SOME CONJECTURES ON ELLIPTIC CURVES
OVER CYCLOTOMIC FIELDS

BY
D. GOLDFELD AND C. VIOLA

ABSTRACT. We give conjectures for the mean values of Hasse-Weil type $L$-functions over cyclotomic fields. In view of the Birch-Swinnerton-Dyer conjectures, this translates to interesting arithmetic information.

1. In 1967 A. Weil [6] showed that if the Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} a(n)n^{-s}, \quad L_1(s, \chi) = \sum_{n=1}^{\infty} a(n)\chi(n)n^{-s} \quad (\chi \mod q)$$

satisfy the functional equations

(1) \((\sqrt{N}/2\pi)^s \Gamma(s)L_1(s) = w_1(\sqrt{N}/2\pi)^{k-s} \Gamma(k-s)L_1(k-s), \quad |w_1| = 1,\)

(2) \((\sqrt{N}/2\pi q)^s \Gamma(s)L_1(s, \chi) = w_\chi(\sqrt{N}/2\pi q)^{k-s} \Gamma(k-s) L_1(k-s, \chi),\)

for "sufficiently many" $q$ such that $(q, N) = 1$ and all primitive characters $\chi \mod q$, where $N$ and $k$ are positive integers and $\varepsilon$ is a primitive character mod $N$, then $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi i n z}$ is a modular form with multiplier $\varepsilon$ of weight $k$ for the congruence subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bigg| a, b, c, d \in \mathbb{Z}, ad - bc = 1, c \equiv 0 \text{ (mod } N) \right\}.$$ 

Our aim is to propose some conjectures on the asymptotic behaviour of the mean value

$$S(X; h) = \sum_{\substack{p \leq X \\text{prime} \atop p \equiv 1 \text{ (mod } \chi_0)}} \prod_{\chi \equiv \chi_0 \text{ (mod } p \chi \neq \chi_0 \text{ for } r<h} L_1(k/2, \chi) \quad (h \geq 2 \text{ a fixed integer})$$

where the sum is restricted to primes $p$, and $\chi_0$ is the principal character mod $p$.

Let $L_1(s, \chi) = \sum_{n=1}^{\infty} a(n)\chi(n)n^{-s}$. If $L_1(s)$ satisfies (1) and (2), then we propose the following

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Main Conjecture. For \( h = 2 \) (\( x^2 = x_0 \neq x_0 \), \( x \mod p \)):

\[
S(X; 2) = \sum_{\chi \in \chi} L_1(k/2, \chi)
\approx \sum_{\chi \in \chi} \frac{1 + \chi(p)(-N)}{\Gamma(k/2)} \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{L_1(k/2 + z)}{\zeta (k/2) \zeta (k/2 + 1)} \frac{(\sqrt{N}/2\pi p)^{k/2}}{z} \chi(z) \text{d}z.
\]

For \( h > 2 \):

\[
S(X; h) \sim \sum_{\chi \in \chi} \frac{1 + \chi(p)\phi(h)}{\Gamma(k/2) \phi(h)} \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{L_1(k/2 + z)}{\zeta (k/2) \zeta (k/2 + 1)} \frac{(\sqrt{N}/2\pi p)^{k/2}}{z} \chi(z) \phi(h)^{1/2} \frac{dz}{z} \quad (c > 0),
\]

where

\[
H(z) = \sum_{n=1}^{\infty} \frac{\alpha(n)^2 \chi_0(n)}{n^{k/2}} + L_1(h(k/2 + z)), \chi_0)^2.
\]

The integrals in the above conjecture can be easily evaluated asymptotically by shifting the line of integration and computing the residues at \( z = 0 \). If we assume that \( L_1(s) \) has an Euler product of the form

\[
L_1(s) = \prod_{p \leq N} \left( 1 - \frac{a(p)}{p^s} \right)^{-1} \cdot \prod_{p \leq N} \left( 1 - \frac{\gamma_p}{p^s} \right)^{-1} \left( 1 - \frac{\gamma_p^*}{p^s} \right)^{-1},
\]

with \( |\gamma_p|^2 = p^{k-1} \) (see [1]), then we have the following

Proposition. Assuming the Main Conjecture and the Euler product (3), it follows that

