BRANCHED COVERINGS. I

BY

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ABSTRACT. This paper analyzes the possible cobordism classes \([ M ] - (\deg \phi)[ N ]\) for \(\phi: M \to N\) a smooth branched covering of closed smooth manifolds. It is assumed that the branch set is a codimension 2 submanifold. The results are a fairly complete description in the unoriented case, a partial description in the oriented case, and a detailed analysis of the case in which \(N\) is a sphere.

1. Introduction. The purpose of this note is to describe the possible cobordism classes \([ M ] - (\deg \phi)[ N ]\) where \(\phi: M \to N\) is a smooth branched covering of closed smooth manifolds.

It is well known that for a genuine covering \(\phi: M \to N\) one has \([ M ] = (\deg \phi)[ N ]\) in unoriented cobordism or in oriented cobordism if \(M\) and \(N\) are oriented manifolds. Thus, the class \([ M ] - (\deg \phi)[ N ]\) depends entirely upon the branching behavior. For this definition, the choice here is to follow Berstein and Edmonds [2] including a smoothness hypothesis or more specifically Brand [3] since the differentiable structures will be assumed to satisfy his regularity condition. Briefly then,

DEFINITION. A branched covering is a smooth map \(\phi: M^n \to N^n\) between smooth manifolds which is finite-to-one and an open map. The singular set \(\Sigma_\phi\) is the set of points of \(M\) at which \(\phi\) is not a local homeomorphism, and the branch set \(B_\phi\) is the image under \(\phi\) of the singular set. \(\phi\) is a branched covering if \(\Sigma_\phi\) is a 2-dimensional submanifold of \(N\).

According to [2], the map \(\phi: \phi^{-1}B_\phi \to B_\phi\) is then an ordinary covering and looks like a union of maps \(\bigcup_j B_{ij} \to B_i\), where \(B_i\) is a component of \(B_\phi\) and each \(B_{ij} \to B_i\) is a covering of degree \(r_{ij}\). If \(v_{ij}\) is the normal bundle of \(B_{ij}\) in \(M\) and \(v_i\) the normal bundle of \(B_i\) in \(N\), then \(\phi^*v_i \to B_{ij}\) looks like a quotient of \(v_{ij}\) by an identification of degree \(d_{ij}\) (the local branching degree) on the fibers; i.e. locally \(\phi\) is the map
\[ R^{n-2} \times \mathbb{C} \to R^{n-2} \times \mathbb{C}: (x, z) \mapsto (x, z^{d_{ij}}). \]

Of course, the local degrees add up, so that
\[ \deg \phi = \sum_j r_{ij}d_{ij} \]
(which is constant on each component of \(N\)). Up to cobordism, the specific differential structure on \(M\) is irrelevant, and so additionally one assumes Brand’s conditions hold.

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Note. For the orthogonal 2-plane bundle \( v_{ij} \) over \( B_{ij} \), one may form a quotient \( \mu_{d_{ij}}(v_{ij}) \) by identifying vectors in fibers which differ by an angle which is an integral multiple of \( 2\pi/d_{ij} \). This is again a 2-plane bundle and is identified with \( \phi^*v_{ij} | B_{ij} \). If one observes that two 2-plane bundles over a space having the same first Stiefel-Whitney class have a tensor product, \( \mu_{d_{ij}}(v_{ij}) \) is just the \( d_{ij} \)th tensor power of \( v_{ij} \).

From a cobordism standpoint, one should first observe that there is a cobordism group of branched coverings of degree \( d \) for closed manifolds, and that the Conner-Floyd \([6]\) methods can actually be used successfully to analyze branched coverings with a fixed degree almost as if one were working with a group action. Specifically, one has an exact sequence

\[
\cdots \to \Omega_n(\text{d-fold cover}) \xrightarrow{i_*} \Omega_n(\text{d-fold branched cover}) \xrightarrow{j_*} \Omega_n(\text{d-fold branched cover, unbranched \partial}) \xrightarrow{\partial} \Omega_{n-1}(\text{d-fold cover}) \xrightarrow{i_*} \cdots
\]

in either oriented or unoriented cobordism. Clearly, \( \Omega_n(\text{d-fold cover}) \) is just the usual bordism of \( BS_d \) and by the work of Brand \([4]\), \( \Omega(\text{d-fold branched cover}) \) is the usual bordism of Brand’s classifying space \( B_d \). The relative group is \( \Omega_n(B_d, BS_d) \) and is the reduced bordism of a certain wedge of very nasty Thom spaces (i.e. \( B_d \) is obtained from \( BS_d \) by attaching a union of disc bundles of 2-plane bundles by means of maps of their sphere bundles into \( BS_d \)).

In the special case when \( d = 2 \), a branched cover is nothing more than an involution with codimension two fixed point set, and, in fact, one completely understands the Conner and Floyd analysis of this case. One has

**Proposition 1.** Assigning to a 2-fold branched cover \( \phi: M^n \to N^n \) the class of \( N \) and the class of \( v \) over \( \phi^{-1}B_\phi \) restricted to the self-intersection of \( \phi^{-1}B_\phi \) in \( M \) defines isomorphisms

\[
\Omega_n(2\text{-fold branched cover}) \cong \Omega_n \oplus \Omega_{n-2}(BO_2)
\]

and

\[
\mathcal{R}_n(2\text{-fold branched cover}) \cong \mathcal{R}_n \oplus \mathcal{R}_{n-4}(BO_2).
\]

In the unoriented case, the analysis is not overly difficult and one finds

**Proposition 2.** If \( \phi: M^n \to N^n \) is a branched cover of closed manifolds, then \( [M^n, \phi] - (\deg \phi)[N^n, \text{identity}] \) in \( \mathcal{R}_n(N^n) \) is the class of the map

\[
\bigcup \left\{ RP(v_{ij} \oplus 1) \mid d_{ij} \text{ is even} \right\} \to B_\phi \subset N,
\]

i.e., one takes the union of the \( D(v_{ij})/(x \sim -x \mid x \in S(v_{ij})) \) for \( d_{ij} \) even, projects onto \( B_{ij} \), composes with \( \phi \) into \( B_i \), and then includes in \( N \).

**Proposition 3.** The set of classes \( [M^n] - d[N^n] \) in \( \mathcal{R}_n \) for \( \phi: M^n \to N^n \) a d-fold branched covering of closed n dimensional manifolds is

\[
\{ \alpha \in \mathcal{R}_n \mid w^*_i(\alpha) = 0 \},
\]

if \( d \geq 2, n > 0 \).
In the oriented case, one has

**Proposition 4.** If $\alpha \in \Omega_n$, $n > 0$, there is an odd integer $k$ and a branched covering $\phi: M^n \rightarrow N^n$ of closed oriented manifolds with

$$[M^n] - (\deg \phi)[N^n] = k\alpha \in \Omega_*.$$

*Note.* There is such an odd integer $k$ for coverings of degree two. If one specifies the degree $d \geq 2$, the $k$ needed may vary.

In the above one cannot take $k = 1$ in general. One has

**Proposition 5.** If $n = 2k(p - 1)$ with $p$ an odd prime and $\alpha \in \Omega_n$ is the class $[M^n] - d[N^n]$ for some $d$-fold branched covering $\phi: M^n \rightarrow N^n$ of closed oriented manifolds with $d \leq 2p$, then one has

$$s_k(p-1)/2(\nu)[\alpha] \equiv 0 \pmod{p}$$

where $s_m(\nu)$ is the primitive characteristic class.

**Remarks.** In particular, for $n = 2k(p - 1)$ and $n/2 + 1$ not a power of $p$, the class $\alpha$ cannot be indecomposable in $\Omega_* / p\Omega_*).

A number of results will be given characterizing the values $s_m(\nu)[\alpha]$ for $\alpha \in \Omega_{4m}$ realized by a $d$-fold branched covering.

In later work, appearing as part II of this paper, it is shown that

$$s_i((p-1)/2, \ldots, (p-1)/2)(\nu)[\alpha] \equiv 0 \pmod{p} \text{ for all } (i_1, \ldots, i_r) \text{ without the restriction } d < 2p, \text{ and precise divisibility of the numbers } s_m(\nu)[\alpha] \text{ is obtained.}

Recently, Edmonds proved [7] that no simply connected closed Spin 4-manifold of nonzero signature can be a 2-fold branched covering of the 4-sphere. His argument can be extended, and one has

**Proposition 6.** If $\phi: M^n \rightarrow N^n$ is a branched covering of closed oriented manifolds with oriented branch set $B_\phi$ and $H^2(N; \mathbb{Q}) = 0$, then $[M^n] - (\deg \phi)[N^n] \in \text{Tor } \Omega_*$.

**Proposition 7.** If $\phi: M^n \rightarrow N^n$ is a branched covering of closed oriented manifolds with $H^4(N; \mathbb{Q}) = 0$, then $[M^n] - (\deg \phi)[N^n] \in \text{Tor } \Omega_*$. 

**Proposition 8.** Let $\phi: M^n \rightarrow S^n$ be a branched covering with $M^n$ closed and if $n = 4$ assume $B_\phi$ orientable. Then $M^n$ is orientable and $[M^n] \in \text{Tor } \Omega_*$. If $M^n$ is a Spin manifold or if $B_\phi$ is orientable, then $[M^n] = 0$ in $\Omega_*$. 

By calculating Stiefel-Whitney numbers one then has

**Proposition 9.** If $\phi: M^n \rightarrow S^n$ is a branched covering then $M^n$ bounds (as oriented manifold) if $n$ is even and greater than 4 or if $n$ is odd and $n + 1$ does not have exactly two ones in its dyadic expansion.

### 2. Involutions.

First, one considers 2-fold branched covers $\phi: M^n \rightarrow N^n$ and defines an involution $t: M \rightarrow M$ by the condition that $\phi^{-1}\phi(m) = \{m, tm\}$. The fixed point set of $t$ is $\phi^{-1}B_\phi \approx B_\phi$. If in the oriented case, one insists that the orientation of $N$ on $N - B_\phi$ lift back to the orientation of $M - \phi^{-1}B_\phi$, and with that choice, $t$ is an orientation preserving involution.
For the unoriented case, one has the Conner and Floyd [6, (28.1)] exact sequence

\[ 0 \to \mathcal{R}^Z_n \xrightarrow{F} \bigoplus_k \mathcal{R}_{n-k}(BO_k) \xrightarrow{\partial} \mathcal{R}_{n-1}(BZ_2) \to 0 \]

and restricting to codimension 2 fixed point set, a corresponding exact sequence

\[ \cdots \to \mathcal{R}^Z_n(cod \ 2) \xrightarrow{F} \mathcal{R}_{n-2}(BO_2) \xrightarrow{\partial} \mathcal{R}_{n-1}(BZ_2) \to \cdots \]

where \( F \) takes the fixed point set with its normal bundle, \( \partial \) assigns to \( \xi^2 \to F \) the antipodal involution on \( S(\xi^2) \) or the double cover \( S(\xi^2) \to RP(\xi^2) \), and \( i \) assigns to a double covering the class of the free involution on the total space.

For an involution with codimension 2 fixed point set \((M^n, t)\), the quotient \( M^n/t \) is again a manifold giving a homomorphism

\[ q: \mathcal{R}^Z_n(cod \ 2) \to \mathcal{R}_n \]

and the composite

\[ qi: \mathcal{R}_n(BZ_2) \to \mathcal{R}_n \]

is the augmentation \( \epsilon[N \to BZ_2] = [N] \), and so image \( \partial \) is contained in \( \mathcal{R}_n(BZ_2) = \ker \epsilon \). Alternatively, \( RP(\xi^2) \) is an \( S^1 \) bundle over \( F \) and hence bounds.

From [6, (26.4)], one has a commutative diagram

\[
\begin{array}{ccc}
\mathcal{R}_{n-2}(BO_2) & \xrightarrow{\partial} & \mathcal{R}_{n-1}(BZ_2) \\
\uparrow I_* & & \downarrow \Delta \\
\mathcal{R}_{n-2}(BO_1) & \xrightarrow{\partial} & \mathcal{R}_{n-2}(BZ_2) \\
\end{array}
\]

where \( I_* \) is the Whitney sum with a trivial line bundle, \( \Delta \) is the Smith homomorphism, and the lower homomorphism \( \partial \) is an isomorphism. Now \( \Delta: \mathcal{R}_{n-1}(BZ_2) \to \mathcal{R}_{n-2}(BZ_2) \) has kernel given by the classes of the trivial double covers (the class of \( P[S^0, A] \)), and maps \( \mathcal{R}_{n-1}(BZ_2) \) isomorphically to \( \mathcal{R}_{n-2}(BZ_2) \) (on bases \( \Delta([S^k, A] - RP^k[S^0, A]) = [S^{k-1}, A]) \).

Thus the Conner-Floyd exact sequence becomes

\[ 0 \to \mathcal{R}_n \xrightarrow{i} \mathcal{R}^Z_n(cod \ 2) \xrightarrow{F} \mathcal{R}_{n-2}(BO_2) \xrightarrow{\partial} \mathcal{R}_{n-1}(BZ_2) \to 0 \]

with \( q: \mathcal{R}^Z_n(cod \ 2) \to \mathcal{R}_n \) splitting \( i \) and with \( \ker \partial \) being identified with the cokernel of

\[ I_*: \mathcal{R}_{n-2}(BO_1) \to \mathcal{R}_{n-2}(BO_2). \]

One has the cofibration \( BO_1 \to BO_2 \to MO_2 \) from which coker \( I_* \approx \mathcal{R}_{n-2}(MO_2) \approx \mathcal{R}_{n-4}(BO_2) \), and recognizing the composite

\[ \mathcal{R}^Z_n(cod \ 2) \xrightarrow{F} \mathcal{R}_{n-2}(BO_2) \to \mathcal{R}_{n-2}(MO_2) \approx \mathcal{R}_{n-4}(BO_2) \]
as being the selfintersection of $F$ with the restriction of the normal bundle, one
obtains

**Proposition 1'**.  \( R_n^{Z_2}(\text{cod} 2) \cong R_n \oplus R_{n-4}(BO_2) \) with the isomorphism assigning to
\((M^n, t)\) the class of $M^n/t$ and of $F^{n-2} \cap F^{n-2}$ with bundle $\nu^2 |_{F \cap F}$.

