APPLICATIONS OF VARIATIONAL INEQUALITIES TO THE EXISTENCE THEOREM ON QUADRATURE DOMAINS

BY

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Abstract. In this paper we shall study quadrature domains for the class of subharmonic functions. By using the theory of variational inequalities, we shall give a new proof of the existence and uniqueness theorem. As an application, we deal with Hele-Shaw flows with a free boundary and show that their two weak solutions, one of which was defined by the author using quadrature domains and the other was defined by Gustafsson [3] using variational inequalities, are identical with each other.

Introduction. In a previous paper [7], the author has defined the quadrature domains of positive measures for the class of subharmonic functions and studied their applications to complex function theory.

Let \( \nu \) be a finite positive measure on the two-dimensional Euclidean space \( \mathbb{R}^2 \). Let \( SV^1(\Omega) \) be the class of subharmonic functions in an open set \( \Omega \) which are integrable with respect to the two-dimensional Lebesgue measure \( m \). A nonempty open set \( \Omega \) is called a quadrature domain of \( \nu \) for class \( SV^1 \) if

(Qi) \( \nu \) is concentrated in \( \Omega \), namely, \( \nu(\Omega^c) = 0 \), where \( \Omega^c \) denotes the complement of \( \Omega \),

(Qii) \( \int_{\Omega} s^+ d\nu < \infty \) and \( \int_{\Omega} s d\nu \leq \int_{\Omega} s^+ d\nu m \) for every \( s \in SV^1(\Omega) \), where \( s^+ = \max\{s, 0\} \).

(Qiii) \( m(\Omega) < \infty \).

Let us denote by \( Q(\nu, SV^1) \) the class of all quadrature domains of \( \nu \) for class \( SV^1 \). The class \( Q(\nu, SV^1) \) may be empty. Let \( \Omega \) be an open set with finite area and let \( f \) be a nonnegative bounded integrable function in \( \mathbb{R}^2 \) satisfying \( f = 0 \) a.e. in \( \Omega^c \). If \( \sup_{W} f < 1 \), then \( Q(fm, SV^1) = \emptyset \). The class \( Q(\chi_{W} m, SV^1) \) consists of all open sets \( \Omega \) satisfying \( \chi_{W} = \chi_{\Omega} \) a.e. in \( \mathbb{R}^2 \), where \( \chi_{W} \) denotes the characteristic function of \( W \), namely, \( \chi_{W}(x) = 1 \) for \( x \in W \) and \( \chi_{W}(x) = 0 \) for \( x \notin W \).

On the contrary, the author has already proved the following theorem (cf. [7, Theorem 3.7]):

Theorem 1. Let \( f \) be a bounded integrable function in \( \mathbb{R}^2 \) such that \( f \geq 1 \) a.e. in a connected open set \( \Omega \) with finite area, \( f = 0 \) a.e. in \( \Omega^c \) and \( \int f d\nu > m(\Omega) \), then \( Q(fm, SV^1) \neq \emptyset \) and there exists a minimum domain \( \Omega \) in \( Q(fm, SV^1) \), namely, \( \Omega \in Q(fm, SV^1) \) if and only if \( \Omega \subseteq \Omega \) and \( m(\Omega \setminus \Omega) = 0 \).

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The main purpose of this paper is to give this theorem a new proof by using variational inequalities.

Recently, Gustafsson [3] has used variational inequalities to solve a moving boundary problem for Hele-Shaw flows. As a corollary, he has proved the existence of quadrature domains of a finite sum of positive point masses for the class of all complex-valued analytic integrable functions [3, Corollary 16.1].

To obtain the result, Gustafsson has used the fact that the boundaries of the above quadrature domains are algebraic curves, so this is a very special case in the theory of quadrature domains. In this paper, we shall deal with a general case stated as in the theorem.

1. Variational inequalities. In this section, we shall show our theorem for a special function \( f \) by using variational inequalities. We assume that \( W \) is a bounded open set \( \mathbb{R}^2 \) and \( f \) is a bounded integrable function with \( f > 1 \) a.e. in \( W \) and \( f = 0 \) a.e. in \( W^c \). The proof will be divided into four steps. Each step is given as a proposition.

For a real-valued bounded integrable function \( g \) in \( \mathbb{R}^2 \) with compact support, we define the logarithmic potential \( U^g \) of \( g \) by

\[
U^g(y) = \int (-\log |x - y|)g(x) \, dm(x),
\]

where \( |x - y| = (\sum_{j=1}^2 (x_j - y_j)^2)^{1/2}, x = (x_1, x_2) \) and \( y = (y_1, y_2) \). It is known that \( U^g \) is of class \( C^1 \) in \( \mathbb{R}^2 \) and \( \Delta U^g = -2\pi g \) in the sense of distributions. First we shall show the following lemma:

**Lemma 1.** Let \( \Omega \in Q(fm, SL^1) \). Then \( \Omega \) is bounded.

**Proof.** Let \( f_1 \) be a nonnegative integrable function in \( \mathbb{R}^2 \) such that \( f_1 \geq 1 \) a.e. in an open set \( W \) and \( f_1 = 0 \) a.e. in \( W^c \). Let \( \Omega \) satisfy \( m(\Omega) < \infty \) and

\[
\int_{W_1} s f_1 \, dm \leq \int_{\Omega} s \, dm
\]

for every \( s \in SL_{\infty}(W_1 \cup \Omega_1) \), where \( SL_{\infty}(W_1 \cup \Omega_1) \) denotes the class of all bounded subharmonic functions in \( W_1 \cup \Omega_1 \).

First we show that if \( \Omega_1 \) is a bounded open set with smooth boundary, then \( W_1 \) is contained in the bounded open set \( G \) whose boundary is the outer boundary of \( \Omega_1 \).

Assume \( W_1 \setminus G \neq \emptyset \). Then \( (\partial W_1) \setminus \overline{G} \neq \emptyset \).

