WEAKENING THE TOPOLOGY OF A LIE GROUP

BY

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Abstract. With any topological group \((G, \mathcal{U})\) one can associate a locally arcwise-connected group \((G, \mathbb{V})\), where \(\mathbb{V}\) is stronger than \(\mathcal{U}\). \((G, \mathcal{U})\) is a weakened Lie \((WL)\) group if \((G, \mathbb{V})\) is a Lie group. In this paper the author shows that the WL groups with which a given connected Lie group \((L, \mathcal{F})\) is associated are completely determined by a certain abelian subgroup \(H\) of \(L\) which is called decisive. If \(L\) has closed adjoint image, then \(H\) is the center \(Z(L)\) of \(L\); otherwise, \(H\) is the product of a vector group \(V\) and a group \(J\) that contains \(Z(L)\). \(J/Z(L)\) is finite (trivial if \(L\) is solvable). We also discuss the connection between these theorems and recent results of Goto.

1. Introduction. Gleason and Palais [1] have shown how to associate with any topological group \((G, \mathcal{U})\) a locally arcwise-connected group \((G, \mathbb{V})\) and proved that the group thus associated with a finite-dimensional metric group must be Lie. In this paper we describe the groups for which the associated locally arcwise-connected group is a connected Lie group. As we shall see in §3, this problem is equivalent to the following question: Given a connected Lie group \((L, \mathcal{F})\), in what ways can \(\mathcal{F}\) be weakened and remain Hausdorff? Our principal result is that \(L\) contains an abelian subgroup \(H\) which is decisive in the sense that the ways in which \(\mathcal{F}\) can be weakened and remain Hausdorff are completely determined by the ways in which the relative topology for \(H\) can be weakened while remaining Hausdorff and keeping finitely many characters of \(H\) continuous. The nature of \(H\) depends upon a crucial distinction between \((CA)\) analytic groups (those with closed adjoint image) and non-\((CA)\) analytic groups. Our proof in the latter case employs a homomorphism used by Goto [3] and relies upon structure theorems of Goto [4] and Zerling [18]. The connection between the present paper and certain results which were recently obtained by Goto [5] is discussed in §8. We also note that Hudson’s examination of arcwise-connected, finite-dimensional groups [7] leads him to consider, from a different perspective, questions similar to those studied here.

2. Notation and conventions. A topology \(\mathcal{U}\) for an abstract group \(G\) will be assumed to make the function \(f: G \times G \to G\) given by \(f(x, y) = xy^{-1}\) continuous, but \(\mathcal{U}\) need not be Hausdorff. For a subgroup \(H\) of \(G\), \(\mathcal{U}_H\) will denote the relative topology.
induced by $\mathcal{U}$. Since several topologies for the same abstract group $G$ may be under simultaneous consideration, topological statements about $G$ will contain references to the particular topology involved. If $(G, \mathcal{U})$ is a topological group, then $T(G, \mathcal{U})$ will denote the collection of all Hausdorff group topologies for $G$ which are weaker than $\mathcal{U}$. Analytic groups, but not Lie groups, are always connected, and Lie groups do not necessarily satisfy the second axiom of countability.

$\mathbb{Z}$, $\mathbb{R}$, $T$, and $\text{GL}(V)$ denote, respectively, the integers, the real numbers, the $q$-dimensional toroid, and the linear automorphism group of a vector space $V$. $T$, $\text{GL}(V)$, and finite groups will be assumed to have their usual topologies, unless stated otherwise. The relative topology for a subgroup $A$ of $\text{GL}(V)$ will be called the full-linear-group ($f\ell g$) topology for $A$. The symbol $\square$ marks the end of a proof.

3. Preliminary definitions and main results. Let $(G, \mathcal{T})$ be a topological group. According to [1, pp. 634–635], the collection of all $\mathcal{T}$-arc-components of $\mathcal{T}$-open subsets of $G$ is the basis for a locally arcwise-connected group topology $\mathcal{T}^*$ for $G$. $(G, \mathcal{T}^*)$ is the locally arcwise-connected (l.a.c.) group associated with $(G, \mathcal{T})$. For easy reference, we list here some of the properties of l.a.c. groups that are proved in [1].

3.1. Theorem (Gleason and Palais [1, 3.2, 4.3, 7.3]). Let $(G, \mathcal{T})$ be a Hausdorff topological group. Then

(i) $\mathcal{T}^*$ is stronger than $\mathcal{T}$, and $(\mathcal{T}^*)^* = \mathcal{T}^*$.

(ii) $(G, \mathcal{T})$ and $(G, \mathcal{T}^*)$ have the same arcs.