The oriented case is slightly more difficult with the analogue of the Conner-Floyd
sequence being

\[
\cdots \to \Omega_{n-2}^Z \overset{F}{\to} \bigoplus_k \Omega_n(MO_{2k}) \overset{\delta}{\to} \Omega_{n-1}(BZ_2) \overset{t^*}{\to} \cdots
\]

with the relative group of orientation preserving involutions which are free on the
boundary being identified with \( \bigoplus_k \Omega_n(MO_{2k}) \) by assigning to \((V^n, t)\) the class of the
map $V^n \to \vee_k MO_{2k}$ sending a tubular neighborhood of $F^{n-2k}$; i.e., $D(\nu^{2k})$, to
$MO_{2k}$ by $F^{n-2k} \to BO_{2k}$ and extending to the bundles $D(\nu^{2k}) \to D(\gamma_{2k}) \to MO_{2k}$
where $c$ is the collapse, and sending the complement of these tubular neighborhoods
to the common basepoint. (Orientation preserving involutions were first analyzed by
Rosenzweig [15], but this description is due to Lee and Wasserman [12, p. 206].)

Restricting to a codimension 2 fixed point set gives

\[
\cdots \to \Omega_{n-2}^Z(\text{cod} 2) \overset{F}{\to} \bigoplus_k \Omega_n(MO_{2k}) \overset{\delta}{\to} \Omega_{n-1}(BZ_2) \overset{t^*}{\to} \cdots
\]

and as before one has $q: \Omega_{n-2}^Z(\text{cod} 2) \to \Omega_n$ sending \((M^n, t)\) to the class of $M^n/t$, with
\( q_i \circ \delta \in \Omega_n(BZ_2) \to \Omega_n \) so that image $\delta \subset \Omega_{n-1}(BZ_2) \cong R_{n-2}$.

*Note*. This isomorphism is due to Atiyah [1] and assigns to $f: P^{n-1} \to BZ_2$ with $\partial P$
mapping to the base point, with $(BZ_{2*})$ being thought of as $(MO_1, \infty)$ the
submanifold $Q^{n-2} \subset P^{n-1}$ obtained by making the map transverse to $BO_1 \subset MO_1$.
$Q$ is an unoriented manifold and its normal line bundle in $P$ is just the orientation
bundle.

Now consider the bundle $\pi^*\gamma_2 \to D(\gamma_2)$ where $\gamma_2$ is the universal 2-plane bundle
over $BO_2$, for which one has a cofibration sequence

\[
T(\pi^*\gamma_2 |_{S(\gamma_2)}) \to T(\pi^*\gamma_2) \to D(\pi^*\gamma_2)/S(\pi^*\gamma_2) \cup D(\pi^*\gamma_2 |_{S(\gamma_2)})
\]

\[
\cup \quad \cup
\]

\[
S(\gamma_2) \to D(\gamma_2)
\]

where $T$ denotes the Thom space. The projection $\pi: D(\gamma_2) \to BO_2$ is a homotopy
equivalence and so $T(\pi^*\gamma_2) \cong MO_2$. The sphere bundle $S(\gamma_2)$ may be identified with
$BO_1$ with the projection onto $BO_2$ pulling $\gamma_2$ back to $\gamma_1 \oplus 1$, so that $T(\pi^*\gamma_2 |_{S(\gamma_2)})$
may be identified with $T(\gamma_1 \oplus 1) \cong \Sigma MO_1$ and so that the map $\Sigma MO_1 \to MO_2$ is
induced by $BO_1 \to BO_2$ classifying the Whitney sum of $\gamma_1$ with a trivial line. Finally,
the disc bundle $D(\pi^*\gamma_2)$ is the disc bundle of $\gamma_2 \oplus \gamma_2$ over $BO_2$ and collapsing
$S(\pi^*\gamma_2) \cup D(\pi^*\gamma_2 |_{S(\gamma_2)}) \cong S(\gamma_2 \oplus \gamma_2)$ makes the cofiber just $M(\gamma_2 \oplus \gamma_2)$. This one
has a cofibration

\[
\Sigma MO_1 \to MO_2 \to M(\gamma_2 \oplus \gamma_2).
\]
Applying the function $\hat{\Omega}_{\cdot}$, one has an exact sequence
\[ \cdots \to \hat{\Omega}_n(\Sigma MO_1) \to \hat{\Omega}_n(MO_2) \to \hat{\Omega}_n(M(\gamma_2 \oplus \gamma_2)) \to \hat{\Omega}_{n-1}(\Sigma MO_1) \to \cdots \]
and since $\gamma_2 \oplus \gamma_2$ is an oriented vector bundle, one has a Thom isomorphism $\hat{\Omega}_n(M(\gamma_2 \oplus \gamma_2)) \cong \Omega_{n-4}(BO_2)$, while $\hat{\Omega}_n(\Sigma MO_1) \cong \Omega_{n-1}(MO_1) \cong R_{n-2}$. One may easily check that the composite
\[ R_{n-2} \cong \hat{\Omega}_n(\Sigma MO_1) \to \hat{\Omega}_n(MO_2) \xrightarrow{\partial} \hat{\Omega}_{n-1}(BZ_2) \cong R_{n-2} \]
is the identity (one quick way to see this is to compare with the unoriented case with $\hat{\Omega}_n(\Sigma MO_1) \to \hat{\Omega}_n(MO_1)$ being the monomorphism $R_{n-2} \to R_{n-2}(BO_1)$ which takes the orientation cover. One has a commutative diagram
\[
\begin{array}{ccc}
R_{n-2} & \rightarrow & \hat{\Omega}_n(MO_2) \\
\downarrow \text{mono} & & \downarrow \text{mono} \\
R_{n-2}(BO_1) & \rightarrow & \hat{\Omega}_n(MO_2) \\
& & \rightarrow \ R_{n-2}(BO_1) \\
& & \ hat{\Omega}_{n-1}(BZ_2) \\
\end{array}
\]
and the composite along the bottom is the identity.

One then has
\[ 0 \rightarrow \Omega_n \xrightarrow{i_*} \Omega^Z_n(\text{cod 2}) \rightarrow \hat{\Omega}_n(MO_2) \xrightarrow{\partial} R_{n-2} \rightarrow 0 \]
with $q: \Omega^Z_n(\text{cod 2}) \rightarrow \Omega_n$ splitting $i_*$ and with kernel $\partial$ being identified with $\Omega_{n-4}(BO_2)$ via the exact sequence
\[ 0 \rightarrow R_{n-2} \xrightarrow{i} \hat{\Omega}_n(MO_2) \rightarrow \Omega_{n-4}(BO_2) \rightarrow 0 \]
with $f$ split by $\partial$.

This gives

**Proposition 1**. Assigning to $(M^n, t)$ the class of $M^n/t$ and $F^{n-2} \cap F^{n-2}$ with normal bundle $v^2 \mid F \cap F$ gives an isomorphism
\[ \Omega^Z_n(\text{cod 2}) \cong \Omega_n \oplus \Omega_{n-4}(BO_2). \]

One could modify this argument by using $BSO_2$ rather than $BO_2$ for involutions preserving orientation and with oriented codimension 2 fixed point set. It is, however, more reasonable to consider actions of $Z_m$, the cyclic group of order $m$ simultaneously with the orientation hypothesis being automatic except for $m = 2$.

If one considers semifree $Z_m$ actions preserving orientation with codimension 2 fixed point set (assumed orientable if $m = 2$, and in fact, oriented) then one has an exact sequence of Conner-Floyd type
\[ \cdots \rightarrow \Omega^Z_n(\text{semifree}) \xrightarrow{F} \bigoplus_{j} \Omega_{n-2}(BSO_2) \rightarrow \Omega_{n-1}(BZ_m) \rightarrow \cdots \]
where the sum on $j$ is for $1 \leq j \leq (m-1)/2$ and $(j, m) = 1$. This indexing by $j$ corresponds to the classification of the nontrivial irreducible real representations,
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which are of the form multiplication by exp(2πij/m) on C with 1 ≤ j ≤ (m - 1)/2, with (j, m) = 1 giving the semifree representations. This choice of j’s gives the normal bundle to F^{n-2} a complex structure or orientation and hence orients F (see Conner and Floyd [6, §38] for m > 2, while for m = 2 the orientation is chosen on F).

One also has a cofibration for the mth tensor power \( \gamma_2^m \) of the bundle \( \gamma_2 \) over \( BSO_2 \):

\[
\begin{array}{ccc}
S(\gamma_2^m) & \rightarrow & D(\gamma_2^m) & \rightarrow & M(\gamma_2^m) \\
\ll & & \ll & & \ll \\
BZ_m & & BSO_2
\end{array}
\]

and applying \( \Omega_* \), one obtains an exact sequence

\[
\cdots \rightarrow \Omega^n \rightarrow (\text{semifree, special}) \rightarrow \Omega_{n-2}(BSO_2) \rightarrow \Omega_{n-1}(BZ^m) \rightarrow \cdots \\
\ll \rightarrow \ll \\
\ll \rightarrow \ll \\
\Omega_n(BSO_2) & & \tilde{\Omega}_n(M(\gamma_2^m))
\]

where “special” means that \( Z_m \) is to act by the standard representation in the (codim 2 assumed) normal bundle to the fixed set. This “special”-sequence maps into the above, and corresponding to a different choice of generator for \( Z_m \) can be mapped in once for each \( j \).

Note. Because image \( \partial \) is finite, it follows that

\[
\theta: \bigoplus_j \Omega_n(BSO_2) \rightarrow \Omega^n(\text{semifree})
\]

has image of finite index, or induces a rational isomorphism

\[
\theta: \bigoplus_j \Omega_n(BSO_2) \rightarrow \Omega^n(\text{semifree})
\]

by identifying the copies of \( \Omega_n \approx \Omega_n(\text{point}) \) for the different \( j \)’s. This says that some multiple of every semifree action is cobordant to a sum of actions, each having the same representation in the normal bundle to each component of \( F \).

For \( \Omega^n(\text{semifree, special}) \equiv \Omega_n(BSO_2) \equiv \Omega_n(D(\gamma_2^m)) \), the classifying space for the appropriate ramified coverings is \( BSO_2 = D(\gamma_2^m) \). The universal ramified covering is given by the infinite \( m \)-dric \( \{ z \in CP^\infty \mid \sum z^m = 0 \} \) ramified over \( CP^\infty = BSO_2 \) (see [16, §4, and particularly p. 308]). The standard basis over \( \Omega_* \) of \( \Omega_*(BSO_2) \) is given by the inclusions \( CP^r \rightarrow CP^\infty = BSO_2 \) classifying the Hopf bundles and the induced \( m \)-fold ramified cover of the \( m \)-dric in \( CP^{r+1} \), \( \tilde{Q}_m^{2r} = \{ z \in CP^{r+1} \mid \sum_{i=0}^{r+1} z^m = 0 \} \), over \( CP^r \).

Note. These ramified coverings were studied by Hirzebruch [9] and Hattori [8]. Both incorrectly indicate that the \( BSO_2 \) classifies semifree \( Z_m \) actions, but one needs a single normal representation. The error is on line - 2, p. 260 of [9]; there is more than one way to include \( G_n \) in \( C^* \) corresponding to the different \( j \) values.
If one wishes to consider these semifree \( Z_m \) actions as \( m \)-fold branched coverings with a single local branching degree \( m \), i.e. \( \bigcup_j B_{ij} \cong B_i \), and with \( B_\# \) oriented, one has a corresponding exact sequence

\[
\cdots \to \Omega_n(\text{m-fold, special}) \xrightarrow{F} \Omega_{n-2}(BSO_2) \xrightarrow{\partial} \Omega_{n-1}(B\Sigma_m) \to \cdots
\]

where "special" refers to the local degree only. The map \( \partial \) factors through \( \Omega_{n-1}(BZ_m) \), but one cannot distinguish a generator of \( Z_m \) and hence has no dependence on the representation \( j \). One has \( \Omega_n(\text{m-fold special}) \cong \Omega_n(X_m) \) where \( X_m \) is obtained by sewing \( D(\widetilde{\eta}^m) \) to \( B\Sigma_m \) along \( S(\widetilde{\eta}^m) \cong BZ_m \), and is a special Brand classifying space.

**Curiosity.** In the case \( m = 2 \), the quadric \( Q_2^r \subset CP^{r+1} \) may be identified as the Grassmannian of oriented 2-planes in \( R^{r+2} \) (see [11]). One may also observe that in the case \( r \) even, \( [Q_2^r] = 2[H^P] \) in \( \Omega_* \), while for \( r \) odd, both \( Q_2^r \) and \( CP^r \) bound.

**Note.** It would appear that Brand's classifying space for 2-fold branched covers might be identifiable with \( CP^\infty/\text{conjugation} \). Inside \( CP^\infty \) one has the quadric \( BSO_2 \) and \( RP^\infty \), with the normal bundle of \( BSO_2 \) being \( \widetilde{\eta}_2^2 \), and with \( CP^\infty \) being the union of tubular neighborhoods of these subsets. Conjugation fixes \( RP^\infty \) and acts on \( BSO_2 \) as the standard free involution reversing orientation. Thus inside \( CP^\infty/\text{conjugation} \) one has copies of \( BZ_2 = RP^\infty \) and \( BO_2 = BSO_2/Z_2 \) with the complement of \( RP^\infty \) being the disc bundle of a 2-plane bundle over \( BO_2 \).

**Comment.** The ideas about \( m \)-fold covers above derive from my joint efforts with Larry Smith on cobordism of ramified covers.

### 3. Unoriented branchings.

In order to describe the classes \([M^n] - (\deg\phi)[N^n] \) in cobordism, one must be able to compute the characteristic numbers, hence, needs to describe the characteristic classes. For this, one follows Brand [3].