Choose a point \( x_0 \in (\partial W_1) \setminus G \) and \( r > 0 \) so that \( \text{Cap}(W_1 \cap B(x_0; r)) > 0 \) and \( \overline{G} \cap B(x_0; r) = \emptyset \), where \( B(x_0; r) = \{ x \in \mathbb{R}^2 ||x - x_0|| < r \} \). Let \( \mu \) be the equilibrium distribution of \( E = W_1 \cap B(x_0; r) \) and let \( u \) be the conductor potential of \( E \), namely,

\[
u(y) = \int_E (-\log |x - y|) \, d\mu(x).
\]

Then \( u \) is bounded from above and harmonic in \( E^c \). Set \( \alpha = \sup G u \) and \( s = \max(\alpha, \alpha) - \alpha \). Then \( s \in SL_{\infty}(W_1 \cup \Omega_1) \), \( \int_{W_1} s f_1 \, dm > 0 \) and \( \int_{\Omega} s \, dm = 0 \). This contradicts (1) and hence \( W_1 \subset G \).
Since $W$ is bounded, we can choose a ball $B$ centered at the origin and $M > 1$ so that $f \leq M\chi_B$ a.e. in $\mathbb{R}^2$. Set $f_1 = \chi_W + M\chi_B - f$ and $W_1 = \Omega \cup B$. Let $\Omega_1$ be a ball centered at the origin such that $m(\Omega_1) = Mm(B)$. We shall show that $\Omega_1$ satisfies (1). Then, by the above argument, we see that $W_1 = \Omega \cup B$ is contained in $G = \Omega_1$, namely, $\Omega$ is bounded.

To show that $\Omega_1$ satisfies (1), let $s \in SL^\infty(W_1 \cup \Omega_1)$. Let $s^*$ be a function in $SL^\infty(W_1 \cup \Omega_1)$ which is harmonic in $\Omega_1$ and satisfies $s \leq s^*$ in $\Omega$ and $s = s^*$ a.e. in $\Omega_1 \setminus \Omega$; note here that $W_1 \cup \Omega_1 = \Omega \cup \Omega_1$. Then

$$\int s^*(\chi_W + M\chi_B - f) \, dm = \int s^* M\chi_B \, dm \leq \int_{\Omega_1} s^* \, dm.$$  

Subtracting $\int (s^* - s)\chi_{\Omega \cap \Omega_1} \, dm$ from both sides, we obtain

$$\int_{W_1} sf \, dm \leq \int \left( s^* \chi_{\Omega_1} + s\chi_{\Omega \cap \Omega_1} + s^*(M\chi_B - f) \right) \, dm \leq \int_{\Omega_1} s \, dm.$$  

This completes the proof.

Proposition 1. Let $\Omega \in Q(fm, SL^1)$ and set $u = -1/(2\pi)\chi\Omega - f$. Then $u$ and $\Omega$ satisfy

(i) $u \geq 0$ in $\mathbb{R}^2$,
(ii) $u = 0$ in $\Omega^c$,
(iii) $\Delta u = \chi_{\Omega} - f$ in the sense of distributions.

Proof. Since $W$ and $\Omega$ are both bounded, $\chi_{\Omega} - f$ has a compact support. Hence $u$ is well defined and (iii) is evident.

For every $v \in \mathbb{R}^2$, $\log |x - v| \in SL^1(\Omega)$ and so

$$U^{x\Omega - f}(y) = \int_{W} (\log |x - y|) f \, dm(x) - \int_{\Omega} \log |x - y| \, dm(x) \leq 0.$$  

Hence $u \geq 0$ in $\mathbb{R}^2$. If $y \not\in \Omega$, then both $\log |x - y|$ and $-\log |x - y|$ belong to $SL^1(\Omega)$. Hence $u(y) = -1/(2\pi)U^{x\Omega - f}(y) = 0$.

Let $B$ be a large open ball centered at the origin such that $W \subset B$, and let $g_B(x, y)$ be the Green function in $B$ of the Laplacian relative to the first boundary condition with pole at $y$.

Set

$$\psi(y) = -\frac{1}{2\pi} \int_B g_B(x, y) (f - \chi_B)(x) \, dm(x).$$  

Then $\psi \in C^1(B)$ and $\psi$ can be extended onto a neighborhood of $\partial B$ so that the extension, we also write it by $\psi$, is of class $C^1$ in the neighborhood. It is easy to show that $\psi = 0$ on $\partial B$ and $\Delta \psi = f - \chi_B$ in $B$ in the sense of distributions.

Let us denote by $H^1(B)$ the Sobolev space $H^{1,2}(B)$ with the norm

$$\|u\|_{H^{1,2}(B)} = \sum_{0 \leq |\alpha| \leq 1} \|D^\alpha u\|_{L^2(B)}$$  

and denote by $H^1_0(B)$ the closure of $C^\infty_0(B)$ in the above norm. According to Poincaré’s inequality, it is well known that $\|\nabla u\|_{L^2(B)}$ is a norm equivalent to the
above norm for $H^1_0(B)$. In what follows, we shall understand that $H^1_0(B)$ is the Hilbert space with the norm $\|u\| = \|\nabla u\|_{L^2(B)}$ (see, e.g. Kinderlehrer and Stampacchia [5, Chapter II, §4]). We note here that $\psi \in H^1_0(B)$.

Let us consider the following variational problem: Minimize $\|h\|$ in the closed convex set $K = \{ h \in H^1_0(B) \mid h \geq \psi \text{ a.e. in } B \}$. The extremal function $v(\psi)$ exists and is determined uniquely. It is easy to show that $v = v(\psi)$ can be characterized by

(Vi) $v \in K$,
(Vii) $\int_B \nabla (h - v) \nabla v \, dm \geq 0$ for every $h \in K$.

**Proposition 2.** If $u \in H^1_0(B)$ and an open subset $\Omega$ of $B$ satisfy

(i)' $u \geq 0$ a.e. in $B$,
(ii)' $u = 0$ a.e. in $B \setminus \Omega$,
(iii)' $\Delta u = \chi_\Omega - f$ in $B$ in the sense of distributions,

then $v = u + \psi$ satisfies (Vi) and (Vii).