(iii) The $\mathcal{T}^*$-connected components of a $\mathcal{T}^*$-open subset $X$ of $G$ are the $\mathcal{T}$-arc-components of $X$.

(iv) If $(G, \mathcal{T})$ is a second-countable Lie group and $\mathcal{U} \in T(G, \mathcal{T})$, then $\mathcal{U}^* = \mathcal{T}$.

(v) If $(G, \mathcal{T})$ is separable, metrizable, and of finite topological dimension, then $(G, \mathcal{T}^*)$ is a Lie group.

Hudson observes that the assumption of metrizability in (v) may be dropped if by “dimension” one means cohomological dimension [6, p. 68]. We will say that $(G, \mathcal{T})$ is a weakened Lie (WL) group if $(G, \mathcal{T}^*)$ is a Lie group and $\mathcal{T}$ is Hausdorff. From (iv) we see that the WL groups with which a given analytic group $(L, \mathcal{T})$ is associated are all those of the form $(L, \mathcal{U})$, where $\mathcal{U} \in T(L, \mathcal{T})$.

We now introduce the notion of “decisiveness,” which is central to our main result. Let $(A, \mathcal{A})$ be a topological group with an abstract subgroup $B$, and let $I(b)$ denote conjugation by an element $b$ of $B$. (Recall that group topologies are not assumed to be Hausdorff.) If $\mathcal{B}$ is a topology for $B$ that makes the function $f: A \times B \to A$, defined by $f(a, b) = bab^{-1}$, $(\mathcal{A} \times \mathcal{B}, \mathcal{A})$-continuous, then $\mathcal{A} \times \mathcal{B}$ is a group topology for the semidirect product $A \times _{I} B$. We define a homomorphism $\alpha$ of $A \times _{I} B$ onto $A$ by $\alpha(a, b) = ab$, and the unique topology for $A$ which makes $\alpha$ continuous and open is called the standard extension of $\mathcal{B}$ to $A$, denoted $\mathcal{E}(\mathcal{B})$. We observe that a basis for the neighborhoods of the identity in $\mathcal{E}(\mathcal{B})$ is the collection of all $PN$, where $P$ and $N$ are, respectively, $\mathcal{A}$- and $\mathcal{B}$-neighborhoods of $e$. $\mathcal{E}(\mathcal{B})$ will be Hausdorff if $\mathcal{A}$ is Hausdorff, $B$ is $\mathcal{A}$-closed, and $\mathcal{B} \in T(B, \mathcal{A}_B)$.

When $\mathcal{B}$ is weaker than $\mathcal{A}_B$, $\mathcal{E}(\mathcal{B})$ may be thought of as “weakening the topology
of $A$ by weakening that of $B$; we are interested in situations where this is the only way in which $\mathcal{A}$ can be weakened. If $B$ is $\mathcal{A}$-closed and if $\mathcal{U} = \mathcal{E}(U_B)$ for every $U \in T(A, \mathcal{A})$, we will say that $B$ is decisive in $(A, \mathcal{A})$. We may now state our main result, the proof of which is contained in §§4-6.

3.2. MAIN THEOREM. Let $(L, \mathcal{T})$ be a connected Lie group with Lie algebra $l$.

(i) $L$ contains an abelian decisive subgroup $H$ of the form $\mathbb{R}^p \times T^q \times \mathbb{Z}^r \times D$, where $p, q, r$ are nonnegative integers and $D$ is finite. The adjoint image of $H$ is contained in a toroid $Q$.

(ii) The WL groups with which $(L, \mathcal{T})$ is associated are precisely those of the form $(L, \mathcal{E}(\mathcal{B}))$, where $\mathcal{B} \in T(H, \mathcal{T}_H)$ and the restriction of the adjoint representation $\text{Ad}: H \to Q$, is $\mathcal{B}$-continuous.

(iii) If $(L, \mathcal{T})$ is (CA)---that is, if $\text{Ad}(L)$ is a $\mathcal{U}$-closed subgroup of $\text{Gl}(l)$---then $H$ is the center of $L$.

(iv) If $(L, \mathcal{T})$ is not (CA), then $H = J \times V$, where $V$ is a vector group, $J$ contains the center $Z(L)$ of $L$, and $J/Z(L)$ is finite (trivial if $L$ is solvable).

We conclude this section with an important lemma about standard extensions.