Let \( \phi: M^n \to N^n \) be a branched covering, and let \( B = B_\# \subset N \) be the branch set with normal bundle \( \nu \). Let \( \phi^{-1}B_\# \) be written as the disjoint union of the submanifolds \( \tilde{B}_k \), where \( \tilde{B}_k \) is the set of points with local branching degree \( k \), and let \( \tilde{\nu}_k \) be the normal bundle of \( \tilde{B}_k \) in \( M \). If one chooses disjoint tubular neighborhoods \( D(\tilde{\nu}_k) \) of the sets \( \tilde{B}_k \), one may collapse the complement to obtain a map

\[
c: M^n \to \bigvee_k T(\tilde{\nu}_k)
\]

and by classification of \( \tilde{\nu}_k \) one has \( \tilde{B}_k \to BO_2 \) covered by a map \( T(\tilde{\nu}_k) \to MO_2 \), and wedging these maps together, one obtains a composite

\[
\tilde{g}: M^n \to \bigvee_k MO_2.
\]

(Note. Brand's map is defined using only those terms \( k \geq 2 \), but here the wedge is for \( k > 1 \), not that it makes a significant difference.) One also has a composite

\[
g: N^n \to T(\nu) \to MO_2
\]

obtained by collapsing onto the tubular neighborhood of \( B \) and then classifying.
Beginning with the bundle $\gamma_2$ over $BO_2$, one has maps

$$
\begin{array}{ccc}
D(\gamma_2) & \xrightarrow{a} & D(\mu_k(\gamma_2)) & \xrightarrow{b} & D(\gamma_2) \\
\downarrow & & \downarrow & & \downarrow \\
BO_2 & \xrightarrow{b'} & BO_2
\end{array}
$$

where $a$ is the degree $k$ wrapping on fibers and $b$ is the bundle map covering $b'$ to classify $\mu_k(\gamma_2)$. The composite $b \circ a$ then induces a map $b \circ a : MO_2 \to MO_2$ which may be wedged together to give a map $\vee^k MO_2 \to MO_2$. If the tubular neighborhoods of the $\tilde{B}_k$ are taken as the inverse images of a small tubular neighborhood of $B$, one obtains a commutative diagram

$$
\begin{array}{ccc}
M^n & \xrightarrow{\tilde{g}} & \vee^k MO_2 \\
\phi \downarrow & & \downarrow \\
N^n & \xrightarrow{g} & MO_2
\end{array}
$$

(up to homotopy of the classifying maps). (Note. This requires the wedge for $k > 1$.)

One then has a certain collection of cohomology classes. One has $U \in H^2(MO_2; \mathbb{Z}_2)$, the Thom class, and the Thom class $U_k \in H^2(\vee^k MO_2; \mathbb{Z}_2)$ coming from the $k$th wedge summand. Rather corrupting notation one has classes $w_1^kU$ and $w_1^kU_k$ obtained by applying the Thom isomorphism to $w_1^k \in H^*(BO_2)$. There is also a unique class $v_1 \in H^4(MO_2; \mathbb{Z})$ mapped to the Pontrjagin class $v_1 \in H^4(BO_2; \mathbb{Z})$ under the map $BO_2 \to MO_2$ including the base space and one lets $v_{1,k} \in H^4(\vee^k MO_2; \mathbb{Z})$ by taking the Pontrjagin class in the $k$th wedge summand.

One then has the results of Brand [3]:

**Proposition.** One has

$$
w(t(M) - \phi^*t(N)) = 1 + g^* \left( \sum_{k \text{ even}} (U_k + w_1U_k + w_1^2U_k + \cdots) \right) \in H^*(M^n; \mathbb{Z}_2)
$$

and

$$
v(t(M) - \phi^*t(N)) = 1 + g^* \left( \sum_{k} \sum_{i=1}^{\infty} (-1)^i (k^2 - 1)k^{2i-2}v_{1,k}^{i} \right) \in H^*(M^n; \mathbb{Z}).
$$

Note. Brand only refers to the classes $v_{1,k}$ in rational cohomology, and asserts the formula for the Pontrjagin class rationally. This all works integrally. If you consider the cofibration $BO_1 \to BO_2 \to MO_2$, $H^*(BO_1; \mathbb{Z})$ is isomorphic to the polynomial ring on the integral Bockstein of $w_1$ (of order 2), i.e. $\mathbb{Z}[\beta w_1]/\{2\beta w_1 = 0\}$ and $i^*(\beta w_1) = \beta w_1$ so $i^*$ is epic, and $j^*$ is monic. Since $i^*(v_1) = 0$, there is a unique integral class hitting $v_1$. Using Brand's arguments one does the calculation by pulling back to $BO_2$, where he uses the Whitney sum formula for Pontrjagin classes. Thus, the formula for the class $v(t(M) - \phi^*t(N))$ is actually correct in integral cohomology modulo 2-torsion. To see that the formula is correct integrally one must
check in the $BO_2$'s that the purported Pontrjagin class has the correct reduction to mod 2 cohomology and that is sufficient because all torsion in $H_*(BO_2; \mathbb{Z})$ has order 2. However, mod $2\Sigma_k\Sigma_{l=1}^\infty(-1)(k^2 - 1)k^{2l-2}\psi_{1,k}$ is $\Sigma_k\Sigma_{l=1}^\infty\psi_{1,k}$ and has mod 2 reduction $\Sigma_{k,\text{even}}U_k^2 = (\Sigma_{k,\text{even}}U_k)^2$.

One then has, almost trivially

**Proposition 3.** The set of classes $[M^n] - d[N^n]$ in $\mathcal{R}_n$ for $d$-fold branched coverings of closed $n$-dimensional manifolds is

$$\{\alpha \in \mathcal{R}_n \mid w^i(\alpha) = 0\},$$

if $d \geq 2$, $n > 0$.

**Proof.** If $\phi: M^n \to N^n$ is a $d$-fold branched cover, one has $w_\omega([M^n] - d[N^n]) = w_\omega[M] - dw_\omega[N] = w_\omega[M] - w_\omega(N)\phi_\ast[M] = (w_\omega(\tau(M)) - \phi_\ast w_\omega(\tau N))[M]$. By Brand's formula, $\phi_\ast(w_\omega(N)) = w_\omega(M)$ and so $w_\omega(M)^d[M] = (\phi_\ast w_\omega(N))^d[M]$, and so $w^i_\omega([M^n] - d[N^n]) = 0$.

From [17, Proposition 9.2], a class $\alpha \in \mathcal{R}_n$ with $w^i(\alpha) = 0$ is the class of a manifold $M^n$ having an involution $T$ with fixed point set $F$ of codimension 2. Letting $\phi: M^n \to M^n/T$ be the quotient map, one has a branched covering of degree 2 with $[M^n] - 2[N^n] = [M^n] = \alpha$. For $d > 2$, let $\phi': M^n \cup (d - 2)N^n \to N^n$ by using $\phi$ on $M$ and the trivial cover for $d - 2$ copies of $N$ and then $[M^n \cup (d - 2)N^n] - d[N^n] = [M^n] - 2[N^n] = \alpha$. Thus, obtains all classes $\alpha$ with $w^i_\omega(\alpha) = 0$ from coverings. \(\square\)

**Note.** This trick of replacing a branched cover $\phi: M^n \to N^n$ by $\phi': M^n \cup (d - \deg \phi)N^n \to N^n$ to increase the degree of the cover without changing the class $[M^n] - (\deg \phi)[N^n]$ will be used repeatedly.

Now consider

**Proposition 2'.** If $\phi: M^n \to N^n$ is a branched cover of closed manifolds, then $[M^n, \phi] - (\deg \phi)[N^n, \text{identity}]$ in the bordism of $N$ is the class of the map $\RP(\psi_{\text{even}} + 1) \to B \subset N$.

**Proof.** One considers a class $x \in H^i(N; \mathbb{Z}_2)$, and Stiefel-Whitney class $w_\omega$, and wishes to compute $w_\omega,\phi_\ast(x)[M] - (\deg \phi)w_\omega,x[N] = (w_\omega,\phi_\ast(x) - \phi_\ast(w_\omega(N) \cdot x))[M]$. For this one uses Brand's formula to write

$$w(M) = w((\tau(M) - \phi_\ast \tau(N)) \oplus \phi_\ast \tau(N))$$

$$= \phi_\ast(w(N))\left(1 + \sum_{k, \text{even}} (U_k + w_1U_k + w_1^2U_k + \cdots)\right),$$

where notationally one deletes $\tilde{g}$. If one expands out $w_\omega(M)$ one obtains $\phi_\ast(w_\omega(N))$ + terms involving factors $w_1U_k$, and the first term in that expression, when multiplied by $\phi_\ast(x)$ and evaluated on $[M]$ gives $\phi_\ast(w_\omega(N) \cdot x)[M]$. Thus the characteristic number remaining is the value on the fundamental class of $[M]$ of the part of $w_\omega(M)\phi_\ast(x)$ involving the classes $w_1U_k$. 

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If you now consider $RP(\tilde{\nu}_k \oplus 1) \to \tilde{B}_k \to B \subset N$, and let $c \in H^1(RP(\tilde{\nu}_k \oplus 1); \mathbb{Z}_2)$ be the first Stiefel-Whitney class of the double cover by the sphere bundle, one has
\[
w(RP(\tilde{\nu}_k \oplus 1)) = \pi_*w(\tilde{B}_k) \left\{ (1 + c)^3 + w_1(\tilde{\nu}_k)(1 + c)^2 + w_2(\tilde{\nu}_k)(1 + c) \right\}
\]
(where actually $w_i(\tilde{\nu}_k)$ should have a $\pi_*$ with a relation $c^3 + w_1(\tilde{\nu}_k)c^2 + w_2(\tilde{\nu}_k)c = 0$.
Also $\phi: \tilde{B}_k \to B$ is a covering so $\pi_*w(\tilde{B}_k) = \pi_*\phi_*w(B) = \pi_*\phi_*w(N)/w(\nu)) = \phi_*w(N)/\pi_*\phi_*w(\nu)$, and assuming $k$ is even, $\phi_*w(\nu) = 1 + w_1(\tilde{\nu}_k)$, so
\[
w(RP(\tilde{\nu}_k \oplus 1)) = \phi_*w(N) \cdot \left\{ (1 + c)^2 + w_1(\tilde{\nu}_k)(1 + c) + w_2(\tilde{\nu}_k) \right\}(1 + c)
\]
\[
= \phi_*w(N) \cdot \left\{ 1 + w_1(\tilde{\nu}_k) + \left( c^2 + w_1(\tilde{\nu}_k)c + w_2(\tilde{\nu}_k) \right) \right\}(1 + c)
\]
\[
= \phi_*w(N) \cdot \left\{ 1 + U_k + w_1(\tilde{\nu}_k)U_k + w_1(\tilde{\nu}_k)^2U_k + \cdots \right\}(1 + c),
\]
where $U_k = c^2 + w_1(\tilde{\nu}_k)c + w_2(\tilde{\nu}_k)$.

Note. One has a cofibration $RP(\tilde{\nu}_k) \to RP(\tilde{\nu}_k \oplus 1) \to T(\tilde{\nu}_k)$ and $U_k$ is the pull back of the Thom class in $T(\tilde{\nu}_k)$. It is the “same” class as the Brand class, but considered in a different space. In homology, $M \to \bigvee_{k \text{ even}} T(\tilde{\nu}_k)$ sends the fundamental class of $M$ to the same class as the image of the fundamental class under
\[
\bigcup_{k \text{ even}} RP(\tilde{\nu}_k \oplus 1) \to \bigvee_{k \text{ even}} T(\tilde{\nu}_k).
\]

Define
\[
\hat{w}(RP(\tilde{\nu}_k \oplus 1)) = w(RP(\tilde{\nu}_k \oplus 1))/(1 + c)
\]
\[
= \phi_*w(N) \left\{ 1 + U_k + w_1(\tilde{\nu}_k)U_k + w_1(\tilde{\nu}_k)^2U_k + \cdots \right\}.
\]
Noting that the evaluation of $\hat{w}_\omega(RP(\tilde{\nu}_k \oplus 1))\phi^*(x)[RP(\tilde{\nu}_k \oplus 1)]$ annihilates the term $\phi_*w_\omega(N)\phi^*(x)$, which comes from $H^n(\tilde{B}_k; \mathbb{Z}_2) = 0$, only the terms involving classes $w_1(\tilde{\nu}_k)U_k$ give nonzero value. One then has

Observation. $w_\omega\phi^*(x)[M] - (\deg \phi)w_\omega x[N] = \hat{w}_\omega\phi^*(x)[RP(\tilde{\nu}_{\text{even}} \oplus 1)]$, where
\[
\hat{w} = w(RP(\tilde{\nu}_{\text{even}} \oplus 1))/(1 + c).
\]

Of course, what this means is that one calculates with the class $\hat{w}$ just as if it were a Stiefel-Whitney class and one was computing a Stiefel-Whitney number. To complete the proof of the proposition one has

Lemma. Let $N$ be a space and $f: P^n \to N$, $g: Q^n \to N$ two maps of closed manifolds into $N$. Suppose there is a class $c \in H^1(P; \mathbb{Z}_2)$ and that $\hat{w} = w(P)/(1 + c)$. If for all $x \in H^1(N; \mathbb{Z}_2)$ and all $\omega$ one has
\[
\hat{w}_\omega f^*(x)[P] = w_\omega g^*(x)[Q],
\]
then $[P^n, f] = [Q^n, g]$ in $\mathcal{R}_\omega(N)$; i.e. $\hat{w}_\omega f^*(x)[P] = w_\omega f^*(x)[P]$.

Note. What this says is that if the modified Stiefel-Whitney number of a bordism element is again a bordism element, then the modification was irrelevant.
modification simply does not give the characteristic numbers of a bordism element in general.

