E_\tau \cap \{ |x| \leq r \} \). For \( x \in E_{\tau, r} \), we have a cube \( Q_x \), center \( x \), \( \text{diam} \, Q_x \leq r \) such that \( \tau \leq \int \phi_Q f \, dv \). We can now apply the Besicovitch covering theorem [1] and select \( \{ Q_j \} \subset \{ Q_x : x \in E_{\tau, r} \} \) such that \( E_{\tau, r} \subset \bigcup \{ Q_j \} \) and \( \sum \chi_{Q_j}(t) \leq C \), where \( C \) depends only upon \( n \). Then \( \mu(Q_j)\tau \leq \int \mu(Q_j) \phi_Q f \, dv \). We set \( H_N = \sum_{j=1}^N \mu(Q_j) \), \( \Phi_N(y) = \sum_{j=1}^N \mu(Q_j)\phi_Q(y) \). Then

\[
H_N \leq \frac{1}{\tau} \int_{\mathbb{R}^n} \Phi_N(y) f(y) \, dv \leq \frac{1}{\tau} \int_0^\infty \Phi_N^*(t) f^*(t) \, dt.
\]

We claim now that \( \Phi_N^*(\xi) \leq c \Phi(\xi/H_N) \), where \( c \) is the Besicovitch constant. If \( \Phi_N(y) > \alpha, \alpha > 0 \), then \( y \in \bigcup_{j=1}^N Q_j \). Thus the number of \( Q_j \)'s containing \( y \) is at most \( c \), and hence for some \( j, \mu(Q_j)\phi_Q(y) > \alpha/c \). Thus

\[
\{ y : \Phi_N(y) > \alpha \} \subset \bigcup_{j=1}^N \{ y : \mu(Q_j)\phi_Q(y) > \alpha/c \}.
\]

We now show that for \( \beta > 0 \),

\[
\nu \{ y : \mu(Q)\phi_Q(y) > \beta \} \leq \mu(Q) \nu \{ t : \Phi(t) > \beta \}.
\]

To prove this we may assume that \( \mu(Q) > 0 \). Then

\[
\nu \{ t : \Phi(t) > \beta \} \geq \nu \{ t : \mu(Q)\phi_Q^*(\mu(Q)t) > \beta \} = \frac{1}{\mu(Q)} \nu \{ y : \mu(Q)\phi_Q(y) > \beta \}.
\]

All this gives us \( \nu \{ y : \Phi_N(y) > \alpha \} \leq H_N \nu \{ t : \Phi(t) > \alpha/c \} \). Consequently, \( \Phi_N^*(\xi) \leq \inf \{ \alpha : \nu \{ t : \Phi(t) > \alpha/c \} \leq \xi/H_N \} = c \Phi(\xi/H_N) \).

Thus

\[
\tau \leq \frac{c}{H_N} \int_0^\infty \Phi \left( \frac{t}{H_N} \right) f^*(t) \, dt \leq \frac{c}{H_N} \int_0^\infty \Phi \left( \frac{t}{H} \right) f^*(t) \, dt,
\]

since \( H_N \leq H \), and \( H = \sum \mu(Q_j) < \infty \). Since \( H_N \uparrow H \), we get

\[
\tau \leq \frac{c}{H} \int_0^\infty \Phi \left( \frac{t}{H} \right) f^*(t) \, dt = c \int_0^\infty \Phi(t) f^*(tH) \, dt,
\]
and since \( \mu(E_{r,R}) \leq H \), we see that 
\[
\tau \leq c_{r,s}^{\Phi}(t) f_s^{\Phi}(\mu(E_{r,R})) dt.
\]
Finally, let \( \tau_0 = (M_t f)_H^{\Phi}(\xi) \) be the inf \( \tau : \mu(E_t) \leq \xi \). Then \( 0 < \tau < \tau_0 \) implies that \( \mu(E_t) > \xi \), and hence for some \( R, \mu(E_{r,R}) > \xi \). From this we get 
\[
\tau \leq c_{r,s}^{\Phi}(t) f_s^{\Phi}(t \xi) dt,
\]
and letting \( \tau \to \tau_0 \), completes the proof.

Remark. Theorem 1 contains many of the known maximal inequalities.

(i) The choice \( \phi_0(y) = \chi_{Q_0}(y)/|Q| \), \( \mu = \nu = \text{Lebesgue measure} \), gives the ordinary Hardy-Littlewood maximal function. In this case \( \Phi(t) = \chi_{[0,1]}(t) \) and so 
\[
(Mf)^*(\xi) \leq A_{t,0}^{\Phi}(t \xi) dt.
\]

(ii) Let \( Q_0 \) be the unit cube centered at the origin, and let \( Q(x, h) \) be the cube with center \( x \), side-length \( h \). Let \( \sup p \subset Q_0 \), and set \( \phi_Q(y) = \phi((x - y)/h)/h^n \), \( Q = Q(x, h) \). If \( \mu = \nu = \text{Lebesgue measure} \), we consider the maximal “approximate identity” operator \( Mf(x) = \sup_{h > 0}((1/h^n) \phi((x - y)/h)f(y) dy \) [4]. In this case 
\[
\lambda_{\phi_Q}(y) = |Q||\{x: \phi(x) > \gamma \}|(|Q|),
\]
and hence \( \phi_Q^*(t) = \phi^*(t/|Q|)|Q| \). Thus \( \Phi(t) = \phi^*(t) \), and we get 
\[
(Mf)^*(\xi) \leq A_{t,0}^{\Phi}(t \xi) dt.
\]
This maximal inequality is due to Jurkat and Troutman [4] and our proof of Theorem 1 is a refinement of theirs.