3.3. LEMMA. Let $(A, \mathcal{A})$ be a topological group with an abstract subgroup $B$. Let $\mathcal{C}$ be the collection of all topologies $\mathcal{A}'$ for $A$ that are weaker than $\mathcal{A}$ and such that the function $h: A \times A \to A$ given by $h(a, g) = gag^{-1}$ is $(\mathcal{A} \times \mathcal{A}', \mathcal{A})$-continuous. If $\mathcal{U} \in \mathcal{C}$ and $\mathcal{U} = \mathcal{E}(U_B)$, then $\mathcal{A}' = \mathcal{E}(\mathcal{A}'_B)$ for every $\mathcal{A}'$ in $\mathcal{C}$ which is stronger than $\mathcal{U}$.

PROOF. By hypothesis, the homomorphism $\alpha: A \times B \to A$ given by $\alpha(a, b) = ab$ is $(\mathcal{A} \times \mathcal{U}_B, \mathcal{U})$-open and continuous. Let $\mathcal{A}' \in \mathcal{C}$ be stronger than $\mathcal{U}$, and let $\{g_i: i \in I\}$ be a net in $A$ which $\mathcal{A}'$-converges to $e$. Then $\{g_i\}$ also $\mathcal{U}$-converges to $e$ and thus has a subnet $\{g_{i(j)}: j \in J\}$ which lifts to a net $\{(a_j, b_j): j \in J\}$ in $A \times B$ that is $\mathcal{A} \times \mathcal{U}_B$-convergent to $e$. Since $b_j = a_j^{-1}g_{i(j)}$ and $\mathcal{A}'$ is weaker than $\mathcal{A}$, it follows that $b_j \to e$ in $\mathcal{A}'$, and thus $\alpha$ is $(\mathcal{A} \times \mathcal{A}'_B, \mathcal{A}')$-open. $\square$

4. DECISIVE SUBGROUPS. The purpose of this section is to prove the first three parts of Main Theorem 3.2; the proof of part (iv) is reserved for §§5 and 6. We begin by proving that every WL group has a continuous adjoint representation.

Let $(G, \mathcal{U})$ be a WL group, and let $L$ be the $\mathcal{U}$-arc-component of the identity. By 3.1, $(L, (\mathcal{U}^*)_L)$ is an analytic group (whose Lie algebra we denote by $l$) and $(\mathcal{U}^*)_L = (\mathcal{U}_L)^*$. $G$ acts by inner automorphism on $L$, and we denote by $I$ and $\text{Ad}$ the corresponding homomorphisms of $G$ into $\text{Aut}(L)$ and $\text{Gl}(l)$. If $\text{Aut}(L)$ has the generalized compact-open topology, then $I$ and $\text{Ad}$ are $\mathcal{U}^*$-continuous. We now show that they are also $\mathcal{U}$-continuous.

4.1. LEMMA. Let $\lambda: [0, 1] \to L$ be a $\mathcal{U}^*_L$-arc with initial point $e$ and let $\{x_j: j \in J\}$ be a net in $G$ which $\mathcal{U}$-converges to $x \in G$. Then $\{x_j \lambda(t)x_j^{-1}: j \in J\}$ $\mathcal{U}^*_L$-converges to $x\lambda(t)x^{-1}$, uniformly in $t$.

PROOF. Let $N$ be the $\mathcal{U}_L$-arc-component of $e$ in a $\mathcal{U}_L$-open neighborhood $U$ of $e$, and let $U_1$ be a $\mathcal{U}_L$-open neighborhood of $e$ such that $U_1 \subseteq U$. For each $j \in J$, define
a $\mathcal{U}$-arc $\beta_j : [0, 1] \to L$ by

$$\beta_j(t) = x\lambda(-t)x^{-1}x_j\lambda(t)x_j^{-1}.$$  

For each $s$ in $[0, 1]$, there exist an open neighborhood $I_s$ of $s$ and an element $j_s$ of $J$ such that

1. $x\lambda(-s)x^{-1}x_j\lambda(t)x_j^{-1} \in U_1$

and

2. $x\lambda(-t)\lambda(s)x^{-1} \in U_1$

whenever $t \in I_s$ and $j \geq j_s$. From (1) and (2) we see that $\beta_j(t)$ is in $U$ if $t \in I_s$, $j \geq j_s$. By compactness of the unit interval, there is a $j^* \in J$ such that if $j > j^*$, then $\beta_j(t) \in U$ for all $t \in [0, 1]$ and for all $j \geq j^*$. For such $j$, $\beta_j$ will then be an arc in $U$ that contains the identity, and therefore $\beta_j$ is in fact an arc in $N$. Thus $\beta_j(t) \in N$ for all $t \in [0, 1]$ and for all $j \geq j^*$, and the lemma is proved. □