**Proof.** Let \( \overline{w} = 1/w \), \( \overline{\omega} = 1/\hat{\omega} \) for the dual Stiefel-Whitney classes. One then has

\[
\langle \overline{w}_i \hat{\omega}_j f^*(x), [P] \rangle = \langle \overline{w}_i \omega g^*(x), [Q] \rangle = \langle \chi(Sq^i)\omega g^*(x), [Q] \rangle,
\]

and

\[
\chi(Sq^i)\hat{\omega}_j f^*(x) = \sum_j \chi(Sq^j)(\hat{\omega}_j) f^*(\chi(Sq^j)x)
\]

\[
= \sum_j \left( \sum_{\omega} a_{\omega} \hat{\omega}_j \right) f^*(\chi(Sq^j)x)
\]

where \( \chi(Sq^j)(\hat{\omega}_j) = \sum_{\omega} a_{\omega} \hat{\omega}_j \) is the universal formula for the Steenrod operation on a Stiefel-Whitney class of a bundle, and \( \hat{\omega} = w(\tau(P) - l) \) where \( w_1(l) = c \) is the class of a bundle, so

\[
\langle \chi(Sq^i)\hat{\omega}_j f^*(x), [P] \rangle = \langle \chi(Sq^j)(\hat{\omega}_j) f^*(\chi(Sq^j)x), [P] \rangle
\]

\[
= \sum_j \left( \sum_{\omega} a_{\omega} \hat{\omega}_j \right) f^*(\chi(Sq^j)x), [Q]
\]

\[
= \langle \chi(Sq^i)\omega g^*(x), [Q] \rangle.
\]

Thus one has

\[
\langle \overline{w}_i \hat{\omega}_j f^*(x), [P] \rangle = \langle \chi(Sq^i)\hat{\omega}_j f^*(x), [P] \rangle = \langle \overline{w}_i \omega f^*(x), [P] \rangle
\]

or

\[
\langle (\overline{\omega} + \overline{w})\hat{\omega}_j f^*(x), [P] \rangle = 0.
\]

Summing over all \( i \), one has

\[
\langle (\overline{w} + \overline{\omega})\hat{\omega}_j f^*(x), [P] \rangle = 0.
\]

Noting that \( \hat{\omega} = w/(1 + c) \), \( \omega = \hat{\omega}(1 + c) \) so

\[
\overline{w} = \overline{\omega}(1 + c + c^2 + c^3 + \cdots)
\]

and

\[
\overline{\omega} + \overline{w} = \overline{\omega}(c + c^2 + c^3 + \cdots).
\]

Thus, one has

\[
\langle (c + c^2 + c^3 + \cdots)\hat{\omega}_j f^*(x), [P] \rangle = 0
\]

for all \( \omega \) and \( x \), so that powers of \( c \) annihilate all expressions \( w_\omega f^*(x) \) when evaluated on \( [P] \). Since \( w = \hat{\omega}(1 + c) \), \( w_\omega = \hat{\omega}_j + \sum_{i>0} c^i \cdot b_{\omega}^i \hat{\omega}_j \) in a universal formula, and so

\[
\langle w_\omega f^*(x), [P] \rangle = \langle \left( \hat{\omega}_j + \sum_{i>0} c^i b_{\omega}^i \hat{\omega}_j \right) f^*(x), [P] \rangle
\]

\[
= \langle \hat{\omega}_j f^*(x), [P] \rangle = \langle w_\omega g^*(x), [Q] \rangle.
\]

Thus, the maps \( f \) and \( g \) have the same Stiefel-Whitney numbers. \( \square \)
Special Note. If one reverses this, one sees that \( c^i w_\phi \phi^*(x) [RP(\tilde{\phi}_{\text{even}} \oplus 1)] = 0 \), which is equivalent to the assertion that \( RP(\tilde{\phi}_{\text{even}}) \to N \times RP^\infty \), with the map induced by \( \phi \) and the class \( c \), is cobordant to zero. For involutions, this is a crucial feature of Conner and Floyd's work with involutions \([6, (24.1)]\) and is the observation \( S(\tilde{\phi}) \to N \) freely bounds \( M \)-interior \( (D(\tilde{\phi}) \to N) \). The above argument shows that the analogue holds for branched covers, but this is certainly not a direct geometric argument.

Remark. These results do not agree with Theorem 3.2 of \([2]\), which is valid only with the additional unstated hypothesis that \( w(N) |_{B^\phi} = 1 \). In line 1 of the proof, \( \tilde{w}(B^\phi) \) is the normal class of \( B^\phi \) in \( N \), while on line 4 it is the normal class in Euclidean space. In the applications only this special case was used. With the hypotheses given the correct conclusion is \( w(M) |_{\phi^{-1}B^\phi} = \phi^*w(N) |_{\phi^{-1}B^\phi} \). One should also remark that the hypothesis that \( M^n \) have even Euler characteristic is unnecessary in Corollary 3.5 of \([2]\), since \( w_\nu(M^n) = (\nu/2(M^n))^2 \) and is also a product, where \( \nu \) is the Wu class.

Corollary. If \( \phi: M^n \to N^n \) is a branched covering of closed manifolds with \( w_\nu(\tilde{\phi}_{\text{even}}) \in \text{image}\{i^*\phi^*: H^*(N^n, Z_2) \to H^*(\tilde{\phi}_{\text{even}}; Z_2)\} \), then \([M^n] - (\deg \phi)[N^n] = 0 \) in \( \mathcal{M}^\ast \).

Note. This condition is satisfied if \( B^\phi \) is orientable, if \( \tilde{\phi}_{\text{even}} \) is orientable, if \( \nu \) is orientable, or \( \tilde{\phi}_{\text{even}} \) is orientable, for one has either \( w_\nu(\tilde{\phi}_{\text{even}}) = i^*\phi^*w_\nu(N) \) or \( w_\nu(\tilde{\phi}_{\text{even}}) = i^*\phi^*(0) \). In particular, Theorem (4.4) of Hattori \([8]\) is a special case of this.

Proof. As noted above, \( (\phi \circ \pi) \times c: RP(\tilde{\phi}_{\text{even}}) \to N \times BZ_2 \) bounds, and

\[
\begin{align*}
0 &= c \left\{ c^2 + c\phi^*(x) \right\} \phi^*(x)^i \left( \frac{w(RP(\tilde{\phi}_{\text{even}}))}{1 + \phi^*(x)} \right)_\omega [RP(\tilde{\phi}_{\text{even}})] \\
&= cw_2(\tilde{\phi}_{\text{even}})^i w_1(\tilde{\phi}_{\text{even}})^i w_\omega(\tilde{B}_{\text{even}})[RP(\tilde{\phi}_{\text{even}})] \\
&= w_2(\tilde{\phi}_{\text{even}})^i w_1(\tilde{\phi}_{\text{even}})^i w_\omega(\tilde{B}_{\text{even}})[\tilde{B}_{\text{even}}],
\end{align*}
\]

and hence the map \( \tilde{\phi}_{\text{even}}: \tilde{B}_{\text{even}} \to BO_2 \) bounds, and so \( RP(\tilde{\phi}_{\text{even}} \oplus 1) \) bounds. □

Note. By including a factor \( \phi^*(y) \) with \( y \in H^*(N^n; Z_2) \) one may conclude that \( (\phi \circ i) \times \tilde{\phi}_{\text{even}}: \tilde{B}_{\text{even}} \to N \times BO_2 \) bounds to see that \([M^n, \phi] - (\deg \phi)[N^n, \text{identity}] = 0 \) in \( \mathcal{M}^\ast(n(N^n)) \).

4. Oriented branched covers. To begin the study of the oriented case, one has

Lemma 1. Every class \( \alpha \in \text{Tor}(\Omega_n) \) is of the form \([M^n] - (\deg \phi)[N^n] \) for some branched covering of closed oriented manifolds of degree \( d \), if \( d \geq 2 \).

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Proof. One has a homomorphism \( \partial : \mathbb{N}_{n+1} \to \Omega_n \) assigning to \( P^{n+1} \) the class of the submanifold dual to \( w_1 \). According to Wall [18], \( \partial \) maps onto \( \text{Tor}(\Omega_n) \) and in fact \( \partial : \mathbb{N}_{n+1} \to \Omega_n \) maps onto the torsion where \( \mathbb{N}_{n+1} \) is the cobordism group of manifolds with \( w_1 \) reduced integral.

Being given \( d > 2 \) and \( \alpha \in \text{Tor}(\Omega_n) \), there is a class \( \beta \in \mathbb{N}_{n+1} \) having all numbers divisible by \( w_1^2 \) zero, i.e., coming from \( \mathbb{N}_{n+1} \), and so that \( \partial \beta = \alpha \). By Proposition 3, there is a branched covering \( \theta : P^{n+1} \to Q^{n+1} \) of degree \( d \) for which \( [P] - d[Q] = \beta \).

Let \( f : Q^{n+1} \to RP^N \) for some large integer \( N \) with \( f^*(i) = \omega_i(Q) \) where \( i \in H^1(RP^N, \mathbb{Z}_2) \) is the nonzero class, and deform \( f|_{B_\theta} \) to be transverse to \( RP^{N-1} \) and then deform \( f \) to be the projection

\[
D(v) \to B_\theta \to RP^N
\]
on a tubular neighborhood of \( B_\theta \), \( f \) is then transverse to \( RP^{N-1} \) on a neighborhood of \( B_\theta \) and without changing the map on a smaller tubular neighborhood of \( B_\theta \) one may further deform \( f \) to be transverse to \( RP^{N-1} \subset RP^N \). Thus, one assumes \( f \) has this form; i.e. \( f \) and \( f|_{B_\theta} \) are transverse to \( RP^{N-1} \) and on a tubular neighborhood of \( B_\theta \), \( f \) is given by projection on \( B_\theta \) followed by \( f|_{B_\theta} \). The composite \( f \circ \theta \) is then also transverse to \( RP^{N-1} \) with \( f \circ \theta|_{\theta^{-1}B_\theta} \) being transverse to \( RP^{N-1} \) and being given by \( (f \circ \theta) \circ \text{projection} \) on a tubular neighborhood of \( \theta^{-1}B_\theta \), and further \( (f \circ \theta)^*(i) = \theta^*(\omega_i(Q)) = \omega_i(P) \).

Letting \( \widetilde{P}^n \subset P^{n+1} \) and \( \widetilde{Q}^n \subset Q^{n+1} \) be \( (f \circ \theta)^{-1}(RP^{N-1}) \) and \( f^{-1}(RP^{N-1}) \), \( \widetilde{\theta} : \widetilde{P}^n \to \widetilde{Q}^n \) is then a branched covering of degree \( d \), where \( \widetilde{\theta} = \theta|_{\widetilde{P}^n} \). Further \( \widetilde{P} \) and \( \widetilde{Q} \) are orientable, being the duals to \( w_1 \) and one has \( [\widetilde{P}] - d[\widetilde{Q}] = (\partial[P] - d\partial[Q]) = \partial([P] - d[Q]) = \partial \beta = \alpha \). (Note. Identification of the normal bundle of \( Q \) in \( Q \) with \( \det \tau(Q) \) gives a choice of compatible orientations by the \( \partial \) process. The only indeterminacy is to completely reverse orientation in the \( \partial \) process, i.e. \( Q \approx -Q \), which does not change \( \alpha \).) Thus the class \( \alpha \) is represented in the desired form.

This reduces the problem of realizing classes in \( \Omega_* \) entirely to a question of possible Pontrjagin numbers and the realization of classes in \( \Omega_*/\text{Tor}(\Omega_*) \).

**Lemma 2.** Let \( n = 4m \) and \( s_m(v) \) the primitive Pontrjagin class. If \( \phi : M^n \to N^n \) is a branched covering of closed oriented manifolds, then

\[
s_m(v)[[M^n]] - (\deg \phi)[N^n]] = \sum_{k \geq 2} (1 - k^2m)v_{1,k}[M^n]
\]

= \( \sum_{k \geq 2} (1 - k^2m)v_1(\tilde{v}_k)^{m-1}[\tilde{B}_k \cap \tilde{B}_k] \).

**Proof.** Clearly,

\[
s_m(v)[[M^n]] - (\deg \phi)[N^n]] = s_m(v)[M^n] - (\deg \phi)s_m(v)[M^n]
\]

= \( \{s_m(v)(\tau(M)) - \phi s_m(v)(\tau(N))\}[M^n] \)

= \( \{s_m(v)(\tau(M) - \phi^\ast \tau(N))\}[M^n] \)

by primitivity. To compute this class universally, following Brand [4], one has \( p = \frac{(1 + p_1)/(1 + k^2p_1)}{(1 + \cdots)^2} \in H^*(BO_2; \mathbb{Z}) \) to give \( s_m(p) = p_1^m - k^{2m}p_1^m = (1 - k^{2m})p_1^m \). The rest of the result is the observation that \( \tilde{B}_k \cap \tilde{B}_k \) is the submanifold dual to \( p_{1,k} \), and \( \mathbb{C}_k \cap \mathbb{C}_k = \mathbb{C}_k \). □

Combining results, one then has easily

**Proposition 4'**. The set classes \( \alpha \in \Omega_n \) of the form \([M^n] - 2[N^n]\) with \( \phi: M^n \to N^n \) a degree 2 branched covering of closed oriented manifolds is a subgroup of \( \Omega_n \) of odd index, if \( n > 0 \).

**Proof.** If \( \phi: M^n \to N^n \) is a degree 2 branched covering so is \( \phi \times \text{(identity)}: M^n \times P^m \to N^n \times P^m \), and so the set of classes \( \alpha \) of the form \([M^n] - 2[N^n]\) in \( \Omega_n \) forms an ideal \( \Lambda_n \) (i.e. \( \Omega_n \) submodule).