3. Minkowski’s integral inequality and Theorem 1 show that
\[
\|Mf\|_{p,u} \leq A_{t,0}^{\Phi}(t^{1/p}) \|f\|_{p,v},
\]
and hence \( \int_0^{\infty} \Phi(t)/t^{1/p} dt < \infty \) implies that \( Mf \) is strong \( (p, p) \). In the setting of Lorentz spaces \( L(p, q) \) [2], this says that \( \Phi \in L(p', 1) \), \( 1/p + 1/p' = 1 \), implies strong \( (p, p) \) for \( Mf \). A major part of this paper is devoted to the converse, i.e., when does strong \( (p, p) \) for \( Mf \) imply \( \Phi \in L(p', 1) \)? Simple examples show that this need not be the case in general. For, if we consider the “approximate identity” example of the previous section and assume that \( \phi \) is radially nonincreasing, then 
\[
Mf(x) \leq \|\phi\|_1 M_0 f(x),
\]
where \( M_0 \) is the ordinary Hardy-Littlewood maximal operator. Simply take \( \phi \in L^1 \), \( \phi \notin L(p', 1) \) to obtain an example.

We let now \( (u,v) \) be a pair of nonnegative functions (weights), i.e., \( u \in L_{1,\infty}^1 \) and \( 0 < v < \infty \), a.e. \( x \). This last restriction is made in order to avoid the special cases arising from division by zero, etc. Then
\[
\frac{1}{|Q|} \int_Q f \, dx = \frac{1}{|Q|} \int f \cdot \frac{\chi_Q}{v} \, v \, dx = \int f \cdot \phi_Q \, dv,
\]
where \( \phi_Q(x) = \chi_Q(x)/|Q| v(x) \), and \( dv = v \, dx \). If we let \( d\mu = u \, dx \) and
\[
\Phi(t) \equiv \Phi_{u,v}(t) = \sup \{\mu(Q) \phi_Q^*(\mu(Q)t)\},
\]
then \( \Phi \in L(p', 1) \) gives the double weight strong \( (p, p) \) for the ordinary Hardy-Littlewood maximal operator, which from now on we will denote by \( Mf \).

4. The single weight problem, i.e., \( u = v \), and the double weight problem are different and the \( \Phi \) reflects this.

Theorem 2. Let \( 1 < p < \infty \) and \( \|Mf\|_{p,u} \leq A \|f\|_{p,v} \). Then \( \Phi = \Phi_{u,v} \) satisfies (i) \( \Phi(t) = O(t^{-1/p}) \), as \( t \to 0 \) or \( \infty \), (ii) \( \Phi(t) = O(t^a) \) for \( 0 > a > -1 \) as \( t \to \infty \).

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Proof. It is known that \((u, v) \in A_p\), i.e., \(\int Q u \cdot (\int Q v^{1-p'})^{p-1} \leq c \left| Q \right|^p\) [5]. We note that

\[
\left( \frac{X_Q}{v} \right)^* (\mu(Q)t) \leq \left[ \frac{1}{\mu(Q)t} \int_0^{\mu(Q)t} \left( \frac{X_Q}{v} \right)^* (u) \, du \right]^{1/p'} \leq \left[ \frac{1}{\mu(Q)t} \int_Q \left( \frac{1}{v} \right)^{p'} v \, dx \right]^{1/p'}.
\]

From this we get

\[
\frac{\mu(Q)}{|Q|} \left( \frac{X_Q}{v} \right)^* (\mu(Q)t) \leq \frac{1}{t^{1/p'}} \frac{\mu(Q)^{1/p}}{|Q|} \left( \int_Q v^{1-p'} \right)^{(p-1)/p} \leq \frac{c}{t^{1/p'}},
\]

and this proves (i).

For (ii) simply note that \(\|Mf\|_{q,u} \leq A_q \|f\|_{q,v}, p \leq q\), so that by (i), \(\Phi(t) = O(t^{-1/q'})\).

Remark. (i) The above result shows that the behavior of \(\Phi\) about 0 is much more critical than that about \(\infty\). (ii) We have shown that \((u, v) \in A_p\) implies that \(\Phi \in L(p', \infty)\).

Theorem 3. \(\|Mf\|_{p,u} \leq A\|f\|_{p,u}\) for some \(p > 1\) if and only if \(\Phi \in L(p', 1)\), i.e., \(\Phi \in L(p', 1)\) and \(u \in A_p\) are equivalent.

Proof. Note that now \(\Phi_Q(x) = X_Q(x) / |Q| u(x)\), so that \(\Phi_Q^* (\mu(Q)t)\) is zero for \(t > 1\), and hence \(\Phi(t) = 0, t > 1\). Thus we have to show that \(\int_0^t \Phi(t) / t^{1/p} \, dt < \infty\).