**Proof.** It is evident that (Vi) follows from (i)'. Since $\Delta v = \Delta u + \Delta \psi = \chi_\Omega - \chi_B \in L^2(B)$, we have

$$\int_B \nabla (h - v) \nabla v \, dm = -\int_B (h - v) \Delta v \, dm = \int_{B \setminus \Omega} (h - v) \, dm$$

for every $h \in H^1_0(B)$. The condition (Vii) follows from the following equalities:

$$\int_{B \setminus \Omega} (h - v) \, dm = \int_{B \setminus \Omega} \{ (h - \psi) - u \} \, dm = \int_{B \setminus \Omega} (h - \psi) \, dm.$$

**Proposition 3.** If $v \in H^1_0(B)$ satisfies (Vi) and (Vii), then $u = v - \psi \in C^1(\overline{B})$ and $u = 0$ on $\partial B$. The function $u$ and $\Omega = \{ x \in B \mid u(x) > 0 \}$ satisfy (i)' to (iii)' in Proposition 2.

**Proof.** The condition (i)' follows from (Vi).

Since $\psi \in H^1_0(B)$ and $\Delta \psi = f - \chi_B \in L^\infty(B)$, $\psi \in H^{2,\lambda}(B)$ for every $s$ with $1 < s < \infty$ (see, e.g. Kinderlehrer and Stampacchia [5, Chapter II, Theorem 4.10]). Hence $v \in H^{2,\lambda}(B) \cap C^{1,\lambda}(\overline{B})$ for every $s$ with $2 < s < \infty$, where $\lambda = 1 - 2/s$ (cf. e.g. [5, Chapter IV, Theorem 2.3]). Hence $u = v - \psi \in C^1(\overline{B})$ and $u = 0$ on $\partial B$. Set $\Omega = \{ x \in B \mid u(x) > 0 \}$. Then (ii)' is satisfied evidently.

Let $\rho$ be a function of class $C^\infty_0$ with $0 \leq \rho \leq 1$ in $B$. Since $v \pm \rho u \in K$ and $\Delta v \in L^2(B)$, by (Vii), we have

$$\int_B \rho u \Delta v \, dm = \int_B \nabla (-\rho u) \nabla v \, dm = 0$$

for every $\rho$. Hence $u \Delta v = 0$ a.e. in $B$ and so $\Delta u + \Delta \psi = \Delta v = 0$ a.e. in $\Omega$. This implies that $\Delta u = 1 - f$ a.e. in $\Omega$.

On $I = B \setminus \Omega$, by definition, $u = 0$ and so $\Delta u = 0$ a.e. (see, e.g. [5, Chapter II, Appendix A, Lemma A4]). By (Vii), we have

$$-\int_I \rho \Delta v \, dm \geq 0$$

for every $\rho \in H^1_0(B)$ with $\rho \geq 0$. Hence $\Delta v \leq 0$ a.e. in $B$ and so $f - \chi_B = \Delta \psi = \Delta v \leq 0$ a.e. on $I$. This implies that $m(W \setminus \Omega) = 0$ since $f > 1$ a.e. in $W$. Hence $\Delta u = 0 = -f$ a.e. on $I$. Combining this with $\Delta u = 1 - f$ a.e. in $\Omega$, we obtain (iii)'.
Lemma 2. Let $\Omega$ be an open set stated as in Proposition 3. Then we can choose a large open ball $B$ so that $\overline{\Omega} \subset B$.

Proof. Take a ball $B_0$ and $M > 1$ so that $f \leq M \chi_{B_0}$. Then it is easily verified that $Q(M \chi_{B_0}, m, S_{L^1})$ consists of the ball $B_1$ which satisfies $m(B_1) = Mm(B_0)$ and has the same center as $B_0$ (see [7, §1]). Choose a ball $B$ so that $B_1 \subset B$ and fix it.

As before Proposition 2, let us consider the obstacle problem and write $\psi = \psi(f)$, $K = K(f)$ and $v = v(f)$. For the corresponding function and the open set stated as in Proposition 3, we write $u = u(f)$ and $\Omega = \Omega(f)$, respectively. Then, by Propositions 1 and 2, $\Omega(M \chi_{B_0}) = B_1$. Hence it is sufficient to show that if $f \leq f_1$, then $u(f) \leq u(f_1)$.

First we show that if $h \in K(f)$ and $\Delta h \leq 0$ a.e. in $B$, then $v(f) \leq v(f_1)$ a.e. in $B$. Set $\omega = h - v(f)$. Then, as we have seen in the proof of Proposition 3, $\Delta v(f) = 0$ a.e. in $\Omega$. Hence $\Delta \omega = \Delta h \leq 0$ a.e. in $\Omega$ and so $\omega$ is superharmonic in $\Omega$. Since $\omega = h - \psi(f) \geq 0$ a.e. in $B \setminus \Omega$ and $\omega \in H^1_0(B)$, we have $\omega \geq 0$ a.e. in $B$, namely, $v(f) \leq v(f_1)$ a.e. in $B$.

Now we shall show that if $f \leq f_1$, then $u(f) \leq u(f_1)$. Let $h = u(f_1) + \psi(f)$. Then $h \in K(f)$ and $\Delta h = \Delta u(f_1) + \Delta \psi(f) \leq \Delta u(f_1) + \Delta \psi(f_1) = \Delta v(f_1) \leq 0$ a.e. in $B$. Hence, by the above argument, we see that $u(f) + \psi(f) = v(f) \leq h = u(f_1) + \psi(f)$. Therefore $u(f) \leq u(f_1)$. This completes the proof.

Proposition 4. If $u \in H^1_0(B)$ and an open set $\Omega$ with $\overline{\Omega} \subset B$ satisfy (i)' to (iii)' in Proposition 2, then $u \in C^1(B)$, and $\bar{W} = \{x \in B \mid u(x) > 0\}$ is the minimum open set in $Q(fm, S_{L^1})$.

Proof. The function $u(x) + 1/(2\pi)\int_B g_B(y, x)(\chi_\Omega - f)(y) \, dm(y)$ belongs to $H^1_0(B)$ and is harmonic to $B$. This implies that it is identically equal to zero and so $u(x) = -1/(2\pi)\int_B g_B(y, x)(\chi_\Omega - f)(y) \, dm(y)$. Since $\bar{W} \cup \overline{\Omega} \subset B$, by (iii)',

$$
\int_B \left\{ g_B(y, x) - \log \frac{1}{|y - x|} \right\} (\chi_\Omega - f)(y) \, dm(y)
= -\int_B \nabla \left\{ g_B(y, x) - \log \frac{1}{|y - x|} \right\} \nabla u(y) \, dm(y).
$$

The above is equal to

$$
\int_B \Delta \left\{ g_B(y, x) - \log \frac{1}{|y - x|} \right\} u(y) \, dm(y),
$$

because $u \in H^1_0(B)$. Since $g_B(y, x) - \log(1/|y - x|)$ is harmonic, the above integral is equal to zero. Hence $u = -1/(2\pi)U^{xu-}$, $u \in C^1(B)$ and $u \geq 0$ in $B$.