By Proposition 1', \( \Omega_n^{[2]}(\text{cod} 2) \cong \Omega_n \oplus \Omega_{n-4}(BO_2) \) and one has \([\xi^2 \to CP^{2r}] \in \Omega_n(BO_2)\), with \( \xi^2 \) the Hopf bundle, having \( p^l([\xi^2]) = 1 \). Since this is the cobordism class of the self-intersection \([\mathbb{P}_2 \to \tilde{B}_2 \cap \tilde{B}_2]\) for some 2-fold branched cover \( \theta: P^{4r+4} \to Q^{4r+4} \), one has a class \( \alpha = [P^{4r+4}] - 2[Q^{4r+4}] \in \Lambda_{4r+4} \) for which \( s_{r+1}(\alpha) = (1 - 2^{r+2}) \). This \( \alpha \) is a suitable polynomial generator for \( \Omega_n \). Since \( \Lambda_n \) contains \( \text{Tor} \Omega_n \) and maps onto \( \Omega_n \) in positive dimensions, \( \Lambda_n \subset \Omega_n \) has odd index if \( n > 0 \). □

**Remark.** One may actually write down branched coverings for the low dimensional classes in \( \Omega_n \). Specifically, one has \( \sigma: CP^2 \to CP^2/\text{conjugation} \cong S^4 \), with the identification to \( S^4 \) due to Kuiper [10], for which the self-intersection class is the inclusion of a point in \( BO_2 \). One also has \( \sigma: P(1, 2) \to S^1 \times S^4 \), where \( P(1, 2) = S^1 \times CP(2)/(1 \times \text{conjugation}) \) is the Dold manifold and \( \sigma \) is the quotient by dividing out the involution \(-1 \times 1 \sim 1 \times \text{conjugation} \), with the self-intersection being the nonzero element in \( \Omega_1(BO_2) \cong \mathbb{Z}_2 \). From §2, one also has the branched covering \( \phi: Q^{2r} \to CP^{2r} \), where \( Q^{2r} \) is the quadric for which \([Q^{2r}] - 2[CP^{2r}] \) is \( 2([HP^r] - [CP^{2r}]) \). Since an odd multiple of this class may also be hit, one has \([HP^r] - [CP^{2r}] \in \Lambda_{4r} \), and this class has \( s \)-number \( 1 - 2^{2r} \), to give an explicit choice of generators for \( \Lambda_n \).

**Proposition 5'.** If \( n = 4m = 2k(p - 1) \) with \( p \) an odd prime and \( \alpha = [M^n] - d[N^n] \) is the class of a \( d \)-fold branched cover with \( d < 2p \), then \( s_m(p)[\alpha] \equiv 0 \mod p \).

**Proof.** Let \( \phi: M^n \to N^n \) be the branched covering. By Lemma 2, \( s_m(p)[\alpha] = \sum_{j \geq 2} (1 - j^{2m})p_1^m[M^n] \) where the sum is for \( j \leq d \) only. For \( j \equiv 0 \), \( j^{2m} = j^{k(p - 1)} \equiv 1 \mod p \), and hence mod \( p \), \( s_m(p)[\alpha] = p_1^m[M^n] = p_1^{m-1}(\tilde{\mathbb{P}}_p \cap \tilde{\mathbb{P}}_p) \). If one considers a component \( \tilde{\mathbb{P}}_p = \tilde{B}_p \), with \( B' = \phi \tilde{B}_p \) being the corresponding component of the branch set with \( \phi: \tilde{B} \to B' \) being an \( r \)-fold cover, \( 2p > d > r \cdot p \) and hence \( r = 1 \), and \( \phi: \tilde{B} \to B' \) is an isomorphism. If \( \tilde{v} \) and \( v' \) are the normal bundles, then \( v' = \mu_\phi(\tilde{v}) \) and \( \phi: S(\tilde{v}) \to S(v') \) is a \( p \)-fold cover, forming a portion of the \( d \)-fold covering \( \phi: \phi^{-1}(S(v')) \to S(v') \) classified by a map \( S(v') \to B\Sigma_d \). This local covering actually factors through a map \( S(v') \to B\Sigma_d \times B\Sigma_p \) corresponding to the two parts of the covering \( \phi^{-1}(S(v')) \to S(\tilde{v}) \) and \( S(\tilde{v}) \).
Claim. The number $\nu_p^{-1}(\bar{p}[B \cap B]) \in \mathbb{Z}_p$ is precisely the class $f_*[S(\nu')] \in p$-torsion part of $H_{2k(p-1)-1}(B \Sigma_d; \mathbb{Z}) = \mathbb{Z}_p$.

Since the local branchings for $\phi: M^n \to N^n$ give rise to the zero element in $\Omega_{n-1}(B \Sigma_d)$, the total homology class in the $p$-torsion part of $H_{n-1}(B \Sigma_d; \mathbb{Z})$ is zero, and hence one must have $\nu_p^{-1}(\bar{p}_p) = 0$.

Note. For the necessary information about $H_* (B \Sigma_d; \mathbb{Z})$, one should recall the work of Nakaoka [13, 14].

There are undoubtedly many ways to see this claim, and one rather unsophisticated way is to consider the diagram,

\[
\begin{array}{cccccc}
\Omega_{*+d}(BSO_2) & \xleftarrow{b} & \hat{\Omega}_*(MSO_2) & \xrightarrow{a} & \Omega_{*-1}(BZ_p) & \to & H_*(BZ_p; \mathbb{Z}) \\
\downarrow c & & \downarrow & & \downarrow & & \downarrow e \\
\Omega_{*+d}(BO_2) & \xleftarrow{d} & \hat{\Omega}_*(MO_2) & \xrightarrow{a} & \Omega_{*-1}(B \Sigma_p) & \to & H_*(B \Sigma_p; \mathbb{Z}) \\
& & \downarrow & & \downarrow f & & \downarrow \\
& & \Omega_{*-1}(B \Sigma_d) & \to & H_*(B \Sigma_d; \mathbb{Z}) & &
\end{array}
\]

where, for a reduced bordism element, the map to the left takes the self-intersection and to the right one takes the $p$-fold ramified cover or local branching, and one takes the usual maps from bordism to homology and the maps of classifying spaces induced by the inclusion of $\mathbb{Z}_p$, the Sylow $p$ subgroup, in $\Sigma_p$ and $\Sigma_d$, with any two inclusions being conjugate.

Now $\hat{\Omega}_*(MSO_2) = \Omega_{*-2}(BSO_2)$ is the free $\Omega_*$ module on the classes $CP^r \to BSO_2$ classifying the Hopf bundle. Applying $a$ takes the class of $[S^{2r+1}, \exp(2\pi i/p)]$ as free $\mathbb{Z}_p$ action or the map of the standard lens space $L^{2r+1}(p)$ into $BZ_p$, giving the standard generator in $H_{2r+1}(BZ_p; \mathbb{Z}) = \mathbb{Z}_p$. Further, decomposables in the $\Omega_*$ module structure of $\hat{\Omega}_*(MSO_2)$ give zero in homology and hence for $V^{2r+2} \to MSO_2$, $U_{r+1}^{-1}([V, \partial V] \in \mathbb{Z}_p$ is just the multiple of the standard generator in homology which is hit by the class of $V$. Now $U_{r+1}^{-1}([V, \partial V] = X^{r+1}[B]$ where $X$ is the Euler class in $H^2(BSO_2; \mathbb{Z})$ and $B \to BSO_2$ is obtained by applying $b$ to the class of $V \to MSO_2$, i.e. taking the appropriate self-intersection.

If one ignores the prime 2, $d$ is an isomorphism, for the third term in the exact sequence with $d$ is $\hat{\Omega}_n(*MO_1) = \mathfrak{H}_{*-2}$ which is a 2 group. Also ignoring the prime 2 and taking $* = 0 \mod 4$, $c$ becomes an isomorphism. (Ignoring 2, $\Omega_*$ is entirely concentrated in dimensions a multiple of 4 and the $CP^r \to BO_2$ form a base of $\Omega_*(BO_2)$ ignoring 2. Similarly, the $CP^r \to BSO_2$ form an $\Omega_*$ base for $\Omega_{4*}(BSO_2)$ and the $CP^{2r+1} \to BSO_2$ form an $\Omega_*$ base for $\Omega_{4*+2}(BSO_2)$.)

Commutativity of the diagram then gives the claim, since $e$ and $f$ are epimorphisms on the $p$-primary part of the homology. □

One may obtain fairly precise information about the possible $s$-numbers with

**Proposition.** The set of possible $s$-numbers $s_*(M^{4m}) - d[N^{4m}]$ for $d$-fold branched coverings of closed oriented manifolds is the subgroup $s_* Z$ of the integers where

\[
s^d_m = a \cdot \gcd \{(1 - 2^2m), (1 - 3^2m), \ldots, (1 - d^2m)\}
\]
and $a = p_1 \cdot p_2 \cdots p_r$, $p_1 < p_2 < \cdots < p_r$, is a product of odd primes with $p_i \leq d$ and $p_i - 1$ dividing $2m$. If $p$ is an odd prime with $p \leq d$ and $p - 1$ dividing $2m$, then $p$ occurs in $a$ if either $2m + 1$ is a power of $p$ or $d < 2p$.

**Proof.** By taking the disjoint union of $d$-fold covers and by reversing orientation one sees that the set of $s_m(p)[\alpha]$ forms a subgroup of $\mathbb{Z}$, and so is $s_m^d\mathbb{Z}$ for some integer $s_m^d$.

Let $h_m^d = \gcd\{(1 - 2^{2m}), (1 - 3^{2m}), \ldots, (1 - d^{2m})\}$, and

$$g_m^d = \gcd\{(1 - 2^{2m}), 3(1 - 3^{3m}), \ldots, d(1 - d^{2m})\}.$$

For any $d$-fold branched cover, Lemma 2 gives

$$s_m(p)[\alpha] = \sum_{d = k \geq 2} (1 - k^{2m})^*_{1, k} [M^{4m}]$$

and since each $^*_{1, k} [M^{4m}]$ is integral, $h_m^d$ divides $s_m(p)[\alpha]$, and hence $h_m^d | s_m^d$. One also has a 2-fold, and hence $d$-fold, branched covering with $s_m(p)[\alpha] = (1 - 2^{2m})$, and for $j \leq d$ one has the $j$-fold covering $Q_j^{2m} \to CP^{2m}$ by the $j$-adic having $s_m(p)[\alpha] = j(1 - j^{2m})$, hence also a $d$-fold branched covering with the same number. Thus $s_m^d$ divides $(1 - 2^{2m})$ and $j(1 - j^{2m})$ for $3 \leq j \leq d$, and hence their greatest common divisor, so $s_m^d$ divides $g_m^d$.

Now, for $p$ dividing $h_m^d$, one has $p > d$ since for $p \leq d$, $p$ does not divide $1 - p^{2m}$. Then $p$ divides $g_m^d$ and if $p^r$ is the power of $p$ dividing $g_m^d$, then $p^r | j(1 - j^{2m})$ and $j \leq d < p$, $p | j$ so $p | (1 - j^{2m})$, and so $p^r$ divides $h_m^d$. Thus $g_m^d = bh_m^d$ where $b$ is divisible only by primes less than or equal to $d$ and $h_m^d$ only by primes larger than $d$.

For $p \leq d$, $p^2$ is not a factor of $p(1 - p^{2m})$ and so $b$ cannot be divisible by $p^2$. If $p - 1$ divides $2m$, then for $j \equiv 0 (p)$, $p$ divides $j - 2^{2m}$, while for $j \equiv 0 (p)$, $p$ divides $j$ and so $p$ divides each $j(1 - j^{2m})$, $3 \leq j \leq d$, and also $1 - 2^{2m}$ and hence $p$ divides $b$. If $p - 1$ does not divide $2m$, then taking $j$ to be a primitive root for $p$, $j < p \leq d$, one has that $p$ does not divide $j(1 - j^{2m})$, and so does not divide $b$.

Thus $b = g_m^d/h_m^d$ is the product of those odd primes $p$ with $p \leq d$ and $p - 1$ dividing $2m$. Since $h_m^d | s_m^d | g_m^d = b \cdot h_m^d$, $s_m^d = a \cdot h_m^d$ for some $a$ dividing $b$, giving the desired form for $a$.

For an odd prime $p$ with $p \leq d$ and $p - 1$ dividing $2m$, and either $2m + 1 = p^s$ or $d < 2p$, one must have $p$ dividing $a$. For the case $2m + 1 = p^s$, $s_m(p)[M^{4n}] \equiv 0 \mod p$ for all manifolds and hence for all classes $\alpha$. For the case $d < 2p$, Proposition 5 gives the divisibility. □

**Note.** I am indebted to Gordon Keller for the argument using primitive roots in the above. For the following comments I am indebted to my son, Richard Stong.

**Comment 1.** If $p$ divides $h_m^d = \gcd\{(1 - 2^{2m}), \ldots, (1 - d^{2m})\}$, which is true for example if $p > d$ and $p - 1$ divides $2m$, and if $p < 3 \times 10^9$ then

$$r_p(h_m^d) = \begin{cases} 
1 + r_p(m) & \text{for } d \geq 3 \text{ or } d = 2 \text{ and } p \neq 1093, 3511, \\
2 + r_p(m) & \text{for } d = 2 \text{ and } p = 1093 \text{ or } 3511.
\end{cases}$$

**Proof.** If $p$ is an odd prime dividing $1 - j^{2m}$, then $p$ divides $1 - j^{p-1}$ and so $1 - j^l$ where $l = \gcd(p - 1, 2m)$. Letting $j^l = x = 1 + sp^r$, $r > 0$ and $s \equiv 0 \mod p$,
one has
\[ x^p = 1 + \left( s + \sum_{i=2}^{p} \frac{p}{i} s^i p^{-i} \right) x^{p+1} \]
\[ = 1 + s' p^{r+1} \text{ with } s' \equiv s \mod p \]
and for \( q \equiv 0 \mod p \),
\[ x^q = 1 + \left( qs + \sum_{i=2}^{q} \frac{q}{i} s^i p^{-i} \right) x^{p+1} \]
\[ = 1 + s'' p^r \text{ with } s'' \equiv qs \mod p, \]
and so \( x^{p/q} = 1 + tp^{r+j} \) with \( t \equiv qs \equiv 0 \mod p \). Thus \( v_p(j^{p-1} - 1) = v_p(j^l - 1) \) and \( v_p(j^{2m-1} - 1) = v_p(j^l - 1) + v_p(2m) \). From [5] one has \( v_p(2^{p-1} - 1) = 1 \) for \( p < 3 \times 10^9 \) and \( p \neq 1093, 3511 \), and in these exceptional cases \( v_p(2^{p-1} - 1) = 2 \) and \( v_p(3^{p-1} - 1) = 1 \).

\[ \square \]

\textbf{Comment 2.} The argument is also valid for \( p > 3 \times 10^9 \) if \( p^2 \) does not divide \( 2^{2m} - 1 \). In order that \( p^2 \) divide \( 2^{2m} - 1 \), \( m \) must be large, and in fact \( m \geq 63 \).