Since \(u \in A_{p-\varepsilon}\) for some \(\varepsilon > 0\) [5], we get from Theorem 2 that \(\Phi \in L((p - \varepsilon)', \infty)\) from which \(\Phi(t)/t^{1/p} \leq c / t^{1/(p-\varepsilon)' + 1/p}\).

Remark. Later we will show that \(\Phi \in L(p', 1)\) implies \(\Phi \in L((p - \varepsilon)', 1)\) in the single weight case without recourse to \(u \in A_{p-\varepsilon}\).

5. From now on we assume that \(n = 1\), and we will denote by \(I, J\) intervals in \(\mathbb{R}\).

In this section we will present a partial converse of Theorem 2, i.e., we ask whether \(\Phi \in L(p', \infty)\) implies some norm inequality for \(Mf\).

For \(\Phi_1, \Phi_2\) two nonincreasing functions on \((0, \infty)\) we write \(\Phi_1 \prec \Phi_2\) provided there are constants \(c_i, c_i', i = 1, 2\), such that \(c_1 \Phi_1 \left( c_1' t \right) \leq \Phi_2(t) \leq c_2 \Phi_1 \left( c_2' t \right), 0 < t < 1\).

Theorem 4. Let \(\Phi_0 \geq 0\) be nonincreasing on \((0, \infty)\) such that \(t \Phi_0(t) > 0\) as \(t \downarrow 0\). Then \(\Phi_0 \in L(p', \infty)\) on \([0, 1]\) if and only if there exists a pair of weights \((u, v)\) such that \(\Phi_{u,v} \prec \Phi_0\) and \(\|Mf\|_{p,u} \leq A\|f\|_{p,v}\).

Remark. The condition \(t \Phi_0(t) > 0\) as \(t \downarrow 1\) can always be achieved by replacing \(\Phi_0\) by \(\Phi(t) = (1/t) \Phi_0(1/t)\) for \(t > 1\) and \(\Phi_0\) is in the same \((p > 1)\) integrability class as \(\Phi_0\).

Proof. By Theorem 2 we only need to show that \(\Phi_0 \in L(p', \infty)\) implies the existence of \((u, v)\). We may assume that \(\Phi_0(1) = 1\) and \(\Phi_0(t) \uparrow \infty\) as \(t \downarrow 0\) (otherwise let \(u = v = 1\)). Let \(\alpha_N = \Phi_0(2^{-N})\). Then \(\alpha_N \leq A 2^{N/p'}\), and since \(2^{-N} \alpha_N \rightarrow 0\), we may assume that \(\alpha_N 2^{-N} \leq 1/4, N = 1, 2, \ldots\). Also note that \(2^{-k} \alpha_k \leq \alpha_j\).

Let \(J_N = [2^{N_1}, 2^{N_1 + 2}, \alpha_N 2^{-N}]\). \(N = 1, 2, \ldots, J_0 = \mathbb{R} \setminus \bigcup_{N=1}^\infty J_N\). \(K_N = [2^{N_1} + \frac{1}{4}, 2^{N_1 + 1}]\). Define \(v_N(t) = \alpha_N^{-1}, t \in J_N\), and \(v_N(t) = 0, t \not\in J_N\). Let \(u_N(t) = 4^{1/2} t, t \in J_N\), and \(v_N(t) = 0, t \not\in J_N\). Let \(j_N(t) = 4\).
\[ t \in K_N, \text{ and } u_N(t) = 0, t \notin K_N. \] The desired pair of weights will be \( v(t) = \Sigma v_N(t) + 4\chi_J(t), u(t) = \Sigma u_N(t). \) We note that \( \nu(J_N) = 2^{-N}, \) from which \( (\chi_J)^*_v(t) = \chi_{[10,2^{-N}]}(t). \) Also \( \mu(K_N) = 1. \)

We wish to estimate
\[ \Phi(t) \equiv \Phi_{u,v}(t) = \sup_I \frac{\mu(I)}{|I|} \left( \frac{\chi_I}{v} \right)^* \left( \mu(I) \right)^{1/2} \left( \chi_I \right)^* \left( \mu(I) \right)^{1/2}, \]
and show that \( \Phi \sim \Phi_0. \) This will follow if for some constants \( c', c'' \), \( c'\alpha_l \leq \Phi(2^{-l}) \leq c''\alpha_l, l = 1, 2, \ldots. \) Our first observation is that
\[ \left( \frac{\chi_I}{v} \right)^* \left( 2^{-l} \right) \leq c\alpha_l, \quad l = 1, 2, \ldots. \]
To see this, note that if \( I \cap J_N = \emptyset \) for every \( N \), then \( (\chi_I/v)^*(t) \leq \frac{1}{2} \leq \alpha_l. \) Otherwise, let \( J_N, J_{N+1}, \ldots, J_M \) be all the \( J_i \)'s with \( I \cap J_i \neq \emptyset \). If \( I_0 = [0,2^{-M}], \) and for \( j > 1, I_j = [2^{-M} + 2^{-M+1} + \ldots + 2^{-M+j-1}, 2^{-M} + 2^{-M+1} + \ldots + 2^{-M+j}], \) then
\[ (\chi_I/v)^*(t) \leq \alpha_{M-j}, \quad t \in I_j. \] Thus, if \( 2^{-l} \in I_j, \ M - j \leq l + 1, \) and since \( 2^{-N}\alpha_n \leq 0, \ 2\alpha_l \geq \alpha_{l+1} \geq \alpha_{M-j}. \)