4.2. PROPOSITION. $\text{Ad} : G \to \text{Gl}(l)$ and $I : G \to \text{Aut}(L)$ are $\mathcal{U}$-continuous.

Proof. Let $\{x_j : j \in J\}$ be a net in $G$ which $\mathcal{U}$-converges to $e$, and let $W$ be an open neighborhood of $0$ in $l$ such that the exponential mapping $\exp : l \to L$ is a diffeomorphism on $2W$. To prove that $\text{Ad}$ is $\mathcal{U}$-continuous, it is sufficient to show that $\text{Ad}(x_j)(w) \to w$ for any $w \in W$. From 4.1 we know that there is an element $j^*$ of $J$ such that, if $j \geq j^*$, then

$$\exp(\text{Ad}(x_j)(w)) = x_j(\exp tw)x_j^{-1} \in \exp W$$

for all $t \in [0, 1]$, and it follows that $\text{Ad}(x_j)(w) \in W$ if $j \geq j^*$. Lemma 4.1 also implies that, in $\mathcal{U}_l^*$,

$$\exp(\text{Ad}(x_j)(w)) = x_j(\exp w)x_j^{-1} \to \exp w.$$  

Since $\exp$ is a diffeomorphism on $W$, we conclude that $\text{Ad}(x_j)(w) \to w$ in $l$, as desired.

To prove that $I$ is $\mathcal{U}$-continuous, we simply observe that the differential operator $d$ is a topological group isomorphism of $\text{Aut}(L)$ onto a subgroup of $\text{Gl}(l)$. This completes the proof of 4.2. □

We now deduce an important criterion by which decisive subgroups of analytic groups may be identified.

4.3. LEMMA. Let $(L, \mathcal{S})$ be an analytic group, $B$ a $\mathcal{S}$-closed subgroup of $L$, and $\mathcal{U}$ the weakest (not necessarily Hausdorff) topology for $L$ that makes $\text{Ad} \mathcal{U}$-continuous. If $\mathcal{U} = \mathcal{S}(\mathcal{U}_B)$, then $B$ is decisive in $L$.

Proof. According to 3.1 and 4.2, every topology in $T(L, \mathcal{S})$ is stronger than $\mathcal{U}$. We may then apply 3.3, with $A = L$ and $\mathcal{A} = \mathcal{S}$. □

If $(L, \mathcal{S})$ is a (CA) analytic group, then $\text{Ad}(L)$ is closed in $\text{Gl}(l)$. If $(L, \mathcal{S})$ is not (CA), then $\text{Gl}(l)$ contains a toral subgroup $Q$ such that the $\text{flg}$-closure $C$ of $\text{Ad}(L)$ equals $\text{Ad}(L) \cdot Q$. (See, for example, Goto [2, Theorem 1].) Thus in either case, $C = \text{Ad}(L) \cdot Q$, where $Q$ is a (possibly trivial) toral subgroup of $\text{Gl}(l)$.  

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4.4. Proposition. Let \((L, \mathcal{F})\) be an analytic group with Lie algebra \(l\), let \(C\) be the flg-closure of \(\text{Ad}(L)\), and let \(Q\) be a (possibly trivial) toral subgroup of \(\text{Gl}(l)\) such that \(C = \text{Ad}(L) \cdot Q\). Then \(\text{Ad}^{-1}(Q)\) is an abelian decisive subgroup of \(L\) and has the form \(\mathbb{R}^p \times T^q \times \mathbb{Z}^r \times D\), where \(p, q,\) and \(r\) are nonnegative integers and \(D\) is finite.

Proof. Let \(H = \text{Ad}^{-1}(Q)\). Because \(d: \text{Aut}(L) \rightarrow d(\text{Aut}(L))\) is a topological isomorphism whose image is flg-closed and includes \(\text{Ad}(L)\), there is a toral subgroup \(K\) of \(\text{Aut}(L)\) such that \(C = dK\), and thus \(H = I^{-1}(K)\). According to Goto [3, Lemma 4], each automorphism in \(K\) leaves \(H\) pointwise fixed. In particular, since \(I(H) \subseteq K\), \(H\) is abelian and therefore has the form \(\mathbb{R}^p \times T^q \times E\), where \(p\) and \(q\) are nonnegative integers and \(E\) is discrete. We now invoke Theorem 1' and the subsequent remark in Mostow [8] to show that \(E\) is finitely generated and thus equals \(\mathbb{Z}^r \times D\), where \(r\) is a nonnegative integer and \(D\) is finite.