(i) \( S(X; 2) \sim \frac{48\pi}{N} \prod_{p \leq N} \left( 1 - p^{k-2} \right)^{-1} \cdot \langle f, f \rangle \cdot \frac{X}{\log X} \),

where

\[
\langle f, f \rangle = \int \left( \sum_{n=1}^{\infty} a(n) e^{2\pi i n x} \right)^2 y^{k-2} \text{d}x \text{d}y
\]

is the Petersson inner product of \( f \) with itself;

(ii) For \( h > 2 \),

\[
S(X; h) \sim 2 \left( \frac{24\pi}{N} \prod_{p \leq N} \left( 1 + p^{k-1} \right)^{-1} \cdot \langle f, f \rangle \right)^{\phi(h)^{1/2}} \frac{X(\phi(h) \log X)^{\phi(h)/2 - 1}}{(\phi(h))!}.
\]

In the case that the weight \( k \) is 2, that (1), (2) and (3) are satisfied and the \( a(n) \) are rational, it is known (see [4, Theorems 7.14, 7.15]) that \( L_1(s) \) is the Hasse-Weil \( L \)-function of some elliptic curve \( E \), and \( \langle f, f \rangle \) can always be expressed as the product of an algebraic number, a power of \( \pi \), and the two periods of \( E \). For example (see [5]), when \( k = 2, N = 11 \) there is a unique cusp form of weight 2:

\[
f(z) = e^{2\pi i z} \sum_{n=1}^{\infty} \frac{(1 - e^{2\pi in})^2}{n^2} \prod_{n=1}^{\infty} (1 - e^{2\pi in})^2 = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z},
\]
and $L_1(s) = L_E(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ is the Hasse-Weil $L$-function of the elliptic curve $E: y^2 + y = x^3 - x^2$. For this curve, our conjectures take the following form:

$$S(X; 2) \sim \frac{11}{\pi} \Omega^+ \Omega^- \frac{X}{\log X},$$

$$S(X; h) \sim 2 \left( \frac{\Omega^+ \Omega^-}{\pi} \right)^{(h)/2} \frac{X(\phi(h) \log X)^{\phi(h)/2 - 1}}{(\frac{1}{2} \phi(h))!} (h > 2).$$

Here $\Omega^+ = 0.6346047 \ldots$ and $\Omega^- = 1.4588166 \ldots$ are the real period and the absolute value of the imaginary period of $E$, respectively.

Now let $p$ be an odd prime. The Hasse-Weil $L$-function of $E$ over the cyclotomic field $\mathbb{Q}(\sqrt[h]{T})$ is given by

$$L_{\mathbb{Q}(\sqrt[h]{T}); E}(s) = \prod_{\chi \mod p} L_E(s, \chi).$$

It is reasonable to expect that the average value (as $p$ varies) is given by

$$\text{average value of } L_{\mathbb{Q}(\sqrt[h]{T}); E}(1) \sim L_E(1) \cdot \frac{11}{\pi} \Omega^+ \Omega^- \prod_{\chi \mod p} \frac{2 \left( \frac{\phi(h)}{\pi} \Omega^+ \Omega^- \log p \right)^{(h)/2}}{(\frac{1}{2} \phi(h))!}.$$

Sharper forms of part (ii) of the Proposition can be derived on assuming the analytic continuation of $L_h(s)$ for $h > 2$. Then there will be extra terms involving lower powers of $\log X$, whose coefficients are expressible in terms of the special values of $L_h(s)$, some of which can be given by the conjectures of Deligne [2].

2. In order to lend credence to our Main Conjecture, we give the following arguments. Firstly, the conjecture for $S(X; 2)$ has already been dealt with in [3]. We therefore consider $S(X; h)$ for $h > 2$. For any prime $p \equiv 1 \pmod{n}$ there are $\phi(h)$ characters $\mod p$ of exact order $h$, and moreover these characters are all primitive. It follows by (2) that, for $p \nmid N$,

$$W = W(p, h) = \prod_{\chi \mod p} L_1(k - s, \chi),$$

where $\prod'$ means that the product is taken over all characters of exact order $h$, which we denote by $\chi_1, \ldots, \chi_{\phi(h)}$. By Lavrik's method (see [3]), we have

$$\prod' L_1(k/2, \chi) = \frac{1}{\Gamma(k/2)^{\phi(h)}} \sum_{n_1} \cdots \sum_{n_{\phi(h)}} a(n_1) \cdots a(n_{\phi(h)}) \chi_1(n_1) \cdots \chi_{\phi(h)}(n_{\phi(h)}) (n_1 \cdots n_{\phi(h)})^{-k/2}$$