It is clear that \( \mu(I) \leq 4|I|, \) and if \( \mu(I) > 0 \) and \( I \cap J_N \neq \emptyset \) for some \( N, \) then \( |I| \leq \frac{1}{2} - \alpha_N/2^N \geq \frac{1}{2}. \) From this we see that, if \( \mu(I) \geq 1, \) then
\[ \frac{\mu(I)}{|I|} \left( \frac{\chi_I}{v} \right)^* \left( \mu(I) \right)^{1/2} \leq c\alpha_l, \]
and if \( 1/2^{k+1} \leq \mu(I) \leq 1/2^k, \) and \( I \cap J_N \neq \emptyset \) for some \( N, \) then
\[ \frac{\mu(I)}{|I|} \left( \frac{\chi_I}{v} \right)^* \left( \mu(I) \right)^{1/2} \leq c2^{-k} \left( \frac{\chi_I}{v} \right)^* \left( 2^{-l-l-1} \right) \leq c2^{-k} \alpha_{k+l+1} \leq c\alpha_l. \]
This shows that \( \Phi(2^{-l}) \leq c\alpha_l, \) and since for \( I = [2^{l/2}, 2^{l/2} + 1], \)
\[ \frac{\mu(I)}{|I|} \left( \frac{\chi_I}{v} \right)^* \left( \mu(I) \right)^{1/2} = \alpha_l, \quad \alpha_l \leq \Phi(2^{-l}). \]

We proceed now with the proof of \( \|Mf\|_{p,u} \leq A \|f\|_{p,v}. \) Let \( f \geq 0, f_N = f\chi_{J_N}, \ N = 0, 1, \ldots. \) Then
\[ \int (Mf)^p dx = \int \left[ M(\sum f_N) \right]^p dx \leq 2^{p-1} \int (Mf_0)^p dx + 2^{p-1} \int \left[ M(\sum f_N) \right]^p dx. \]
Note that \( \int (Mf_0)^p dx \leq c\|f_0\|_{p,v}, \)
We next claim that \( \int (Mf_N)^p u_N dx \leq c\|f_N\|_{p,v} \). For \( x \in K_N = \text{supp } u_N \) we have
\[ (Mf_N)^p(x) \leq 2 \left[ 2^{N+1} + \alpha_N 2^{-N} f_N dx \right]^p \leq 2^p (\alpha_N 2^{-N})^{p'/p} \|f_N\|_p. \]
Since \( \alpha_N = O(2^{N/p'}) \) we obtain \( (\alpha_N 2^{-N})^{p'/p} \leq c\alpha^{-1} \) from which \( (Mf_N)^p \leq c(1/\alpha_N) \|f_N\|_p = c\|f_N\|_{p,v}. \)
We next observe that \( \int M^p(\sum f_N)u_k \, dx = \sum \int M^p(\sum f_N)u_k \, dx \), and thus, using \((\sum a_j)^p \leq 2^{jp-1}a^p\), we get
\[
\int M^p(\sum f_N)u_k \, dx \leq 2^{p-1} \sum_{N > k} 2^{(N-k)(p-1)} \int M^p f_N u_k \, dx \\
+ 2^{p-1} \sum_{N < k} 2^{(k-N)(p-1)} \int M^p f_N u_k \, dx.
\]

For \( k < N \) we get
\[
\int M^p f_N u_k \, dx \leq c \int \left( \frac{1}{2N^2 - 2k^2} \int f_N \right)^p u_k(x) \, dx \\
\leq \frac{c}{2pN^2} \left( \int f_N \right)^p \int u_k \, dx \leq \frac{c}{2pN^2} \int M^p f_N u_N \, dx,
\]
since \( \int u_k \, dx = \int u_N \, dx \).

Similarly, if \( k > N \), \( \int M^p f_N u_k \, dx \leq (c/2 pk^2) \int M^p f_N u_N \, dx \), and thus we get
\[
\int M^p(\sum f_N)u_k \, dx \leq 2^{p-1} \int M^p f_k \cdot u_k \, dx \\
+ c 2^{p-1} \left\{ \sum_{N > k} \frac{2^{N(p-1)}}{2pN^2 \cdot 2k^{p-1}} \int M^p f_N \cdot u_N \, dx + \sum_{N < k} \frac{2^{k(p-1)}}{2p^2 \cdot 2N^{p-1}} \int M^p f_N \cdot u_N \, dx \right\}.
\]