Set $\bar{W} = \{x \in B \mid u(x) > 0\}$. Then, by (i)' and (ii)', we have $\chi_{\bar{W}} \leq \chi_\Omega$ a.e. in $B$. Since $\Delta u = 0$ a.e. in $B \setminus \bar{W}$ (see, e.g. Kinderlehrer and Stampacchia [5, Chapter II, Appendix A, Lemma A4]) and $f > 1$ a.e. in $\bar{W}$, by (iii)', we see that $\chi_{\bar{W} \cup \overline{\Omega}} \leq \chi_{\bar{W}}$ a.e. in $B$. Hence $\chi_{\bar{W}} = \chi_B$ a.e. in $B$.

Next let us show $\bar{W} \in Q(fm, S_{L^1})$. In what follows, for the sake of simplicity, set $g = \chi_{\bar{W}} - f$. Let $y \in B \setminus \bar{W}$. Then $u(y) = 0$. Since $u$ is of class $C^1$ and $u$ attains its minimum at $y$, $\partial u/\partial x_j(y) = 0$, $j = 1, 2$. Hence $U^x = -2\pi u = 0$ and $\partial U^y/\partial x_j = -2\pi \partial u/\partial x_j = 0$ in $B \setminus \bar{W}$.
Let \( \{ \omega_n \}_{n=1}^{\infty} \) be a sequence of \( C^\infty \)-functions in \( \tilde{W} \) such that \( 0 \leq \omega_n \leq 1 \), \( \omega_n = 0 \) in a neighborhood of \( \partial \tilde{W} \), \( \omega_n = 1 \) outside a neighborhood of \( \partial \tilde{W} \), \( \lim_{n \to \infty} \omega_n(x) = 1 \) for all \( x = (x_1, x_2) \in \tilde{W} \), and

\[
| D^a \omega_n(x) | \leq A_n \delta(x)^{-|a|} \left( \log \frac{1}{\delta(x)} \right)^{-1}
\]

for all \( x \in \tilde{W} \) and all multi-indices \( \alpha \), where \( A_n \) denotes a constant depending only on \( \alpha \), and \( \delta(x) \) denotes the minimum of \( e^{-2} \) and the distance from \( x \) to \( \partial \tilde{W} \). For the existence of the above sequence \( \{ \omega_n \} \), see Hedberg [4, p. 13, Lemma 4].

It follows that

\[
\Delta U^g = \sum_j \frac{\partial^2}{\partial x_j^2} U^g = -2\pi g
\]

in the sense of distributions. Since

\[
\frac{\partial U^g}{\partial x_j}(x) - \frac{\partial U^g}{\partial x_j}(y) = O \left( |x - y| \log \frac{1}{|x - y|} \right), \quad j = 1, 2,
\]

for every pair of points \( x \) and \( y \) with \( |x - y| < e^{-2} \),

\[
U^g(x) = O \left( \delta(x) \log \frac{1}{\delta(x)} \right),
\]

\[
\frac{\partial U^g}{\partial x_j}(x) = O \left( \delta(x) \log \frac{1}{\delta(x)} \right), \quad j = 1, 2,
\]

in a neighborhood of each boundary point of \( \tilde{W} \). Hence

\[
\int_{\tilde{W}} sg \, dm = \lim_{n \to \infty} \int_{\tilde{W}} sg \omega_n \, dm = -\frac{1}{2\pi} \lim_{n \to \infty} \int_{\tilde{W}} s \Delta (U^g \omega_n) \, dm
\]

for every \( s \in L^1(\tilde{W}) \). If \( s \) is subharmonic in \( \tilde{W} \), then \( \Delta s \geq 0 \) in the sense of distributions. Let \( \varphi \) be a nonnegative \( C^\infty \)-function of \( |x| \) in \( \mathbb{R}^2 \) such that \( / \varphi \, dm = 1 \) and set \( s_\varepsilon(x) = \int s(x - \varepsilon y) \varphi(y) \, dm(y) \) for \( \varepsilon > 0 \). Then \( s_\varepsilon \) is a subharmonic \( C^\infty \)-function on a given compact subset of \( \tilde{W} \) for every sufficiently small \( \varepsilon > 0 \) and \( s_\varepsilon \downarrow s \) as \( \varepsilon \downarrow 0 \) on the compact set. Since \( U^g = -2\pi u \leq 0 \), by letting \( \varepsilon \) tend to 0, we see that

\[
\int_{\tilde{W}} sg \, dm \geq 0
\]

for every \( s \in SL^1(\tilde{W}) \). Hence \( \tilde{W} \in Q(fm, SL^1) \). Let \( \Omega \in Q(fm, SL^1) \). Then, by Proposition 1 and the above argument, we see that \( \chi_\Omega = \chi_{\tilde{W}} \) a.e. in \( \mathbb{R}^2 \). If \( y \notin \Omega \), then \( -\log |x - y| \in SL^1(\Omega) \) and so

\[
0 \leq \int (-\log |x - y|)(\chi_\Omega - f)(x) \, dm(x) = -2\pi u(y).
\]

Hence \( u(y) = 0 \), namely, \( y \notin \tilde{W} \). Therefore \( \tilde{W} \subset \Omega \) for every \( \Omega \in Q(fm, SL^1) \). The proof is now complete.
Thus we have proved our theorem for the function $f$ given at the beginning of this section. From (2), we have an additional result which is also true for the function $f$ as in Theorem 2.

**Corollary.** Let $\Omega \in Q(fm, SL^1)$ and $s \in SL^1(\Omega)$. Then

$$\int_W sf \, dm = \int_\Omega s \, dm$$

if and only if $s$ is harmonic in $\tilde{W}$.