\textbf{Proof.} If \( p^2 \) divides \( 2^{2m} - 1 = (2^m - 1)(2^m + 1) \), the two factors are relatively prime and so \( p^2 \) divides either \( 2^m - 1 \) or \( 2^m + 1 \). Thus \( 2^m + 1 \geq p^2 > 9 \times 10^{18} \).

Now \( 2^{63} > 9 \times 10^{18} > 2^{62} \) so one must have \( m \geq 63 \) at the minimum. \( \square \)

\textbf{Comment 3.} There are examples of primes \( p \) dividing both \( (1 - 2^{2m}) \) and \( (1 - 3^{2m}) \) without having \( (p - 1)|2m \). Specifically, 2 and 3 are both quadratic residues of 73, so 73 divides \( 1 - 2^{36} \) and \( 1 - 3^{36} \).

\section{5. Edmonds' theorem.}
The arguments given by Edmonds actually prove considerably more, and this section will show how these arguments work.

\textbf{Proposition 6.} If \( \phi : M^n \to N^n \) is a branched covering of closed oriented manifolds with oriented branch set \( B_\phi \) and \( H^2(N; \mathbb{Q}) = 0 \), then \([M^n] - (\deg \phi)[N^n] \in \text{Tor}(\Omega_*) \).

\textbf{Proof.} If \( B \) is oriented, then its covering \( \phi^{-1}B_\phi \) is also orientable. Thus one has a factorization for \( g \) and \( \bar{g} \),
\[
M^n \xrightarrow{h} \bigvee_k \text{MSO}_2 \xrightarrow{\bar{u}} \bigvee_k \text{MO}_2 \\
N^n \xrightarrow{h} \text{MSO}_2 \xrightarrow{u} \text{MO}_2
\]
One now has \( \bar{u}^*(v_{1,k}) = U_k^2 \) where \( U_k \) is the Thom class in the \( k \)th wedge summand of \( \bigvee_k \text{MSO}_2 \), and \( \bar{h}^*(U_k^2) = X(v_k)h^*(U_k) \) where \( X(v_k) = c_1(v_k) \in H^2(B_k; \mathbb{Z}) \) is the Euler class or first Chern class of the normal bundle \( v_k \) of \( B_k \) in \( M \). One then has
\[
\tilde{B} \xrightarrow{i} M^n \xrightarrow{\phi} N^n \xrightarrow{h} \text{MSO}_2
\]
with \( (h \circ \phi \circ i)^*(U) = X(v_k^i) = c_1(v_k^i) = kc_1(v_k) \), where \( v_k^i \) is the \( k \)th tensor power of the complex line bundle \( v_k \), and \( U \) is the Thom class. Since \( H^2(N, \mathbb{Q}) = 0 \) one has
\( h^*(U) = 0 \) and so \( k\gamma_1(B_k) = 0 \) and \( X(B_k) = 0 \) in \( H^2(B_k; \mathbb{Q}) \). Thus \( \nu_{1,k} = \eta^*(\nu_{1,k}) = 0 \) in \( H^*(M^n; \mathbb{Q}) \) and by Brand's formula \( \nu(\tau(M) - \phi^*\tau(N)) = 1 \) in \( H^*(M; \mathbb{Q}) \). Thus \( \phi^*\nu(\tau(N)) = \nu(\tau(M)) \) rationally, so that \( M \) and \( (\deg \phi)N \) have the same Pontrjagin numbers. \( \square \)

**Proposition 7.** If \( \phi: M^n \to N^n \) is a branched covering of closed oriented manifolds with \( H^4(N; \mathbb{Q}) = 0 \), then \( [M^n] - (\deg \phi)[N^n] \in \operatorname{Tor}(\Omega_\bullet) \).

**Proof.** Two proofs will be given, with the first being a bit sophisticated.

**Proof Number 1.** Let \( N \to B_{\deg \phi} \) be a classifying map for the branched covering \( \phi \). According to Brand [4], \( (B_{\deg \phi} \otimes \mathbb{Q}) \) is a wedge of Eilenberg-Mac Lane spaces \( K(\mathbb{Q}, 4) \), and since \( H^4(N; \mathbb{Q}) = 0 \), the map \( f \) in \( \Omega_\bullet(B_{\deg \phi} \otimes \mathbb{Q}) \) lies in the image of \( \Omega_\bullet(\text{point}) \otimes \mathbb{Q} \). Thus some multiple of \( \phi \) is cobordant to a trivial unbranched covering. \( \square \)

**Proof Number 2.** Essentially duplicating the argument of Proposition 6, one has \( k^2\nu_{1}(B_k) = (g \circ \phi \circ i)^*(\nu_{1}) = 0 \) in \( H^*(M; \mathbb{Q}) \) and so \( \nu_{1,k}^2 = \nu_{1}(B_k) \cdot \nu_{1,k} = 0 \) in \( H^*(M, \mathbb{Q}) \). Thus Brand's formula becomes

\[
\nu(\tau(M) - \phi^*\tau(N)) = 1 + \sum_k (1 - k^2)\nu_{1,k} \in H^*(M; \mathbb{Q}).
\]

For any partition \( \omega \), one then has

\[
s_\omega(\nu)(\tau(M)) = s_\omega(\nu)((\tau(M) - \phi^*\tau(N)) \oplus \phi^*\tau(N))
\]

\[
= \sum_{\omega' \cup \omega'' = \omega} s_{\omega'}(\nu(\tau(M) - \phi^*\tau(N)) \cup s_{\omega''}(\nu)(\phi^*\tau(N)))
\]

\[
= \begin{cases} 
\phi^*(s_\omega(\nu)(\tau(N))) & \text{if } \omega \neq (\omega', 1), \\
\phi^*(s_\omega(\nu)(\tau(N))) + \phi^*(s_{\omega}(\nu)(\tau(N))) \cdot \left( \sum_k (1 - k^2)\nu_{1,k} \right) & \text{if } \omega = (\omega', 1),
\end{cases}
\]

since \( s_\omega(\nu)(\tau(M) - \phi^*\tau(N)) \) is nonzero only for \( \omega = (0) \) or \( (1) \), in rational cohomology. Thus \( s_\omega(\nu)([M]) = (\deg \phi)s_\omega(\nu)([N]) \) except for \( \omega = (\omega', 1) \), and

\[
s_{(\omega, i)}(\nu)([M]) - (\deg \phi)s_{(\omega, i)}(\nu)([N]) = \sum_k (1 - k^2)\phi^*(s_{\omega}(\nu)(\tau(N)))\nu_{1,k}[M]
\]

\[
= \sum_k (1 - k^2)i^*\phi^*(s_{\omega}(\nu)(\tau(N)))[B_k \cap B_k]
\]

\[
= \sum_k (1 - k^2)<s_{\omega}(\nu)(\tau(N)), (\phi \circ i)_*([B_k \cap \tilde{B}_k])>. 
\]

However, \( (\phi \circ i)_*([B_k \cap \tilde{B}_k]) \in H_{n-4}(N^n; \mathbb{Q}) \equiv H^4(N^n; \mathbb{Q}) \equiv 0 \) and so this number is zero. Thus \( [M^n] - (\deg \phi)[N^n] \) has all Pontrjagin numbers zero. \( \square \)

Collecting together everything one knows, one then has

**Proposition 8.** Let \( \phi: M^n \to S^n \) be a branched covering with \( M^n \) closed and if \( n = 4 \) assume \( B_\phi \) orientable. Then \( M^n \) is orientable and \( [M^n] \in \operatorname{Tor}(\Omega_\bullet) \). If \( M^n \) is a Spin manifold or if \( B_\phi \) is orientable, then \( [M^n] = 0 \) in \( \Omega_\bullet \).
PROOF. \( \omega_1(M) = \phi^*\omega_1(S^n) = 0 \), so \( M \) is orientable. Let \( \phi: M \to S^n \) be the restriction to the component \( M_i \) of \( M \) of \( \phi \). Then \( M_i \) with the orientation induced by the covering satisfies the conditions of Proposition 6 for \( n = 4 \) and otherwise Proposition 7. Thus \([M_i] = [M_i] - (\deg \phi_i)[S^n] \in \text{Tor } \Omega^*_\text{e}. \) Reversing the orientation of \( M_i \) does not change that, and hence the class of \( M \) belongs to \( \text{Tor } \Omega^*_\text{e} \) no matter how orientations are chosen.

If \( M^n \) is a Spin manifold, one applies Corollary 3.5 of Bernstein and Edmonds [2], remarking as in §3 that the hypothesis that \( M \) have even Euler characteristic is unnecessary since \( \omega_n(M^n) \) is the square of the Wu class \( v_{n/2}(M^n) \). This gives \([M^n] = 0 \) in \( \Omega^*_\text{e} \). If \( B_\phi \) is orientable, one applies the corollary from §3 to obtain \([M^n] = 0 \) in \( \mathcal{R} \). Finally, one recalls the Tor \( \Omega^*_\text{e} \) injects into \( \mathcal{R}^*_\text{e} \), and hence \( M^n \) is an oriented boundary in both of these cases. □

Note. For \( n = 4 \), \( \pi: \mathbb{CP}^2 \to \mathbb{CP}^2/\text{conjugation} = S^4 \) has nonorientable branch set. One also has the \( k \)-dric \( \rho: Q^2_k \to \mathbb{CP}^2 \) branched along the \( k \)-dric \( Q^2_k = \{ z \in \mathbb{CP}^2 \mid z_0^k + z_1^k + z_2^k = 0 \} \) with \( Q^2_k \) a Spin manifold for \( k \) even and \( Q^2_k \) does not meet \( RP^2 \). Thus the composite \( \rho \circ \pi: Q^2_k \to S^4 \) is a branched cover with \( Q^2_k \) being Spin, provided \( k \) is even and \( v_1(Q^2_k) = (4 - k^2)k \neq 0 \) for \( k = 4 \).

To see that there is a branched covering \( \phi: \mathbb{CP}^5 \to S^5 \) with \([P^5] \neq 0 \), one may proceed as follows. One has a cofibration \( RP^\infty \to B_2 \to M(\mu_2(\gamma_2)) \) where \( B_2 \) is Brand’s classifying space for 2-fold branched covers, and \( \gamma_2 \) is the 2-plane bundle over \( BO_2 \), and a cofibration \( M(\mu_2(\gamma_2) | S\gamma_2) \to M(\mu_2(\gamma_2) \oplus \gamma_2) \). Now \( M(\mu_2(\gamma_2) \oplus \gamma_2) = M(\mu_2(\gamma_1 + 1)) \approx M(\mu_1 + 1) \approx \Sigma RP^\infty \). The composite \( \beta \circ \alpha: B_2 \to M(\mu_2(\gamma_2) \oplus \gamma_2) \) is then a homotopy equivalence. It is readily seen to induce an isomorphism on unoriented bordism, hence on \( Z_2 \) homology, while \( \alpha \) and \( \beta \) are isomorphisms in \( Z_2 \) homology for odd \( p \). Finally, both spaces are simply connected. One then observes that \( \mu(\gamma_2) \oplus \gamma_2 \) is orientable to produce a map \( \theta: M(\mu_2(\gamma_2) \oplus \gamma_2) \to K(Z; 4) \times K(Z_2, 5) \) via the classes \( \phi(1) \) and \( \phi(w_1) \), where \( \phi \) is the Thom isomorphism. \( \theta \) induces an isomorphism in mod 2 cohomology through dimension 7. Thus \( \pi_3(B_2) \approx Z_2 \) (plus possible odd torsion) with nontrivial image in \( H_5(B_2; \mathbb{Z}_2) \). The branch set for this map is the nonzero class in \( \mathcal{R}_1(BO_2) \) with \( w_1 \) being the nonzero number.

6. Coverings of spheres. Since all classes of manifolds branched over \( S^n \) with \( n > 5 \) belong to \( \text{Tor } \Omega^*_n \subset \mathcal{R}^*_n \), one should analyze the possible Stiefel-Whitney numbers for manifolds branched over \( S^n \). This section will do so.

Observation. The set \( B(S^n) \) of classes \([M^n] \in \Omega^*_n \), with \( \phi: M^n \to S^n \) a branched covering, is a subgroup of \( \Omega^*_n \).

Proof. If \( \phi: M^n \to S^n \) is a branched covering, \( B_\phi \subset S^n \) is a proper closed subset of \( S^n \) and hence one may find a closed disc contained in \( S^n - B_\phi \). By reparametrizing \( S^n \), one may assume that disk is the “southern” hemisphere \( D^n \) and hence that \( B_\phi \subset \text{interior}(D^n) \). If \( \psi: N^n \to S^n \) is a second branched covering one may similarly suppose \( B_\psi \subset \text{interior}(D^n) \). The union \( \phi \cup \psi: M^n \cup N^n \to S^n \) is then a branched cover and gives the sum of the classes in \( B(S^n) \). □

Note. If \( \phi: M^n \to S^n \) and \( \psi: N^n \to S^n \) both have degree \( d \), one may realize the sum by a branching of degree \( d \). One simply joins \( \phi^{-1}(D^n_+) \) and \( \psi^{-1}(D^n_-) \) along their
common boundaries which are copies of $S^{n-1} \times \{1, 2, \ldots, d\}$. The resulting manifold is obtained by surgery on $d$ copies of $D^n \times S^0$ in $M \cup N$. This phenomenon is much more general since one could sew together two $d$-fold coverings over $M^n$ and $N^n$ to obtain a $d$-fold covering of $M^n \# N^n$.