We sum this over \( k \) and interchange the order of summation to get
\[
\int M^p f u \, dx \leq 2^{p-1} \sum_{k=1}^{\infty} \int M^p f_k \cdot u_k \, dx \\
+ c 2^{p-1} \left\{ \sum_{N=1}^{\infty} \sum_{k=1}^{N} \frac{2^{N(p-1)}}{2pN^2 \cdot 2k^{p-1}} \int M^p f_N \cdot u_N \, dx \\
+ \sum_{N=1}^{\infty} \sum_{k=N}^{\infty} \frac{2^{k(p-1)}}{2p^2 \cdot 2N^{p-1}} \int M^p f_N \cdot u_N \, dx \right\}
\]
\[
\leq A \sum_{k=1}^{\infty} \int M^p f_k \cdot u_k \, dx \leq A \sum_{k=1}^{\infty} \int f_k^p v_k \, dx \leq A \int f^p v \, dx.
\]

**Remark.** Under the hypothesis of Theorem 4, \( \Phi_0 \in L(p', \infty) \) if and only if there exists \((u, v)\) for which \( \mu(x; M\Phi(x) > y) \leq c ||f||_{p', y}^p \) and \( \Phi_{u,v} \sim \Phi_0 \).

6. In order to make a more detailed study of \( \Phi = \Phi_{u,v} \) for a pair of weights \((u, v)\) we need some preliminary results. Again our analysis will take place on \( \mathbb{R} \).

For \( f: \mathbb{R} \to [0, \infty) \), and \( I, J \) compact intervals let
\[
M_{j,I} f(x) = \sup_{x \in J \subset I} \frac{1}{|J|} \int_J M_{j-1,I} f(y) \, dy,
\]
the \( j \)th iterated maximal function relative to \( I \). Set \( M_{j,I} f(x) = 0, x \notin I \). \( M_{0,I} f(x) = f(x) \chi_I(x) \).
Lemma 1. Let \( f \geq 0 \) be in \( L^1(I) \), \( \text{supp } f \subset I \), and let \( g \geq 0 \) be in \( L^1(J) \), \( \text{supp } g \subset J \), and assume \( |I| = |J| \). Assume there are constants \( C \geq 1, A \geq 1 \) with
\[
C | \{ x \in I: f(x) > \alpha / A \} | \geq | \{ y \in J: g(y) > \alpha \} | , \quad \alpha > 0.
\]
Then there is a constant \( B \) so that
\[
ACB \int_I M_{\alpha, I} f \geq \int_J M_{\alpha, J} g.
\]

Proof. We will first establish that for \( \alpha > 0 \)
\[
2BC \left| \left\{ x \in I: M_{\alpha, I} f(x) > \frac{\alpha}{2A} \right\} \right| \geq \left| \left\{ y \in J: M_{\alpha, J} g(y) > \alpha \right\} \right| ,
\]
where \( B \) is the Besicovitch covering constant. To do this, we may assume that \( \alpha / 2A > (1/|I|) \int_I f \), as otherwise \( M_{\alpha, I} f(x) > \alpha / 2A, x \in I \), and (1) follows. We have
\[
\left| \left\{ y \in J: M_{\alpha, J} g(y) > \alpha \right\} \right| \leq \frac{2B}{\alpha} \int_{(g > \alpha / 2)} g = \frac{2B}{\alpha} \left[ \frac{\alpha}{2} \int_{\lambda_g} ^{\alpha} \lambda_g(\tau) d\tau \right] ,
\]
where \( \lambda_g(\tau) = \left| \{ x \in J: g(x) > \tau \} \right| \). By hypothesis this is majorized by
\[
\frac{2BC}{\alpha} \left[ \frac{\alpha}{2} \int_{\lambda_g} ^{\alpha} \lambda_g(\tau) d\tau \right] = \frac{2BCA}{\alpha} \int_{(g > \alpha / 2A)} f
\]
\[
\leq 2BC \left| \left\{ x \in I: M_{\alpha, I} f(x) > \frac{\alpha}{2A} \right\} \right| ,
\]
(see [8, p. 23]).

We now iterate (1) and get
\[
(2B)'C \left| \left\{ x \in I: M_{\alpha, I} f(x) > \frac{\alpha}{2A} \right\} \right| \geq \left| \left\{ y \in J: M_{\alpha, J} g(y) > \alpha \right\} \right| ,
\]
Consequently, \( \int_I M_{\alpha, I} f = \int_J \lambda_{M_{\alpha, J} g}(\alpha) d\alpha \leq (2B)'CA \int_I M_{\alpha, I} f \).

Lemma 2. Let \((u, v) \in A_p\) for some \( p > 1 \), i.e., \( \int_I u \cdot (\int_J v^{1-p'})^{p-1} \leq c |I|^p \), and form \( \Phi = \Phi_{u,v} \). Assume that \( v^{-1}(t) = 0, t > 0 \). Then for each \( N \) there exists \( \alpha_N \) and compact intervals \( I_N \supset J_N, I_N \supset I_N \), such that \( J_N \cap I_N = \emptyset \) and \( J_N, I_N \) have an endpoint in common with \( I_N \), and there is \( S_N \subset J_N \) such that

(i) \( \Phi(2^{-N}) \leq c\mu(I_N)\alpha_N / |I_N| \leq c2^{N/p'} \),
(ii) \( \alpha_N \leq 1/v(x) \leq 5\alpha_N, x \in S_N \),
(iii) \( \mu(I_N)/(5 \cdot 2^N) \leq \nu(S_N) \leq \mu(I_N)/2^N \),
(iv) \( \alpha_N \leq (\chi_{S_N}/v)^* (\mu(I_N)/(5 \cdot 2^N)) \leq 5\alpha_N \).