**2. Proof of the theorem.** In this section, we assume that $W$ is an open set in $\mathbb{R}^2$ with finite area and $f$ is a bounded integrable function with $f \geq 1$ a.e. in $W$, $f = 0$ a.e. in $W^c$ and $\int f \, dm > m(O)$ for every connected component $O$ of $W$. We shall show the following as our main theorem:

**Theorem 2.** Let $f$ and $W$ be as above. Then $Q(fm, SL^1) \neq \emptyset$ and there exists a minimum domain $W$ in $Q(fm, SL^1)$.

First we show the following two lemmas:

**Lemma 3.** Let $f_i$, $i = 1, 2$, be bounded integrable functions in $\mathbb{R}^2$ such that $f_i \geq 1$ a.e. in open sets $W_i$ and $f_i = 0$ a.e. in $W_i^c$, and let $\beta_i \in Q(f_i m, SL^1)$, $i = 1, 2$. If $\beta_1 \leq \beta_2$ a.e. in $\mathbb{R}^2$, then $\chi_{\Omega_1} \leq \chi_{\Omega_2}$ a.e. in $\mathbb{R}^2$.

**Proof.** Assume that $\Omega_1 \setminus \Omega_2 \neq \emptyset$. Take a point $y \in \Omega_1 \setminus \Omega_2$ and set

$$s(x) = \begin{cases} g_{\Omega_1}(x, y) & \text{in } \Omega_1, \\ 0 & \text{in } \Omega_2 \setminus \Omega_1, \end{cases}$$

where $g_{\Omega_1}(x, y)$ denotes the Green function in $\Omega_1$ with pole at $y$. Then $s \geq 0$ in $\Omega_2 \cup \Omega_1$, $-s|_{\Omega_1} \in SL^1(\Omega_1)$ and $s|_{\Omega_2} = s^* \; a.e.$ in $\Omega_2$ for some $s^* \in SL^1(\Omega_2)$, because $m(\Omega_1) \leq m(\Omega_2) < \infty$. Hence

$$\int_{\Omega_1} s \, dm \leq \int_{\Omega_1} sf_1 \, dm \leq \int_{\Omega_2} sf_2 \, dm \leq \int_{\Omega_2} s \, dm$$

and so

$$\int_{\Omega_1 \setminus \Omega_2} s \, dm \leq \int_{\Omega_2 \setminus \Omega_1} s \, dm = 0.$$

This implies that $m(\Omega_1 \setminus \Omega_2) = 0$, namely, $\chi_{\Omega_1} \leq \chi_{\Omega_2}$ a.e. in $\mathbb{R}^2$.

**Corollary.** Let $f$ be a bounded integrable function in $\mathbb{R}^2$ such that $f \geq 1$ a.e. in an open set $W$ and $f = 0$ a.e. in $W^c$. Let $\Omega_i \in Q(fm, SL^1)$, $i = 1, 2$. Then $\chi_{\Omega_1} = \chi_{\Omega_2}$ a.e. in $\mathbb{R}^2$.

**Lemma 4.** Let $g$ be a bounded nonnegative integrable function in $\mathbb{R}^2$ with compact support which is contained in a connected open set $W$. Let $\int g \, dm > 0$ and $K$ be a compact subset of $W$. Then there are a bounded nonnegative integrable function $f_{g,K}$ in $\mathbb{R}^2$ and a bounded connected open set $W_{g,K}$ such that $f_{g,K} > 0$ in $W_{g,K}$, $f_{g,K} = 0$ in $W_{g,K}^c$, $K \cup \text{supp } g \subset W_{g,K} \subset W_{g,K}^c \subset W$ and $\int sf \, dm \leq \int sf_{g,K} \, dm$ for every $s \in SL^1(W)$. 

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Proof. We may assume that $\inf_{x \in L} g(x) > 0$ for a compact subset $L$ of $W$ with $m(L) > 0$. Let $\delta$ be a number such that $0 < \delta < d(L, \partial W)/2$, where $d(L, \partial W)$ denotes the distance between $L$ and $\partial W$, and define a bounded nonnegative integrable function $g_1$ in $\mathbb{R}^2$ by

$$g_1(x) = \int_{B(x; \delta)} g(y) \chi_L(y) \, dm(y)/m(B(x; \delta)).$$

Then $g_1$ is continuous, $\text{supp } g_1$ is compact and $\int s g \, dm \leq \int s(g \chi_L + g_1) \, dm$ for every $s \in SL^1(W)$. Take a ball $B_1$ and a number $\alpha_1 > 0$ so that $B_1 \subset W$ and $g_1 \geq \alpha_1$ in $B_1$. For every $x \in (K \cup \text{supp } g \cup \text{supp } g_1)$, we can find balls $B_j$, $j = 2, 3, \ldots, n$, with centers $p_j$ such that $p_j = x$, $B_j \subset W$ and $p_j \in B_{j-1}$ for every $j$. Let $v_1 = \alpha_1 \chi_{B_1}$.

Assume that there are a bounded nonnegative integrable function $v_{j-1}$ in $\mathbb{R}^2$ and a number $\alpha_{j-1} > 0$ such that $\text{supp } v_{j-1} \subset \bigcup_{i=1}^{j-1} B_i$, $v_{j-1} \geq \alpha_{j-1}$ in $\bigcup_{i=1}^{j-1} B_i$ and $\int s v_{j-1} \, dm \geq \int s v_j \, dm$ for every $s \in SL^1(W)$. Take a ball $B$ with center $p_j$ such that $B \subset B_{j-1} \cap B_j$. Then

$$\int s v_{j-1} \, dm = \int s(v_{j-1} - \alpha_{j-1} \chi_B) \, dm + \alpha_{j-1} \int_B s \, dm \\ \leq \int s(v_{j-1} - \alpha_{j-1} \chi_B) \, dm + \alpha_{j-1} \frac{m(B)}{m(B_j)} \int_B s \, dm$$

for every $s \in SL^1(W)$. Set $v_j = v_{j-1} - \alpha_{j-1} \chi_B + (\alpha_{j-1} m(B)/m(B_j)) \chi_{B_j}$ and $\alpha_j = \alpha_{j-1} m(B)/m(B_j)$. The function $v_j$ and a number $\alpha_j$ satisfy the above conditions for $j$. Thus, by induction, we can construct $v_n$ and $\alpha_n > 0$ such that $\text{supp } v_n \subset \bigcup_{j=1}^n B_j$, $v_n \geq \alpha_n$ in $\bigcup_{j=1}^n B_j$ and $\int s v_n \, dm \geq \int s v_j \, dm$ for every $s \in SL^1(W)$.