If one now considers a branched covering $\phi: M^n \to S^n$ with $n > 5$, one has by Brand's formula

$$w(M^n) = \phi^*w(S^n) \{1 + U_{ev} + w_1 U_{ev} + \cdots + w_l U_{ev} + \cdots\}$$

$$= \{1 + U_{ev} + w_1 U_{ev} + \cdots + w_l U_{ev} + \cdots\},$$

where $U_{ev} = \sum_k U_k$ and induced homomorphisms are ignored. Letting $B_{ev}$ be the points of $\phi^{-1}(B_n)$ of even local branching degree, $B_{ev}$ is the submanifold of $M^n$ dual to the class $U_{ev}$. One then has

$$w(B_{ev}) = \frac{1}{1 + w_1} = 1 + w_1 + w_1^2 + w_1^3 + \cdots + w_1^{n-2}$$

and

$$w(\tilde{B}_{ev}) = 1 + w_1 + w_2$$

where $w_2$ is the restriction of $U_{ev}$ to $B_{ev}$.

**Lemma 3.** The Stiefel-Whitney numbers of $M^n$ are given by

$$w_i \cdots w_l [M^n] = \begin{cases} 0 & \text{if any } i_a = 1, \\
\frac{B_{ev}}{B_{ev}} & \text{if each } i_a > 1.
\end{cases}$$

**Proof.** Since $w_i(M) = 0$, the first formula is obvious. For the second, one has from the proof of Proposition 2' that $w_i(M^n) = \hat{w}_i[RP(\tilde{B}_{ev} \oplus 1)]$, where

$$\hat{w}(RP(\tilde{B}_{ev} \oplus 1)) = 1 + U + w_1 U + \cdots + w_l U + \cdots$$

and $U = c^2 + w_1 c + w_2$ with $c U = 0$, and here the classes $w_l U$ are actual products. Thus

$$\hat{w}_i \cdots \hat{w}_l [RP(\tilde{B}_{ev} \oplus 1)] = \hat{w}_i^{i_1 + \cdots + i_r} [RP(\tilde{B}_{ev} \oplus 1)]$$

$$= \hat{w}_i^{n-r} w_2^{r-1} [RP(\tilde{B}_{ev} \oplus 1)] = \hat{w}_1^{n-2r} w_2^{r-1} [B_{ev}].$$

**Note.** One may obtain the second formula directly by considering the map $g: M \to M(\tilde{B}_{ev})$ by collapsing. Then $w_i \cdots w_l [M] = g^*(\Phi(w_i^{i-2}) \cdots \Phi(w_l^{r-2})) [M] = g^*(\Phi(w_i^{n-2r} w_2^{r-1})) [M]$, where $\Phi$ is the Thom isomorphism. This is $\langle \Phi(w_i^{n-2r} w_2^{r-1}), \hat{w}_i \cdots \hat{w}_l [M] \rangle = \langle w_i^{n-2r} w_2^{r-1}, \hat{w}_i \cdots \hat{w}_l [M] \rangle$ where $\phi$ is the homology Thom isomorphism, and $\Phi g_\ast (M) = [B_{ev}]$ gives the result.

**Note.** This argument does not depend on the use of $S^n$, and shows that $\chi(w_i U_{ev}) [M] = \chi w_i [B_{ev}]$ for any branching and class $\chi$, i.e. $w_i U$ acts like a product.

**Note.** This gives an alternative proof that $\phi: M^n \to S^n$ with $M$ Spin implies $M$ bounds without using a category argument. Use $w_n = \epsilon_n^{2/2}$ to get an equivalent number $w_n [M] = \sum w_i \cdots w_l [M]$ with $r > 0$. Then $w_i \cdots w_l [M] = w_{n-2r+r} w_2^{r-1} [M] = 0$ whenever $r > 0$, so all numbers of $M$ are zero.
Lemma 4. The Wu class of $B_{ev}$ is given by
\[ v = 1 + w_1 + w_1^3 + \cdots + w_1^{2^r-1} + \cdots, \]
and
\[ 0 = \bar{w}_{n-2}(\bar{v}_{ev})[B_{ev}] = \sum_{k=0}^{[(n-2)/2]} \left( \begin{array}{c} n - 2 - k \\ k \end{array} \right) w_1^{n-2-2k} w_2^k [B_{ev}]. \]

Proof. The first formula is obtained by $v = Sq^{-1} w$ with $w = 1 + w_1 + w_1^3 + \cdots$. One calculates the dual Stiefel-Whitney class $\bar{w} = 1/w$ of $M$ by calculation in $MO_2$, and to do that calculation one may calculate in $BO_2$, following Brand. One has
\[ \frac{1 + w_1}{1 + w_1 + w_2} = \frac{1 + w_1 + w_2 + w_2}{1 + w_1 + w_2} = 1 + \frac{w_2}{1 + w_1 + w_2}, \]
and so
\[ \bar{w}(M) = 1 + U_{ev}/(1 + w_1 + U_{ev}) \]
expanded in the usual formal way. Thus
\[ 0 = \bar{w}_n[M^n] = \left( \frac{1}{1 + w_1 + w_2} \right)[B_{ev}] = \bar{w}_{n-2}(\bar{v}_{ev})[B_{ev}]. \]
Finally, the degree $i$ component of $1/(1 + w_1 + w_2)$ is $\Sigma_{j=0}^{[(i-2)/2]} (i-2j+j) w_1^{i-2j} w_2^j$, which is easily seen by induction on $i$. \[ \square \]

Note. The condition $\bar{w}_{n-2}(\bar{v}_{ev})[B_{ev}] = 0$ is equivalent to the assertion that $RP(\bar{v}_{ev}) \to S^n \times RP^\infty$ bounds. One has $w(RP(\bar{v}_{ev})) = 1$ and the only relation is $0 = c^{n-1}(RP(\bar{v}_{ev}))$ which is this relation on $B_{ev}$.

Lemma 5. If $n$ is even, $n > 4$, then $M^n$ bounds.

Proof. One considers the numbers $w_1^a w_2^b [B_{ev}]$ with $a + 2b = n - 2$. Thus $a$ must be even.

For $b$ odd and $a > 0$, one has
\[ w_1^{2p+2} w_2^{2q+1} [B_{ev}] = w_1 \left( (w_1 w_2) \cdot (w_1^p w_2^q)^2 \right) [B_{ev}] = Sq^1((Sq^1 w_2) \cdot x^2) [B_{ev}] = 0 \]
for $Sq^1 Sq^1 = 0$ and $Sq^1(x^2) = 0$.

For $b > 0$ and even, one has
\[ w_1^{2p} w_2^q [B_{ev}] = Sq^{(n-2)/2} (w_1^p w_2^q) [B_{ev}] = \begin{cases} 0 & \text{if } (n - 2)/2 \neq 2t - 1, \\
 w_1^{2t-1+p} w_2^q [B_{ev}] & \text{if } (n - 2)/2 = 2t - 1, \end{cases} \]
by the formula for $v$. One notes that $2t - 1 + p > 0$ and that the power of 2 dividing $b$ has been reduced in the second case. Inductively, on this power of 2, these numbers are zero.
The only possible nonzero numbers are then those with $a = 0$ and $b$ odd or with $b = 0$, with the latter only for $(n - 2)/2 = 2^r - 1$ by the same Wu class argument.

Now

$$w_2^{2q+1}[B_{ev}] = w_2^{2q+2}[M^n] = (w_2^2)^{q+1}[M^n] \equiv (v_1)^{q+1}[M^n] \mod 2$$

and since $[M^n] \in \text{Tor}(\Omega_n)$, this is zero. (Note. This does not hold for $n = 4$, and is the crucial number for the case.)

If $(n - 2)/2 = 2^r - 1$, $n = 2^{r+1}$, and the coefficient of $w_1^{2^{r+1}-2}$ in $\bar{w}_{n-2}(B_{ev})$ is 1. Since all other $w_i^a w_j^b [B_{ev}]$ are zero, one must have $w_1^{2^{r+1}-2}[B_{ev}] = 0$. □

The situation for $n$ odd is much harder. Since $\Omega_1 = \Omega_3 = \Omega_7 = 0$ and since $\Omega_5 \cong \mathbb{Z}_2$ has a nonzero class known to branch over $S_5$, one may suppose $n > 9$. One may then divide up into the cases $2^{k+1} < n > 2^k$, where with no loss $k \geq 3$. (Everything could be checked for smaller $k$.) Further, it is convenient to consider

$$2^k + 2^r + 1 - 3 \geq n > 2^k + 2^r - 3,$$

with $1 \leq r \leq k$. (Note. For $r = 1$, $n = 2^k + 1$ only, and for $r = k$, $n = 2^{k+1} - 1$ only.)

**Lemma 6.** For $n$ odd, $w_1^{n-2p} w_2^{p-1}[B_{ev}] = 0$ if $p$ is odd.

**Proof.** $w_1^{n-2p} w_2^{p-1}[B_{ev}] = \text{Sq}^1(w_1^{n-2p-1}/2 w_2^{p-1}/2)[B_{ev}] = 0$ for $\text{Sq}^1 x^2 = 0$. □

**Lemma 7.** For $n$ odd, $2^{k+1} > n > 2^k$, $w_1^{n-2p} w_2^{p-1}[B_{ev}] = 0$ except for $((n - 1)/4) - (2^k - 1) \leq p/2 \leq ((n - 1)/4)$.

**Proof.** To have $w_1^{n-2p} w_2^{p-1}[B_{ev}] \neq 0$ one must have $2(p - 1) \leq n - 2$ and since $n$ is odd, $2(p - 1) \leq n - 3$ or $p/2 \leq ((n - 1)/4)$. From $2^{k+1} > n > 2^k$, one has $2^k - 1 \geq (n - 2)/2 > 2^{k-1} - 1$. Since $v = 1 + w_1 + w_1^2 + \cdots + w_1^{2^k-1} + \cdots$ and $v_i = 0$ if $i > [(n - 2)/2]$, one has $v = 1 + w_1 + w_1^3 + \cdots + w_1^{2^k-1} - 1$ and $w_1^{2^k-1} = 0$. To have $w_1^{n-2p} w_2^{p-1}[B_{ev}] \neq 0$, one must then have $n - 2p < 2^k - 1$ or $(n + 1 - 2^k)/2 < p$ and $(n + 1 - 2^k)/2 + 1 \leq p$. Thus $(n - 1)/2 - (2^{k-1} - 2) \leq p$ and dividing by 2 gives the result. □

**Lemma 8.** For $n$ odd, the numbers $w_1^{n-2p} w_2^a [B_{ev}]$ depend only on the numbers $w_1^{n-2p} w_2^{2^i-1}[B_{ev}]$ with $2^i \leq p$.

**Proof.** If $p \neq 2^r$, there are integers $a, b$ with $a + b = p - 1$ and $(\xi) \equiv 1 \mod 2$, for example, if $p = 2^r(2s + 1)$, $r, s > 0$, one may let $b = 2^r$. One then has

$$0 = \text{Sq}^{2p}(w_1^{n-2p} w_2^a)[B_{ev}],$$

and

$$\text{Sq}(w_1^{n-2p} w_2^a) = w_1^{n-2p} w_2^a (1 + w_1)^{n-2p}(1 + w_1 + w_2)^a.$$

One wishes to examine terms of $(1 + w_1)^{n-2p}(1 + w_1 + w_2)^a$ of dimension $2b$, but

$$(1 + w_1)^{n-2p}(1 + w_1 + w_2)^a = \sum_{i=0}^{a} \binom{a}{i} w_1^i (1 + w_1)^{n-2p+a-i},$$
so that the coefficient of $w_2^k$ is $\binom{\alpha}{\beta}$. Thus one has an equation

$$w_1^{n-2p}w_2^{-1}[B_{ev}] = w_1^{n-2p}w_2^{a} \cdot w_2^b[B_{ev}] = \sum_{q \leq p} \alpha_q w_1^{n-2q}w_2^{q-1}[B_{ev}]$$

where $\alpha_q$ depends only on $n$, $p$, and $a$, and of course $q$. Inductively, the result holds for $q < p$, giving the result. $\square$

**Corollary.** For $n$ odd, $2^k + 2^{r+1} - 3 \geq n > 2^k + 2^r - 3$, $2^{k+1} > n$, \[\dim Z_2(B(S^n)) \leq k - r.\]

**Proof.** From Lemmas 6 and 7, the class of $M^n$ is determined by the numbers $w_1^{n-2p}w_2^{p-1}[B_{ev}]$ with $p = 2^{r+1}$ and

$$\left(\frac{n-1}{4}\right) - (2^{k-2} - 1) \leq 2s \leq \left(\frac{n-1}{4}\right).$$

Now $\left(\frac{n-1}{4}\right) - (2^{k-2} - 1) > (2^k + 2^r - 4)/4 - (2^{k-2} - 1) = 2^{r-2}$, so $s \geq r - 1$. Also $n - 1 < 2^{k+1}$, and so $s < k - 1$. Thus $r - 1 \leq s \leq k - 2$. Thus one has $k - r$ choices for $s$. $\square$

**Corollary.** If $n = 2^{k+1} - 1$, $M^n$ bounds.

**Proof.** This is the case $r = k$, and $B(S^n) = 0$. $\square$

**Lemma 8'.** If $n$ is odd and $w_1^{n-2p}w_2^{p-1}[B_{ev}] = 0$ for $p < 2^{r+1}$, then $w_1^{n-2p}w_2^{p-1}[B_{ev}] = 0$ for $2^{r+1} < p < 2^{r+1} + 2^r$.