Proof. Since
\[
\Phi(2^{-N}) = \sup_I \frac{\mu(I)}{|I|} \left( \frac{X_I}{v} \right)^* \left( \frac{\mu(I)}{2^N} \right)
\]
choose an interval \( \tilde{I}_N = [a_N, b_N] \) for which
\[
\Phi(2^{-N}) \leq 2 \frac{\mu(\tilde{I}_N)}{|I_N|} \left( \frac{X_{\tilde{I}_N}}{v} \right)^* (\mu(\tilde{I}_N)2^{-N}) .
\]
We can pick points $a_N = x_0 < x_1 < x_2 < x_3 < x_4 = b_N$ for which $\int_{x_{i-1}}^{x_i} u \, dx = \mu(I_N)/4$, $i = 1, 2, 3, 4$. If $I_{N,i} = [x_{i-1}, x_i]$, we have, since
\[
\left( \frac{X_{I_N}}{v} \right)^* (\tau) \leq \frac{4}{\tau} \sum_{i=1}^{4} \left( \frac{X_{I_{N,i}}}{v} \right)^* (\tau/4),
\]
an $i$ such that for each $j$, \[\Phi(2^{-N}) \leq \frac{\mu(I_{N,i})}{|I_N|} \left( \frac{X_{I_{N,i}}}{v} \right)^* \left( \frac{\mu(I_{N,i})}{2^N} \right).
\]
Select now a $j$ so that $I_{N,j} \cap I_{N,i} = \emptyset$, and write $J_N = I_{N,i}$, $J_N^* = I_{N,j}$ and let $I_N$ be the smallest interval containing $J_N \cup J_N^*$. Then
\[
\Phi(2^{-N}) \leq \frac{\mu(J_N^*)}{|I_N|} \left( \frac{X_{J_N^*}}{v} \right)^* \left( \frac{\mu(J_N^*)}{2^N} \right).
\]
Let us denote by / an interval in $J_N^*$ which has that endpoint in common with $J_N^*$ which $J_N$ has in common with $I_N$, and set
\[
\overline{\Phi}(2^{-N}) = \sup_{J'} \frac{\mu(J')}{|I_N|} \left( \frac{X_{J_N^*}}{v} \right)^* \left( \frac{\mu(J')}{2^N} \right).
\]
Select now $J'$ for which the sup is “attained”, i.e.
\[
\overline{\Phi}(2^{-N}) \leq 2 \frac{\mu(J')}{|I_N|} \left( \frac{X_{J_N^*}}{v} \right)^* \left( \frac{\mu(J')}{2^N} \right),
\]
and let $\alpha_N = \left( \frac{X_{J_N^*}}{v} \right)^*(\mu(J')/2^N)$.

We define $S_N \subset \{x \in J_N' : 5\alpha_N \geq 1/v(x) \geq \alpha_N\}$, and $S_N' = \{x \in J_N' : 1/v(x) \geq \alpha_N\}$. Since $|v^{-1}(t)| = 0$, $t > 0$, we see that \[v(S_N') = v\{x \in J_N : 1/v(x) > \alpha_N\} = \frac{\mu(J')}{2^N}.
\]
We claim now that $v(S_N') \geq \frac{1}{\alpha_N} v(S_N')$. To prove this we may assume that $v(S_N') > v(S_N)$. If $v(S_N') < \frac{1}{\alpha_N} v(S_N')$, then
\[
\frac{\mu(J')}{2^N} \geq v(S_N') > \frac{1}{\alpha_N} v(S_N') = \frac{4}{5} \frac{\mu(J')}{2^N}.
\]
We can now choose an interval $J'' \subset J_N^*$ for which $\mu(J'')2^{-N} \leq v(S_N' \setminus S_N) \leq \mu(J'')2^{-N+1}$, and $J''$ is a candidate for the sup of $\Phi$. Then $\mu(J'') > \frac{3}{2} \mu(J')$, and since \[(X_{S_N' \setminus S_N}/v)^*(\mu(J')/2^N) \geq 5\alpha_N
\]
we get
\[
\overline{\Phi}(2^{-N}) \geq \frac{\mu(J'')}{|I_N|} \left( \frac{X_{S_N' \setminus S_N}}{v} \right)^* \left( \frac{\mu(J'')}{2^N} \right) \geq 2\alpha_N \frac{\mu(J')}{|I_N|} \geq \overline{\Phi}(2^{-N}).
\]
Hence $\frac{1}{2\alpha_N} \mu(J')/2^N \leq v(S_N') \leq \mu(J')/2^N$.

If we let $I_N' = J'$, the properties (ii), (iii), and (iv) of the lemma follow, and the only thing that remains is $\mu(I_N')\alpha_N / |I_N| \leq c2^{N/\rho}$. This can be done by the same argument used in Theorem 2 for (i) since $(u, v) \in A_p$.