It remains to show that \(H\) is decisive in \(L\). If \(\mathcal{U}\) is the weakest topology for \(L\) which makes \(\text{Ad} (\mathcal{U}, \text{flg})\)-continuous, then it suffices, by 4.3, to show that \(\mathcal{U} = \mathcal{E}(\mathcal{U}_H)\). To do so, we form the semidirect product \(L \rtimes K\) and define a homomorphism \(\phi: L \rtimes K \rightarrow C\) by \(\phi(x, k) = \text{Ad}(x) \cdot dk\) for \(x \in L, k \in K\). If \(\mathcal{W}\) is the topology which \(K\) inherits from the generalized compact-open topology, then \(\phi\) is surjective, \((\mathcal{F} \times \mathcal{W}, \text{flg})\)-continuous, and thus \((\mathcal{F} \times \mathcal{W}, \text{flg})\)-open onto its image \(C\). It follows that a basis for the neighborhoods of \(e\) in the flg-topology for \(C\) is the collection of all \(\text{Ad}(P) \cdot dN\), where \(P\) is a \(\mathcal{F}\)-neighborhood of \(e\) in \(L\) and \(N\) is a \(\mathcal{W}\)-neighborhood of \(e\) in \(K\), and therefore a basis for the \(\mathcal{U}\)-neighborhoods of \(e\) is the collection of all \(\text{Ad}^{-1}(\text{Ad}(P) \cdot dN)\). Now for each such \(P\) and \(N\), the fact that \(\text{Ad}\) is a homomorphism implies that

\[
\text{Ad}^{-1}(\text{Ad}(P) \cdot dN) = P \cdot \text{Ad}^{-1}(dN).
\]

Since \(\text{Ad}^{-1}(dN)\) is simply a \(\mathcal{U}_H\)-neighborhood of \(e\), we have shown that \(\mathcal{U} = \mathcal{E}(\mathcal{U}_H)\). □

Proposition 4.4 proves part (i) of the Main Theorem, and we may now proceed to prove parts (ii) and (iii). Let \(H\) be any abelian decisive subgroup of \((L, \mathcal{F})\). We will call a topology \(\mathcal{B}\) for \(H\) allowable if \(\mathcal{B} \in T(H, \mathcal{F}_H)\) and the restriction \(\text{Ad}: H \rightarrow \text{Gl}(l)\) is \(\mathcal{B}\)-continuous. If \((L, \mathcal{U})\) is a Hausdorff topological group and \(\mathcal{U}^* = \mathcal{F}\), then \(\mathcal{U} = \mathcal{E}(\mathcal{U}_H)\), and by 4.2 \(\mathcal{U}_H\) is an allowable topology for \(H\). On the other hand, if \(\mathcal{B}\) is such a topology then \(\mathcal{E}(\mathcal{B})\) is a group topology for \(L\) which is in \(T(L, \mathcal{F})\) and by 3.1(iv) \((L, \mathcal{F})\) is the l.a.c. group associated with \((L, \mathcal{E}(\mathcal{B}))\). To prove part (iii) of 3.2, we simply note that when \((L, \mathcal{F})\) is (CA) we may choose the trivial toroid for \(Q\), so that the decisive subgroup \(\text{Ad}^{-1}(Q)\) will be the center of \(L\).

5. Subgroups of \(\text{Gl}(n, \mathbb{R})\). Before proving part (iv) of the Main Theorem, which is the purpose of §6, we must examine in detail the structure of the flg-closure of \(\text{Ad}(L)\) when \((L, \mathcal{F})\) is not (CA). The basis for our discussion is a result in Goto [4], which we now describe. Let \((G, \mathcal{G})\) be an analytic subgroup of \(\text{Gl}(n, \mathbb{R})\) which is not flg-closed, with \(C\) denoting the flg-closure of \(G\). Let \(N\) be any subgroup of \(G\) which is maximal among those that are \(\mathcal{G}\)-connected, flg-closed, and contain the commutator subgroup \(D\) of \(G\). (Such groups exist because, by Lemma 7 in [2], \(D\) is flg-closed.) If \(T_1\) is the radical of a maximal flg-compact subgroup of \(C\), let \(T_2\) be the flg-connected component of the identity in \(N \cap T_1\) and \(T\) a toroid such that \(T_1 = T_2 \cdot T, T_2 \cap T = \{e\}\).
In our notation, Goto's theorem may be stated as follows:

5.1. **Theorem (Goto [4, p. 197]).** $N \cap T$ is finite, $C = N \cdot T$, and $G$ contains a $\mathfrak{V}$-closed vector subgroup $W$, the $\mathfrak{flg}$-closure of which is $T$, such that $G = N \cdot W$, $N \cap W = \{e\}$. $C$ is $(\mathfrak{flg}, \mathfrak{flg} \times \mathfrak{V})$-diffeomorphic with $T \times N$.