(4)

$$= \frac{1}{\pi i} \int_{c-i\infty}^{c+i\infty} \left( \Gamma \left( \frac{k}{2} + z \right) \left( \frac{\sqrt[N]{N}}{2\pi p} \right)^z \right)^{\phi(h)} (n_1 \cdots n_{\phi(h)})^{-z} \frac{dz}{z}.$$

Now, summing over $p$ should give a lot of cancellation except for those $\phi(h)$-tuples
\[(n_1, \ldots, n_{\phi(h)})\] for which
\[
\prod_{i=1}^{\phi(h)} \chi_i(n_i) = 1
\]
for all \(p \nmid n_i \ (i = 1, \ldots, \phi(h))\). We can arrange the characters so that \(\chi_i = \bar{\chi}_{\phi(h) - i+1}\). It follows that
\[
\chi_i(n_i)\chi_{\phi(h)-i+1}(n_{\phi(h)-i+1}) = 1
\]
whenever \(n_i = n_{\phi(h) - i+1}\) and \(p \nmid n_i\). Also \(\chi_i(n_i) = 1\) whenever \(n_i = m_i^h\) is an \(h\)th-power and \(p \nmid m_i\). Combinations of these two cases are the only ways in which the aforementioned tuples can be constructed. Hence, every tuple \((n_1, \ldots, n_{\phi(h)})\) satisfying (5) is given as follows. Let \(0 \leq r \leq \frac{1}{2} \phi(h)\); choose an \(r\)-tuple \((i_1, \ldots, i_r)\) with \(1 \leq i_1 < \cdots < i_r \leq \frac{1}{2} \phi(h)\). Put \(n_{i_j} = n_{\phi(h) - i_j + 1} \ (j = 1, \ldots, r)\). Also let \(n_i = m_i^h\) be a perfect \(h\)th-power for \(i \neq i_j\). Since there are exactly \((\phi(h)/2)^r\) such \(r\)-tuples, it is reasonable to expect that, after summing (4) over primes \(p \equiv 1 \pmod{h}\), \(S(X; h)\) should be given asymptotically as
\[
S(X; h) \sim \sum_{p \equiv 1(h)} \frac{1 + W}{\Gamma(k/2)^{\phi(h)}} \sum_{\tau=0}^{\phi(h)/2} \left(\frac{1}{r}\right) \chi_{\phi(h)-i+1}(n_{\phi(h)-i+1}) \cdot \left(\sum_{n=1}^{\infty} \frac{a(n)^2 \chi_0(n)}{n^{k+2a}}\right) \cdot \left(\sum_{m=1}^{\infty} \frac{a(m^h)^2 \chi_0(m)}{m^{k(1/2+\varepsilon)}}\right)^{\phi(h)/2-2r} dz \frac{dz}{z}
\]
In order to derive the Proposition from the Main Conjecture, note that the Rankin-Selberg \(L\)-function \(\sum_{n=1}^{\infty} |a(n)|^2 n^{-s}\) has a simple pole at \(s = k\) with residue
\[
\frac{48\pi}{N} \prod_{p \mid N} (1 + p^{-1})^{-1} \cdot \langle f, f' \rangle.
\]
If \(L_1(s)\) satisfies (3), then the coefficients \(a(n)\) are real, and \(w_1(p) = \pm 1\). Also, \(L_2(s)\) is regular for \(\text{Re}(s) > 1 + h(k-1)/2\). Hence, for \(h > 2\), \(H(z)\) is regular for \(\text{Re}(z) > \frac{1}{h} - \frac{1}{2} - \frac{1}{h} \) except for a simple pole at \(z = 0\), with residue
\[
\frac{24\pi}{N} \prod_{p \mid N} (1 + p^{-1})^{-1} \cdot \langle f, f' \rangle.
\]
Shifting the line of integration to \(c = \frac{1}{h} - \frac{1}{2} + \varepsilon \ (0 < \varepsilon < \frac{1}{2} - \frac{1}{h})\) and using standard estimates for the growth of the Rankin-Selberg \(L\)-function, the proposition follows on computing the main term of the residue and applying the prime number theorem for arithmetic progressions.

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DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139

ISTITUTO DI MATEMATICA, UNIVERSITÀ DI PISA, PISA, ITALY