Let us write $v_n$ and $V_n$ for $v_n$ and $\bigcup_{j=1}^n B_j$, respectively. Since $K \cup \text{supp } g \cup \text{supp } g_1$ is compact, we can find a finite number of open sets $V_{x_1}, \ldots, V_{x_n}$ such that $(K \cup \text{supp } g \cup \text{supp } g_1) \subset \bigcup_{j=1}^n V_{x_j}$. Set

$$f_{g, k} = g \chi_{L^c} + g_1 - \alpha_1 \chi_{B_1} + \frac{1}{k} \sum_{j=1}^k v_{x_j}, \quad W_{g, k} = \bigcup_{j=1}^k V_{x_j}.$$

These satisfy the required condition.

Proof of Theorem 2. At first, let us construct an open set $G \subset Q(fm, SL^1)$. For every connected component $O_i$ of $W$, let $L_i$ be a compact subset of $O_i$ such that $f((f - 1) \chi_{L_i}) \, dm > 0$. Let $g_i = (f - 1) \chi_{L_i}$ and let $(O_{i,j})$ be an exhaustion of $O_i$ such that $O_{i,j}$ is compact for every $j$. By using Lemma 4, we can find $f_{i,j} = f_{g_i/2, O_{i,j}}$ and $W_{i,j} = W_{g_i/2, O_{i,j}}$ such that $f_{i,j} > 0$ in $W_{i,j}$, $f_{i,j} = 0$ in $W_{i,j}$, $O_{i,j} \cup L_i \subset W_{i,j} \subset W_{i,j} \subset W$ and $\int s g_i \, dm \leq \int s f_{i,j} \, dm$ for every $s \in SL^1(W)$. Set

$$f_0 = f - \sum_{i=1}^{\infty} g_i, \quad f_n = f_0 x_{W_n} + \sum_{1 \leq i < n} \sum_{1 \leq j < n-i+1} f_{i,j}, \quad n = 1, 2, \ldots,$$

where $W_n = \bigcup_{1 \leq i < n} \bigcup_{1 \leq j < n-i+1} W_{i,j}$. Then $f_n$ is a bounded integrable function in $\mathbb{R}^2$ with $f_n > 1$ in a bounded open set $W_n$ and $f_n = 0$ in $W_n^c$.

From the argument given in §1, we can construct the minimum open set $\hat{W}_n \subset Q(fm, SL^1)$ for every $n$. Since $f_n \leq f_{n+1}$, from the proof of Lemma 2, we
obtain \( u(f_n) \leq u(f_{n+1}) \) (for the notation, see the proof of Lemma 2). Hence \( \tilde{W}_n \subset \tilde{W}_{n+1} \). Set \( G = \bigcup \tilde{W}_n \). By the proof of Proposition 3, we have \( m(W_n \setminus \tilde{W}_n) = 0 \). Hence it follows that \( m(W \setminus G) = 0 \).

Next let us show
\[
\int s f \, dm \leq \int_s G \, dm
\]
for every \( s \in SL^1(G) \). For every \( \varepsilon > 0 \), we can take a number \( n \) so that
\[
\int_{W_n} s \, dm + \varepsilon \geq \int_{\tilde{W}_n} s \, dm
\]
and
\[
\int s f \, dm - \varepsilon \leq \int s \left( f_0 W_n + \sum_{1 \leq i \leq n} \sum_{1 \leq j \leq n - i + 1} g_i / 2^j \right) \, dm.
\]
Since
\[
\int s \left( f_0 W_n + \sum_{1 \leq i \leq n} \sum_{1 \leq j \leq n - i + 1} g_i / 2^j \right) \, dm \leq \int s \left( f_0 W_n + \sum_{1 \leq i \leq n} \sum_{1 \leq j \leq n - i + 1} f_{i,j} \right) \, dm
\]
\[
\leq \int_{W_n} s f_n \, dm \leq \int_{\tilde{W}_n} s \, dm,
\]
we have
\[
\int s f \, dm \leq \int_s G \, dm + 2\varepsilon
\]
for every \( \varepsilon > 0 \). Hence
\[
\int s f \, dm \leq \int_s G \, dm
\]
for every \( s \in SL^1(G) \).

For \( s = 1 \), we have
\[
\int f \, dm = \lim_{n \to \infty} \int \left( f_0 W_n + \sum_{1 \leq i \leq n} \sum_{1 \leq j \leq n - i + 1} g_i / 2^j \right) \, dm
\]
\[
= \lim_{n \to \infty} \int f_n \, dm = \lim_{n \to \infty} m(\tilde{W}_n) = m(G).
\]
Hence \( m(G) < \infty \). Thus we have proved that \( G \in Q(fm, SL^1) \).

From the corollary to Lemma 3, \( \chi_G = \chi_G \) a.e. for every \( \Omega \in Q(fm, SL^1) \). Since \( \chi_{\Omega} - f \) has not necessarily compact support, take two distinct points \( \xi_1 \) and \( \xi_2 \) in \((\cup \Omega)^c\), where \( \cup \Omega \) denotes the union of all \( \Omega \in Q(fm, SL^1) \), consider the generalized logarithmic potential \( U^{x_0 - f}(x; \xi_1, \xi_2) \) (see [8, §3]) and set
\[
u(x) = -\frac{1}{2\pi} U^{x_0 - f}(x; \xi_1, \xi_2).
\]
The function \( u \) is determined independently of the choice of \( \Omega \in Q(fm, SL^1) \). Let \( \tilde{W} = \{ x \in \mathbb{R}^2 \mid u(x) > 0 \} \). If \( \Omega \in Q(fm, SL^1) \) and \( x \notin \Omega \), then \( u(x) = 0 \) and so
x \not\in \tilde{W}. Therefore \tilde{W} \subset \Omega for every \Omega \in Q(fm, SL^1). Since u(x) = 0 in \tilde{W}^c, \Delta u = 0 a.e. on \tilde{W}^c. Hence 0 = \Delta u = \chi_{\Omega} - f a.e. in \tilde{W}^c and so \chi_{\Omega} \leq \chi_{\tilde{W}} a.e. in \mathbb{R}^2. This implies that \tilde{W} \subset \Omega and m(\Omega \setminus \tilde{W}) = 0 for every \Omega \in Q(fm, SL^1).