Although the groups $N$ and $T$ are not in general unique, we will prove the following.

5.2. **Proposition.** If $N$ and $T$ are chosen in the manner described, then

(i) the dimensions of $N$, $W$, and $T$ are uniquely determined;

(ii) the finite groups $N \cap T$ are all isomorphic;

(iii) $N \cap T$ is trivial if $G$ is solvable.

The proposition depends upon a lemma about abelian analytic groups, the straightforward proof of which we omit.

5.3. **Lemma.** Let $(A, \mathfrak{A})$ be an abelian analytic group with a dense analytic subgroup $(B, \mathfrak{B})$. Let $C$ be a maximal $\mathfrak{B}$-connected and $\mathfrak{A}$-closed subgroup of $B$. Then the dimension of $C$ equals the sum of the dimensions of the vector part of $(A, \mathfrak{A})$ and the compact part of $(B, \mathfrak{B})$, and the intersection of $C$ with the compact part of $(A, \mathfrak{A})$ is independent of the choice of $C$.

**Proof of 5.2.** Let $\mathcal{U}'$ and $\mathcal{F}'$ denote the quotient topologies for $G/D$ and $C/D$ obtained from $\mathcal{U}$ and the $\mathfrak{flg}$-topology. To prove (i) it will suffice to show that $N/D$ is maximal among the subgroups of $G/D$ which are $\mathcal{U}'$-connected and $\mathcal{F}'$-closed, for then 5.3 will assure that the dimension of $N/D$, and thus of $N$, is uniquely determined ($N/D$ must be $\mathcal{F}'$-closed because it is the inverse image of the identity under the projection $C/D \to C/N$, which is continuous if each group is given the quotient topology from the $\mathfrak{flg}$-topology.) That $N/D$ is, in fact, maximal follows from the maximality of $N$ and the $\mathfrak{V}$-connectedness of $D$.

To prove (ii), we first show that the intersection $N \cap T_1$ is, for given $T_1$, independent of the choice of $N$. For if $N_1$ and $N_2$ are two choices for $N$, then $N_1/D$ and $N_2/D$ are, as we have seen, maximal among the $\mathcal{U}'$-connected subgroups of $G/D$ which are $\mathcal{F}'$-closed, and so 5.3 implies that the intersections of $N_1/D$ and $N_2/D$ with the image of $T_1$ in $C/D$ are equal. Therefore $N_1 \cap DT_1 = N_2 \cap DT_1$, whence $N_1 \cap T_1 = N_2 \cap T_1$.

Since $T_1$, as the radical of a maximal $\mathfrak{flg}$-compact subgroup of $C$, is determined up to conjugation by an element of $C$, and since $N$ is normal in $C$, the groups $N \cap T_1$ are thus all $\mathfrak{flg}$-isomorphic, regardless of the choices of $N$ and $T_1$. We complete the proof of (ii) by showing that, for fixed $N$ and $T_1$, the isomorphism class of the finite group $N \cap T$ is independent of the choice of $T$. This follows from the fact that, for any choice of $T$, $N \cap T_1 = T_2(N \cap T)$ and $T_2 \cap (N \cap T) = \{e\}$.

Finally, we prove (iii). It clearly suffices to show that $N \cap T_1$ is $\mathfrak{flg}$-connected when $G$ is solvable. Since $G$ and $C$ have the same commutator subgroup $D$, $C$ also is solvable. Applying Proposition 2.4 of Van Est [17], we find that $G = B \cdot A$, where $B$ is a $\mathfrak{flg}$-closed and simply-connected group containing $D$, the $\mathfrak{flg}$-closure $F$ of $A$ is a toroid, $C = B \cdot F$, and $B \cap F = \{e\}$. We may choose $N$ and a maximal $\mathfrak{flg}$-compact subgroup
K of C in such a way that $B \subseteq N, F \subseteq K$. That the latter inclusion is in fact an equality follows from the easily verified equation $K = (B \cap K) \cdot F$ and the fact that $B$, as a simply-connected and solvable Lie group in the flg-topology, contains no nontrivial compact subgroups. We may therefore let $T_1$ equal $F$, and the proof is completed by observing that $N \cap F$ must be flg-connected, since $N = B(N \cap F), N$ and $B$ are flg-connected, and $B \cap (N \cap F) = B \cap F$ is trivial. □

Applying 5.1 and 5.2 to the adjoint image of a non-(CA) analytic group, we obtain the following.