7. It is well known that $u \in A_p$, $p > 1$, implies that $u \in A_{p-\varepsilon}$, for some $\varepsilon > 0$, and that this is no longer the case for $(u, v) \in A_p$ [5, 6]. If we want $(u, v) \in A_{p-\varepsilon}$, then
in terms of $\Phi = \Phi_{u,v}$ we need to prove that $\Phi \in L((p - \epsilon)', 1)$. Here is where the behavior of the iterated maximal operator $M_j f$ comes into the picture.

**Theorem 5.** Let $(u, v) \in A_p$, $1 < p$, and let $\Phi = \Phi_{u,v}$. Assume that $\| M_j f \|_{p,u} \leq A_j \| f \|_{p,v}$, $f \in L_v^p$, $j = 1, 2, \ldots$. Then there are constants $c > 0$, $B > 0$ such that for every $j$, $N$,

$$\Phi(2^{-N}) \leq c \frac{A_{j+1}}{B^j} \left( \frac{j!}{N!} \right)^{2N/p'}.$$

**Proof.** We will first show that we may assume that $|\tilde{v}^{-1}(t)| = 0$, $t > 0$. Since our overall assumption on $v$ is $0 < v < \infty$ a.e., we choose $v(x) < \tilde{v}(x) < 2v(x)$ such that $|\tilde{v}^{-1}(t)| = 0$, $t > 0$. Then $(u, \tilde{v}) \in A_p$ and $\Phi_{u,\tilde{v}}(t) \leq 2\Phi_{u,v}(t)$.

We now choose $I_N \supset J_N$, $I_N \subset J_N$, and $\alpha_N$ as in Lemma 2. Then

$$\alpha_N \approx \left( \frac{\chi_{S_N}}{v} \right)^* \left( \frac{\mu(I_N)}{5 \cdot 2^N} \right)$$

and

1. $\alpha_N \leq l/v(x) \leq 5\alpha_N$, $x \in S_N$,
2. $|S_N|/5\alpha_N \leq \mu(S_N) \leq |S_N|/\alpha_N$,
3. $\mu(I_N)/(5 \cdot 2^N) \leq \mu(S_N)/(5 \cdot 2^N)$,
4. $\Phi(2^{-N}) \leq c \mu(I_N)\alpha_N/|I_N| \leq c \cdot 2^N/p'$.

We begin with

$$\int_{I_N} \left\{ M_{j+1}(v^{1-p'}\chi_{S_N}) \right\}^p u \ dx \geq \frac{\mu(I_N)}{|I_N|^p} \left\{ \int_{I_N} M_j(v^{1-p'}\chi_{S_N}) \right\}^p$$

$$\geq \frac{\mu(I_N)}{|I_N|^p} \left\{ \int_{I_N} M_{j+1}(\alpha_n^{-1}X_{S_N}) \right\}^p.$$

By Lemma 1 this is

$$\geq B^p \frac{\alpha_N^{p-1} \mu(I_N)}{|I_N|^p} \left( \int_{|S_N|}^{l/|S_N|} M_{j+1}(X_{[t, l]}(x)) \right)^p,$$

where $H_N = [0, l]$. Since for $|S_N| \leq t_1 \leq |I_N|$, $M_{j+1}(X_{[0, |S_N|]})(t_1) \geq |S_N|/t_1$ we see that

$$\int_{|S_N|}^{t_1} M_{2,j+1}(X_{[0, |S_N|]}(X_{[0, |S_N|]}(t_1) \geq \int_{|S_N|}^{t_1} \frac{1}{t_2} \int_{|S_N|}^{t_2} M_{1,j+1}(X_{[0, |S_N|]}) \ dt_1$$

$$\geq \int_{|S_N|}^{t_2} \frac{|S_N|}{t_2} \log \left( \frac{t_2}{|S_N|} \right) \ dt_2 = \frac{|S_N|}{2} \log^2 \frac{|I_N|}{|S_N|}.$$ 

Thus in general,

$$\int_{I_N} \left\{ M_{j+1}(v^{1-p'}\chi_{S_N}) \right\}^p u \ dx \geq \frac{\alpha_N^{p-1} \mu(I_N)}{|I_N|^p} \left( \frac{\alpha_N^{p-1} \mu(I_N)}{|I_N|^p} \left[ B^p \log^2 \left( \frac{|I_N|}{|S_N|} \right) \right] \right)^p.$$
Since $\|M_{j+1} f\|_{p, u} \leq A_{j+1} \| f \|_{p, v}$ we get with $f = v^{1-p'}^\ast \chi_{S_N}$,

$$\alpha_{\mathcal{N}}^{(p'-1)p} \left| \frac{\mu(I'_N)}{|I_N|^p} \right|^p \left[ \frac{B^j \log \left( \frac{|I_N|}{|S_N|} \right)}{j!} \right]^p \leq A_{j+1} \int_{S_N} v^{1-p'}^* \, dx \leq 5^{p'-1} A_{j+1} \alpha_{\mathcal{N}}^{p'-1} \left| \frac{S_N}{|S_N|} \right|,$$

or

$$\frac{\alpha_{\mathcal{N}} \mu(I'_N)}{|I_N|^p} \left| \frac{\mu(I'_N)}{|S_N|^p} \right|^{p-1} \leq c \left[ \frac{A_{j+1} \cdot j!}{B^j \log \left( \frac{|I_N|}{|S_N|} \right)} \right]^p.$$