Finally, the fact that \tilde{W} \in Q(fm, SL^1) follows from the similar argument given in the proof of Proposition 4. In contrast with the proof of Proposition 4, the open set \tilde{W} is not necessarily bounded. For the generalized logarithmic potential and the similar argument given in the proof of Proposition 4, see [8, §3].

3. The case of higher dimensions. Our theorem is also valid for the case of higher dimensions. In the case of dimension \(d \geq 3\), let us write by \(S_d\) the surface area of the \((d - 1)\)-dimensional unit hypersphere, namely, \(S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}\). We replace \(-\log |x - y|\) by \(|x - y|^{2-d}\) and consider the Newton potential

\[ U^g(y) = \int |x - y|^{2-d}g(x)\,dm(x) \]

instead of the logarithmic potential which we have used in the case of dimension 2. In the above definition, \(g\) is a real-valued bounded integrable function defined in \(\mathbb{R}^d\) and \(m\) denotes the \(d\)-dimensional Lebesgue measure.

It is known that

1. \(U^g\) is of class \(C^1\),
2. \(\frac{\partial U^g(x)}{\partial x_j} - \frac{\partial U^g(y)}{\partial x_j} = O(|x - y| \log(1/|x - y|)), j = 1, 2, \ldots, d,\) for every pair of points \(x\) and \(y\) with \(|x - y| < e^{-2}\).
3. \(\Delta U^g = -(d - 2)S_d g\) in the sense of distributions.

Therefore our arguments are also valid if we replace \(-1/(2\pi)\) and \(-\log |x - y|\) by \(-1/((d - 2)S_d)\) and \(|x - y|^{2-d}\), respectively.

Let us give here a remark on the generalized logarithmic potential used in the proof of Theorem 2. It is unnecessary to consider "generalized" in the case of dimension \(d \geq 3\). Because we can define the Newton potential \(U^g\) of a bounded integrable function \(g\) which has not necessarily a compact support.

4. Hele-Shaw flows with a free boundary. As an application of the new proof of our theorem, we deal with Hele-Shaw flows with a free boundary produced by the injection of fluid into the narrow gap between two parallel planes (for the mathematical formulation, see Richardson [6] and Sakai [7]).

In [7], the author has defined a weak solution of a free boundary problem of Hele-Shaw flows with the initial connected open set \(\Omega(0)\). It is a family \(\{\Omega(t)\}_{t > 0}\) of quadrature domains \(\Omega(t)\) such that \(\Omega(t)\) is the minimum domain in \(Q(\chi_{\Omega(0)}m + t\delta_c, SL^1)\) for every \(t > 0\), where \(\delta_c\) denotes the Dirac measure at the injection point \(c \in \Omega(0)\) of the fluid.

Recently, Gustafsson [3] has defined another weak solution of Hele-Shaw flows by using variational inequalities (for the case having the container wall, see Elliott and Janovský [2]).

Let \(f_t = \chi_{\Omega(0)} + t(1/m(B(c; r)))\chi_{B(c; r)}\) (in [3], Gustafsson has used \(2\pi t\) and \(B(0; r)\) for \(t\) and \(B(c; r)\), respectively), where \(\Omega(0)\) denotes a bounded connected open set and \(B(c; r)\) satisfies \(B(c; r) \subset \Omega(0)\), and consider the variational problem given before Proposition 2 for large ball \(B_t\) (which depends on \(t\)) and for a function
ψ = ψ(𝑓). Then Gustafsson’s weak solution \( \{Ω(𝑡)\}_{𝑡>0} \) is, in our notation given in the proof of Lemma 2, a family of domains \( Ω(𝑡) = Ω(0) ∪ Ω(𝑓) \) for every \( t > 0 \).

In this section, we shall note first that \( Ω(𝑡) = Ω(𝑓) \), namely, \( Ω(0) ⊂ Ω(𝑓) \) (this result is also given by Gustafsson [3, Lemma 14(iv)]) and next show that the above two weak solutions are identical with each other.

The first assertion follows immediately from the following lemma:

**Lemma 5.** Let \( f, W \) and \( \tilde{W} \) be as in Theorem 2. Then \( W \subset \tilde{W} \).

**Proof.** Since \( f \geq 1 \) a.e. in \( W \), \( Δu = χ_{\tilde{W}} - f \leq 0 \) a.e. in \( W \). Hence \( u \) is a nonnegative superharmonic function in \( W \). If \( u(x) = 0 \) for some \( x \in W \), then \( u \equiv 0 \) in the connected component of \( W \) containing \( x \). This contradicts \( m(W \setminus \tilde{W}) = 0 \) and so \( u(x) > 0 \) in \( W \), namely, \( W \subset \tilde{W} \).

The next corollary guarantees that \( Ω(f) \) is connected.

**Corollary.** If \( W \) is connected, then \( \tilde{W} \) is also connected.

**Proof.** Assume that \( \tilde{W} \) is disconnected. Since \( W \subset \tilde{W} \) and \( W \) is connected, we can find a connected component \( O \) of \( \tilde{W} \) such that \( W \cap O = \emptyset \). For every \( s \in SL^1(\tilde{W} \setminus O) \), let \( \tilde{s} \) be a function defined by \( \tilde{s}(x) = s(x) \) in \( \tilde{W} \setminus O \) and \( \tilde{s}(x) = 0 \) in \( O \). Then \( \tilde{s} \in SL^1(\tilde{W}) \) and

\[
\int_W sf \, dm = \int_W \tilde{s}f \, dm \leq \int_{\tilde{W} \setminus O} \tilde{s} \, dm = \int_{\tilde{W} \setminus O} s \, dm.
\]

Hence \( \tilde{W} \setminus O \in Q(fm, SL^1) \). This contradicts the fact that \( \tilde{W} \) is the minimum domain in \( Q(fm, SL^1) \).

To show the second assertion, by the argument given in §1, it is sufficient to show that \( Q(χ_{W}m + α, SL^1) = Q(f, m, SL^1) \) for every \( t > 0 \). This follows immediately from the proposition below.