5.4. Corollary. Let $(L, \mathcal{F})$ be a non-(CA) analytic group, $\mathcal{U}$ the Lie topology for $\text{Ad}(L)$, and $C$ the flg-closure of $\text{Ad}(L)$. If $N$ is a maximal $\mathcal{U}$-connected and flg-closed subgroup of $\text{Ad}(L)$ containing the commutator subgroup of $\text{Ad}(L)$, $T_1$ is the radical of a maximal flg-compact subgroup of $C$, and $T$ is a toroid in $T_1$ complementary to the flg-connected component of the identity in $N \cap T_1$, then $\text{Ad}(L)$ contains a $\mathcal{U}$-closed vector subgroup $W$, whose flg-closure is $T$, such that $\text{Ad}(L) = N \cdot W$, $C = N \cdot T$, and $N \cap W$ is trivial. Moreover, $N \cap T$ is a finite group whose isomorphism class does not depend on the particular choices of $N$ and $T$. $N \cap T$ is trivial if $\text{Ad}(L)$ is solvable, and the dimensions of $N, W,$ and $T$ are independent of the choices of $N$ and $T$.

6. The non-(CA) case. The proof of part (iv) of the Main Theorem, which is contained in this section, relies not only upon 5.4 but also upon Zerling's structure theorem for non-(CA) analytic groups [18], which says that every such group is the semidirect product of a (CA) analytic group and a vector group. More precisely, the relevant portion of Zerling's results may be stated as follows.

6.1. Theorem (Zerling [18, Theorem 2.1]). With notation as in 5.4, let $P$ be the $\mathcal{F}$-connected component of the identity in $\text{Ad}^{-1}(N)$. Then $(P, \mathcal{F}_P)$ is (CA) and contains the center of $L$, and $\text{Aut}(P)$ contains a vector subgroup $V$ with compact closure such that, if $\mathcal{W}$ is the vector topology for $V$, then $P \otimes V$ is $(\mathcal{F}_P \times \mathcal{W}, \mathcal{F})$-isomorphic with $L$. If we identify $L$ with $P \otimes V$, then $\text{Ad}(V) = W$, and $V$ and $W$ have the same dimension.

Adopting the notation of 5.4 and 6.1, we now combine these results with 4.4 to prove part (iv) of the Main Theorem. Since $C = \text{Ad}(L) \cdot T$ by 5.4, it follows from 4.4 that $H = \text{Ad}^{-1}(T)$ is an abelian decisive subgroup of $L$. If we let $J$ equal $P \cap \text{Ad}^{-1}(N \cap T)$, a trivial computation verifies that $H = J \otimes V$. Since $H$ is abelian, $V$ must act trivially on $J$, whence $H = J \times V$. To complete the proof, we note that $\text{Ad}(P)$ must equal all of $N$, since $N \cap \text{Ad}(V) = N \cap W$ is trivial. Therefore $\text{Ad}(J) = N \cap T$ and $J/\text{Z}(L)$ is isomorphic to the finite group $N \cap T$. If $L$ is solvable, then so is $\text{Ad}(L)$, and 5.4 assures that $J/\text{Z}(L)$ is trivial. This completes the proof of the Main Theorem. We may also note that, according to 5.4 and 6.1, the dimension of $V$ and the isomorphism class of $J/\text{Z}(L)$ are independent of the particular choices of $N$ and $T$.

7. Examples. We now give an example of a non-(CA) analytic group whose center, in the notation of the Main Theorem, has index two in the group $J$. Define an action of $\text{SU}(2) \times T^2$ on $\mathbb{C}^3$ by letting $(A, e^{i\theta}, e^{i\phi})$, where $A \in \text{SU}(2)$, correspond to the $3 \times 3$ complex matrix in Figure 1. The semidirect product $G = \mathbb{C}^3 \otimes (\text{SU}(2) \times T^2)$
is then a Lie group in its usual topology. If \( \mu \) is some fixed irrational number, then \( L = C^3 \otimes (SU(2) \times \mathbb{R}) \) becomes a dense Lie subgroup of \( G \) by means of the injection \( \alpha: (z, A, t) \mapsto (z, A, e^{it}, e^{i\mu t}) \), \( z \in C^3, A \in SU(2), t \in \mathbb{R} \). \( L \) has trivial center and thus cannot, by Theorem 2.2.1 of Van Est [16], be (CA).