**Proof.** This requires more precision in the proof of Lemma 8. Assume $w_1^{n-2p}w_2^{p-1}[B_{ev}] = 0$ for $2^{r+1} < p' < p$, which is true for $p' = 2^{r+1} + 1$ since $p'$ is then odd. One then has the formula

$$w_1^{n-2p}w_2^{p-1}[B_{ev}] = \alpha_{2s+1}w_1^{2^{r+2}-1}w_2^{2^{r+1}-1}[B_{ev}]$$

since all other terms are zero. The coefficient of $w_2^{2^{r+1}-1-a}$ in

$$(1 + w_1)^{n-2p}(1 + w_1 + w_2)^a$$

is

$$\left(\frac{a}{2^{r+1} + 1 - a}\right)(1 + w_1)^{n-2p+a-(2^{r+1}-1-a)}$$

and the binomial coefficient can be nonzero only when $2^{r+1} - 1 - a = 0$. If one can choose $a \neq 2^{r+1} - 1$ one then has $\alpha_{2s+1} = 0$ and so $w_1^{n-2p}w_2^{p-1}[B_{ev}] = 0$. For $p = 2^{r+1} + t$, $t < 2^r$, one may let $b = 2^s$, $a = 2^s + t - 1$ to obtain $a < 2^{r+1} - 1$. $\square$

**Lemma 9.** If $n \equiv (2^q - 1) \mod 2^s$ and $w_1^{n-2p}w_2^{p-1}[B_{ev}]$ is zero for $p < 2^s$, nonzero for $p = 2^s$ and $s > q$, then $n < 2^{s+1} + 2^q$.

**Proof.** Suppose $n - 2^{s+1} - 2^q \geq 0$, and consider

$$0 = v_{2^s}\left(w_1^{n-2^{s+1}-2^q}w_2^{2^s-1}\right)[B_{ev}] = Sq^{2^s}\left(w_1^{n-2^{s+1}-2^q}w_2^{2^s-1}\right)[B_{ev}].$$
One has

\[
\text{Sq}\left( w_1^{n-2^{s+1}-2^s} w_2^{2^s-1} \right)
= w_1^{n-2^{s+1}-2^s} w_2^{2^s-1} \sum_{j=0}^{2^s-1} \binom{2^s-1}{j} w_2^j (1 + w_1)^{n-2^{s+1}-2^s+2^s-1-j}
\]

and in the terms for \( \text{Sq}^{2^s} \) the powers \( w_2 \) occur for \( 0 \leq j \leq 2^{q-1} < 2^s-1 \). By Lemma 8', only the term with \( j = 0 \) can be nonzero, giving

\[
0 = \left( n - 2^{s+1} - 2^q + 2^s - 1 \right) w_1^{n-2^{s+1}+2^s-1} \text{[Bv]}. \]

Now \( n - 2^{s+1} - 2^q + 2^s - 1 = n - 2^q - 2^q - 1 \geq 2^s - 1 > 0 \) is congruent to \(-2 \) mod \( 2^{q+1} \), and the binomial coefficient is \( 1 \) mod \( 2 \). Thus one has a contradiction, and so \( n < 2^{s+1} + 2^q \). \( \square \)

**Lemma 10.** If \( n \equiv (2^q - 1) \mod 2^{q+1} \), then \( \text{B}(S^n) = 0 \) except for \( n = 2^k + (2^q - 1) \), \( k > q \), and for \( n = 2^k + (2^q - 1) \) with \( k > q \),

\[
\dim_{Z_2} \text{B}(S^n) \leq \begin{cases} 
1, & k = q + 1, \\
2, & k > q + 1.
\end{cases}
\]

**Proof.** If \( n = 2^q - 1 \), \( \text{B}(S^n) = 0 \) by the second corollary to Lemma 8. Thus, one may suppose \( n = 2^{q+1} t + (2^q - 1) \) with \( t > 0 \).

If \( t = 1 \), one has \( 2^s + 2 > n > 2^{q+1} \) and \( 2^{q+1} + 2^s + 1 - 3 > n > 2^{q+1} + 2^q - 3 \), i.e. \( k = q + 1 \), \( r = q \). By the first corollary to Lemma 8, \( \dim_{Z_2} \text{B}(S^n) \leq k - r = (q + 1) - 1 = 1 \).

Now suppose \( t > 1 \), and choose \( k, r \) with \( 2^k + 1 > n > 2^k, 2^k + 2^s + 1 - 3 > n > 2^{k+1} + 2^q - 3 \). Because \( t > 1 \), \( k \geq q + 2 \), and \( n \neq 2^{k+1} - 1 \) so \( k > r \). For \( r < q \), the interval \( (2^k + 2^s - 3, 2^k + 2^s + 2^s + 1 - 3] \) contains no integer congruent to \( 2^q - 1 \) mod \( 2^{q+1} \), and hence \( r \geq q \).

If \( r > q \), then from the argument for the first corollary of Lemma 8, \( w_1^{n-2^s+1} w_2^{2^s-1} \text{[Bv]} = 0 \) only for \( r \leq s \leq k - 1 \), and let \( s' \) be the smallest such \( s \) giving a nonzero value, so that \( s' \geq r > q \). By Lemma 9, \( n < 2^{s'-1} + 2^s < 2^k + 2^q \), so \( 2^k + 2^s - 1 < n < 2^k + 2^s - 1 \) contradicting the assumption \( r > q \), or that \( s' \) exists. Thus \( B(S^n) = 0 \).

For \( r = q \), \( 2^k + 2^s + 1 - 3 > n > 2^k + 2^s - 3 \) gives \( n = 2^k + (2^q - 1) \). Consider the subspace of \( \text{B}(S^n) \) consisting of those manifolds for which \( w_1^{n-2^s+1} \text{[Bv]} = 0 \). On this subspace one has \( w_1^{n-2^s+1} w_2^{2^s-1} \text{[Bv]} = 0 \) only for \( q + 1 \leq s < k - 1 \), and letting \( s' \) be the smallest such \( s \), one has \( s' > q \). By Lemma 9, \( n < 2^{s'-1} + 2^s \) and so \( s' + 1 = k \), i.e. \( s' = k - 1 \). Thus, the subspace of \( \text{B}(S^n) \) for which \( w_1^{n-2^s+1} w_2^{2^s-1} \text{[Bv]} = 0 \) has dimension at most one and is detected by \( w_1^{n-2^k} w_2^{2^k-1} \text{[Bv]} \). Hence, \( \dim_{Z_2} \text{B}(S^n) \leq 2 \). \( \square \)
Combining all of the pieces, one has

**Proposition 9'.** For \( n \) even, \( B(S^n) = \Omega_n = \mathbb{Z} \) if \( n = 0 \) or 4, and \( B(S^n) = 0 \) otherwise. For \( n \) odd, \( n \equiv (2^q - 1) \mod 2^{q+1}, B(S^n) = 0 \) except possibly for \( n = 2^k + (2^q - 1), k > q \), and for \( n = 2^k + (2^q - 1) \) one has

\[
\dim_{\mathbb{Z}} B(S^n) \leq \begin{cases} 
1, & \text{if } k = q + 1 \\
2, & \text{if } k > q + 1.
\end{cases}
\]

**Notes.** (1) The arguments actually work for manifolds \( S^n \) more general than the sphere. One could assume \( w(S^n) = 1 \) and take \( B(S^n) \) to be the subgroup of \( \Omega_n \) (or \( \mathfrak{M}_n \)) generated by the classes \( [M^n] - (\deg \phi)[S^n] \) (i.e. use \( \phi: M^n \to S^n \) a union of branched coverings). In the proof of Lemma 5 one has \( w_2^{2q+1}[B_{ev}] = \overline{w}_{n-2}(\overline{\nu}_{ev})[B_{ev}] = 0 \) if \( n \neq 2^{q+1} \), while \( n = 2^{q+1} \) gives only \( w_1^{2^{q+1}-2}[B_{ev}] = w_2^{2^{q+1}-1}[B_{ev}] \). Assuming \( H^4(S^n; Q) = 0 \), one has the full result for the classes \( [M^n] - (\deg \phi)[S^n] \), and if \( [S^n] = 0 \) in \( \Omega_n \) the full result for the classes \( [M^n] \). With no extra assumption one has an unoriented result with an extra case \( n = 2^{q+1} \), with \( \dim_{\mathbb{Z}} (B(S^n)) \leq 1 \).

(2) For \( n = 2^k + 2^q - 1, k > q \), consider the Dold manifold

\[
P^n = P(2^q - 1, 2^{k-1}) = S^{2q-1} \times CP^{2k-1} / (-1 \times \text{conjugation})
\]

and the Milnor hypersurface

\[
H^n = H(2^q, 2^k) = \left\{ ([x], [y]) \in RP^{2q} \times RP^{2k} \mid \sum_{i=0}^{2q} x_i y_i = 0 \right\}.
\]

Over each of these manifolds one has a 2-plane bundle \( \eta \),

\[
S^{2q-1} \times (\text{Hopf bundle}) / (-1 \times \text{conjugation})
\]

or \( \xi_1 \oplus \xi_2 \mid H^n \) respectively, and hence a composite map \( M^n \to BO_2 \to MO_2 \). One has

\[
w(P^n) = (1 + c)^{2q-1} - (1 + c + d)^{2^{q+1} - 1} = (1 + d + cd + \cdots + c^{2q-1}d)(1 + d^{2^{q-1}})
\]

and

\[
w(H^n) = \frac{(1 + a)^{2^{q+1}}(1 + b)^{2^{q+1}}}{(1 + a + b)} = \left( 1 + \frac{a \beta}{1 + a + b} \right) \left( 1 + a^2 + b^2 \right)
\]

with the classes \( d^{2^{k-1}}, a^{2q} \), and \( b^{2q} \) making no contributions to Stiefel-Whitney numbers. Thus, the elements \( (M^n, i \circ \eta) \) in \( \mathfrak{M}_n(MO_2) \) have the same characteristic numbers as if \( w(M^n) \) were \( (i \circ \eta)^*(1 + \Sigma w_i(U_i)) \). Making the maps \( i \circ \eta \) transverse to \( BO_2 \subset MO_2 \), one obtains codimension 2 submanifolds \( B_{ev}(M^n) \), and this can be done explicitly to give

\[
B_{ev}(P) = P(2^{q-1}, 2^{k-1} - 1) \quad \text{and} \quad B_{ev}(H) = H(2^{q-1}, 2^{k-1})
\]

with

\[
w(B_{ev}(P)) = 1 / (1 + c), w(\tilde{\nu}_{ev}) = 1 + c + d,
\]

\[
w(B_{ev}(H)) = 1 / (1 + \alpha + \beta), w(\tilde{\nu}_{ev}) = 1 + (\alpha + \beta) + \alpha \beta.
\]

One then has

\[
w_{1}^{n-2^{q+1}} w_{2}^{2^{q+1}-1}[B_{ev}(P)] = 0 \quad \text{if } k > q + 1, \quad w_{1}^{n-2^{q}} w_{2}^{2^{q+1}-1}[B_{ev}(P)] \neq 0,
\]
and
\[ w_{1}^{n-2^q+1}w_{2}^{2^q-1}[B_{ev}(H)] \neq 0, \quad w_{1}^{n-2^q}w_{2}^{2^q-1}B_{ev}(H) = 0 \quad (if \ k > q + 1). \]

These provide examples of manifolds with the correct Stiefel-Whitney numbers, and with \( B_{ev}(M) \) actually having the correct Stiefel-Whitney class structure, to all of the exceptional cases in Proposition 9'. Thus the argument cannot be improved at the Stiefel-Whitney number level.

In the case \( n = 5 \), there are maps \( f : M^5 \to MO_2 \), not lifting to \( BO_2 \), for which Brand’s stable bundle actually pulls back to \( t(A^5) \). For \( P^5 \), one has the branching \( \phi : P^5 \to S^1 \times S^4 = P(1,2)/(1 \times \text{conjugation}) \), and for \( H^5 \) one has the involution \( T([x, y_0, y_1, y_2, y_3, y_4]) = ([x, y_0, y_1, y_2, -y_3, -y_4]) \) giving a branched cover \( H^5 \to N^5 = H(2,4)/T \). In order to identify \( N^5 \), consider \( RP^2 \times RP^4 \) as \( RP(5) \), the projective space bundle of a trivial 5-plane bundle over \( RP^2 \), and observe that the defining relation \( \Sigma_0^3 x_j y_j = 0 \) gives orthogonality, i.e. \( H(2,4) = RP(\lambda^4 + 2) \) where \( \lambda \) is the orthogonal complement of the line bundle \( \lambda \) inside 3. \( T \) is multiplication by \(-1\) in the fibers of 2, and so \( N^5 \) is the quotient of \( S(\lambda^4 + 2) \) by the \( Z_2 \times Z_2 \) given by \(-1\) in the fibers of \( \lambda \) and by \(-1\) in the fibers of 2, the product of these being \(-1\) on the sphere. Thinking of \( S(\lambda^4 + 2) \) as the fiberwise join \( S(\lambda^4) \star S(2) \), \( S(\lambda^4 + 2)/Z_2 \times Z_2 \) is \( S(\mu_2 \lambda^2) \star S(\mu_2 2) = S(\mu_2 \lambda^4 + \mu_2 2) = S(\mu_2 \lambda^4 + 2) \). Now \( \mu_2 \lambda^4 + 2 \equiv \lambda + 3 \), being 4-plane bundles with the same Stiefel-Whitney class and so \( N^5 \) is the normal sphere bundle of \( RP^2 \) imbedded in \( R^6 \), hence a framed manifold.

One may generalize this construction for \( H^5 \). Noting that \( 4\lambda = 4 \) over \( RP^2 \), \( H^5 \equiv RP(\lambda^4 + 2) = RP(3\lambda + 1) = RP(\lambda + 3) \) and branches over \( S(\mu_2(\lambda + 1) + \mu_2 2) = S(\lambda + 1 + 2) = S(\lambda + 3) \).

Specifically, for \( n = 2^k + 2^q - 1 \), one may consider \( Q^n = RP((\lambda_1 + 1) + (\lambda_2 + 1)) \) over \( RP^{2^k-2} \times RP^{2^q-2} \) as branching over \( U^n = S((\lambda_1 + 1) + 1) \). One notes that \( w(U^n) = 1 \), and that \( U^n \) is frameable for \( 3 \geq k > q \geq 1 \). By [17, Lemma 3.4] \( Q^n \) is indecomposable in \( \mathfrak{M} \), and one may check very painfully that \( w_{n+2-2^q+1}w_{2^q-1}[Q^n] \neq 0 \) to see that \( Q^n \) is cobordant to \( H^n \). Thus, one actually has branchings over manifolds having \( w = 1 \) in every exceptional dimension, and over framed manifolds when \( n = 5, 9, \) and 11.

References


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