For the sake of simplicity, we assume that \( W \) is a connected open set. Let \( μ \) be a positive finite measure with compact support contained in \( W \). For a number \( α \) with \( 0 < α < d(supp μ, ∂W)/2 \), where \( d(supp μ, ∂W) \) denotes the distance between \( supp μ \) and \( ∂W \), let us define a bounded function \( M_αμ \) by

\[
(M_αμ)(x) = \frac{μ(B(x; α))}{m(B(x; α))}.
\]

The support of \( M_αμ \) is contained in \( W \).

**Lemma 6.** \( Q(χ_{W}m + μ, SL^1) = Q((χ_{W} + M_αμ)m, SL^1) \).

**Proof.** We may assume that \( μ ≠ 0 \). If \( Ω \in Q((χ_{W} + M_αμ)m, SL^1) \), then, by Lemma 5, \( W \subset Ω \). Since

\[
\int s \, dμ ≤ \int s(M_αμ) \, dm
\]

for every \( s \in SL^1(W) \), \( Ω \in Q(χ_{W}m + μ, SL^1) \).
Conversely, assume that $\Omega \in Q(X_wm + \mu, SL^1)$. Let $G = \{x \in W| (M_a\mu)(x) > 0\}$. Then, since $M_a\mu$ is lower semicontinuous, $G$ is an open set containing supp $\mu$. We shall show $\Omega \subset G$. If $y \in G \setminus \Omega$, then $\mu(B(y; \beta)) > 0$ for $\beta$ with $\alpha < \beta < d(\text{supp} \mu, \partial W)/2$. Set

$$s(x) = \max\{\log(1/|x - y|), \log(1/\beta)\} - \log(1/\beta).$$

Then $s \mid \Omega \in SL^1(\Omega)$. Since $m(W \setminus \Omega) = 0,$

$$\int_W s(X_w dm + d\mu) > \int_W s dm = \int_\Omega s dm.$$

This contradicts $\Omega \in Q(X_wm + \mu, SL^1)$. Hence $G \subset \Omega$.

Let $s \in SL^1(\Omega)$, and let $s^* \in SL^1(\Omega)$ be harmonic in $G$ and satisfy $s^* = s$ a.e. in $\Omega \setminus G$. Since

$$\int_W s^*(X_w + M_a\mu) dm = \int_W s^*(X_w dm + d\mu) \leq \int_\Omega s^* dm$$

and $s \leq s^*$ in $G$, we have

$$\int_W s(X_w + M_a\mu) dm \leq \int_W s^*(X_w + M_a\mu) dm + \int_G (s - s^*) dm$$

$$\leq \int_\Omega s^* dm + \int_G (s - s^*) dm = \int_\Omega s dm.$$

Therefore $\Omega \in Q((X_w + M_a\mu)m, SL^1)$.

**Proposition 5.** Let $\mu_i, i = 1,2,$ be positive finite measures with compact support contained in a connected open set $W$. If there is an open subset $G$ of $W$ such that $G \supset \text{supp} \mu_1 \cup \text{supp} \mu_2$ and $\int h d\mu_1 = \int h d\mu_2$ for every harmonic function in $G$, then $Q(X_wm + \mu_1, SL^1) = Q(X_wm + \mu_2, SL^1)$.

**Proof.** By Lemma 6, it is sufficient to show that $Q((X_w + M_a\mu_1)m, SL^1) = Q((X_w + M_a\mu_2)m, SL^1)$ for small $\alpha > 0$. We obtain this equality by using Lemma 5 and the argument as in the proof of Lemma 6.

**5. Quadrature domains for harmonic and analytic functions.** In [7], quadrature domains for harmonic and analytic functions are introduced. Let $\nu$ be a positive finite measure in $\mathbb{R}^2$ and let $HL^1(\Omega)$ (resp. $AL^1(\Omega)$) be the class of all real-valued (resp. complex-valued) harmonic (resp. analytic) integrable functions in $\Omega$. A non-empty open set $\Omega$ is called a quadrature domain of class $HL^1$ (resp. $AL^1$), if $\Omega$ satisfies (Qii), (Qiii) and

$$(Qii)^{'} \quad \int_\Omega |h| d\nu < \infty \quad \text{and} \quad \int_\Omega h d\nu = \int_\Omega h dm$$

for every $h \in HL^1(\Omega)$ (resp. $h \in AL^1(\Omega)$). We denote by $Q(\nu, HL^1)$ (resp. $Q(\nu, AL^1)$) the class of all quadrature domains of $\nu$ for class $HL^1$ (resp. $AL^1$).
By using the generalized logarithmic potential, we obtain the following proposition:

**Proposition 6.** Let $f$ and $W$ be as in Theorem 2. Let $\Omega$ be an open set with finite area, let $\xi_1$ and $\xi_2$ be two distinct points in $\Omega^c$ and set $u(x) = -1/(2\pi)U^{x_0-}(x; \xi_1, \xi_2)$. Then

1. $\Omega \subseteq Q(fm, SL^1)$ if and only if $u = 0$ in $\Omega^c$ and $u > 0$ in $\Omega$,
2. $\Omega \subseteq Q(fm, HL^1)$ if and only if $u = 0$ and $\partial u/\partial x_j = 0, j = 1, 2,$ in $\Omega^c$,
3. $\Omega \subseteq Q(fm, AL^1)$ if and only if $\partial u/\partial x_j = 0, j = 1, 2,$ in $\Omega^c$.

**Proof.** The assertions (1) and (2) are proved from the argument similar to the proof of Proposition 4. Let $(\chi_\Omega - f)$ be the generalized Cauchy transform of $\chi_\Omega - f$ (for the definition, see [8]). Then $(\chi_\Omega - f)^* = (\partial/\partial x_1 - i\partial/\partial x_2)U^{x_0-}$. Hence $\partial u/\partial x_j = 0, j = 1, 2$, in $\Omega^c$ implies that $(\chi_\Omega - f)^* = 0$ in $\Omega^c$. Let $z = x_1 + ix_2$. Since the subclass of $AL^1(\Omega)$ which consists of all linear combinations of $1/(z - \xi_k)$ with $\xi_k \in \Omega^c$ is dense in $AL^1(\Omega)$ (see Bers [1]), the assertion (3) follows.

**References**


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