By considering the action of \( G \) on \( L \) by inner automorphism, one verifies that \( G/Z(G) \) is, in the quotient topology, isomorphic with the flg-closure of \( \text{Ad}(L) \), so that we may regard the adjoint representation of \( L \) as the composition of \( \alpha \) with the projection \( \pi: G \to G/Z(G) \). Letting \( N = \pi(C^3 \times SU(2) \times 1 \times 1) \) and \( T_1 = \pi(0 \times I \times T^2) \), we find that \( N \cap T_1 \) is a two-element group. Therefore \( T = T_1 \) and \( \text{Ad}^{-1}(T) = 0 \times J \times \mathbb{R} \), where \( J \) is the group generated by \(-I\).

\[
\begin{pmatrix}
e^{it}A & 0 & 0 \\
0 & 0 & e^{i\phi} \\
0 & 0 & 0
\end{pmatrix}
\]

**Figure 1**

We also observe that the reduction to the abelian case which the Main Theorem effects does not prevent WL groups from having rather peculiar topologies. Although the author will undertake a systematic study of "unusual" topologies for abelian groups in a subsequent paper [15], we may note here that any sequence in \( \mathbb{R}^n \) or \( \mathbb{Z}^n \) which "goes to infinity sufficiently fast" will, in an appropriately weakened topology, converge to 0. For example,

\[
d(n, m) = \inf \{ \sum |c_i|/|n - m| = \sum c_i(i! + 1), c_i \in \mathbb{Z} \}
\]

defines a metric on \( \mathbb{Z} \) in which \( i! + 1 \to 0 \). Other examples of unusual topologies for \( \mathbb{Z} \) and \( \mathbb{R}^n \) can be found in [9-14].

8. Related results. After writing this paper, the author learned of related but independent results obtained by Goto [5], and in this section we will sketch the connection between his work and ours. Let \((L, \mathcal{T})\) and \((G, \mathcal{U})\) be, respectively, an analytic and a topological group, and let \( f: L \to G \) be a continuous, injective homomorphism. Although the course of Goto's analysis parallels our own in certain respects, his primary interest is a description of the set \( f(L) \), while our Main Theorem can be viewed as a characterization of the topology of \( f(L) \). Changing Goto's notation to distinguish the \( gm \)-torus in [5] from our own decisive subgroup \( H \), we can summarize his principal results as follows: If \( \nu(J) \) is the vector part of the \( gm \)-torus \( J \), then \( \overline{f(L)} = f(L)\overline{f(\nu(J))} \), and \( f \) is an imbedding (i.e., a homeomorphism of \( L \) onto \( f(L) \)) if and only if \( f|\nu(J) \) is. We note that the equation \( \overline{f(L)} = f(L)\overline{f(\nu(J))} \) is a statement about sets, not topologies, and that it is trivially valid when \( f(L) = G \).

Now our decisive subgroup \( H \), besides revealing whether \( f \) is an imbedding, also
explicitly determines the topology of \( f(L) \) when \( f \) is not an imbedding; identifying \( L \) with \( f(L) \), we know that \( \mathcal{U}_L = \mathcal{S}(\mathcal{U}_H) \). The following example illustrates the relationship among \( H, J, \) and \( \nu(J) \). If \( L = \mathbb{R} \times T^1 \times SU(2) \), then \( H = \mathbb{R} \times T^1 \times \{ \pm I \} \), \( J = \mathbb{R} \times T^1 \times O(2) \), and \( \nu(J) = \mathbb{R} \times 1 \times I \). As the author will show in [15], there is a topology \( \mathcal{U} \) for \( L \), weaker than the usual topology, in which \( \{(n!, (-1)^n, I)\} \) converges to the identity. Clearly \( \mathcal{U} \) cannot equal \( \mathcal{S}(\mathcal{U}_v(J)) \), and thus \( \nu(J) \) is "too small" to determine the topology of \( L \); indeed, a theorem in [15] shows that in this case \( H \) is the smallest subgroup that will suffice. On the other hand, \( J \) is "too big", because it properly contains \( H \).

Finally, we note that Theorem 1 in [5] provides a somewhat more elegant proof of our 4.2 and that Theorem 2 in [5] is a sharper form of the theorem in [4] which we cited in §5. The latter makes possible the following improvement in our Main Theorem: When \( L \) is not (CA), we can alter the choice of \( Q \) to assure that \( H = Z(L) \times V \), even if \( L \) is not solvable. The details are contained in [15].

REFERENCES

7. ———, On the topology and geometry of arcwise connected finite-dimensional groups, Pacific J. Math. 82 (1979), 429–450.