Since $|S_N| \leq 5 \alpha_{\mathcal{N}} p(S_N) \leq 5 \alpha_{\mathcal{N}} \mu(I'_N)/2^N \leq c |I_N|/2^N \cdot 2^{N/p'} = c |I_N|/2^N$, we get $|I_N|/|S_N| = c 2^N/p'$ from which

$$\frac{\alpha_{\mathcal{N}} \mu(I'_N)}{|I_N|^p} \left[ \frac{\alpha_{\mathcal{N}} \mu(I'_N)}{5 \cdot 2^N} \right]^{p-1} \leq c \left[ \frac{A_{j+1} \cdot j!}{B^j \log (c 2^{N/p'})} \right]^p.$$

From this we finally obtain

$$\frac{\alpha_{\mathcal{N}} \mu(I'_N)}{|I_N|^p} \leq c \frac{A_{j+1} \cdot j!}{B^j |N|^j \cdot 2^{N/p'}},$$

and the proof is complete.

We can replace in Theorem 5 the strong $(p, p)$ for $M_j f$ by weak $(p, p)$ and obtain the same result. We state this as

**Corollary.** If $(u, v) \in A_p$, $1 < p$, $\Phi = \Phi_{u,v}$, and $\mu(x: M_j f(x) > y) \leq A_p \| f \|_{p, v}^p$, $j = 1, 2, \ldots$, then there are constants $c > 0$, $B > 0$ such that for every $j, N$,

$$\Phi(2^{-N}) \leq c \frac{A_{j+1}}{B^j} \left( \frac{j!}{N^j} \right) 2^{N/p'}.$$

**Proof.** Start out exactly as in Theorem 5, and note that for $x \in I_N$,

$$\left\{ \frac{M_{j+1} \left( v^{1-p'} \chi_{S_N} \right)}{I_N} \right\}^p(x) \geq \left\{ \frac{1}{|I_N|^p} \int_{I_N} M_j \left( v^{1-p'} \chi_{S_N} \right) \right\}^p.$$

If we let $y^p$ be the right side of this inequality, then

$$y^p \mu \left\{ x: M_{j+1} \left( v^{1-p'} \chi_{S_N} \right) > y \right\} \geq \frac{\mu(I_N)}{|I_N|^p} \left\{ \int_{I_N} M_j \left( v^{1-p'} \chi_{S_N} \right) \right\}^p.$$

The rest of the proof is exactly as that of Theorem 5.

**Theorem 6.** Let $(u, v) \in A_p$ for some $p > 1$, and form $\Phi = \Phi_{u,v}$.

(i) If $\mu(x: M_j f(x) > y) \leq A \| f \|_{p, v}^p$, then $\Phi \in L(\frac{1}{p'}, 1)$ and hence $\| Mf \|_{p, u} \leq A \| f \|_{p, v}^p$. 


(ii) If \( \sup_{1 \leq f \in L^p} \| M_f \|_{p, u} = a(A^j) \), then there exists \( \varepsilon > 0 \) such that \( \Phi \in L((p - \varepsilon)', 1) \), and hence \( \| Mf \|_{p - \varepsilon, u} \leq A \| f \|_{p - \varepsilon, u} \).

Proof. To prove (i) we use the corollary and obtain \( \Phi(2^{-N}) \leq c 2^{N/p'} / N^2 \) and so \( \Sigma \Phi(2^{-N}) / 2^{N/p'} < \infty \). Hence \( \Phi \in L((p', 1)) \).

For (ii) we use Theorem 5 and get \( \Phi(2^{-N}) \leq c (A/N)^{j! 2^{N/p'}} \), for some constant \( A \). Since by Stirling’s formula \( j! \sim \sqrt{2\pi} e^{-j + 1/2} \), we get

\[
\Phi(2^{-N}) \leq c \left( \frac{A j}{e N} \right)^{j/2} 2^{N/p'}.
\]

If we now let \( a = e/2A \) and \( j = [\alpha N] \), then

\[
\Phi(2^{-N}) \leq c \frac{N^{1/2}}{2^{\alpha N}} 2^{N/p'} \leq \frac{c}{N^2} 2^{N/p' - \alpha N/2} \leq \frac{c}{N^2} 2^{N/(p - \varepsilon')},
\]

for some \( \varepsilon > 0 \). Thus \( \Sigma \Phi(2^{-N}) / 2^{N/(p - \varepsilon')} < \infty \) and so \( \Phi \in L((p - \varepsilon)', 1) \).

Remark. Theorem 6 provides us with a different proof of \( u \in A_p \) implies \( u \in A_{p - \varepsilon} \) for some \( \varepsilon > 0 \). From [7,3] we know that \( u \in A_p \) implies \( \| Mf \|_{p, u} \leq A \| f \|_{p, u} \) without recourse to \( A_{p - \varepsilon} \). But then \( \| Mf \|_{p, u} \leq A \| f \|_{p, u} \), and thus from (ii), \( \| Mf \|_{p - \varepsilon, u} \leq B \| f \|_{p - \varepsilon, u} \) from which we get \( u \in A_{p - \varepsilon} \).

References